

# GELFAND-ZETLIN POLYTOPES AND FLAG VARIETIES.

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Abstract: I construct a correspondence between the Schubert cycles on the variety of complete flags in  $\mathbb{C}^n$  and some faces of the Gelfand-Zetlin polytope associated with the irreducible representation of  $SL_n(\mathbb{C})$  with a strictly dominant highest weight. The construction is based on a geometric presentation of Schubert cells by Bernstein–Gelfand–Gelfand [2] using Demazure modules. The correspondence between the Schubert cycles and faces is then used to interpret the classical Chevalley formula in Schubert calculus in terms of the Gelfand-Zetlin polytopes. The whole picture resembles the picture for toric varieties and their polytopes.

## 1. INTRODUCTION

Let  $G$  be the group  $SL_n(\mathbb{C})$ , and  $X = G/B$  the flag variety for  $G$  (here  $B \subset G$  denotes a Borel subgroup). The main goal of this paper is to translate to the flag variety some of the rich interplay that exists between geometry of toric varieties and combinatorics of convex polytopes. As in the case of toric varieties, there is a polytope  $P_H$ , namely *Gelfand-Zetlin polytope*, naturally associated with each very ample divisor  $H$  on  $X$ . For a toric variety, an analogous polytope associated with a divisor  $H$  gives information about torus orbits in the toric variety and their intersection products with  $H$ . For the flag variety, I will show how to extract similar information about *Schubert cycles* in  $X$  and their intersection products with  $H$  using the Gelfand-Zetlin polytope  $P_H$ . In particular, the classical Chevalley formula can be reformulated nicely in terms of Gelfand-Zetlin polytopes (see Theorem 1.1 and Theorem 5.5). Toric and flag varieties are most studied examples of *spherical varieties*. The further goal and motivation for the present paper is to use the relation between the geometry of flag varieties and Gelfand-Zetlin polytopes developed here to get new insights into geometry of more general spherical varieties as outlined in [8].

Recall that a *Schubert or Bruhat cell* is defined as an orbit of  $B$  in  $X$  under the left action, and *Schubert cycles* are the cycles in the Chow ring of  $X$  represented by the closures of Schubert cells. Schubert cycles provide a basis in the Chow ring of  $X$ , and the latter is isomorphic to the cohomology ring  $H^*(X, \mathbb{Z})$  of  $X$  (see e.g. [3, 1.3]). On the other hand, the cohomology ring of the flag variety is generated by the degree two classes (see, for instance, [13, Theorem 3.6.15]). The group  $H^2(X, \mathbb{Z})$  is isomorphic to the Picard group of  $X$  and can be identified with the

weight lattice of  $G$  so that very ample divisors are identified with strictly dominant weights (see [3, 1.4.3]). Recall that the weight lattice of  $G$  is by definition the character lattice  $\mathbb{Z}^{n-1}$  of a maximal torus in  $G$ . The central formula in Schubert calculus is the Chevalley formula for the intersection product of a Schubert cycle with a divisor (see Subsection 2.3 for more details). The Chevalley formula was proved independently by Bernstein–Gelfand–Gelfand [2] and Demazure [6] and was already contained in a manuscript of Chevalley [5], which for many years remained unpublished. This formula allows to express Schubert cycles in terms of divisors thus relating two different descriptions of the cohomology ring of the flag variety [2].

Fix the upper-triangular Borel subgroup  $B^+$ . Let  $\lambda$  a strictly dominant (with respect to  $B^+$ ) weight, and  $H_\lambda$  the divisor corresponding to  $\lambda$ . We now assign to  $H_\lambda$  a convex polytope  $Q_\lambda$ . Recall that with each strictly dominant weight  $\lambda$  one can associate the Gelfand-Zetlin polytope  $Q_\lambda$  (note that Zetlin is sometimes also transliterated as Cetlin or Tsetlin). This is a convex polytope in  $\mathbb{R}^d$  whose vertices lie in the integral lattice  $\mathbb{Z}^d \subset \mathbb{R}^d$  (see Subsection 2.1 for the definition). Here  $d = n(n-1)/2$  denotes the dimension of  $X$ . Let  $T$  be the diagonal maximal torus. The integral points inside and at the boundary of  $Q_\lambda$  parameterize a natural basis of  $T$ -eigenvectors (introduced in [7]) in the irreducible representation  $V_\lambda$  of  $G$  with the highest weight  $\lambda$ .

I will assign to each Schubert cycle in  $X$  a face of the Gelfand-Zetlin polytope (see Section 3). My construction depends on a choice of a Borel subgroup  $B$  containing the maximal torus  $T$  (so in fact, I provide  $n!$  different correspondences between Schubert cycles and faces). For each choice of  $B$ , we first construct a correspondence between  $B$ -orbits and faces and then use the one-to-one correspondence between Schubert cycles and  $B$ -orbits. The correspondence between  $B$ -orbits and faces preserves dimensions. The faces obtained for a given  $B$  correspond to *Demazure  $B$ -modules* in the representation space  $V_\lambda$ . The freedom in the choice of a Borel subgroup allowed by this construction is very useful. In many cases, it allows us to choose a face whose combinatorics captures geometry of a given Schubert cycle especially well (see Theorem 1.1 below). It might also lead to an interesting realization of Schubert calculus in terms of Gelfand-Zetlin polytopes (this is work in progress with Evgeny Smirnov and Vladlen Timorin). See Section 4 for an example of such calculus in the case  $G = SL_3(\mathbb{C})$ .

