SCHUBERT CALCULUS FOR ALGEBRAIC COBORDISM

JENS HORNBOSTEL AND VALENTINA KIRITCHENKO

Abstract. We give an explicit formula for the push-forward morphism in algebraic cobordism for projective line fibrations. Using this formula, we establish a Schubert calculus for Bott-Samelson resolutions in the algebraic cobordism ring of a complete flag variety G/B.

1. Introduction

We fix a base field k of characteristic 0. Algebraic cobordism $\Omega^*(-)$ has been invented some years ago by Levine and Morel [12] as the universal oriented algebraic cohomology theory on smooth varieties over k. In particular, its coefficient ring $\Omega^*(k)$ is isomorphic to the Lazard ring $\mathbb{L}$ (introduced in [10]). In a recent article [13], Levine and Pandharipande show that algebraic cobordism $\Omega^n(X)$ allows a presentation with generators being projective morphisms $Y \to X$ of relative codimension $n(:= \dim(X) - \dim(Y))$ between smooth varieties and relations given by a refinement of the naive algebraic cobordism relation (involving double point relations). A recent result of Levine [11] which relies on unpublished work of Hopkins and Morel asserts an isomorphism $\Omega^n(-) \cong MGL^{2n,n}(-)$ between Levine-Morel and Voevodsky algebraic cobordism for smooth quasiprojective varieties. In particular, algebraic cobordism is representable in the motivic stable homotopy category.

In short, algebraic cobordism is to algebraic varieties what complex cobordism is to topological manifolds.

The above fundamental results being established, it is high time for computations, which have been carried out only in a very small number of cases (see e.g. [18] and [19]). The present article focuses on cellular varieties X, for which the additive structure of $\Omega^*(X)$ is easy to describe: it is the free $\mathbb{L}$-module generated by the cells (see the next section for more precise definitions, statements, proofs and references). So additively, algebraic cobordism for cellular varieties behaves exactly as Chow groups do. Of course, algebraic K-theory also behaves in a similar way, but we will restrict our comparisons here and below to Chow groups.

Things become interesting when investigating the ring structure of $\Omega^*(X)$. Let us concentrate on complete flag varieties $X = G/B$ where B is a Borel subgroup of a connected reductive group G over k. In the case where $G = GL_n(k)$, the cobordism ring $\Omega^*(X)$ may be described as the quotient of a free polynomial ring over $\mathbb{L}$ with generators x_i being the first Chern classes of certain line bundles on X and explicit relations. More precisely, we show (see Theorem 2.7):

Theorem 1.1. *The cobordism ring $\Omega^*(X)$ is isomorphic to the graded ring $\mathbb{L}[x_1, \ldots, x_n]$ of polynomials with coefficients in the Lazard ring $\mathbb{L}$ and* $\deg x_i = 1$*, quotient by the ideal S*

The second author would like to thank Jacobs University Bremen, the Hausdorff Center for Mathematics and the Max Planck Institute for Mathematics in Bonn for hospitality and support.

generated by the homogeneous symmetric polynomials of strictly positive degree:

$$\Omega^*(X) \simeq \mathbb{L}[x_1, \dots, x_n]/S.$$

This generalizes a theorem of Borel [2] on the Chow ring (or equivalently the singular cohomology ring) of a flag variety to its algebraic cobordism ring.

The Chow ring of the flag variety has a natural basis given by the *Schubert cycles*. The central problem in Schubert calculus was to find polynomials (later called Schubert polynomials) representing the Schubert cycles in the Borel presentation. This problem was solved independently by Bernstein–Gelfand–Gelfand [1] and Demazure [8] (most of the ingredients were already contained in a manuscript of Chevalley [6], which for many years remained unpublished). Explicit formulas for Schubert polynomials give an algorithm for decomposing the product of any two Schubert cycles into a linear combination of other Schubert cycles with integer coefficients. A still open problem in Schubert calculus for complete flag varieties is to give a meaningful combinatorial description of these coefficients similar to the one in the case of Grassmannians [14] (where analogous coefficients coincide with the celebrated Littlewood-Richardson coefficients).

The algebraic cobordism ring of the flag variety also has a natural basis given by the *Bott-Samelson resolutions* of the Schubert cycles (note that the latter are not always smooth and so, in general, do not define any cobordism classes). We give explicit formulas for the polynomials (now with coefficients in the Lazard ring $\mathbb{L}$) representing the classes of Bott-Samelson resolutions. Note that each such polynomial contains the respective Schubert polynomial as the lowest degree term (but in most cases also has non-trivial higher order terms). We also give an algorithm for decomposing the product of two Bott-Samelson resolutions into a linear combination of other Bott-Samelson resolutions with coefficients in $\mathbb{L}$.

Our main theorem is the following (compare Theorem 3.2). Let $I = (\alpha_1, \dots, \alpha_l)$ be an l-tuple of simple roots of G, and R_I the corresponding Bott-Samelson resolution of the Schubert cycle X_I (see Section 3 for the precise definitions). Recall that there is an isomorphism between the Picard group of the flag variety and the weight lattice of G such that very ample line bundles map to strictly dominant weights (see, for instance, [4, 1.4.3]). We denote by $L(\lambda)$ the line bundle on X corresponding to a weight λ, and by $c_1(L(\lambda))$ its first Chern class in algebraic cobordism. For each α_i, we define the operator A_i on $\Omega^*(X)$ in a purely algebraic way (see Section 3 for the rigorous definition for arbitrary reductive group). Informally, the operator A_i can be defined in the case $G = GL_n$ by the formula

$$A_i = (1 + \sigma_{\alpha_i})\frac{1}{c_1(L(\alpha_i))},$$

where σ_{α_i} acts on the variables $(x_1, \dots, x_n)$ by the transposition corresponding to α_i. Here we use that the Weyl group of GL_n can be identified with the symmetric group S_n so that the simple reflections s_{α_i} correspond to elementary transpositions (see Sections 3 and 5 for more details). Note that the $c_1(L(\alpha_i))$ can be written explicitly as polynomials in $x_1, \dots, x_n$ using the formal group law (see Section 5).

Theorem 1.2. *For any complete flag variety $X = G/B$ and any tuple $I = (\alpha_1, \dots, \alpha_l)$ of simple roots of G, the class of the Bott-Samelson resolution R_I in the algebraic cobordism ring $\Omega^*(X)$ is equal to*

$$A_l \dots A_1 R_e,$$

where R_e is the class of a point.

This theorem reduces the computation of the products of the geometric Bott-Samelson classes to the products in the polynomial ring given by the previous theorem. Note that in the cohomology case analogously defined operators A_i coincide with the operators defined in [1, 8], so our theorem generalizes [1, Theorem 4.1] and [8, Theorem 4.1] for Schubert cycles in cohomology and Chow ring, respectively, to Bott-Samelson classes in algebraic cobordism.

There is an article of Bressler-Evens [3] (we thank Burt Totaro from whom we first learned about this reference) which provides an algorithm for computing products of Bott-Samelson resolutions in (topological) complex cobordism. Some of its ingredients rely on results from homotopy theory. Using the techniques explained in the next section, it might be possible to deduce some of our results on algebraic cobordism from their results on complex cobordism. We will not attempt to do this. Instead, all our proofs are purely algebraic or algebro-geometric. Conversely, we note that all our proofs concerning algebraic cobordism ring of the flag variety may be easily translated to proofs for the analogue statements concerning the complex cobordism ring.

The main tool is our formula for the push-forward in algebraic cobordism for projective line fibrations. This formula, which is interesting in itself and might have further applications is stated in Proposition 2.1. Push-forwards (also called "transfers") for algebraic cobordism are considerably more intricate than the ones for Chow groups. Consequently, their computation, which applies to any orientable cohomology theory, is more complicated. One of its ingredients is the double point relation for algebraic cobordism established by Levine and Pandharipande [13].

The article [3] does not contain any computations. It would be interesting to do some computation using their algorithm and then comparing them with our approach, which we consider to be the easier one. (Note also that the notations of [3] are essentially consistent with [1], but not always with [14]. We rather stick to the former than to the latter.)

This paper is organized as follows. In the next section, we give some further background on algebraic cobordism and establish the formula for the push-forward mentioned above. We also describe the additive structure of the algebraic cobordism ring of cellular spaces. In the case of the flag variety for GL_n, we describe the multiplicative structure of its algebraic cobordism ring. In the third section, we recall the definition of Bott-Samelson resolutions and then state and prove our main theorem expressing the classes of Bott-Samelson resolutions as polynomials with coefficients in the Lazard ring. Section 4 contains some old and new results related to Chevalley-Pieri formulas and a short discussion of why the proof of [1] for singular cohomology does not carry over to algebraic cobordism. The final section contains some examples and explicit computations using the results of the previous sections. In particular, we give an explicit formula for the product of a Bott-Samelson class with the first Chern classes and an algorithm for computing the products of Bott-Samelson classes in terms of other Bott-Samelson classes.

Our main results are valid for the flag variety of an arbitrary reductive group G, but can be made more explicit in the case $G = GL_n$ using Borel presentation given by Theorem 2.7. So we will use the flag variety for GL_n as the main illustrating example whenever possible. One might conjecture that the algebraic cobordism rings of flag varieties with respect to other reductive groups G also allow a Borel presentation as polynomial rings over $\mathbb{L}$ in certain first Chern classes modulo the polynomials fixed by the appropriate Weyl groups (at least when passing to rational coefficients), because the corresponding statement is valid for singular cohomology resp. Chow groups (compare [2] resp. [7]).

After most of this preprint was finished, we learned that Calmes, Petrov and Zainoulline are also working on Schubert calculus for algebraic cobordism. It will be interesting to compare their results and proofs to ours.

We are grateful to Paul Bressler and Nicolas Perrin for useful discussions.

2. ALGEBRAIC COBORDISM GROUPS, PUSH-FORWARDS AND CELLULAR VARIETIES

We briefly recall the geometric definition of algebraic cobordism [13] and some of its basic properties as established in [12]. For more details see [12, 13]. Recall that (up to sign) any element in the algebraic cobordism group $\Omega^n(X)$ for a scheme X (separated, of finite type over k) may be represented by a projective morphism $Y \rightarrow X$ with Y smooth and $n = \dim(X) - \dim(Y)$, the relations being the "double point relations" which we explain further below. In particular, $\Omega^*(X)$ only lives in degrees $\leq \dim X$, which we will use several times throughout the paper. Similar to the Chow ring CH^*, algebraic cobordism Ω^* is a functor on the category of smooth varieties over k, covariant for projective and contravariant for smooth and more generally lci morphisms, which allows a theory of Chern classes. However, the map from the Picard group of a smooth variety X to $\Omega^1(X)$ given by the first Chern class is neither a bijection nor a homomorphism anymore (unlike the corresponding map in the Chow ring case). Its failure of being a group homomorphism is encoded in a *formal group law* that can be constructed from Ω^*. More precisely, any algebraic *orientable* cohomology theory allows by definition a calculus of Chern classes, and consequently the construction of a formal group law. A formal group law is a formal power series $F(x,y)$ in two variables such that for any two line bundles L_1 and L_2 we have the following identity relating their first Chern classes:

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

E. g. the formal group law for CH^* is additive, that is, $F(x+y) = x+y$. Algebraic cobordism is the universal one among the algebraic orientable cohomology theories. In what follows, $F(x,y)$ will always denote the universal formal group law corresponding to algebraic cobordism unless otherwise mentioned.

In this and in many other ways - as the computations below will illustrate - algebraic cobordism is a refinement of Chow ring, and one has a natural isomorphism of functors $\Omega^*(-) \otimes_{\mathbb{L}} \mathbb{Z} \cong CH^*(-)$ (see [12] where all these results are proved). Here and in the sequel, $\mathbb{L}$ denotes the Lazard ring, which classifies one-dimensional commutative formal group laws and is isomorphic to the graded polynomial ring $\mathbb{Z}[a_1, a_2, \ldots]$ in countably many variables [10], where we put a_i in degree $-i$. When considering polynomials $p(x_1, \ldots. x_n)$ over $\mathbb{L}$ with $\deg(x_i) = 1$, we will distinguish the (total) *degree* and the *polynomial degree* of $p(x_1, ..., x_n)$.

We are going to use a geometric interpretation of the formal group law, namely, the *double point relation.* This is an equality for elements in the algebraic cobordism ring established in [13]. We recall the definition for the reader's convenience.

Double point relation:

Assume that we have three smooth hypersurfaces A, B and C on a smooth variety Z such that the following conditions hold

(1) C is linearly equivalent to $A+B$
(2) A, B and C have transverse pairwise intersections
(3) C does not intersect $A \cap B$

Then we have the following *double point relation.* Denote by D the intersection $A \cap B$. We have

$$[C \to Z] = [A \to Z] + [B \to Z] - [\mathbb{P}_D \to Z]$$

in $\Omega^*(X)$, where $\mathbb{P}_D = \mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) = \mathbb{P}_D(N_{B/D} \oplus \mathcal{O}_D)$ and the map $\mathbb{P}_D \to Z$ is the the composition of the natural projection $\mathbb{P}_D \to D$ with the embedding $D \subset Z$. Here $N_{A/D}$ and $N_{B/D}$ are the normal bundles to D in A and B, respectively. The second condition ensures that $\mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) = \mathbb{P}_D(N_{B/D} \oplus \mathcal{O}_D)$ (since $L(C)|_D = (L(A) \otimes L(B))|_D = N_{A/D} \otimes N_{B/D}$ is trivial).

