Images of Galois representations associated to Hida families

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Preliminaries

Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for each prime p.
Fix a classical Hecke eigenform $f \in S_k(\Gamma_0(N), \chi)$.

Modular form

$f = \sum_{n=1}^{\infty} a_n q^n$

$\mathcal{O}$: integral closure of $\mathbb{Z}[a_n : n \in \mathbb{Z}^+]$

$\mathfrak{p}$: prime of $\mathcal{O}$

$(p) = \mathfrak{p} \cap \mathbb{Z}$

Galois represenation

$\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$

- unramified outside Np
- $\mathrm{tr}\, \rho_{\mathfrak{p}}(\mathrm{Frob}_\ell) = a_\ell$ for all primes $\ell \nmid Np$
- $\det \rho_{\mathfrak{p}}(\mathrm{Frob}_\ell) = \chi(\ell)\ell^{k-1}$ for all primes $\ell \nmid Np$

Note: If $f = f_E$ for an elliptic curve $E/\mathbb{Q}$ then ρ_p is just the p-adic Tate module of E.

The Question

Question

What is the image of $\rho_{\mathfrak{p}}$?

Heuristic

The image of a Galois representation (such as $\rho_{\mathfrak{p}}$) should be as large as possible subject to the symmetries of the geometric object it arises from (such as f).

Symmetries

- We say f has *CM* if there is a non-trivial Dirichlet character η such that

$$a_\ell = \eta(\ell) a_\ell \text{ for almost all primes } \ell.$$

Henceforth we assume f does not have CM.

- We say an automorphism σ of $\mathcal{O}$ is a *conjugate self-twist* of f if there is a non-trivial Dirichlet character η_σ such that

$$a_\ell^\sigma = \eta_\sigma(\ell) a_\ell \text{ for almost all primes } \ell.$$

Ribet and Momose showed that these symmetries, together with the determinant of $\rho_\mathfrak{p}$, determine the image of $\rho_\mathfrak{p}$ up to finite error.

Images of classical modular Galois representations

Notation:
$\Gamma : \{\sigma \in \operatorname{Aut} \mathcal{O} : \sigma$ is a conjugate self-twist for $f\}$
$\mathcal{O}_0$: integral closure of $\mathbb{Z}$ in the field fixed by Γ
$H = \bigcap_{\sigma \in \Gamma} \ker \eta_\sigma$

Theorem (Ribet $k = 2$, Momose)

If f as above does not have CM then for all primes $\mathfrak{p}$ of $\mathcal{O}$

1. $\rho_{\mathfrak{p}}|_H$ *takes values in* $\mathrm{GL}_2(\mathcal{O}_{0,\mathfrak{p}})$*;*
2. $\operatorname{Im} \rho_{\mathfrak{p}}|_H$ *contains an open subgroup of* $\mathrm{SL}_2(\mathcal{O}_{0,\mathfrak{p}})$*; i.e.*

$$\operatorname{Im} \rho_{\mathfrak{p}}|_H \supseteq \Gamma_{\mathcal{O}_{0,\mathfrak{p}}}(\pi^r) = \{x \in \mathrm{SL}_2(\mathcal{O}_{0,\mathfrak{p}}) : x \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod \pi^r\}$$

for a uniformizer π of $\mathcal{O}_{0,\mathfrak{p}}$ and $r \geq 0$.

Hida Families

Fix a prime $p \geq 5$.
$\Lambda = \mathbb{Z}_p[[T]]$ (base ring; analogous to $\mathbb{Z}$)
For integers $k \geq 2$ we define the k-th *arithmetic prime* of Λ

$$P_k = (1 + T - (1+p)^k)\Lambda.$$

$\mathbb{I}$: integral domain finite flat over Λ (analogous to $\mathcal{O}$)

Definition (Hida family)

A formal power series $F = \sum_{n=1}^{\infty} A_n q^n \in \mathbb{I}[[q]]$ is a *Hida family* if $A_p \in \mathbb{I}^\times$ and for every $k \geq 2$ and every prime $\mathfrak{P}$ of $\mathbb{I}$ lying over P_k

- $F \bmod \mathfrak{P}$ has coefficients in $\overline{\mathbb{Q}}$ (rather than just $\overline{\mathbb{Q}}_p$)
- $F \bmod \mathfrak{P}$ gives the q-expansion of a classical modular form $f_{\mathfrak{P}}$ of weight k.