For a special choice of a Borel subgroup, namely for the lower-triangular Borel subgroup  $B^-$ , my construction gives the correspondence between some of the Schubert cells and faces of the Gelfand-Zetlin polytope constructed by Kogan using the moment map  $X \rightarrow Q_\lambda$  [11] (see Section 3 for more details). In [12], Kogan and Miller extended this correspondence to all Schubert cycles: they assigned to each Schubert cycle a union of faces using Caldero’s toric degenerations of flag varieties [4]. Both approaches (with moment map and toric degenerations) only allow to work with  $B^-$ -orbits, that is, there is only one way to assign a face or a union of faces to a given Schubert cycle.

For some of the faces of the Gelfand-Zetlin polytope that correspond to the Schubert cycles, the Chevalley formula for the intersection product of a Schubert cycle with the divisor  $H_\lambda$  admits the following interpretation in terms of the respective face (cf. Theorem 3.4). We fix a Borel subgroup  $B$  containing  $T$ , and hence fix a correspondence between Schubert cycles and faces. Denote by  $\mathcal{O}_\Gamma$  the  $B$ -orbit corresponding to a face  $\Gamma$ , and by  $Z_\Gamma$  the Schubert cycle defined by  $\mathcal{O}_\Gamma$ . In what follows, we only consider those faces that do correspond to Schubert cycles. We say that a face  $\Gamma$  is *admissible* if for each codimension one orbit  $\mathcal{O}_\Delta$  in the closure of the orbit  $\mathcal{O}_\Gamma$  the face  $\Gamma$  contains the face  $\Delta$ . In other words, the *Bruhat order* on Schubert cycles agrees with the natural order on faces given by inclusion.

**Theorem 1.1.** *For any admissible face  $\Gamma$  we have*

$$H_\lambda Z_\Gamma = \sum_{\Delta \subset \Gamma} d(v, \Delta) Z_\Delta,$$

where the sum is taken over the facets  $\Delta$  of  $\Gamma$  (that correspond to the Schubert cells  $\mathcal{O}_\Delta$  of codimension one at the boundary of  $\mathcal{O}_\Gamma$ ). Here  $v$  is a fixed vertex of the face  $\Gamma$  and  $d(v, \Delta)$  denotes the integral distance from  $v$  to the face  $\Delta$  (see Section 2.2 for the definition).

Note that in this form the formula is completely analogous to the well-known formula for toric varieties (e.g. see [8]). There is a generalization of Theorem 1.1 that holds for all faces (see Theorem 5.5).

Many Schubert cycles can be represented by an admissible face for different choices of  $B$ , but not all of them. E.g. for  $G = SL_3$  all Schubert cycles can be represented by admissible faces. For  $G = SL_4$ , exactly two Schubert cycles can not be represented by an admissible face. These two cycles are given by the Schubert cells whose closures in the flag variety are not smooth. I conjecture that all Schubert cycles defined by Schubert cells with smooth closures can be represented by admissible faces. Note also that if we only take  $B^-$  (as in [11, 12]) then already for  $SL_3$  there will be a Schubert cycle such that the corresponding face is not admissible (see Remark 4.1).

It might be possible to extend the correspondence between Schubert cycles and faces constructed in this paper to the complete flag varieties for other reductive groups by replacing the Gelfand-Zetlin polytope with appropriate *string polytopes*.

This paper is organized as follows. In Section 2, we recall the definition of the Gelfand-Zetlin polytope and the notion of integral distance. We also state the classical Chevalley formula. Section 3 contains the main results: the construction of correspondences between faces of the Gelfand-Zetlin polytope and Schubert cycles and Chevalley formula in terms of the Gelfand-Zetlin polytope (Theorem 3.4). In Section 4, we consider in detail the example  $G = SL_3$ . In Section 5, we study combinatorics of the Gelfand-Zetlin polytope and prove Theorem 3.4. We also formulate and prove an extension of Theorem 3.4 to non-admissible faces (Theorem 5.5).

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## 2. GELFAND-ZETLIN POLYTOPES AND CHEVALLEY FORMULA

In this section, we recall the definition of the Gelfand-Zetlin polytope and the Chevalley formula for the intersection product of a Schubert cycle with a divisor. We also discuss the notion of integral distance.