This formulation is a special case of the extended double point relation in [13, Lemma 5.2]. The double point relation allows to express geometrically the discrepancy between the additive formal group law and the universal one. Namely, since $C = F(A,B)$, we get

$$A + B - F(A,B) = [\mathbb{P}_D \to Z].$$

We will use this equation when proving Proposition 2.1.

We will also use repeatedly the projective bundle formula, which we recall below for the reader's convenience. For more details see [12, Section 1.1] and [14, 3.5.2].

Projective bundle formula: Let $E \to X$ be a vector bundle of rank r over X. Denote by $Y = \mathbb{P}(E^*)$ the variety of hyperplanes of E, and by π the natural projection $\pi : Y \to X$. The variety $\mathbb{P}(E^*)$ is a fibration over X with fibers isomorphic to $\mathbb{P}^{r-1}$. Note that equivalently $\mathbb{P}(E^*)$ can be defined as the variety of one-dimensional quotients of E since there is a canonical isomorphism between the variety of hyperplanes and the variety of quotients by hyperplanes in a vector space. This is how $\mathbb{P}(E^*)$ is defined in [12, Section 1.1] (where it is denoted by $\mathbb{P}(E)$). Let $A^*(-)$ be any oriented cohomology theory. Denote by ξ the first Chern class of the tautological quotient line bundle $\mathcal{O}_E(1)$ on Y whose restriction on each fiber of Y over X coincides with $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. The first Chern can be defined as $\xi = s^*s_*(1_Y)$ where $s : Y \to \mathcal{O}_E(1)$ is the zero section and $1_Y \in A^0(Y)$ is the multiplicative unit element. Then there is a ring isomorphism:

$$A^*(Y) = A^*(X)[\xi]/(\sum_{j=0}^{r}(-1)^j c_j(\pi^*E)\xi^{r-j}).$$

The isomorphism identifies a polynomial $a_0 + a_1\xi + \ldots + a_{n-1}\xi^{n-1}$ in $A^*(X)[\xi]$ with the element $\pi^*a_0 + (\pi^*a_1)\xi + \ldots + (\pi^*a_{n-1})\xi^{n-1}$ in $A^*(Y)$. In particular, $A^*(Y)$ splits into the direct sum $\pi^*A^*(X) \oplus \xi\pi^*A^*(X) \oplus \ldots \oplus \xi^{n-1}\pi^*A^*(X)$.

Note that the relation

$$\sum_{j=0}^{r}(-1)^j c_j(\pi^*E)\xi^{r-j} = 0$$

admits the following alternative description. Consider a short exact sequence of vector bundles on Y:

$$0 \to \tau_E \to \pi^*E \to \mathcal{O}_E(1) \to 0,$$

where τ_E is the tautological hyperplane bundle on Y. By the Whitney sum formula we have that the total Chern class $c(\pi^*E)$ is equal to the product $c(\tau_E)c(\mathcal{O}_E(1))$. Since $c(\mathcal{O}_E(1)) = 1+\xi$ we have $c(\pi^*E) = c(\tau_E)(1+\xi)$. We now divide this identity by $(1+\xi)$ (that is, multiply by $\sum_{j=0}^{r+\dim X-1}(-1)^j\xi^j$) and get that $c(\tau_E) = c(\pi^*E)(\sum_{j=0}^{r+\dim X-1}(-1)^j\xi^j)$. In particular,

$$c_r(\tau_E) = (-1)^r \sum_{j=0}^{r}(-1)^j c_j(\pi^*E)\xi^{r-j},$$

so we can interpret the relation above as the vanishing of the r-th Chern class of the bundle τ_E (which has rank $r-1$). We will use this interpretation in the proof of Theorem 2.7.

2.1. **A formula for the push-forward.** Let X be a smooth algebraic variety, and $E \to X$ a vector bundle of rank two on X. Consider the projective line fibration $Y = \mathbb{P}(E)$ defined as the variety of all lines in E. We have a natural projection $\pi : Y \to X$ which is projective and hence induces a *push-forward* (or *transfer*, sometimes also called *Gysin map*) $\pi_* : \Omega^*(Y) \to \Omega^*(X)$. We now provide a formula for this push-forward. Note that this formula is true not only for algebraic cobordism but for any orientable cohomology theory, as the proofs remain true in this more general case.

Consider the ring of formal power series in two variables y_1 and y_2 with coefficients in $\Omega^*(X)$. Define the operator A on this ring by the formula

$$A(f) = (1+\sigma)\frac{f}{F(y_1,\chi(y_2))},$$

where $[\sigma(f)](y_1,y_2) := f(y_2,y_1)$. Here F is the universal formal group law (or more generally, the one of the orientable cohomology theory one considers) and χ is the inverse for the formal group law F, that is, χ is uniquely determined by the equation $F(x,\chi(x)) = 0$ (we use notation from [12, 2.5]). The operator A is an analog of the *divided difference operator* introduced in [1, 8]. In the case of Chow rings, our definition coincides with the classical divided difference operator, since the formal group law for Chow ring is additive, that is, $F(x,y) = x+y$ and $\chi(x) = -x$. Though $A(f)$ is defined as a fraction, it is easy to write it as a formal power series as well (see Section 5). Such a power series is unique since $F(y_1,\chi(y_2)) = y_1 - y_2 + \dots$ is clearly not a zero divisor. E.g. we have

$$A(1) = \frac{x+\chi(x)}{x\chi(x)} = q(x,\chi(x)) = -a_{11} - a_{12}(x+\chi(x)) + \dots,$$

where $x = F(y_1,\chi(y_2))$, and $q(x,y)$ is the power series uniquely determined by the equation $F(x,y) = x+y-xyq(x,y)$. In particular, since $F(x,\chi(x)) = 0$ by definition of the power series $\chi(x)$, we have $x+\chi(x)-x\chi(x)q(x,\chi(x)) = 0$ which justifies the second equality. For the last equality, we used computation of the first few terms of $F(x,y)$ and $\chi(x)$ from [12, 2.5]. We also have

$$A(y_1) = y_2A(1) + \frac{F(x,y_2)-y_2}{x} = y_2q(x,\chi(x)) - y_2q(x,y_2) + 1 = 1 + a_{12}y_1y_2 + \dots.$$

The pull-back $\pi^* : \Omega^*(X) \to \Omega^*(Y)$ gives $\Omega^*(Y)$ the structure of an $\Omega^*(X)$-module. Recall that by the projective bundle formula we have an isomorphism of $\Omega^*(X)$-modules

$$\Omega^*(Y) \cong \pi^*\Omega^*(X) \oplus \xi\pi^*\Omega^*(X),$$

where $\xi = c_1(\mathcal{O}_E(1))$. Since the push-forward is a homomorphism of $\Omega^*(X)$-modules, it is enough to determine the action of π_* on 1_Y and on ξ.

Proposition 2.1. *Let c_1 and c_2 be the Chern roots of E, that is, formal variables satisfying the conditions $c_1 + c_2 = c_1(E)$ and $c_1c_2 = c_2(E)$. Then the push-forward acts on 1_Y and ξ as follows:*

$$\pi_*(1_Y) = [A(1)](c_1, c_2),$$

$$\pi_*(\xi) = [A(y_1)](c_1, c_2),$$

where $A(1)$ and $A(y_1)$ are the formal power series in two variables defined above.

Since $A(1)$ and $A(y_1)$ are symmetric in y_1 and y_2, they can be written as power series in $y_1 + y_2$ and y_1y_2. Hence, the right hand sides are power series in $c_1(E)$ and $c_2(E)$ and even polynomials (as all terms of degree greater than $\dim X$ *will vanish by* [13]*). So the right hand sides indeed define elements in $\Omega^*(X)$.*

For the Chow ring and K_0, analogous statements were proved in [8, Propositions 2.3,2.6] for certain morphisms $Y \to X$. Note that for both of these theories, the formula for $\pi_*(\xi)$ reduces to $\pi_*(\xi) = 1$ since the corresponding formal group laws do not contain terms of degree greater than two. Note also that there is a less explicit formula for arbitrary orientable theories and E of arbitrary rank established by Quillen, Panin and Shinder [17].

If we identify $\Omega^*(Y)$ with the polynomial ring $\Omega^*(X)[\xi]/(\xi^2 - c_1(E)\xi + c_2(E))$ by the projective bundle formula, we can reformulate Proposition 2.1 as follows:

$$\pi_*(f(\xi)) = [A(f(y_1))](c_1, c_2)$$

for any polynomial f with coefficients in $\Omega^*(X)$ (where $f(y_1)$ in the right hand side is regarded as an element in $\Omega^*(X)[[y_1, y_2]]$). In this form, Proposition 2.1 is consistent with the classical formula for the push-forward in the case of Chow ring (cf. [14, Remark 3.5.4]). Indeed, since the formal group law for Chow ring is additive we have $A(1) = \frac{1}{y_1 - y_2} + \frac{1}{y_2 - y_1} = 0$ and $A(y_1) = \frac{y_1}{y_1 - y_2} + \frac{y_2}{y_2 - y_1} = 1$.

Definition 2.2. *We define an $\Omega^*(X)$-linear operator A_π on $\Omega^*(Y)$ as follows. We have an isomorphism*

$$\Omega^*(X)[[y_1, y_2]]/(y_1 + y_2 - c_1(E), y_1y_2 - c_2(E)) \cong \Omega^*(Y)$$

given by $f(y_1, y_2) \mapsto f(\xi, c_1(E) - \xi))$. Then the operator A on $\Omega^*(X)[[y_1, y_2]]$ descends to an operator A_π on $\Omega^*(Y)$, which can be described using the above isomorphism as follows

$$A_\pi : f(\xi, c_1(E) - \xi) \to [A(f(y_1, y_2))](\xi, c_1(E) - \xi).$$

We also define a $\Omega^*(X)$-linear endomorphism σ_π of $\Omega^*(Y)$ by the formula:

$$\sigma_\pi : f(\xi, c_1(E) - \xi) = f(c_1(E) - \xi, \xi).$$

The operator A_π is well-defined since A preserves the ideal $(y_1+y_2-c_1(E), y_1y_2-c_2(E))$. Indeed, for any power series $f(y_1,y_2)$ symmetric in y_1 and y_2 (in particular, for $y_1+y_2-c_1(E)$ and $y_1y_2-c_2(E)$) and any power series $g(y_1,y_2)$ we have $A(fg)=fA(g)$. The operator A_π decreases degrees by one, and its image is contained in $\pi^*\Omega^*(X)\subset\Omega^*(Y)$, which can be identified using the above isomorphism for $\Omega^*(X)$ with the subring of symmetric polynomials in y_1 and y_2. Proposition 2.1 tells us that the push-forward $\pi_*:\Omega^*(Y)\to\Omega^*(X)$ is the composition of A_π with the isomorphism $\pi^*\Omega^*(X)\cong\Omega^*(X)$, which sends (under the above identifications) a symmetric polynomial $f(y_1,y_2)$ into the polynomial $g(c_1(E),c_2(E))$ such that $g(y_1+y_2,y_1y_2)=f(y_1,y_2)$. Hence, we get the following corollary, which we will use in the sequel.

Corollary 2.3. *The composition* $\pi^*\pi_*:\Omega^*(Y)\to\Omega^*(Y)$ *is equal to the operator* A_π*:*

$$\pi^*\pi_*=A_\pi.$$

We now prove Proposition 2.1.

Proof. Note that the proof can be reduced to the case where E is the direct sum of line bundles by the splitting principle [12, Remark 4.1.2]. So we may assume that $E=L_1\oplus L_2$ splits into the sum of two line bundles.

First, note that replacing E with $E_M=E\otimes M$ for an arbitrary line bundle M does not change the variety Y and the map π. However, this does change the tautological quotient line $c_1(\mathcal{O}_E(1))$. More precisely, we have the following isomorphism of line bundles on Y (compare e.g. [13, Proof of Lemma 7.1]):

$$\pi^*M\otimes\mathcal{O}_E(1)=\mathcal{O}_{E_M}(1).$$

Let us denote by ξ_M the first Chern class of the tautological quotient line bundle $\mathcal{O}_{E_M}(1)$. The identity above implies that $\xi_M=F(\xi,\pi^*c_1(M))$ or equivalently $\xi=F(\xi_M,\pi^*c_1(M^*))$. Hence, to compute $\pi_*\xi$ it is enough to compute π_*1_Y and $\pi_*\xi_M$ for some M (see an explicit computation of $\pi_*\xi$ after the proof of Lemma 2.4). It is convenient to choose $M=L_1^*$ so that E_M has a trivial summand, and hence one of the Chern roots of E_M is zero. In this case, the second formula of Proposition 2.1 reduces to $\pi_*\xi_M=1$, which is easy to show (see Lemma 2.4) by similar methods as for the Chow ring. It is more difficult to compute π_*1_Y, which is the cobordism class of $[\pi:Y\to X]$. For the Chow ring, it is zero by degree reasons, but for cobordisms it is not. E.g. even for a trivial bundle E we have $[\pi:Y\to X]=[\pi:X\times\mathbb{P}^1\to X]=-a_{11}1_X$. However, using a suitable double point relation we can replace π_*1_Y with a power series in push-forwards of the first Chern classes of certain tautological quotient line bundles. The latter can be computed using that $\pi_*\xi_M=1$ and other similar statements (see the second part of Lemma 2.4).