Hida Theory

Theorem (Hida)

1. *Every (p-ordinary) classical modular form of weight at least 2 can be put into a unique such family.*
2. *Furthermore, there is a representation $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I})$ such that for all $k \geq 2$ and every prime $\mathfrak{P}$ of $\mathbb{I}$ lying over P_k we have*

$$\rho_F \bmod \mathfrak{P} \cong \rho_{f_{\mathfrak{P}}}.$$

We can define CM and conjugate self-twist as in the classical case in terms of q-expansions:

- $A_\ell = \eta(\ell) A_\ell$ a.a. ℓ
- $A_\ell^\sigma = \eta_\sigma(\ell) A_\ell$ a.a. ℓ and $\sigma \in \mathrm{Aut}\,\mathbb{I}$

Notation:
Γ : {conjugate self-twists of F}
$\mathbb{I}_0$: integral closure of Λ in the field fixed by Γ
$H = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma$

Theorem (L.)

Let F be a non-CM Hida family such that $\rho_F \bmod \mathfrak{m}_{\mathbb{I}}$ is absolutely irreducible (+ small technical condition). Then

1. $\rho_F|_H$ *takes values in* $\mathrm{GL}_2(\mathbb{I}_0)$*;*
2. *There is a non-zero* $\mathbb{I}_0$*-ideal* $\mathfrak{a}$ *such that*

$$\operatorname{Im} \rho_F|_H \supseteq \Gamma_{\mathbb{I}_0}(\mathfrak{a}) = \{x \in \mathrm{SL}_2(\mathbb{I}_0) : x \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) \bmod \mathfrak{a}\}.$$

Proof: Lifting Twists

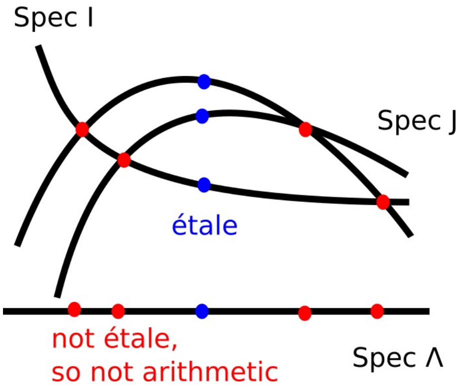
Theorem (L.)

Let $\mathfrak{P}$ be an arithmetic prime of $\mathbb{I}$ and σ a conjugate self-twist of $f_{\mathfrak{P}}$. If σ preserves the local field generated by the Fourier coefficients of $f_{\mathfrak{P}}$, then σ can be lifted to a conjugate self-twist $\tilde{\sigma}$ of F.

- Lift σ to a conjugate self-twist Σ of the (unrestricted) universal deformation of $\bar{\rho}_F$.
- Show that Σ preserves an appropriate Hida Hecke algebra. Thus $\operatorname{Spec} \mathbb{I}$ and $\Sigma^* \operatorname{Spec} \mathbb{I}$ are modular irreducible components intersecting at the arithmetic point $\mathfrak{P}$.

Étatleness of the Hecke algebra

Use the fact that the Hecke algebra is étale over Λ at arithmetic points to conclude that Σ descends to the desired automorphism $\tilde{\sigma}$ of $\mathbb{I}$.



Proof: Reduction Steps

P: arithmetic prime of Λ

Big image for classical Gal. reps. (Ribet/Momose)

$\Downarrow$ Lifting Theorem

$\operatorname{Im} \rho_F|_H \bmod \mathfrak{P}_0$ is open in $\mathrm{GL}_2(\mathbb{I}_0/\mathfrak{P}_0), \quad \forall \mathfrak{P}_0 | P$

$\Downarrow$ Goursat's Lemma argument

$\operatorname{Im} \rho_F|_H \bmod P$ is open in $\mathrm{GL}_2(\mathbb{I}_0/P\mathbb{I}_0) \sim \prod_{\mathfrak{P}_0|P} \mathrm{GL}_2(\mathbb{I}_0/\mathfrak{P}_0)$

Pink's Lie algebra $\Downarrow$ (ρ_F is p-ordinary)

Big image for Gal. reps. of Hida families (L.)

Thank you!