**2.1. Gelfand-Zetlin polytope.** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a strictly increasing collection of  $n$  integer numbers. To each such collection we assign the irreducible representation

$$\pi_\lambda : G \rightarrow GL(V_\lambda)$$

with the strictly dominant highest weight  $(\lambda_2 - \lambda_1)\omega_1 + \dots + (\lambda_n - \lambda_{n-1})\omega_{n-1}$  (which will also be denoted by  $\lambda$ ), where  $\omega_1, \dots, \omega_{n-1}$  are the fundamental weights of  $G$ . To define the fundamental weights we fix the diagonal maximal torus  $T$  and the upper-triangular Borel subgroup  $B^+$ . The Gelfand-Zetlin polytope  $Q_\lambda$  associated with  $\lambda$  is a convex polytope in  $\mathbb{R}^d$  (recall that  $d = n(n-1)/2$ ) defined by the inequalities

$$\begin{array}{ccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\ & x_{1,1} & & x_{1,2} & & \dots & & & x_{1,n-1} \\ & & x_{2,1} & & \dots & & & & x_{2,n-2} \\ & & & \ddots & & \dots & & & \\ & & & & x_{n-2,1} & & & & x_{n-2,2} \\ & & & & & & & & x_{n-1,1} \end{array}$$

where  $(x_{1,1}, \dots, x_{1,n-1}; x_{2,1}, \dots, x_{2,n-2}; \dots; x_{n-2,1}, x_{n-2,2}; x_{n-1,1})$  are coordinates in  $\mathbb{R}^d$  and the notation

$$\begin{array}{cc} a & b \\ & c \end{array}$$

means  $a \leq c \leq b$ . See Figure 1 for a picture of the Gelfand-Zetlin polytope for  $G = SL_3$ .

There is a  $T$ -eigenbasis in  $V_\lambda$  such that its vectors are in one-to-one correspondence with the integral points inside  $Q_\lambda$  (see for instance [12, Section 5] for the description of this basis). We will denote by the same letter  $v$  an integral point in  $Q_\lambda$  and the corresponding basis vector in  $V_\lambda$ . There is a natural map  $p$  that assigns to each integral point  $v$  the weight of the corresponding basis vector  $v \in V_\lambda$ . Let us extend this map by linearity to the map  $p : \mathbb{R}^d \rightarrow \mathbb{R}^{n-1}$ . Denote by  $P_\lambda \subset \mathbb{R}^{n-1}$  the weight polytope of the representation  $V_\lambda$ . The map  $p$  sends the Gelfand-Zetlin polytope  $Q_\lambda$  to the weight polytope  $P_\lambda$  and can be written in coordinates as follows [11, 2.1.2]. Let  $\alpha_1, \dots, \alpha_{n-1}$  be the simple roots of  $G$  (so they form a basis in  $\mathbb{R}^{n-1}$  dual with respect to the Cartan-Killing form to the basis of the fundamental weights  $\omega_1, \dots,$

$\omega_{n-1}$ ). Then we have

$$p : (x_{ij}) \rightarrow \left( \sum_{i=1}^{n-1} x_{1,i} \right) \alpha_1 + \left( \sum_{i=1}^{n-2} x_{2,i} \right) \alpha_2 + \dots + (x_{n-2,1} + x_{n-2,2}) \alpha_{n-2} + x_{n-1,1} \alpha_{n-1} + \\ + \text{constant vector.}$$

**Remark 2.1.** Note that for any two strictly dominant weights  $\lambda$  and  $\mu$  the corresponding Gelfand-Zetlin polytopes  $Q_\lambda$  and  $Q_\mu$  are *analogous*, that is, have the same normal fan. In particular, there is a bijective correspondence between their faces. This is similar to the toric case, where polytopes corresponding to any two very ample divisors are analogous.

**2.2. Integral distance.** Below we recall the notion of *integral distance*. Consider the integral lattice  $\mathbb{Z}^d \subset \mathbb{R}^d$  in the affine space  $\mathbb{R}^d$ . Let  $H$  be a hyperplane spanned by lattice vectors, and  $v \in \mathbb{Z}^d$  an integral point. Then the *integral distance*  $d(v, H)$  from  $v$  to the hyperplane  $H$  is the index in  $\mathbb{Z}^d$  of the subgroup spanned by the vectors  $v - u$  for all  $u \in H$ . To compute the integral distance we first find a primitive integral equation  $f(x) = 0$  defining  $H$ , that is,  $f(x) = a_0 + a_1x_1 + \dots + a_dx_d$  where  $a_i \in \mathbb{Z}$  and the greatest common divisor of  $a_0, \dots, a_d$  is 1. It is then easy to check that the integral distance between  $v$  and  $H$  is equal to the absolute value of  $f(v)$ .

In the sequel, we will use the notion of integral distance in the following setting. Let  $P$  be a convex lattice polytope of dimension  $d$  in  $\mathbb{R}^d$ . Recall that a vertex  $u$  of  $P$  is called *simple* if exactly  $d$  facets intersect in  $u$  (or equivalently, exactly  $d$  edges meet at  $u$ ). In other words, in the neighborhood of  $u$  the polytope  $P$  looks like a  $d$ -dimensional simplex. Let  $\Gamma \subset P$  be a face of  $P$ , and  $\Delta \subset \Gamma$  a facet of  $\Gamma$  that contains at least one simple vertex of  $P$ . This ensures that there is a unique hyperplane  $H$  such that  $H \cap P$  is a facet of  $P$  and  $H \cap \Gamma = \Delta$ . For any integral point  $v \in \Gamma$  we can now define the integral distance  $d(v, \Delta)$  as the integral distance from  $v$  to the hyperplane  $H$ . Such distances arise naturally in toric geometry when one computes products of toric orbits with divisors.