Lemma 2.4. *Suppose that* $E=\mathcal{O}_X\oplus L$ *contains a trivial summand. Then we have* $\pi_*(\xi)=1_X$*. More generally,*

$$\pi_*(\xi^i)=c_1(L)^{i-1}$$

for all $i>0$*. Hence, for any formal power series* $f(x)$ *in* x *with coefficients in* $\Omega^*(X)$ *such that* $f(x)=xg(x)$ *for some power series* $g(x)$ *we have* $\pi_*f(\xi)=g(c_1(L))$*. We also have*

$$\pi_*(1_Y)=q(c_1(L),c_1(L^*))$$

where q *is the power series defined above.*

Proof. Consider the natural embedding $i : X = \mathbb{P}(\mathcal{O}_X) \to Y = \mathbb{P}(E)$. Then $\xi = i_* 1_X$ by [12, Lemma 5.1.11]. Hence, $\pi_*(\xi) = \pi_* i_* 1_X = 1_X$ since $\pi \circ i = id_X$.

We can now compute $\pi_*(\xi^i)$ for all $i > 0$ using that $\xi^2 = \pi^* c_1(L)\xi$ by the projective bundle formula and that π_* is $\Omega^*(X)$-linear. We get that $\pi_*(\xi^i) = c_1(L)^{i-1}$ for all $i > 0$.

We are now going to use the double point relation to compute $\pi_* 1_Y = [\pi : Y \to X]$. Note that by the equality $\xi = i_* 1_X$ we have an isomorphism of line bundles $\mathcal{O}(X) \cong \mathcal{O}_E(1)$ on Y. Since the normal bundle $N_{Y/X}$ to X in Y is isomorphic to $i^*\mathcal{O}(X)$, and $i^*\mathcal{O}(X) \cong i^*\mathcal{O}_E(1) \cong L$, we have that $Y = \mathbb{P}(\mathcal{O}_X \oplus N_{Y/X})$. Consider the variety $Z = \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(X))$ and denote by $p : Z \to Y$ the natural projection. Take the smooth hypersurfaces $A = \mathbb{P}(\mathcal{O}_Y(X))$, $B = p^*X$ and $C = \mathbb{P}(\mathcal{O}_Y)$ in Z. It is easy to check that A, B and C satisfy the conditions (1), (2) and (3) for the double point relation, that the embedding $D = A \cap B \subset A$ can be identified with the embedding $i : X \to Y$ and hence, $\mathbb{P}_D = \mathbb{P}(\mathcal{O}_D \oplus N_{A/D})$ can be identified with $Y = \mathbb{P}(\mathcal{O}_X \oplus N_{Y/X})$. Then the map $[\mathbb{P}_D \to D]$ identifies with the map $\pi : Y \to X$ and we have the equality

$$[\pi : Y \to X] = \pi_* p_* [\mathbb{P}_D \to Z],$$

where the map $\mathbb{P}_D \to Z$ is the composition of the projection $\mathbb{P}_D \to D$ with the embedding $D \subset Z$.

Note that our choice of Z, A, B and C is a partial case of *twisting* construction considered in [13, Section 7.2] (namely, take $E = 0$, $L = \mathcal{O}_X$, $H = X$ and keep in mind that what is called $\mathbb{P}(L)$ in [13] we call $\mathbb{P}(L(X))$ and vice versa). E.g., the fact that $A + B \sim C$ follows from [13, Lemma 7.1]. The geometric picture behind this construction in our case becomes clear if one considers the simplest possible example: $X = Spec(k)$ is a point, $Y = \mathbb{P}^1$ is a projective line and $Z = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ is the blow-up of $\mathbb{P}^2$ at a point $D \in \mathbb{P}^2$. Then A is the exceptional divisor (which contains D), B is a line passing through D and C is a line not passing through D.

We now apply the double point relation and get that $[\mathbb{P}_D \to Z] = [A \to Z] + [B \to Z] - [C \to Z]$. Note that the cobordism class $[A \to Z]$ is the first Chern class ξ_V of the tautological quotient line bundle $\mathcal{O}_V(1)$ on Z, if we regard Z as the projectivization $\mathbb{P}(V)$ of the vector bundle $V = \mathcal{O}_Y(-X) \oplus \mathcal{O}_Y$ on Y. The cobordism class $[B \to Z]$ is equal to $p^*\xi$, and then $[C \to Z] = F(\xi_V, p^*\xi)$ since $C \sim A + B$. We have

$$[\pi : Y \to X] = \pi_* p_* [\mathbb{P}(D) \to Z] = \pi_* p_* (\xi_V + p^*\xi - F(\xi_V, p^*\xi)).$$

Note that the power series $x + y - F(x, y) = xyq(x, y)$ only contains monomials $x^i y^j$ for $i, j > 0$. Hence, we can apply the first part of the lemma to compute $\pi_* p_* (xyq(x, y))$ for $x = \xi_V$ and $y = p^*\xi$ and get exactly $q(\chi(c_1(L)), c_1(L))$.

□

We now return to the general case $E = L_1 \oplus L_2$. Denote by L_π the line bundle $L_1^* \otimes L_2$. By Lemma 2.4 applied to $E_{L_1^*} = \mathcal{O}_X \oplus L_\pi$, we get the desired formula for $\pi_* 1_Y$. It remains to prove the formula for $\pi_*(\xi)$. We use that $\xi = F(\xi_{L_1^*}, \pi^* c_1(L_1))$ and $F(x, y) = y + x(1 - yq(x, y))$. By Lemma 2.4 applied to $E_{L_1^*}$ we have

$$\pi_*(F(\xi_{L_1^*}, \pi^* c_1(L_1))) = c_1(L_1)\pi_* 1_Y + 1_X - c_1(L_1) q(c_1(L_\pi), c_1(L_1)).$$

Since the Chern roots of E are exactly $c_1(L_1)$ and $c_1(L_2)$ and $c_1(L_\pi) = F(\chi(c_1(L_1)), c_1(L_2))$, the proposition follows. □

2.2. **Algebraic cobordism groups of cellular varieties.** We start with the definition of a cellular variety. The following definition is taken from [9, Example 1.9.1], other authors sometimes consider slight variations.

Definition 2.5. *We say that a smooth variety X over k is "cellular" or "admits a cellular decomposition" if X has a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset ... \subset X_n = X$ by closed subvarieties such that the $X_i - X_{i-1}$ are isomorphic to a disjoint union of affine spaces $\mathbb{A}^{d_i}$ for all $i = 0, ..., n$, which are called the "cells" of X.*

Examples of cellular varieties include projective spaces and more general Grassmannians, and complete flag varieties G/B where G is a reductive group and B is a Borel subgroup.

We mention the following theorem, for which we make no claim concerning originality. Note however that it is not contained in [12], where a stronger condition (CD') which requires the closures of the cells to be smooth is used (see Remark 5.1.7 of loc. cit.) to deduce a weaker statement. To prove the theorem, note that the corresponding theorem for $MGL^{**}(-)$ is proved in [15], so applying [11] which relies on (unfortunately still unpublished) work of Hopkins and Morel yields the claim. Our results do not rely on this theorem.

Theorem 2.6. *Let X be a cellular variety with a cellular decomposition as in the definition above. Then we have an isomorphism of graded abelian groups (and even of $\mathbb{L}$-modules)*

$$\Omega^*(X) \cong \oplus_i \mathbb{L}[d_i]$$

where the sum is taken over the cells of X.

In particular, this theorem tells us that $\Omega^*(X)$ is generated by the classes labeled by the cells of X. This fact alone has a purely algebraic proof, which we now give:

Proof. We use the following filtration of the algebraic cobordism ring from [12, 4.5.2]. Define the graded subgroup $F^{(n)}\Omega^*(X)$ to be the one generated by classes $[f : Y \to X]$ with Y smooth, irreducible and $\dim(Y) - \dim f(Y) \geq n$. This is in fact an $\mathbb{L}$-submodule of $\Omega^*(X)$, and by [12, Theorem 4.5.7], we have that $F^{(n)}\Omega^*(X) = \mathbb{L}^{\leq -n} \cdot \Omega^*(X)$. We will also use that $F^{(1)}\Omega^*(X)$ is the kernel of the homomorphism $\theta : \Omega^*(X) \to CH^*(X)$ [12, Remark 4.5.6]. The homomorphism θ exists by the universality property of algebraic cobordism [12, Theorem 7.1.3], and sends $[f : Y \to X]$ to $f_*^{CH}(1_Y^{CH})$.

We now construct a generating set (over the Lazard ring $\mathbb{L}$) for $\Omega^*(X)$ labeled by the cells in X. For the closure $X_\alpha \subset X$ of each cell take a resolution of singularities R_α and take its cobordism class $Z_\alpha = [R_\alpha \to X_\alpha \subset X]$. Then every element $c \in \Omega^*(X) \setminus F^{(1)}\Omega^*(X)$ of degree d can be written as a linear combination of elements Z_α with coefficients in $\mathbb{L}$. Indeed, write $\theta(c)$ in $CH^*(X)$ as a linear combination $\sum c_\alpha X_\alpha$ (note that X_α will all have degree d). Then $c - \sum c_\alpha Z_\alpha$ lies in the kernel of θ. Indeed, by definition of θ we have that if $f : Y \to X$ maps Y birationally onto its image then $\theta([f : Y \to X]) = [f(Y)]$ (see [12, Section 4.5]). Hence, $c - \sum c_\alpha Z_\alpha \in F^{(1)}\Omega^*(X) = \mathbb{L}^{\leq -1} \cdot \Omega^*(X)$ is the sum of terms of the form ac' where $a \in \mathbb{L}^{\leq -1}$ and hence c' has degree at least $d + 1$. We then can proceed by a downward induction on the degree of c, which will stop at $d = \dim(X)$.

Note also that all relations between the generators Z_α must lie in $F^{(1)}\Omega^*(X)$. Indeed, the image of such a relation under the map $\theta : \Omega^*(X) \to CH^*(X)$ should be zero since there are no relations between X_α in $CH^*(X)$. ☐

For complex cobordism of topological complex cellular spaces, the corresponding theorem simply follows from an iterated use of the long exact localization sequence which always splits as everything in sight has MU^*-groups concentrated in even degrees only. Note also that in the topological case, the Atiyah-Hirzebruch spectral sequence degenerates for these spaces, which allows to transport information from singular cohomology to complex cobordism. As Morel points out, the analogous motivic spectral sequence invented by Hopkins-Morel (unpublished) converging to algebraic cobordism does not in general degenerate even for the point $Spec(k)$, because the one converging to algebraic K-theory does not.

We now turn to the ring structure. First, we note that if $k = \mathbb{C}$, then there is a map of graded rings $\Omega^*(X) \to MU^{2*}(X(\mathbb{C})^{an})$ by universality of algebraic cobordism [12, Example 1.2.10]. Using [13], we may describe this map explicitly by mapping an element $[Y \to X]$ of $\Omega^*(X)$ to $[Y(\mathbb{C})^{an} \to X(\mathbb{C})^{an}]$ in $MU^{2*}(X(\mathbb{C})^{an})$. As both product structures are defined by taking cartesian products of the geometric representatives and pulling it back along the diagonal of X resp. $X(\mathbb{C})^{an}$, we see that this map does indeed preserve the ring structure. Also, for any embedding $k \to \mathbb{C}$ we obtain a ring homomorphism from algebraic cobordism over k to algebraic cobordism over $\mathbb{C}$. For the flag variety of GL_n, this is a ring isomorphism by Theorem 2.7 below as the base change from k to $\mathbb{C}$ respects products and Chern classes.

For some varieties X, the ring structure of $\Omega^*(X)$ can be completely determined using the projective bundle formula [12, Section 1.1]. This is the case for the variety of complete flags for $G = GL_n$ (see Theorem 2.7 below) and also for Bott-Samelson resolutions of Schubert cycles in a complete flag variety for any reductive group G (see Section 3).

2.3. **Borel presentation for the flag variety of GL_n.**

We now turn to the case of the complete flag variety X for $G = GL_n(k)$. The points of X are identified with *complete flags* in k^n. A *complete flag* is a strictly increasing sequence of subspaces

$$F = \{\{0\} = F^0 \subset F^1 \subset F^2 \subset \ldots \subset F^n = k^n\}$$

with $\dim(F^k) = k$. The group G acts transitively on the set of all flags, and the stabilizer of a point is isomorphic to a Borel subgroup $B \subset G$, which makes $X = G/B$ into a homogeneous space under G. By this definition, X has structure of an algebraic variety.

Note that over $\mathbb{C}$, one may equivalently define the flag variety X to be the homogeneous space K/T under the maximal compact subgroup $K \subset G$, where T is a maximal compact torus in K (that is, the product of several copies of S^1) [2]. E. g., for $G = GL_n(\mathbb{C})$ (resp. $SL_n(\mathbb{C})$), the maximal compact subgroup is $U_n(\mathbb{C})$ (resp. $SU(n)$). This is the language in which many of the definitions and results in [1], [2] and [3] are stated. We sometimes allow ourselves to use those definitions and results which do carry over to the "algebraic" case (reductive groups over k) without mentioning explicitly the obvious changes that have to be carried out.

