A characterization of free finite rank $\mathbb{Z}_p$-modules

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The purpose of these notes is giving the proof of Proposition 1. We recall that, if G is a topological group and N is a normal open subgroup, then X is a set of *topological normal generators* of N if the set ${}^G X = \{gxg^{-1} \mid x \in X,\ g \in G\}$ generates N topologically.

Proposition 1. *Let p be a prime number and let G be a pro-p-group. Let moreover r be a positive integer. Then the following conditions are equivalent.*

1. *For every open normal subgroup N of G, a minimum set of topological normal generators of N has cardinality r.*

2. *The group G is isomorphic to $\mathbb{Z}_p^r$.*

We remark that the implication (2) $\Rightarrow$ (1) follows, among other things, from the fact that $\mathbb{Z}_p$ is a principal ideal domain. We will prove (1) $\Rightarrow$ (2).

Until the end of this paper, p will denote a prime number, r a positive integer, and G a pro-p-group. The following two lemmas are basic facts following from the definition of the commutator subgroup.

Lemma 2. *The following are equivalent.*

1. *For every open normal subgroup N of G, a minimum set of topological normal generators of N has cardinality r.*

2. *For every open normal subgroup N of G, the group $N/[G, N]$ is generated by exactly r elements.*

In particular, if condition (1) is satisfied, the group G generated by exactly r elements.

Lemma 3. *Let N be an open normal subgroup of G such that G/N is cyclic. Then $[G, G] = [G, N]$.*

Lemma 4. *Assume that every open normal subgroup of G has a minimum set of topological normal generators of cardinality r. Then $G/[G,G]$ is torsion-free.*

Proof. Assume by contradiction that the torsion subgroup of $\overline{G} = G/[G,G]$ is non-trivial. As a consequence of Lemma 2, there exists a positive integer n and an element $a = (a_1, \ldots, a_n)$ of $\mathbb{Z}_{>0}^n$ such that $\overline{G}$ is isomorphic to $\mathbb{Z}_p^{r-n} \oplus \bigoplus_{i=1}^n \mathbb{Z}/p^{a_i}\mathbb{Z}$. We call $\overline{N}$ the subgroup $\mathbb{Z}_p^{r-n} \oplus \bigoplus_{i\neq 1} \mathbb{Z}/p^{a_i}\mathbb{Z}$ of $\overline{G}$ and we observe that $\overline{G}/\overline{N}$ is cyclic. Let $\pi : G \to \overline{G}$ be the canonical projection and define $N = \pi^{-1}(\overline{N})$. By the isomorphism theorems, G/N and $\overline{G}/\overline{N}$ are isomorphic and thus N is an open normal subgroup of finite index in G. By Lemma 3, the groups $[G,N]$ and $[G,G]$ are the same and so $N/[G,N]$ is generated by $r-1$ elements. Contradiction to Lemma 2. ∎

Lemma 5. *Assume that every open normal subgroup of G has a minimum set of topological normal generators of cardinality r and let N be an open normal subgroup of G that contains $[G,G]$. Then $[G,N] = [G,G]$.*

Proof. Let $\pi : N/[G,N] \to N/[G,G]$ be the canonical projection. By Lemma 2, both $G/[G,G]$ and $N/[G,N]$ are generated by exactly r elements and, by Lemma 4 and $\mathbb{Z}_p$ being a principal ideal domain, the group $G/[G,G]$ is a free $\mathbb{Z}_p$-module of rank r. The subgroup N having finite index in G, the group $N/[G,G]$ is itself a free $\mathbb{Z}_p$-module of rank r. From the fact that $N/[G,N]$ is r-generated, it follows that $N/[G,N]$ is a free $\mathbb{Z}_p$-module of rank r. Since π is a surjective map between free $\mathbb{Z}_p$-modules of rank r (finite!), it is also injective. In particular, $0 = \ker\pi = [G,G]/[G,N]$. ∎

The following lemma gives the most interesting implication for the proof of Proposition 1.

Lemma 6. *Assume that every open normal subgroup of G has a minimum set of topological normal generators of cardinality r. Then G is isomorphic to $\mathbb{Z}_p^r$.*

Proof. Let $\mathcal{H}$ be the collection of all finite quotients of G and let $H \in \mathcal{H}$. Let moreover $\pi : G \to H$ be the canonical projection and observe that $N = \pi^{-1}([H,H])$ is an open normal subgroup of G. From Lemma 5 it follows that $[G,N] = [G,G]$ and therefore $[H,[H,H]] = \pi([G,N]) = \pi([G,G]) = [H,H]$. Since H is a p-group, the last equality implies that H is abelian and, given the arbitrary choice of H, any element of $\mathcal{H}$ is abelian. Being a profinite group, G is abelian itself and, as a consequence of Lemma 4, it is isomorphic to $\mathbb{Z}_p^r$. ∎