**2.3. Bruhat order and Chevalley formula.** Fix a strictly dominant weight  $\lambda$ . Recall that  $V_\lambda$  denotes the irreducible representation with the highest weight  $\lambda$ . We assume that  $G/B$  is embedded into the projective space  $\mathbb{P}(V_\lambda)$  as the  $G$ -orbit of the line spanned by a highest weight vector  $v \in V_\lambda$ . Denote by  $H_\lambda$  the divisor of hyperplane section on  $G/B$  (this is one of the equivalent ways to identify strictly dominant weights with very ample divisors [3, 1.4]). For each Schubert cell  $\mathcal{O}$  in  $G/B$ , the Chevalley formula computes explicitly the intersection of  $H_\lambda$  with the closure of  $\mathcal{O}$  as a linear combination of the closures of Schubert cells at the boundary of  $\mathcal{O}$ . We will now state this formula.

First, recall that the choice of a Borel subgroup  $B$  in  $G$  defines a one-to-one correspondence between the Schubert cells in  $G/B$  and elements of the Weyl group  $W$  of  $G$ . We identify the Weyl group with  $N(T)/T$ , where  $N(T)$  is the normalizer of  $T$  in  $G$ . Then the Schubert cell  $\mathcal{O}_w$  is the  $B$ -orbit of the line spanned by  $wv \in V_\lambda$ .

Note that the length  $l(w)$  (defined as the minimal number of simple reflections in a decomposition of  $w$ ) is equal to the dimension of  $\mathcal{O}_w$ . Recall that there is a natural partial order on Schubert cells called *Bruhat order*. We say that  $\mathcal{O}_{w'}$  *precedes*  $\mathcal{O}_w$  with respect to the Bruhat order if  $\mathcal{O}_{w'}$  is contained in the closure of  $\mathcal{O}_w$  and  $\dim \mathcal{O}_{w'} = \dim \mathcal{O}_w - 1$ . In other words,  $\mathcal{O}_{w'}$  is a boundary divisor in  $\overline{\mathcal{O}_w}$ . The Bruhat order can also be defined in terms of the Weyl group as follows. Denote by  $s_\alpha$  the reflection in the hyperplane perpendicular to a root  $\alpha$ . Then  $\mathcal{O}_{w'}$  precedes  $\mathcal{O}_w$  if and only if  $w' = ws_\alpha$  for some root  $\alpha$  and  $l(w') = l(w) - 1$  (see e.g [2, Theorem 2.11] or [13, Proposition 3.6.4]).

For each root  $\alpha$ , define the linear function  $(\cdot, \alpha)$  (that is, the *coroot*) on the weight lattice of  $G$  by the property  $s_\alpha \lambda = \lambda - (\lambda, \alpha)\alpha$  for all weights  $\lambda$ . (The pairing  $(a, b)$  is often denoted by  $\langle a, b^\vee \rangle$  or by  $\langle a, b \rangle$ .) Denote by  $Z_w$  the Schubert cycle represented by the closure of the orbit  $\mathcal{O}_w$ . The following result is proved in [2, Proposition 4.1] and [6, Proposition 4.4]:

$$H_\lambda Z_w = \sum_{\alpha} (\lambda, \alpha) Z_{ws_\alpha},$$

where the sum is taken over all positive roots  $\alpha$  of  $G$  such that  $l(ws_\alpha) = l(w) - 1$ . In particular, the coefficients  $(\lambda, \alpha)$  are always nonnegative.

One of our goals is to interpret this formula in terms of the Gelfand-Zetlin polytope  $Q_\lambda$ . In what follows we will use the following equivalent formulation:

$$H_\lambda Z_w = \sum_{\alpha} (w\lambda, \alpha) Z_{s_\alpha w}, \quad (2.1)$$

where the sum is taken over all roots  $\alpha$  such that  $w^{-1}\alpha$  is positive and  $l(s_\alpha w) = l(w) - 1$ .

### 3. CORRESPONDENCE BETWEEN THE SCHUBERT CELLS AND THE FACES OF THE GELFAND-ZETLIN POLYTOPE.

In this section, we will construct a correspondence between Schubert cycles and some of the faces of the Gelfand-Zetlin polytope corresponding to a strictly dominant weight (by Remark 2.1 it does not matter which weight we choose).

**3.1. Schubert cells.** Fix once and for all the diagonal maximal torus  $T \subset G$  and denote by  $\mathfrak{t}$  its Lie algebra. Everything below (weight vectors, Borel subalgebras etc.) are assumed to be compatible with  $T$ . As before we assume that  $G/B$  is embedded into the projectivization  $\mathbb{P}(V_\lambda)$  of the irreducible representation  $V_\lambda$  as the  $G$ -orbit of the line spanned by a highest weight vector.