There are n natural line bundles $L_1,\ldots,L_n$ on X, namely, the fiber of L_i at the point F is equal to F^i/F^{i-1}. Put $x_i = c_1(L_i)$, where the first Chern class c_1 with respect to algebraic cobordism is defined in [12]. Note that our definition of x_i differs by sign from the one in [14]. The following result on the algebraic cobordism ring is an analog of the Borel presentation for the singular cohomology ring of a flag variety. In fact, it holds for any orientable cohomology theory since its proof only uses the projective bundle formula.

Theorem 2.7. *Let $A^*(-)$ be any orientable cohomology theory (e.g. $CH^*(-)$ or $\Omega^*(-)$). Then the ring $A^*(X)$ is isomorphic as a graded ring to the ring of polynomials in $x_1,\dots,\ x_n$ with coefficients in the coefficient ring $A^*(pt)$ and $deg(x_i)=1$, quotient by the ideal S generated by the symmetric polynomials of strictly positive polynomial degree:*

$$A^*(X) \simeq A^*(pt)[x_1,\dots,x_n]/S.$$

More generally, let E be a vector bundle of rank n over a smooth variety Y and $\mathbb{F}(E)$ be the flag variety relative to this bundle. Then we have an isomorphism of graded rings

$$A^*(\mathbb{F}(E)) \simeq A^*(pt)[x_1,...,x_n]/I$$

where I is the ideal generated by the relations $e_k(x_1,..,x_n)=c_k(E)$ for $1\le k\le n$ with e_k denoting the k-th elementary symmetric polynomial.

Proof. We essentially follow the corresponding proof of [14, Theorem 3.6.15] for the Chow ring case with a modification that makes this proof applicable to any other orientable theory A^*. Namely, for an arbitrary oriented cohomology theory A^*, it is more convenient to dualize the geometric argument in [14, Theorem 3.6.15] because we can no longer use that $c_i(E)=(-1)^ic_i(E^*)$ for a vector bundle E (which is used implicitly several times in the proof of [14, Theorem 3.6.15]).

We denote by P_i the variety of partial flags $P_i=\{\ F^{n-i}\subset F^{n-i+1}\subset\ldots\subset F^n=k^n\}$ (e. g. P_1 is the variety of hyperplanes in k^n and $P_{n-1}=X$) and by $\mathcal{W}_j$ the corresponding tautological vector bundle of rank j over P_i where $j\ge n-i$ (that is, the fiber of $\mathcal{W}_j$ over a point $\{\ F^{n-i}\subset F^{n-i+1}\subset\ldots\subset F^n\}$ is equal to F_j). In particular, L_i defined above is equal to $\mathcal{W}_i/\mathcal{W}_{i-1}$. Put $x_i=c_1(L_i)$. As $P_i=\mathbb{P}((\mathcal{W}_{n-i+1})^*)$ is the projective bundle over P_{i-1} and the line bundle $\mathcal{O}_{\mathcal{W}_{n-i+1}}(1)$ is isomorphic to L_{n-i+1}, the projective bundle formula for orientable cohomology theories [12, Section 1.1] yields

$$A^*(P_i)\cong A^*(P_{i-1})[x_{n-i+1}]/(\sum_{j=0}^{n-i+1}(-1)^jc_j(\mathcal{W}_{n-i+1})x_{n-i+1}^{n-i+1-j}).$$

Or, using the above interpretation of the relation in the projective bundle formula and the short exact sequence of vector bundles on P_i

$$0\to\mathcal{W}_{n-i}\to\mathcal{W}_{n-i+1}\to L_{n-i+1}\to 0$$

we get

$$A^*(P_i)\cong A^*(P_{i-1})[x_{n-i+1}]/(c_{n-i+1}(\mathcal{W}_{n-i})).$$

It remains to compute $c_{n-i+1}(\mathcal{W}_{n-i})$. This can be done by induction on i starting from $i=0$ (in which case $\mathcal{W}_n$ is a trivial line bundle) and applying the Whitney sum formula to the short exact sequence of vector bundles above. We get $c(\mathcal{W}_{n-i})=\prod_{j=n-i+1}^n c(L_j)^{-1}=\prod_{j=n-i+1}^n(1+x_j)^{-1}=\sum_{k\ge0}(-1)^kh_k(x_{n-i+1},\dots,x_n)$, where $h_k(x_{n-i+1},\dots,x_n)$ denotes the sum of all monomials of degree k in $x_{n-i+1},\dots,x_n$. In particular, $c_{n-i+1}(\mathcal{W}_{n-i})=(-1)^{n-i+1}h_{n-i+1}(x_{n-i+1},\dots,x_n)$. From this we deduce that

$$A^*(P_i)\cong A^*(P_{i-1})[x_{n-i+1}]/(h_{n-i+1}(x_{n-i+1},\dots,x_n)),$$

and hence

$$A^*(P_n)\cong A^*(pt)[x_1,\dots,x_n]/(h_n(x_n),h_{n-1}(x_{n-1},x_n),\dots,h_1(x_1,\dots,x_n)).$$

The ideal generated by the relations $h_{n-i+1}(x_{n-i+1},\dots,x_n)$ is exactly S, which is easy to check starting with the recurrence relation

$$h_i(x_1,\dots,x_n)=h_i(x_i,\dots,x_n)+\sum_{j<i}x_jh_{i-1}(x_j,\dots,x_n).$$

The more general case follows similar to the proof of [14, Proposition 3.8.1]. $\square$

Remark 2.8. The proof immediately implies that the class of a point in $A^*(X)$ is equal to $x_n^{n-1}x_{n-1}^{n-2}\cdots x_2$. Since $x_n^{n-1}x_{n-1}^{n-2}\cdots x_2=\frac{1}{n!}\prod_{i>j}(x_i-x_j)\mod S$ (which is easy to show by induction on n using that $(x_n-x_{n-1})\cdots(x_n-x_1)=nx_n^{n-1}\mod S$) we also have that the class of a point can be represented by the polynomial $\Delta_n=\frac{1}{n!}\prod_{i>j}(x_i-x_j)$.

Note that the proof also gives an explicit formula for the classes of one-dimensional *Schubert cycles* $X_1=X_{s_{\gamma_1}},\dots,X_{n-1}=X_{s_{\gamma_{n-1}}}$ in X corresponding to the simple roots $\gamma_1,\dots,\gamma_{n-1}$ of GL_n (see the beginning of Section 3 for the definition of the Schubert cycles X_w for w in the Weyl group of G and the beginning of Section 5 for the definition of the γ_i). The cycle X_k consists of flags $F=\{\{0\}=F^0\subset F^1\subset F^2\subset\ldots\subset F^n=k^n\}$ s.t. all F^i except for F^k are fixed. Then the class of X_k is equal to the class of a point divided by x_{k+1}. Indeed, to get the class of $X_k\subset X$ we should take the point in P_{n-k-1} corresponding to the fixed partial flag $\{F^{k+1}\subset F^{k+2}\subset\ldots\subset F^n=k^n\}$ and then take a line in a fiber of the projective bundle $P_{n-k}\to P_{n-k-1}$ over this point. Namely, the line will consist of all hyperplanes in F^{k+1} that contain the fixed codimension two subspace F^{k-1}. Again it is easy to show by induction on n that the polynomial $x_n^{n-1}x_{n-1}^{n-2}\cdots x_2/x_k$ is equal to $2\Delta_n/(x_{k+1}-x_k)$ modulo the ideal S.

Note that the Borel presentation for singular cohomology implies, in particular, that Picard group of the flag variety is freely generated (as an abelian group) by the first Chern classes of the line bundles $L_1,\dots,L_n$. In what follows, we will also use the following alternative description of the Picard group of X. Recall that each strictly dominant weight λ of G defines an irreducible representation $\pi_\lambda:G\to GL(V_\lambda)$ and an embedding $G/B\to\mathbb{P}(V_\lambda)$. Hence, to each dominant weight λ of G we can assign a very ample line bundle $L(\lambda)$ on X by taking the pull-back of the line bundle $\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$ on $\mathbb{P}(V_\lambda)$. The map $\lambda\mapsto L(\lambda)$ extended to non-dominant weights by linearity gives an isomorphism between the Picard group of X and the weight lattice of G [4, 1.4.3]. In particular, for the line bundles above we have $L_i=L(-e_i)$ where e_i is the weight of GL_n given by the i-th entry of the diagonal torus in GL_n.

3. Schubert calculus for algebraic cobordism of flag varieties

In this section, we assume that G is an arbitrary connected reductive group unless we explicitly mention that $G=GL_n(k)$, and $X=G/B$ is the complete flag variety for G. We now investigate the ring structure of $\Omega^*(X)$ in more geometric terms.

3.1. **Schubert cycles and Bott-Samelson resolutions.** Recall that the flag variety X is cellular with the following cellular decomposition into *Bruhat cells.* Let us fix a Borel subgroup B. For each element $w\in W$ of the Weyl group of G, define the *Bruhat (or Schubert) cell* C_w as the B–orbit of the the point $wB\in G/B=X$ (we identify the Weyl group with $N(T)/T$ for a maximal torus T of G inside B). The *Schubert cycle* X_w is defined as the closure of C_w in X. The dimension of X_w is equal to the length of w [1]. Recall that the length of an element $w\in W$ is defined as the minimal number of factors in a decomposition of w into the product of simple reflections. Recall also that for each l-tuple $I = (\alpha_1, \dots, \alpha_l)$ of simple roots of G, one can define the *Bott-Samelson resolution* R_I (which has dimension l) together with the map $r_I : R_I \to X$. Bott-Samelson resolutions are smooth. Consequently, for any I the map $r_I : R_I \to X$ represents an element in $\Omega^*(X)$ which we denote by Z_I.

Denote by $s_\alpha \in W$ the reflection corresponding to a root α, and by s_I the product $s_{\alpha_1} \cdots s_{\alpha_l}$. If the decomposition $s_I = s_{\alpha_1} \cdots s_{\alpha_l}$ defined by I is reduced (that is, s_I can not be written as a product of less than l simple reflections, or equivalently, the length of s_I is equal to l), then the image $r_I(R_I)$ coincides with the Schubert cycle X_{s_I} (which we will also denote by X_I). The dimension of X_I in this case is also equal to l and the map $r_I : R_I \to X_I$ is a birational isomorphism. In this case, the variety R_I is a resolution of singularities for the Schubert cycle X_I.

Bott-Samelson resolutions were introduced by Bott and Samelson in the case of compact Lie groups, and by Demazure in the case of algebraic semisimple groups [8]. There are several equivalent definitions, see e. g. [5, 8, 14]. We will use the definition below (which follows easily from [5, 2.2]), since it is most suited to our needs. Namely, R_I is defined by the following inductive procedure starting from $R_\emptyset = pt = Spec(k)$ (in what follows we will rather denote $R_\emptyset$ by R_e). For each j-tuple $J = (\alpha_1, \dots, \alpha_j)$ with $j < l$, denote by $J \cup \{j+1\}$ the $(j+1)$-tuple $(\alpha_1, \dots, \alpha_j, \alpha_{j+1})$. Define $R_{J\cup\{j+1\}}$ as the fiber product $R_J \times_{G/P_{j+1}} G/B$, where P_{j+1} is the minimal parabolic subgroup corresponding to the root α_{j+1}. Then the map $r_{J\cup\{j+1\}} : R_{J\cup\{j+1\}} \to X$ is defined as the projection to the second factor. In what follows, we will use that R_J can be embedded into $R_{J\cup\{j+1\}}$ by sending $x \in R_J$ to $(x, r_J(x)) \in R_J \times_{G/P_{j+1}} G/B$ (this embedding will be denoted $i_{J,J\cup\{j+1\}}$ below).

In particular, one-dimensional Bott-Samelson resolutions are isomorphic to the corresponding Schubert cycles. It is easy to show that any two-dimensional Bott-Samelson resolution R_I for a reduced I is also isomorphic to the corresponding Schubert cycle. More generally, R_I is isomorphic to X_I if and only if all simple roots in I are pairwise distinct (in particular, the length of I should not exceed the rank of G). The simplest example where R_I and X_I are not isomorphic for a reduced I is $G = GL_3$ and $I = (\gamma_1, \gamma_2, \gamma_1)$ (where γ_1, γ_2 are two simple roots of GL_3).

It is easy to show that $R_{J\cup\{j+1\}}$ is the projectivization of the bundle $r_J^*\pi_{j+1}^*E$, where E is the rank two vector bundle on G/P_{j+1} defined in the next subsection and $\pi_{j+1} : G/B \to G/P_{j+1}$ is the natural projection. This is the definition used in [3]. In the topological setting, the vector bundle $r_J^*\pi_{j+1}^*E$ splits into the sum of two line bundles [3] but in in the algebro-geometric setting this is no longer true (though $r_J^*\pi_{j+1}^*E$ still contains a line subbundle as follows from the proof of Lemma 3.4).