We will use the following description of the Schubert cells from [2]. Let  $v \in V_\lambda$  be a non-zero weight vector with an *extremal weight*. (Recall that a weight is extremal if it is of the form  $w\lambda$  for some element  $w$  in the Weyl group of  $G$ .) Extremal weights are exactly the vertices of the weight polytope of  $V_\lambda$ , and their weight spaces are always one-dimensional. In what follows, we will not distinguish between non-zero proportional vectors with the same extremal weight. Let  $B$  be a Borel subgroup

in  $G$  containing  $T$ , and  $\mathfrak{b}$  its Lie algebra. Note that all such Borel subgroups lie in the same orbit under the action of the Weyl group  $W$  and there are exactly  $|W|$  of them. Denote by  $U(\mathfrak{b})$  the universal enveloping algebra of  $\mathfrak{b}$ . Then the pair  $(v, B)$  defines the Schubert cell  $\mathcal{O}(v, B)$ , which is the  $B$ -orbit of  $v$ , and the closure of this cell in the flag variety can be realized as follows [2, Lemma 2.12]:

$$\overline{\mathcal{O}(v, B)} = X \cap \mathbb{P}(U(\mathfrak{b})v).$$

Note that  $U(\mathfrak{b})v$  is a  $B$ -invariant vector subspace in  $V_\lambda$  (called *Demazure module*). It would be natural to assign to the cell  $\mathcal{O}(v, n)$  a face of the Gelfand-Zetlin polytope  $Q_\lambda$  by taking the convex hull of all basis vectors in the Gelfand-Zetlin basis that lie in the subspace  $U(\mathfrak{b})v$  (we identify the basis vectors in the Gelfand-Zetlin basis with the integral points in  $Q_\lambda$ ). Unfortunately, it might happen that the convex hull is not a face or has wrong dimension. However, this approach still works after some modification (see Subsection 3.2).

Two Schubert cells  $\mathcal{O}(v, B)$  and  $\mathcal{O}(v', B')$  are conjugate by the action of the Weyl group (and hence represent the same cohomology class) if and only if  $B' = wBw^{-1}$  and  $v'$  is proportional to  $wv$  for some element  $w \in W$ . If we fix a Borel subgroup  $B$  then the Schubert cells  $\mathcal{O}(wv, B)$  for all  $w \in W$  give the full set of  $B$ -orbits in the flag variety. In particular,

$$X = \sqcup_{w \in W} \mathcal{O}(wv, B).$$

For all possible choices of  $v$  and  $B$ , we get  $|W|^2$  Schubert cells forming  $|W|$  orbits under the action of the Weyl group. Note that there is no canonical identification (i.e. independent of the choice of a Borel subgroup containing the torus  $T$ ) between the cohomology classes of the cells  $\mathcal{O}(v, B)$  and the elements of the Weyl group. By different choices of a Borel subgroup we assign to each cohomology class  $[\overline{\mathcal{O}(v, B)}]$  different elements of the Weyl group that are conjugate to each other. Since we use simultaneously the cells  $\mathcal{O}(v, B)$  for different choices of  $B$  we will not identify the Schubert cells with elements of the Weyl group.

**3.2. Faces of the Gelfand-Zetlin polytope.** To each cell  $\mathcal{O}(v, B)$  of dimension  $l$  we now assign an  $l$ -dimensional face of the Gelfand-Zetlin polytope  $Q_\lambda$ . Recall that to each extremal vector  $v$  there corresponds a unique vertex of the Gelfand-Zetlin polytope, which we also denote by  $v$ . The vertex  $v$  is a unique preimage of a vertex of the weight polytope  $P_\lambda$  under the map  $p : Q_\lambda \rightarrow P_\lambda$ . It is easy to show that the vertex  $v$  is simple [11, 2.2.2]. The edges coming out of  $v$  are in one-to-one correspondence with the roots  $\alpha$  of  $G$  such that the root space  $\mathfrak{g}_\alpha$  does not annihilate the extremal weight vector  $v$  [11, 2.2.3]. For each such root  $\alpha$  denote by  $e(v, \alpha)$  the edge corresponding to  $\alpha$ . The edge  $e(v, \alpha)$  is uniquely defined by the property that its projection under the map  $p : Q_\lambda \rightarrow P_\lambda$  is parallel to the root  $\alpha$  (see Section 5 for more details on simple vertices  $v$  and edges  $e(v, \alpha)$ ).

Denote by  $R(v, B)$  the set of roots such that the root space  $\mathfrak{g}_\alpha \subset \mathfrak{g}$  is contained in  $\mathfrak{b}$  and does not annihilate  $v$ . The cardinality of  $R(v, B)$  is equal to the dimension of the cell  $\mathcal{O}(v, B)$  [2, Lemma 2.2]. Let  $\{\beta_{i_1}, \dots, \beta_{i_l}\}$  be all roots in the set  $R(v, B)$ .

Assign to the Schubert cell  $\mathcal{O}(B, v)$  the  $l$ -dimensional face  $\Gamma(v, B)$  of the Gelfand-Zetlin polytope spanned by the edges  $e(v, \beta_{i_1}), \dots, e(v, \beta_{i_l})$ . There is a unique such face since the vertex  $v$  is simple. This face can be thought of as a lifting of the Demazure module  $U(\mathfrak{b})v$  to the Gelfand-Zetlin polytope.

**Remark 3.1.** There is an alternative description of the face  $\Gamma(v, B)$  using *Morse theory* on polytopes (such analog of the Morse theory was introduced in [10]). Namely, choose a linear function  $f_B$  on  $\mathbb{R}^{n-1}$  that takes positive values on all roots  $\alpha$  whose root spaces  $\mathfrak{g}_\alpha$  are contained in  $\mathfrak{b}$ . Then the composition  $f_B \circ p$  is a linear function on the Gelfand-Zetlin polytope. The face  $\Gamma(v, B)$  is then precisely the *upper separatrix face* for the function  $f_B \circ p$  at the vertex  $v$ . The *upper separatrix face* is by definition the face spanned by all edges at  $v$  going upward with respect to the function  $f_B \circ p$  (that is,  $f_B \circ p$  increases along these edges).