This description of R_I implies easily (by repeated use of the projective bundle formula) that the algebraic cobordism ring $\Omega^*(R_I)$ is freely generated as an $\mathbb{L}$-module by the cobordism classes $[R_J]$ of R_J for all subsets $J \subset I$. Indeed, let $p : R_I \to R_{I^l}$ be the natural projection. Denote by I^l the $(l-1)$–tuple $(\alpha_1, \dots, \alpha_{l-1})$. Then by the projective bundle formula we have

$$\Omega^*(R_I) \cong p^*\Omega^*(R_{I^l}) \oplus [R_{I^l}]p^*\Omega^*(R_{I^l}),$$

using that $s^*s_*(1) = i_{I^l,I_*}i^*_{I^l,I} = [R_{I^l}]$ where s is the zero section of $\mathcal{O}_{R_I}(1)$. Note that for all subsets $J, K \subset I$ such that $J \cup K = I$ the subvarieties R_J and R_K of R_I intersect transversally and the intersection $R_J \cap R_K$ coincides with $R_{J\cap K}$ [5, 2.2 (10)], which implies that $[R_J][R_K] =$

$[R_{J\cap K}]$ in $\Omega^*(R_I)$. Note also that by definition of Bott-Samelson resolutions we have that $p^*[R_J] = [R_{J\cup\{l\}}]$ for all $J \subset I^l$. Hence, we can proceed by induction on l.

The following proposition is an immediate corollary of Theorem 2.6. We are not going to use it anywhere later in this preprint.

Proposition 3.1. *As an $\mathbb{L}$-module, the algebraic cobordism ring $\Omega^*(X)$ of the flag variety is freely generated by classes that are in one-to-one correspondence with the elements of the Weyl group.*

An analogous statement for complex cobordism is proved in [3, Proposition 1] by using the Atiyah-Hirzebruch spectral sequence (as mentioned in Section 2). Our proof of the surjectivity part of Theorem 2.6 immediatly implies that the cobordism ring of an arbitrary flag variety is generated as an $\mathbb{L}$-module by the classes $Z_{I(w)}$ of Bott-Samelson resolutions for $w \in W$, where $I(w)$ defines a reduced decomposition for w (we choose exactly one $I(w)$ for each w). Moreover, in the case $G = GL_n$, we now provide a proof of this proposition that does not rely on Theorem 2.6 and uses instead the Borel presentation of Theorem 2.7.

Proof. First, note that there is a basis in $\Omega^*(X) \cong \mathbb{L}[x_1, \ldots, x_n]/S$ over $\mathbb{L}$ given by the classical Schubert polynomials S_w (regarded as polynomials in $\mathbb{L}[x_1, \ldots, x_n]$), where w runs over the symmetric group S_n (this statement follows immediately from the respective statement over $\mathbb{Z}$, see [14, Proposition 2.5.3, Corollary 2.5.6]). Schubert polynomials are polynomials with integer coefficients representing the cohomology classes of the Schubert cycles X_w in the Borel presentation. Note that under the homomorphism $\theta : \Omega^*(X) \to CH^*(X)$, Schubert polynomials map to Schubert polynomials because θ respects first Chern classes and thus the x_i. Therefore, we have a basis of $\Omega^*(X)$ whose image in $CH^*(X)$ consists of Schubert polynomials and hence [14, Theorem 3.6.18] of Schubert cycles. Note however that in contrast with the Chow ring case, the Schubert polynomials in the cobordism case do not have any geometric interpretation.

We now construct a basis whose elements are Bott-Samelson classes, and hence do have a geometric meaning. Let $Z_w := Z_{I(w)}$ be the cobordism class of the Bott-Samelson resolution of X_w for some reduced decomposition $I(w)$ of the element w. Then we have that the Z_w form a generating set by our proof of the surjectivity part in Theorem 2.6. It remains to show that there are no nontrivial relations. We know that $Z_w = \sum_{v\in W} a_v S_v$ for some elements $a_v \in \mathbb{L}$. Denote by R_w the polynomial $\sum_{v\in W} a_v S_v$ in $\mathbb{L}[x_1, \ldots, x_n]$. Since the lift $\tilde{\theta} : \mathbb{L}[x_1, ..., x_n] \to \mathbb{Z}[x_1, ..., x_n]$ (which takes all coefficients in $\mathbb{L}^{\leq -1}$ to zero) maps $\sum_{v\in W} a_v S_v$ to $\sum_{v\in W}(a_v \mod \mathbb{L}^{\leq -1})S_v$ and $\tilde{\theta}(R_w) = S_w$ we get that $a_v \in L^{\leq -1}$ unless $v = w$, in which case $a_w = 1 \mod L^{\leq -1}$ (and hence $a_w = 1$ since S_w and R_w are homogeneous with respect to the total grading). This implies that the lowest polynomial degree part of R_w is S_w.

Suppose now that there is a relation between the R_w, that is, some linear combination $R = \sum_{w\in V} a_w R_w$ lies in the ideal S. Here $V \subset W$ is a subset of the Weyl group and $a_w \in \mathbb{L}$ are some nonzero elements. Consider the case where V is not empty and compute the lowest polynomial degree part R^l of R (by l we denote the polynomial degree of this part). We have $R^l = \sum_{w\in V'} a_w S_w$, where $V' \subset V$ consists of all $w \in V$ such that $\deg S_w = l$. Indeed, for all w in $V \setminus V'$ we have $\deg S_w > l$ and hence the corresponding R_w does not contribute to R^l. Since S is the direct sum of abelian groups which are homogeneous with respect to the polynomial degree, we have that $R^l = \sum_{w\in V'} a_w S_w$ lies in S. But there are no nontrivial relations between the S_w so V' and hence V must be empty contrary to our assumption. We deduce that there are no nontrivial relations between the R_w. $\square$

3.2. **Schubert calculus.** We will now describe the cobordism classes Z_I as polynomials in the first Chern classes of line bundles on X. This allows us to compute products of Bott-Samelson resolutions and hence achieves the goal of a Schubert calculus for algebraic cobordism.

We first define operators A_i on $\Omega^*(X)$ following the approach of the previous section (see Definition 2.2). These operators generalize the *divided difference operators* on the Chow ring $CH^*(X)$ defined in [1, 8, 6] to algebraic cobordism.

We first define operators A_i for GL_n since in this case the Borel presentation allows to make them more explicit. We start with the subgroup B of upper triangular matrices and the diagonal torus, which yields an isomorphism $W \cong S_n$. Under this isomorphism, the reflection s_α with respect to a root $\alpha = e_i - e_j$ goes to the transposition $(i\ j)$ (see Section 5). For each positive root α of G, we define the operators σ_α and $\hat{A}_\alpha$ on the ring of formal power series $\mathbb{L}[[x_1, \ldots, x_n]]$ as follows:

$$(\sigma_\alpha f)(x_1, \ldots, x_n) = f(x_{s_\alpha(1)}, \ldots, x_{s_\alpha(n)}),$$

$$\hat{A}_\alpha = (1 + \sigma_\alpha)\frac{1}{F(x_{i+1}, \chi(x_i))}.$$

It is easy to check that $\hat{A}_\alpha$ is well-defined on the whole ring $\mathbb{L}[[x_1, \ldots, x_n]]$ (see Section 5). Note also that under the homomorphism $\mathbb{L}[[x_1, \ldots, x_n]] \to \mathbb{L}[x_1, \ldots, x_n]/S \cong \Omega^*(X)$ the power series $F(x_{i+1}, \chi(x_i))$ maps to $c_1(L(\gamma_i))$ (again, see Section 5), so our definition for additive formal group law reduces to the definition of divided difference operator on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ (see [14, 2.3.1]). Finally, we define the operator $A_\alpha : \Omega^*(X) \to \Omega^*(X)$ using the Borel presentation by the formula

$$A_\alpha(f(x_1, \ldots, x_n)) = \hat{A}_\alpha(f)(x_1, \ldots, x_n)$$

for each polynomial $f \in \mathbb{L}[x_1, \ldots, x_n]$. Again, by degree reasons the right hand side is a polynomial. The operator A_α is well defined (that is, does not depend on a choice of a polynomial f representing a given class in $\mathbb{L}[x_1, \ldots, x_n]/S$) since for any polynomial h and any symmetric polynomial g we have $\hat{A}_\alpha(gh) = g\hat{A}_\alpha(h)$.

We now define $A_i = A_{\alpha_i}$ for an arbitrary reductive group G and a simple root α_i. Denote by $P_i \subset G$ the minimal parabolic subgroup corresponding to the root α_i. Then $X = \mathbb{P}(E)$ is a projective line fibration over G/P_i. Indeed, we have $X = \mathbb{P}(E)$, where the vector bundle E of rank two over G/P_i is defined as in [3, p. 805]. Namely, we think of G/P_i as of the variety of parabolic subgroups in G conjugate to P_i and to each such parabolic $P \subset G$ assign the two-dimensional standard representation of the copy of SL_2 inside P. Denote by π_i the natural projection $X \to G/P_i$. We now use Definition 2.2 to define an $\Omega^*(G/P_i)$-linear operator $A_i := A_{\pi_i}$ on $\Omega^*(X)$. For $G = GL_n$, this definition coincides with the one given above. This is easy to show using that G/P_i for $\alpha_i = \gamma_i$ is the partial flag variety whose points are flags $F = \{\{0\} = F^0 \subset \ldots \subset F^{i-1} \subset F^{i+1} \subset \ldots \subset F^n = k^n\}$.

Let $I = (\alpha_1, \ldots, \alpha_l)$ be an l-tuple of simple roots of G. Define the element $\mathcal{R}_I$ in $\Omega^*(X)$ by the formula

$$\mathcal{R}_I := A_l \ldots A_1 Z_e.$$

In the case $G = GL_n$, we can also regard $\mathcal{R}_I$ as a polynomial in $\mathbb{L}[x_1, \dots, x_n]/S$. Similar to [1, Theorem 3.15] or [3, page 807], one may describe Z_e for general G using the formula

$$Z_e = \mathcal{R}_e := \frac{1}{|W|} \prod_{\alpha \in R^+} c_1(L(\alpha)),$$

where R^+ denotes the set of positive roots of G (recall that $|R^+| = \dim X$). We won't use this below. Note that for GL_n the formula for $\mathcal{R}_e$ reduces to $\mathcal{R}_e = \Delta_n$ since $c_1(L(e_i - e_j)) = x_j - x_i$+higher order terms, and hence the equality $Z_e = \mathcal{R}_e$ follows from Remark 2.8. In particular, by the same remark $\mathcal{R}_e$ modulo S has a denominator-free expression $x_n^{n-1} x_{n-1}^{n-2} \cdots x_2$.

Theorem 3.2. *The cobordism class $Z_I = [r_I : R_I \to X]$ of the Bott-Samelson resolution R_I is equal to $\mathcal{R}_I$.*

Proof. The essential part of the proof is our formula for the push-forward as stated in Corollary 2.3. Once this formula is established it is not hard to show that $A_i Z_I = Z_{I \cup \{i\}}$ for all I by exactly the same methods as in the Chow ring case [14] and in the complex cobordism case [3]. Namely, we have the following commutative square

$$\begin{tikzcd} G/B \times_{G/P_i} G/B \arrow[r, "p_2"] \arrow[d, "p_1"'] & G/B \arrow[d, "\pi_i"] \\ G/B \arrow[r, "\pi_i"] & G/P_i. \end{tikzcd}$$

E.g., if $G = GL_n$ we get exactly the diagram of [14, proof of Lemma 3.6.20]. Using this commutative diagram and the definition of Bott-Samelson resolutions it is easy to show that ${\pi_i}^* \pi_{i*} Z_I = Z_{I \cup \{i\}}$ [3, proof of Proposition 2.1]. We now apply Corollary 2.3 and get that $A_i = {\pi_i}^* \pi_{i*}$. It follows by induction on the length of I that $Z_I = A_l \dots A_1 Z_e$. ☐

Remark 3.3. *Note that if we apply (A2) to the commutative diagram from the proof of Theorem 3.2, we get $p_{1*} p_2^* = \pi_i^* \pi_{i*}$, where the right hand side is precisely the definition of the "geometric" operator denoted A_i in* [3]*, while the left hand side is the operator denoted δ_i in* [14, proof of Theorem 3.6.18]. *Hence Manivel and Bressler–Evens consider the same operators.*

We now compute the action of the operator A_i on polynomials in the first Chern classes (this computation will be used in Sections 4 and 5). Consider the operator $\sigma_i := \sigma_{\pi_i}$ again defined as in Definition 2.2. Note that σ_i corresponds to the simple reflection $s_i := s_{\alpha_i}$ in the following sense.

Lemma 3.4. *For any line bundle $L(\lambda)$ on X, we have*

$$\sigma_i(c_1(L(\lambda))) = c_1(L(s_i \lambda)).$$

Proof. Recall that by $P_i \subset G$ we have denoted the minimal parabolic subgroup corresponding to a simple root α_i, and by E the vector bundle of rank two on G/P_i that assigns to each parabolic subgroup $P \in G/P_i$ the two-dimensional standard representation of the copy of SL_2 inside P. Since $X = \mathbb{P}(E)$, the bundle $\pi_i^* E$ on X admits the usual short exact sequence

$$0 \to \tau_E \to \pi_i^* E \to \mathcal{O}_E(1) \to 0,$$

where τ_E is the tautological line bundle on X. E.g. for $G = GL_n$ we have $\tau_E = L_i$ and $\mathcal{O}_E(1) = L_{i+1}$. Similarly, for arbitrary reductive G we have that $\tau_E = L(-\chi_i)$ and $\mathcal{O}_E(1) = L(-\chi_{i+1})$, where χ_i and χ_{i+1} are characters given by the diagonal entries of B that correspond to the diagonal entries of $SL_2 \subset P_i$ (this follows from an alternative definition of the line bundle $L(\lambda)$ [4, Remark 4.2] in terms of homogeneous line bundles on X). Since $\alpha_i = \chi_i - \chi_{i+1}$, there is an isomorphism of line bundles

$$\tau_E^* \otimes \mathcal{O}_E(1) = L(\alpha_i).$$

By definition, σ_i switches $c_1(\tau_E)$ and $c_1(\mathcal{O}_E(1))$. Hence, $\sigma_i(c_1(L(\alpha_i))) = c_1(L(-\alpha_i))$. Since the Picard group of G/P_i can be identified with with the sublattice $\{\lambda|(\lambda, \alpha_i) = 0\}$ of the weight lattice of G (this follows from [4, remark after Proposition 1.3.6] combined with [4, Proposition 1.4.3]), and hence $\sigma_i(c_1(L(\lambda))) = c_1(L(\lambda))$ for all λ perpendicular to α_i, we get the desired identity. ☐

This lemma allows us to describe explicitly the action of σ_i and hence of A_i on any polynomial in the first Chern classes. Indeed, since for any weight λ we have $s_i\lambda = \lambda + k\alpha_i$ for some integer k, we can compute $c_1(L(\sigma_i\lambda)) = c_1(L(\lambda) \otimes L(\alpha_i)^k)$ as a power series in $c_1(L(\lambda))$ and $c_1(L(\alpha_i))$ using the formal group law. This will be used in the proof of Proposition 4.3 below and in Subsection 5.1.

4. Chevalley-Pieri formulas

A key ingredient for the classical Schubert calculus is the Chevalley-Pieri formula for the product of the Schubert cycle with the first Chern class of the line bundle on X, see e. g. [1, Proposition 4.1] and [8, Proposition 4.2]. We now establish analogous formulas for the products of Z_I and $\mathcal{R}_I$ with $c_1(L(\lambda))$ (without using that $Z_I = \mathcal{R}_I$). At the end of this section, we explain why in the case of algebraic cobordism this alone is not enough to show that $Z_I = \mathcal{R}_I$, hence justifying our different approach of the previous two sections.

By $L(D)$ denote the line bundle corresponding to the divisor D. For each l-tuple I as above, denote by I^j the $(l-1)$–tuple $(\alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_l)$. For each root α, define the linear function $(\cdot, \alpha)$ (that is, the *coroot*) on the weight lattice of G by the property $s_\alpha\lambda = \lambda - (\lambda, \alpha)\alpha$ for all weights λ. (The pairing (a, b) is often denoted by $\langle a, b^\vee\rangle$ or by $\langle a, b\rangle$.) Note that by definition $(\lambda, \alpha) = (w\lambda, w\alpha)$ for all elements w of the Weyl group.

Proposition 4.1. ***Geometric Chevalley-Pieri formula***

(1) (for Bott-Samelson resolutions) *In the Picard group of R_I we have*

$$r_I^* L(\lambda) = \otimes_{j=1}^{l} L(R_{I^j})^{(\lambda, \beta_j)}$$

where $\beta_j = s_l \cdots s_{j+1}\alpha_j$.

(2) (for Schubert cycles)[1, Proposition 4.1], [8, Proposition 4.4], [6] *In the Chow ring of X we have*

$$c_1(L(\lambda))X_I = \sum_j (\lambda, \beta_j)X_{I^j}$$

where the sum is taken over $j \in \{1, \ldots, l\}$ for which the decomposition defined by I^j is reduced.

The first part of this proposition was proved in [3, Proposition 4] in the topological setting (for flag varieties of compact Lie groups). It is not hard to check that the proof carries over to algebro-geometric setting. We instead provide a shorter proof along the same lines. Our proof is based on the following lemma.

Lemma 4.2. [8, Proposition 2.1] *Let $p : R_I \to R_{I^l}$ be the natural projection (coming from the fact that we defined R_I as a projective bundle over R_{I^l}). Then we have an isomorphism*

$$r_I^*L(\lambda) \cong p^* r_{I^l}^* L(s_l\lambda) \otimes L(R_{I^l})^{(\lambda,\alpha_l)}$$

of line bundles on R_I.

Proposition 4.1(1) now follows from Lemma 4.2 by induction on l. The base $l = 1$, that is $r_1^*L(\lambda) = \mathcal{O}_{\mathbb{P}^1}(1)^{(\lambda,\alpha_1)}$, follows from the fact that $r_1 : R_1 \to X$ maps R_1 isomorphically to $P_1/B \cong \mathbb{P}^1$, which is the flag variety for a copy of $SL_2 \subset G$ with the only simple root α_1. Then the weight λ restricted to the SL_2 is equal to (λ, α_1) times the highest weight of the tautological representation of SL_2, which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on the flag variety of SL_2. To prove the induction step plug in the induction hypothesis for $r_{I^l}^*L(s_l\lambda) = \otimes_{j=1}^{l-1} L(R_{I^{j,l}})^{(s_l\lambda, s_{l-1}\cdots s_{j+1}\alpha_j)}$ into the lemma and use that $(s_l\lambda, s_{l-1}\cdots s_{j+1}\alpha_j) = (\lambda, \beta_j)$ (since $s_l^2 = e$) and $p^*R_{I^{j,l}} = R_{I^j}$.

Proposition 4.1(1) was used in [3] to establish an algorithm for computing $c_1(L(\lambda))Z_I$ in $\Omega^*(X)$ [3]. We now briefly recall this algorithm. By the projection formula we have

$$c_1(L(\lambda))Z_I = (r_I)_*(c_1(r_I^*L(\lambda))).$$

Note that the usual projection formula with respect to smooth projective morphisms $f : X \to Y$ holds for algebraic cobordism as well. This follows from the definition of products via pullbacks along the diagonal and the base change axiom ($A2$) of [12] applied to the cartesian square obtained from $Y \overset{diag}{\to} Y \times Y \overset{p\times id}{\leftarrow} X \times Y$.

One can now use Proposition 4.1(1) and the formal group law to compute $c_1(r_I^*L(\lambda))$ in terms of the Bott-Samelson classes in $\Omega^*(R_I)$ by an iterative procedure (since the multiplicative structure of $\Omega^*(R_I)$ can be determined by the projective bundle formula and the Chern classes arising this way again have form $c_1(L(\lambda))$ for some λ). After $c_1(r_I^*L(\lambda))$ is written as $\sum_{J\subset I} a_J[R_J]$ for some $a_J \in \mathbb{L}$ it is easy to find $(r_I)_*(c_1(r_I^*L(\lambda)))$ since $(r_I)_*[R_J] = Z_J$.

However, this procedure is rather lengthy, and we will not use it. Instead, we will prove a more explicit formula for $c_1(L(\lambda))Z_I$ (see formula 5.1 below) using our algebraic Chevalley-Pieri formula together with Theorem 3.2.

Proposition 4.3. ***Algebraic Chevalley-Pieri formula:***

(1) (cobordism version) *Let $A_1 = A_{\alpha_1}, \ldots, A_l = A_{\alpha_l}$ be the operators on $\Omega^*(X)$ corresponding to $\alpha_1, \ldots, \alpha_l$. Then we have*

$$c_1(L(\lambda))A_1 \ldots A_l\mathcal{R}_e = \sum_{j=1}^{l} A_1 \ldots A_{j-1}\frac{c_1(L(\lambda_j)) - c_1(L(s_j\lambda_j))}{c_1(L(\alpha_j))}A_{j+1}\ldots A_l\mathcal{R}_e$$

in $\Omega^*(X)$, where $\lambda_j = s_{j-1}\cdots s_1\lambda$ and $s_j = s_{\alpha_j}$ is the reflection corresponding to the root α_j.

(2) (Chow ring version) *Let $A_1 = A_{\alpha_1}, \ldots, A_l = A_{\alpha_l}$ be the operators on $CH^*(X)$ corresponding to $\alpha_1, \ldots, \alpha_l$. Then*

$$c_1(L(\lambda))A_1 \ldots A_l\mathcal{R}_e = \sum_{j=1}^{l}(\lambda, s_1\cdots s_{j-1}\alpha_j)A_1 \ldots \hat{A}_j \ldots A_l\mathcal{R}_e$$

in $CH^*(X)$.

Proof. First, note that $\frac{c_1(L(\lambda_j))-c_1(L(s_j\lambda_j))}{c_1(L(\alpha_j))}$ is a well-defined element in $\Omega^*(X)$ because $s_j\lambda = \lambda - (\lambda,\alpha_j)\alpha_j$ (and hence $L(\lambda) = L(s_j\lambda)\otimes L(\alpha_j)^{(\lambda,\alpha_j)}$) and the formal group law expansion for $c_1(L_1\otimes L_2^k) - c_1(L_1)$ is divisible by $c_1(L_2)$ for any integer k [12, (2.5.1)]. Next we show that

$$c_1(L(\lambda))A_1 - A_1c_1(L(s_1\lambda)) = \frac{c_1(L(\lambda)) - c_1(L(s_1\lambda))}{c_1(L(\alpha_1))},$$

where both sides are regarded as operators on $\Omega^*(X)$. Indeed, by definition $A_1 = (1+\sigma_1)\frac{1}{c_1(L(\alpha_1))}$ and $c_1(L(\lambda))\sigma_1 = \sigma_1c_1(L(s_1\lambda))$ by Lemma 3.4.

Hence, we can write

$$c_1(L(\lambda))A_1\dots A_l\mathcal{R}_e = \frac{c_1(L(\lambda)) - c_1(L(s_1\lambda))}{c_1(L(\alpha_1))}A_2\dots A_l\mathcal{R}_e + A_1c_1(L(s_1\lambda))A_2\dots A_l\mathcal{R}_e,$$

and then continue moving $c_1(L(s_1\lambda))$ to the right until we are left with with the term $A_1\dots A_lc_1(L(s_l\dots s_1\lambda))\mathcal{R}_e$. This term is equal to zero since $c_1(L(s_l\dots s_1\lambda))\mathcal{R}_e$ is the product of more than $\dim X$ first Chern classes, and hence its degree is greater than $\dim X$. The Chow ring case follows immediately from the cobordism case since

$$\frac{c_1(L(\lambda)) - c_1(L(s_j\lambda))}{c_1(L(\alpha_j))} = (\lambda,\alpha_j)$$

in the Chow ring. The last identity holds because the formal group law for the Chow ring is additive, and hence $c_1(L(\lambda)) - c_1(L(s_j\lambda)) = (\lambda,\alpha_j)c_1(L(\alpha_j))$. □

The second part of this proposition was proved in [1] by more involved calculations. A calculation similar to ours was used in [16] to deduce a combinatorial Chevalley-Pieri formula for K-theory. It would be interesting to find an analogous combinatorial interpretation of our Chevalley-Pieri formula in the cobordism case.

Note that in the case of Chow groups, the algebraic Chevalley-Pieri formula for $A_l\dots A_1\mathcal{R}_e$ is exactly the same as the geometric one for the Schubert cycle X_I. Together with the Borel presentation this easily implies that the polynomial $A_l\dots A_1\mathcal{R}_e$ represents the Schubert cycle X_I whenever I defines a reduced decomposition [1]. Indeed, algebraic and geometric Chevalley-Pieri formulas allow to compute the intersection indices of $A_l\dots A_1\mathcal{R}_e$ and of X_I, respectively, with any nonconstant monomial of complementary dimension, and the result is the same in both cases. Hence, by the non-degeneracy of the intersection form (that is, by Poincaré duality) we have that $A_l\dots A_1\mathcal{R}_e$ must be equal to X_I. Note that the only geometric input in this proof is the geometric Chevalley-Pieri formula.

In the cobordism case, it is not immediately clear why geometric and algebraic Chevalley-Pieri formulas are the same (though, of course, it follows from Theorem 3.2). But even without using that $\mathcal{R}_I = Z_I$ it might be possible to show that both formulas have the same structure coefficients, that is, if $c_1(L(\lambda))Z_I = \sum_{J\subset I} a_JZ_J$ then necessarily $c_1(L(\lambda))\mathcal{R}_I = \sum_{J\subset I} a_J\mathcal{R}_J$ with the same coefficients $a_J\in\mathbb{L}$. However, this does not lead to the proof of $\mathcal{R}_I = Z_I$ as in the case of the Chow ring. The reason is that even if there is an analog of Poincaré duality for cobordism ring (see Lemma 4.4 below for such a statement for $X = GL_n/B$), this only yields an equality $\mathcal{R}_I = Z_I$ up to a multiple of $[pt]$, and this is not enough to carry out the desired induction argument. For the Chow ring, Poincaré duality also only yields an equality up to the class of a point, but unless $I = \emptyset$ (in which case the desired equality can be checked explicitly), the difference $\mathcal{R}_I - Z_I$ (where now Z_I means the Schubert cycle and not the Bott-Samelson class) can not be a non-zero multiple of $[pt]$ because the coefficient ring $CH^*([pt]) = CH^*(k) \cong \mathbb{Z}$ is concentrated in degree zero, hence has no nonzero elements in the corresponding degree $l-d$. However, for algebraic cobordism the coefficient ring $\Omega^*(k) \cong \mathbb{L}$ does contain plenty of elements of negative degree, so one can not deduce $\mathcal{R}_I = Z_I$.