Note that the Bruhat cells  $\mathcal{O}(v, B)$  can be defined in an analogous way. Namely, there is a Morse function on  $X$  given by the composition of the moment map  $X \rightarrow P_\lambda$  with the same function  $f_B$ , and the cells  $\mathcal{O}(v, B)$  are the upper separatrix manifolds for this Morse function (see [1, Section 4]).

We now compare the Bruhat order on the cells  $\mathcal{O}(v, B)$  with the inclusion order on the faces  $\Gamma(v, B)$ . It is easy to see that if  $\Gamma(u, B)$  is a facet in  $\Gamma(v, B)$ , then  $\mathcal{O}(u, B)$  precedes  $\mathcal{O}(v, B)$  with respect to the Bruhat order. The converse is wrong. I.e. it happens already for  $G = SL_3$  that  $\mathcal{O}(u, B)$  lies at the boundary of  $\mathcal{O}(v, B)$  in the flag variety but the face  $\Gamma(u, B)$  does not belong to the face  $\Gamma(v, B)$  (see Section 4). We say that the face  $\Gamma(v, B)$  is *admissible* if it contains all faces  $\Gamma(u, B)$  such that the Schubert cell  $\mathcal{O}(u, B)$  precedes the Schubert cell  $\mathcal{O}(v, B)$  with respect to the Bruhat order.

Denote by  $B^-$  the Borel subgroup opposite to the one used to construct the Gelfand-Zetlin polytope. If we fix  $B = B^-$  and only vary  $v$  then my correspondence between Schubert cycles and faces reduces to the correspondence defined in [11] (see Remark 5.1). Note that the collection of faces assigned to  $\mathcal{O}(v, B^-)$  in [12, Section 4] always contains  $\Gamma(v, B^-)$ . In particular, if this collection consists of just one face (that is, of  $\Gamma(v, B^-)$ ), then the corresponding Schubert cycle is a Kempf variety [11, Proposition 2.3.2], which is a very restrictive condition (see [11, Proposition 2.2.1] for a characterization of such Schubert cycles). Equivalently, the corresponding Schubert polynomial consists of a single monomial. In particular, it is easy to check that in this case  $\Gamma(v, B^-)$  must be admissible. An advantage of my construction is that the freedom in the choice of  $B$  allows us to represent more general Schubert cycles by a single admissible face of the Gelfand-Zetlin polytope (see Remark 4.1).

An interesting problem is to describe all admissible faces as well as the corresponding Schubert cycles. It is not true that for each Schubert cycle there exists a representative  $\mathcal{O}(v, B)$  such that the face  $\Gamma(v, B)$  is admissible. There is a counterexample for the flag variety  $X_4$  of  $SL_4(\mathbb{C})$ . Namely, if the closure of a Schubert cell  $\mathcal{O}$  in  $X_4$  is not smooth then none of the faces corresponding to the cohomology class of  $\overline{\mathcal{O}}$  is admissible (there are two such Schubert cells in  $X_4$ ). For the flag variety

$X_3$  of  $SL_3(\mathbb{C})$  the closure of each Schubert cell is smooth and every Schubert cycle can be represented by an admissible face (see Section 4). These examples suggest the following conjecture:

**Conjecture 3.2.** *If a Schubert cell has smooth closure then its cohomology class can be represented by an admissible face.*

More generally, suppose that the closure of the Schubert cell  $\mathcal{O}_w$  corresponding to an element  $w \in W$  has at most  $k - 1$  irreducible divisors at the boundary  $\overline{\mathcal{O}_w} \setminus \mathcal{O}_w$ , where  $k$  is the number of pairwise distinct simple reflections in a reduced decomposition for  $w$  (in particular,  $k \leq n - 1$ ). This is the case for smooth Schubert cycles by [3, Proposition 2.2.8]). I conjecture that the Schubert cycle  $[\overline{\mathcal{O}_w}]$  can be represented by an admissible face.

We now state the Chevalley formula in terms of the Gelfand-Zetlin polytope. For a weight vector  $u$ , denote by  $p(u)$  the weight of  $u$ .

**Proposition 3.3.** *If  $\Gamma(u, B)$  is a facet of  $\Gamma(v, B)$  (in particular,  $p(v) = s_\alpha p(u)$  for some root  $\alpha$ ), then*

$$|(p(v), \alpha)| = d(v, \Gamma(u, B)),$$

where  $d(v, \Gamma(u, B))$  is the integral distance from  $v$  to the face  $\Gamma(u, B)$  as defined in Section 2.2.

This proposition will be proved in Section 5. If we apply it to formula (2.1) we immediately get the following Chevalley formula for the admissible faces.

**Theorem 3.4.** *If the face  $\Gamma(v, B)$  is admissible then the Chevalley formula for the Schubert variety  $\overline{\mathcal{O}(v, B)}$  and the divisor  $H_\lambda$  can be written as*

$$H_\lambda \overline{\mathcal{O}(v, B)} = \sum d(v, \Gamma(u, B)) \overline{\mathcal{O}(u, B)},$$

where the sum is taken over all Schubert cells  $\mathcal{O}(u, B)$  that precede  $\mathcal{O}(v, B)$ .

#### 4. EXAMPLE: FLAG VARIETY FOR $SL_3(\mathbb{C})$

Figure 1 shows the Gelfand-Zetlin polytope  $Q_\lambda$  for the irreducible representation of  $SL_3(\mathbb{C})$  with the highest weight  $\lambda = a\omega_1 + b\omega_2$ . This is a polytope in  $\mathbb{R}^3$  (with coordinates  $x$ ,  $y$  and  $z$ ) defined by the following six inequalities:

$$0 \leq x \leq a; \quad a \leq y \leq b; \quad x \leq z \leq y.$$

The weight polytope  $P_\lambda$  is a hexagon in  $\mathbb{R}^2$ . The polytope  $Q_\lambda$  has six simple vertices which are mapped bijectively to the vertices of the weight polytope  $P_\lambda$  under the map  $p$ . This bijection is used to label the simple vertices of  $Q_\lambda$ . Namely, we label by  $v$  the vertex that goes to the highest weight  $\lambda$ . A simple vertex  $u$  is then labeled by  $wv$  if  $p(u) = wp(v)$  for some element  $w$  from the Weyl group. Put  $s_1 = s_{\alpha_1}$  and  $s_2 = s_{\alpha_2}$ . We denote by  $[u_1, u_2]$  the edge of the Gelfand-Zetlin polytope connecting vertices  $u_1$  and  $u_2$ .

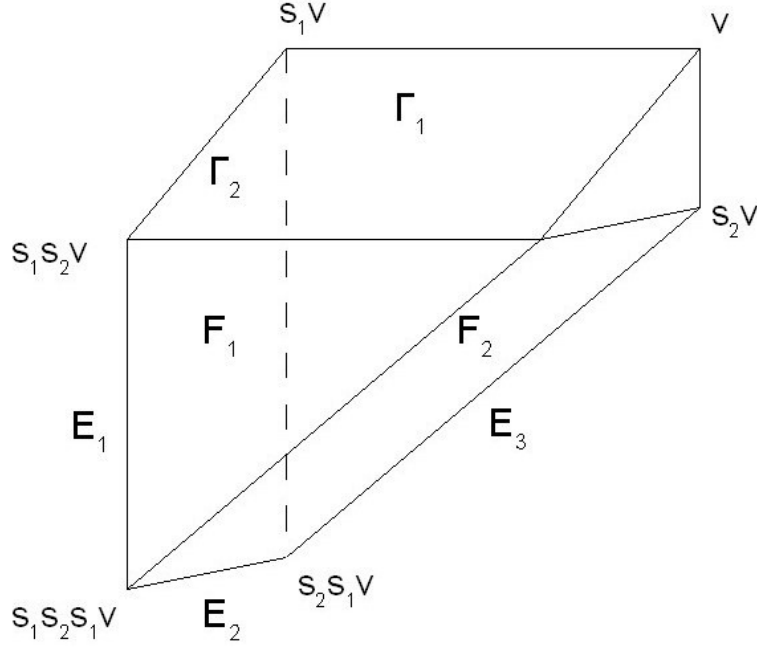


FIGURE 1

All faces of  $Q_\lambda$  except for a unique non-simple vertex can be represented as  $\Gamma(v, B)$  for some choice of a simple vertex  $v$  and a Borel subgroup  $B$ . E.g. if  $B = B^+$  then  $\Gamma(v, B^+) = v$ ,  $\Gamma(s_1 v, B^+)$  is the edge  $[s_1 v, v]$  and  $\Gamma(s_2 s_1 v, B^+)$  is the face  $\{y = b\} \cap Q_\lambda$ . If  $B = B^-$  then  $\Gamma(v, B^-) = Q_\lambda$ ,  $\Gamma(s_1 v, B^-)$  is the face  $\{x = 0\} \cap Q_\lambda$  and  $\Gamma(s_2 s_1 v, B^-)$  is the edge  $[s_2 s_1 v, s_1 s_2 s_1 v]$ .

All faces of  $Q_\lambda$  that do not contain the non-simple vertex are admissible. In particular, there are two 2-dimensional admissible faces  $\Gamma_1 = \Gamma(s_2 s_1 v, B^+)$  and  $\Gamma_2 = \Gamma(s_1 v, B^-)$  corresponding to the cells  $\mathcal{O}(s_2 s_1 v, B^+)$  and  $\mathcal{O}(s_1 v, B^-)$ . It is easy to check that these two cells correspond to different Schubert cycles. Denote these cycles by  $Z_{21}$  and  $Z_{12}$ , respectively (that is, we label the cohomology class of  $\mathcal{O}(wv, B^+)$  by  $Z_w$  and encode  $w = s_1 s_2$  by 12 etc). There are also six admissible edges that connect simple vertices of  $Q_\lambda$ . These correspond to two Schubert cycles of dimension one. Namely, the edges  $[v, s_1 v]$ ,  $[s_1 s_2 v, s_1 s_2 s_1 v]$  and  $[s_2 s_1 v, s_2 v]$  correspond to  $Z_1$ , and the other three edges correspond to  $Z_2$ . Then Theorem 3.4 applied to the two-dimensional admissible faces tells that

$$H_\lambda Z_{12} = bZ_1 + (a + b)Z_2; \quad H_\lambda Z_{21} = (a + b)Z_1 + aZ_2.$$

**Remark 4.1.** Note that if we only considered faces  $\Gamma(u, B^-)$  for the lower-triangular Borel subgroup  $B^-$  (that is, proceeded as in [11, 12]) then we would not be able to represent the Schubert cycle  $Z_{21}$  by a single admissible face. Instead, we would get the union of two faces: the rectangular one  $\{x = z\}$  and the triangular one  $\{y = a\}$ .