Lemma 4.4. *Let $X = GL_n/B$ and let $c \in \Omega^*(X) \cong \mathbb{L}[x_1, \dots, x_n]/S$ be a homogeneous element of degree l such that the product of c with any non-constant monomial in $x_1, \dots, x_n$ is zero. Then c belongs to $\mathbb{L}^{l-d}[pt]$.*

Proof. We will use that the ideal S contains all homogeneous polynomials of degree greater than d with integer coefficients [14, Corollary 2.5.6]. Let $\hat{c}$ be a homogenous element in $\mathbb{L}[x_1, ..., x_n]$ that represents c. Recall that $\mathbb{L}$ is isomorphic to the graded polynomial ring $\mathbb{Z}[a_1, a_2, \dots]$ in countably many variables, where a_i has degree $-i$. Therefore $\hat{c}$ has a unique decomposition as a sum of integral polynomials with coefficients being monomials in the a_i, that is

$$\hat{c} = c_0 + a_1c_1 + a_2c_2 + a_1^2c_{1,1} + a_3c_3 + a_1a_2c_{1,2} + a_1^3c_{1,1,1} + \dots$$

where $c_{i_1,\dots,i_s}$ is a polynomial of degree $l - \sum \deg(a_{i_j})$ with integer coefficients. Note that we might choose a $\hat{c}$ such that the sum is finite since $c_{i_1,\dots,i_s}$ vanishes modulo S if $l - \sum \deg(a_{i_j}) > d$. Now we multiply $\hat{c}$ with an arbitrary monomial m_{d-l} in the x_i of degree $d-l$. Since $m_{d-l}c = 0$ it follows that $m_{d-l}c_0$ is zero modulo S. By algebraic Poincaré duality [14, Proposition 2.5.7] it follows that $c_0 = 0$ modulo S. Next, we multiply with monomials of degree $d-l-1$ to deduce that c_1 equals zero modulo S, and then deduce inductively that all the $c_{i_1,\dots,i_s}$ of degree strictly less than d are zero. It remains to note that each $c_{i_1,\dots,i_s}$ of degree d is equal to an integer multiple of $[pt]$ (since all homogeneous polynomials of degree d with integer coefficients are equal to a multiple of $\mathcal{R}_e$ modulo the ideal S [14, 2.5.2]) Hence $\hat{c} = a[pt]$ for some $a \in \mathbb{L}$, which must have degree $l-d$ by homogeneity of $\hat{c}$. □

5. Computations and examples

Until now, we used the formal group law of algebraic cobordism (i.e., the universal one) as little as possible in order to make our presentation simpler. In this section, we make the results of the previous section more explicit using this formal group law. In particular, we give an explicit formula for the products of a Bott-Samelson resolution with the first Chern class of a line bundle in terms of other Bott-Samelson resolutions (see formula 5.1 below). Using this formula, we give an algorithm for computing the product of two Bott-Samelson resolutions.

First, we tie two loose ends from the previous sections. First, we show that the operator A from Section 2 and the operator $\hat{A}_\alpha$ from Section 3 are well-defined. Second, in the case $G = GL_n$, we provide an explicit formula for $c_1(L(\alpha))$.

We use notation of Subsection 2.1, so $F(u,v)$ is the universal formal group law and $\chi(u)$ is the inverse for the universal formal group law defined by the identity $F(u, \chi(u)) = 0$. To show that the operator $A = (1+\sigma)\frac{1}{F(y_1,\chi(y_2))}$ is well defined on $\Omega^*(X)[[y_1, y_2]]$ it is enough to show that $A(m)$ is a formal power series for any monomial $m = y_1^{k_1}y_2^{k_2}$. We compute $A(y_1^{k_1}y_2^{k_2})$ using that $y_1 = F(x, y_2) = y_2 + \chi(x)p(x, y_2)$ and $y_2 = F(\chi(x), y_1) = y_1 + \chi(x)p(\chi(x), y_1)$ where $x = F(y_1, \chi(y_2))$ and $p(u,v) = \frac{F(u,v)-u}{v}$ is a well-defined power series (since $F(u,v) - u$ contains only terms u^iv^j for $j\geq 1$). We get

$$A(y_1^{k_1}y_2^{k_2})=(1+\sigma)\frac{y_1^{k_1}y_2^{k_2}}{x}=\frac{y_1^{k_1}y_2^{k_2}}{x}+\frac{y_1^{k_2}y_2^{k_1}}{\chi(x)}=\frac{(y_2+p(x,y_2))^{k_1}(y_1+\chi(x)p(\chi(x),y_1))^{k_2}}{x}+\frac{y_2^{k_1}y_1^{k_2}}{\chi(x)}=$$

$$=y_2^{k_1}y_1^{k_2}q(x,\chi(x))+\frac{\text{terms divisible by } x \text{ or by } \chi(x)}{x}.$$

The second term in the last expression is a power series since the formal group law expansion for $\chi(x)$ is divisible by x [12, (2.5.1)].

A similar argument shows that the operator $\hat{A}_\alpha$ from Section 3 is indeed well-defined on the whole ring $\mathbb{L}[[x_1,\ldots,x_n]]$ for any root α. Indeed, by relabeling $x_1,\ldots, x_n$ we can assume that $\alpha=e_1-e_2$. Then for any monomial $m=x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$ we have

$$\hat{A}_\alpha(m)=x_3^{k_3}\ldots x_n^{k_n}\hat{A}_\alpha(x_1^{k_1}x_2^{k_2}).$$

Then exactly the same argument as the one above for A shows that $\hat{A}_\alpha(x_1^{k_1}x_2^{k_2})$ is a power series in x_1 and x_2.

We now compute $c_1(L(\alpha_i))$ as a polynomial in $x_1,\ldots, x_n$. Let $\gamma_1,\ldots,\gamma_{n-1}$ be the simple roots of G. Our choice of simple roots is compatible with the choice of the Borel subgroup B used to define the Schubert cells in the beginning of Section 3. In particular, $L(X_{w_0s_{\gamma_i}})=L(\omega_i)$, where $\omega_1,\ldots,\omega_{n-1}$ are fundamental weights (that is, they are defined by the property $(\omega_i,\gamma_j)=\delta_{ij}$) and $w_0\in W$ is the longest element. This can be checked for instance by applying Proposition 4.1(2) to the line bundle $L(\omega_i)$ and using that $w_0s_{\gamma_1},\ldots,w_0s_{\gamma_{n-1}}$ are the only elements of length $l(w_0)-1$ in W. We can also express the line bundles $L(\omega_i)$ and $L(\gamma_i)$ in terms of the line bundles $L_1,\ldots,L_n$ (used to define $x_1,\ldots,x_n$ in Theorem 2.7). Recall that $L_i=L(-e_i)$ where e_i is the weight of GL_n given by the i-th entry of the diagonal torus in GL_n. We can easily express the simple roots γ_i in terms of e_i, namely, $\gamma_i=e_i-e_{i+1}$. Hence, the line bundles $L(\gamma_i)$ and $L(\omega_i)$ are isomorphic to $L_i^*\otimes L_{i+1}$ and $L_{i+1}\otimes\ldots\otimes L_n$, respectively. In particular, we can compute

$$c_1(L(\gamma_i))=c_1(L_i^*\otimes L_{i+1})=F(\chi(x_i),x_{i+1}).$$

E.g. by formulas for $F(x,y)$ and $\chi(x)$ from [12, 2.5] the first few terms of $c_1(L(\gamma_i))$ look as follows

$$c_1(L(\gamma_i))=-x_i+x_{i+1}+a_{11}x_i^2-a_{11}x_ix_{i+1}+\ldots,$$

where $a_{11}=-[\mathbb{P}^1]$.

Note however that in computations involving the operators A_α it is more convenient not to replace $c_1(L(\alpha))$ with its expression in terms of x_i until the very end.

We now describe the isomorphism $W\cong S^n$ used to define the operator σ_α in Section 3. The simple reflection s_α for any root $\alpha=e_i-e_j$ acts on the weight lattice (spanned by the weights $e_1,\ldots,e_n$, which form an orthonormal basis) by the reflection in the plane perpendicular to e_i-e_j and hence permutes the weights $e_1,\ldots, e_n$ by the transposition $(i\ j)$.

5.1. **Algorithm for computing the products of Bott-Samelson resolutions.** We now produce an explicit algorithm for computing the product of the Bott-Samelson classes Z_I in terms of other Bott-Samelson classes, where $I=(\alpha_1,...,\alpha_l)$. The key ingredient is our algebraic

Chevalley-Pieri formula (Proposition 4.3) which can be reformulated as follows

$$c_1(L(\lambda))A_1\dots A_lZ_e=\sum_{j=1}^{l}A_1\dots A_{j-1}A_j^*(c_1(L(\lambda_j)))A_{j+1}\dots A_lZ_e,$$

where $\lambda_j=s_{j-1}\cdots s_1\lambda$ (in other words, $c_1(L(\lambda_j))=[\sigma_{j-1}\dots\sigma_1](c_1(L)))$ and the operator A_j^* is defined as follows

$$A_j^*=A_{\alpha_j}^*=\frac{1}{c_1(L(\alpha_j))}(1-\sigma_{\alpha_j}).$$

We can compute A_j^* on any polynomial in the first Chern classes by the same methods as A_j (see the end of Section 3). Note that for the Chow ring $A_j=A_j^*$ but for the algebraic cobordism ring this is no longer true.

More generally, for any polynomial $f=f(c_1(L(\mu_1)),\dots,c_1(L(\mu_k)))$ in the first Chern classes of some line bundles on X, exactly the same argument as in the proof of Proposition 4.3 implies

$$fA_1\dots A_lZ_e=\sum_{j=1}^{l}A_1\dots A_{j-1}[A_j^*\sigma_{j-1}\dots\sigma_1](f)A_{j+1}\dots A_lZ_e+A_1\dots A_l[\sigma_l\dots\sigma_1](f)Z_e \quad (5.0)$$

Note that the last term on the right hand side is equal to the constant term of the polynomial $[\sigma_l\dots\sigma_1](f)$ (which is of course the same as the constant term of f) times $A_1\dots A_lZ_e$. In particular, for $f=c_1(L(\lambda))$ this term vanishes modulo S. Here and below, by the "constant term" of a polynomial in $\mathbb{L}[x_1,\dots,x_n]$ we mean the term of polynomial degree zero (the total degree of such a constant term might be negative since the Lazard ring $\mathbb{L}$ contains elements of negative degree). Note that all elements of $\mathbb{L}\subset\mathbb{L}[x_1,\dots,x_n]$ are invariant under the operators σ_i, and hence commute with the operators A_i. For an arbitrary reductive group, the constant term of an element $f\in\Omega^*(X)$ is defined as the product of f with the class of a point.

It is now easy to show by induction on l that

$$fA_1\dots A_lZ_e=\sum_{J\subset I}a_J(f)[\prod_{i\in I\setminus J}A_i]Z_e,$$

where $a_J(f)$ for the k-subtuple $J=(\alpha_{j_1},\dots,\alpha_{j_k})$ of I is the constant term in the expansion for $[\sigma_l\dots\sigma_{j_k+1}A_{j_k}^*\sigma_{j_k-1}\dots\sigma_{j_1+1}A_{j_1}^*\sigma_{j_1-1}\dots\sigma_1]f$, which is invariant under σ_α and hence equal to $[A_{j_k}^*\sigma_{j_k-1}\dots\sigma_{j_1+1}A_{j_1}^*\sigma_{j_1-1}\dots\sigma_1]f$. Indeed, we first use formula (5.0) above and then apply the induction hypothesis to all terms in the right hand side except for the last term, which already has form $a_J(f)[\prod_{i\in I\setminus J}A_i]Z_e$ for $J=\emptyset$. We get

$$A_1\dots A_{j-1}[A_j^*\sigma_{j-1}\dots\sigma_1](f)A_{j+1}\dots A_lZ_e=$$

$$=A_1\dots A_{j-1}\sum_{J\subset I\setminus\{1,\dots,j\}}a_J([A_j^*\sigma_{j-1}\dots\sigma_1](f))[\prod_{i\in I\setminus(J\cup\{1,\dots,j\})}A_i]Z_e=$$

$$=\sum_{J'\subset I}a_{J'}(f)[\prod_{i\in I\setminus J'}A_i]Z_e,$$

where the last summation goes over all subsets J' of I that do contain j but do not contain 1,…,$j-1$. Plugging this back into formula (5.0) we get the desired formula. Combining this with Theorem 3.2, we get the following formula in $\Omega^*(X)$ for the product of the Bott-Samelson class Z_I with the first Chern class $c_1(L(\lambda))$ in terms of other Bott-Samelson classes

$$c_1(L(\lambda))Z_I = \sum_{J\subset I} b_J(\lambda)Z_{I\setminus J}, \tag{5.1}$$

where $b_J(\lambda)$ is the constant term in the expansion for

$$[A^*_{j_1}\sigma_{j_1+1}\dots\sigma_{j_k-1}A^*_{j_k}\sigma_{j_k+1}\dots\sigma_l](c_1(L(\lambda))).$$

We changed the order of the σ_i when passing from a_J to b_J since $Z_I = A_l\dots A_1Z_e$. Note that for $J=\emptyset$ we have $b_J=0$, and for $J=(\alpha_j)$ we have $b_J=(\lambda,\beta_j)$ since the constant term in $A^*_j(c_1(L(s_{j+1}\dots s_l\lambda))) = \frac{c_1(L(s_{j+1}\dots s_l\lambda))-c_1(L(s_js_{j+1}\dots s_l\lambda))}{c_1(L(\alpha_j))}$ is equal to $(s_{j+1}\dots s_l\lambda,\alpha_j)$ (see the proof of Proposition 4.3, and Proposition 4.1 for the definition of β_i), which is equal to (λ,β_j). So the first order terms of this formula give an analogous formula for the Chow ring as expected. Comparing the higher order terms might lead to further interesting identities in the Chow ring. To compute the product Z_IZ_J we follow the usual procedure (namely the same as for the Chow ring). That is, we replace Z_J with the respective polynomial $\mathcal{R}_J$ in the first Chern classes and then compute the product of Z_I with each monomial in $\mathcal{R}_J$ using repeatedly formula (5.1). Note that formula (5.1) allows us to make this algorithm more explicit than the one given in [3] (see an example below).

The naive approach to represent both Z_I and Z_J as fractions of polynomials in first Chern classes and then computing their product is less useful. In particular translating the product of the fractions back into a polynomial of Bott-Samelson classes will be very hard, if possible at all.

5.2. **Examples.** We now compute the Bott-Samelson classes Z_I in terms of the Chern classes x_i for the example $X = SL_3/B$ where B is the subgroup of upper-triangular matrices. We then compute certain products of Bott-Samelson classes in two ways, by hand and then using the algorithm above together with formula (5.1). Note that only the second approach generalizes to higher dimensions.

In SL_3, there are two simple roots γ_1 and γ_2. In X, there are six Schubert cycles $X_e = pt$, X_1, X_2, X_{12}, X_{21} and $X_{121} = X$ (here 12 is a short hand notation for (γ_1,γ_2), etc.). Each X_I except for X_{121} coincides with its Bott-Samelson resolution R_I. Note that in general R_I and X_I do not coincide even when X_I is smooth. (By the way, for $G = GL_n$ the first non-smooth Schubert cycles show up for $n = 4$.)

Computing Z_I as a polynomial in the first Chern classes. We have $L_1 = L(X_{12})^*$, $L_2 = L(X_{12})\otimes L(X_{21})^*$ and $L_3 = L(X_{21})$. This immediately follows from the identities $L(X_{w_0s_i}) = L_{i+1}\otimes\dots\otimes L_n$ established in the beginning of this section for the flag variety of GL_n. We now want to express Z_I as a polynomial in x_1, x_2, x_3 using the formulas

$$Z_{s_{i_1}\dots s_{i_l}} = A_{i_l}\dots A_{i_1}\mathcal{R}_e;\quad \mathcal{R}_e = \frac{1}{6}c_1(L(\gamma_1))c_1(L(\gamma_2))c_1(L(\gamma_1+\gamma_2)).$$

Let us for instance compute R_1 as a polynomial in x_1, x_2, x_3 modulo the ideal S generated by the symmetric polynomials of positive degree:

$$\mathcal{R}_1 = A_1\mathcal{R}_e = \frac{1}{6}(1+s_1)c_1(L(\gamma_2))c_1(L(\gamma_1+\gamma_2)) =$$

$$\frac{1}{3}c_1(L(\gamma_2))c_1(L(\gamma_1+\gamma_2)) = \frac{1}{3}F(\chi(x_2),x_3)F(\chi(x_1),x_3).$$

We have $\chi(u) = -u + a_{11}u^2 - a_{11}^2u^3$ and $F(u,v) = u + v + a_{11}uv + a_{12}u^2v + a_{21}uv^2$, where $a_{11} = -[\mathbb{P}^1]$ and $a_{12} = a_{21} = [\mathbb{P}^1]^2 - [\mathbb{P}^2]$ [12, 2.5]. Thus

$$\frac{1}{3}F(\chi(x_2),x_3)F(\chi(x_1),x_3) = \frac{1}{3}F(-x_2 + a_{11}x_2^2 - a_{11}x_2^3, x_3)F(-x_1 + a_{11}x_1^2 - a_{11}x_1^3, x_3) =$$

$$= \frac{1}{3}(-x_2 + x_3 + a_{11}x_2^2 - a_{11}x_2x_3)(-x_1 + x_3 + a_{11}x_1^2 - a_{11}x_1x_3) = x_3^2,$$

since $(x_3 - x_2)(x_3 - x_1) = 3x_3^2 \mod S$, and $(x_2 + x_1)(x_2 - x_3)(x_1 - x_3) = -3x_3^3 = 0 \mod S$. So the answer agrees with what we got below by a different method. Here are the polynomials for the other Bott-Samelson resolutions:

$$\mathcal{R}_{121} = \mathcal{R}_{212} = 1$$

$$\mathcal{R}_{12} = -x_1 - [\mathbb{P}^1]x_1^2 \quad \mathcal{R}_{21} = x_3 = -x_1 - x_2$$

$$\mathcal{R}_1 = x_3^2 = x_1x_2 \quad \mathcal{R}_2 = x_1^2$$

$$\mathcal{R}_e = x_1^2x_2.$$

So these polynomials coincide with the classical Schubert polynomials (see e.g. [14] and keep in mind that his x_i is equal to our $-x_i$) except for the polynomial for Z_{12}.

Computing products of the Bott-Samelson resolutions. Let us for instance compute $Z_{12}Z_{21}$. First, we do it by hand.

Applying Proposition 4.1(2) to $X_{121} = X$ we get

$$L(\lambda) = L(X_{21})^{(\lambda,\gamma_2)} \otimes L(X_{12})^{(\lambda,\gamma_1)}.$$

Hence, $c_1(L(\omega_1)) = X_{12}$ and $c_1(L(\omega_2)) = X_{21}$. (Note that if we instead applied Proposition 4.1(1) to R_{121}, we would obtain the more complicated expression

$$r_{121}^*L(\lambda) = L(R_{21})^{(\lambda,\gamma_2)} \otimes L(R_{11})^{(\lambda,\gamma_1+\gamma_2)} \otimes L(R_{12})^{(\lambda,\gamma_1)},$$

which does not allow us to express $X_{12} = R_{12}$ as the Chern class of the line bundle $L(\lambda)$ on R_{121}.)

Hence,

$$Z_{12}Z_{21} = c_1(L(\omega_1))Z_{21} = r_{21*}c_1(r_{21}^*L(\omega_1))$$

by the projection formula: $c_1(L(\lambda)) \cdot Z_I = r_{I*}c_1({r_I}^*L(\lambda))$. We now apply Proposition 4.1(1) to R_{21} and $L(\omega_1)$ and get $r_{21}^*L(\omega_1) = L(R_1) \otimes L(R_2)$. Using the formal group law we compute $c_1(L(R_1) \otimes L(R_2)) = R_1 + R_2 - [\mathbb{P}^1]R_e$. Finally, we use that $r_{J*}[R_J] = Z_J$ and get that

$$Z_{12}Z_{21} = Z_1 + Z_2 - [\mathbb{P}^1]Z_e.$$

Similarly, we can easily compute the following products:

$$Z_{12}Z_{12} = Z_2; \quad Z_{21}Z_{21} = Z_1$$

$$Z_{12}Z_1 = Z_{21}Z_2 = Z_e, \quad Z_{12}Z_2 = Z_{21}Z_1 = 0,$$

which in particular gives us another way to compute polynomials $\mathcal{R}_I$.

So the only product that differs from the analogous product in the Chow ring case is the product $Z_{12}Z_{21}$.

We now compute the product $Z_{12}Z_{21}$ by using formula (5.1). We have $Z_{12} = c_1(L(\omega_1))$ by Proposition 4.1(1). Hence, according to formula (5.1)

$$Z_{12}Z_{21} = c_1(L(\omega_1))Z_{21} = b_1(\omega_1)Z_2 + b_2(\omega_1)Z_1 + b_{21}(\omega_1)Z_e,$$

where b_1, b_2 and b_{21} are the constant terms in $A_1^*(c_1(L(\omega_1)))$, $[A_2^*s_1](c_1(L(\omega_1)))$ and $[A_2^*A_1^*](c_1(L(\omega_1)))$, respectively. We already know that $b_1(\lambda) = (\lambda, \gamma_1)$ and $b_2(\lambda) = (\lambda, s_1\gamma_2)$. It remains to compute $b_{21}(\lambda)$. First, by using that $L(\lambda) = L(s_1\lambda) \otimes L(\gamma_1)^{(\lambda,\gamma_1)}$ and the formal group law we write

$$A_1^*(c_1(L(\lambda))) = \frac{c_1(L(\lambda)) - c_1(L(s_1\lambda))}{c_1(L(\gamma_1))} =$$

$$= (\lambda, \gamma_1) + a_{11}(\lambda, \gamma_1)[c_1(L(s_1\lambda)) + \frac{(\lambda, \gamma_1) - 1}{2}c_1(L(\gamma_1))] + \text{terms of deg} \geq 2$$

Hence,

$$[A_2^*A_1^*](c_1(L(\lambda))) = a_{11}(\lambda, \gamma_1)A_2^*[c_1(L(s_1\lambda)) + \frac{(\lambda, \gamma_1) - 1}{2}c_1(L(\gamma_1))] + \text{terms of deg} \geq 1 =$$

$$= a_{11}(\lambda, \gamma_1)[(\lambda, s_1a_2) - \frac{(\lambda, \gamma_1) - 1}{2}] + \text{terms of deg} \geq 1,$$

and $b_{21} = a_{11}(\lambda, \gamma_1)[(\lambda, s_1\gamma_2) - \frac{(\lambda,\gamma_1)-1}{2}]$. We get

$$c_1(L(\lambda))Z_{21} = (\lambda, \gamma_1)Z_2 + (\lambda, s_1\gamma_2)Z_1 + a_{11}(\lambda, \gamma_1)[(\lambda, s_1\gamma_2) - \frac{(\lambda, \gamma_1) - 1}{2}]Z_e.$$

In particular, $c_1(L(\omega_1))Z_{21} = Z_2 + Z_1 + a_{11}Z_e$ (which coincides with the answer we have found above by hand).

Finally, note that it takes more work to compute $c_1(L(\lambda))Z_{21}$ using the algorithm in [3] because apart from certain formal group law calculations (which are more involved than the calculations we used to find b_{21}) one has also to compute the products R_1^2 and R_2^2 in $CH^*(R_{21})$.

References

[1] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand, *Schubert cells, and the cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), no. 3, 1–26.

[2] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math., **57** (1953), 115-207

[3] P. Bressler, S. Evens, *Schubert calculus in complex cobordism*, Trans. Amer. Math. Soc. **331** (1992), no. 2, 799–813.

[4] M. Brion, *Lectures on the geometry of flag varieties,* Topics in cohomological studies of algebraic varieties, 33–85, Trends Math., Birkhäuser, Basel, 2005

[5] M. Brion, S. Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory,* Progress in Mathematics, **231** Birkhäuser Boston, Inc., Boston, MA, 2005

[6] C. Chevalley, *Sur les décompositions cellulaires des espaces G/B*, With a foreword by Armand Borel. Proc. Sympos. Pure Math., **56**, Part 1, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), 1–23, Amer. Math. Soc., Providence, RI, 1994

[7] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. math. **21** (1973), 287–301.

[8] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I. Ann. Sci. École Norm. Sup. 4 **7** (1974), 53–88.

[9] W. Fulton, *Intersection theory*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin, 1998.

[10] M. Lazard, *Sur les groupes de Lie formels à un paramètre,* Bull. Soc. Math. France **83** (1955), 251-274.

[11] M. Levine, *Comparison of cobordism theories*, preprint 2008 arXiv:0807.2238v1 [math.KT].

[12] M. Levine, F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics. Springer, Berlin, 2007.

[13] M. Levine, R. Pandharipande, *Algebraic cobordism revisited*, Invent. math. **176** (2009), 63–130

[14] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, translated from the 1998 French original by John R. Swallow. SMF/AMS Texts and Monographs, 6. Cours Spécialisés, 3. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001.

[15] A. Nenashev, K. Zainoulline, *Oriented cohomology and motivic decompositions of relative cellular spaces*, J. Pure Appl. Algebra **205** (2006), no. 2, 323–340.

[16] H. Pittie, A. Ram, *A Pieri-Chevalley formula in the K-theory of a G/B-bundle*, Electron. Res. Announc. Amer. Math. Soc. **5** (1999), 102–107 (electronic)

[17] E. Shinder, *Mishchenko and Quillen formulas for oriented cohomolgy pretheories*, St. Petersburg J. Math. **18** (2007), 671–678.

[18] A. Vishik, N. Yagita, *Algebraic cobordisms of a Pfister quadric*, J. Lond. Math. Soc. (2) **76** (2007), no. 3, 586–604.

[19] N. Yagita, *Algebraic cobordism of simply connected Lie groups*, Math. Proc. Cambridge Philos. Soc. **139** (2005), no. 2, 243–260.