The union of these two faces looks like the admissible face  $\Gamma_1$  (corresponding to  $Z_{21}$  by my construction) broken into two pieces.

We now describe heuristic Schubert calculus on the faces of  $Q_\lambda$ . We can represent Schubert cycle  $Z_{21}$  by faces in two different ways: as  $\Gamma_1$  and as  $F_1 + F_2$ , where  $F_1$  and  $F_2$  denote the faces given by the equations  $y = a$  and  $x = z$ , respectively. The latter representation comes from [12]. We also represent  $Z_{12}$  by  $\Gamma_2$ . Finally, we represent the one-dimensional Schubert cycle  $Z_1$  in two ways, by the edge  $E_1 = [s_1s_2v, s_1s_2s_1v]$  and the edge  $E_3 = [s_2v, s_2s_1v]$ , and represent  $Z_2$  by the edge  $E_2 = [s_2s_1v, s_1s_2s_1v]$  (see Figure 1). We can now compute  $Z_{21}Z_{12}$  and  $Z_{12}^2$  by intersecting the corresponding faces:

$$(F_1 + F_2) \cap \Gamma_2 = E_1 + E_2,$$

which is exactly the identity  $Z_{21}Z_{12} = Z_1 + Z_2$ . Similarly,

$$(F_1 + F_2) \cap \Gamma_1 = E_3$$

gives the identity  $Z_{21}^2 = Z_1$ . We can also get the identities  $Z_1Z_{12} = Z_2Z_{21} = [pt]$  and  $Z_1Z_{21} = Z_2Z_{12} = 0$  by choosing the edges representing  $Z_1$  and  $Z_2$  so that they have transverse intersection with  $\Gamma_1$  or  $\Gamma_2$ . E.g. to find  $Z_1Z_{12}$  we represent  $Z_1$  by  $E_3$  and  $Z_{12}$  by  $\Gamma_2$  and get that  $\Gamma_2 \cap E_3 = pt$ . Similarly, to find  $Z_1Z_{21}$  we represent  $Z_1$  by  $E_1$  and  $Z_{21}$  by  $\Gamma_1$ , which yields  $\Gamma_1 \cap E_1 = \emptyset$ .

An analogous Schubert calculus on the Gelfand-Zetlin polytope can be done for arbitrary  $n$  [9]. It can be rigorously justified using the concept of the *polytope ring* whose elements are linear combinations of faces modulo some relations.

## 5. GEOMETRY AND COMBINATORICS OF THE GELFAND-ZETLIN POLYTOPE.

To prove Proposition 3.3 we have to study the faces of the Gelfand-Zetlin polytope  $Q_\lambda$ . First, we describe explicitly the simple vertices of  $Q_\lambda$  and the edges going out of simple vertices mostly following [11]. Brief explanations are provided for the reader's convenience, for more details see [11, 2.1-2.3]. Next, we will find out under which conditions two simple vertices are connected by the edge (see Lemma 5.2). Finally, we prove Proposition 3.3 and formulate and prove a Chevalley formula for arbitrary faces  $\Gamma(v, B)$  (see Theorem 5.5).

We describe the faces of  $Q_\lambda$  by triangular *diagrams* following [11]. Put  $x_{0,i} := \lambda_i$  for  $i = 1, \dots, n$ . It is easy to see that each face of  $Q_\lambda$  is defined by the equations of the form  $x_{i,j} = x_{i-1,j}$  or  $x_{i,j} = x_{i-1,j+1}$  for some  $i = 1, \dots, n-1, j = 1, \dots, n-i$ . For a face  $\Gamma$ , encode all the equations defining  $\Gamma$  by the following graph  $D(\Gamma)$ . Draw  $n$  rows indexed by  $1, \dots, n$  with  $n-i+1$  points  $p_{i,1}, \dots, p_{i,n-i+1}$  in the  $i$ -th row. These are the vertices of the graph  $D(\Gamma)$  (each vertex  $p_{i,j}$  corresponds to the coordinate  $x_{i-1,j}$ ). For each equality  $x_{i,j} = x_{i-1,j}$  and  $x_{i,j} = x_{i-1,j+1}$  defining the face  $\Gamma$  we draw the edge  $e_{i+1,j}^L$  of *type L* between the vertices  $p_{i+1,j}$  and  $p_{i,j}$  and the edge  $e_{i+1,j}^R$  of *type R* between  $p_{i+1,j}$  and  $p_{i,j+1}$ , respectively. The resulting graph is the *diagram* of the face  $\Gamma$ . Figure 2 shows the diagrams for the vertices  $v, s_1v$  and  $s_2v$  of the Gelfand-Zetlin polytope for  $SL_3$  considered in Section 4.

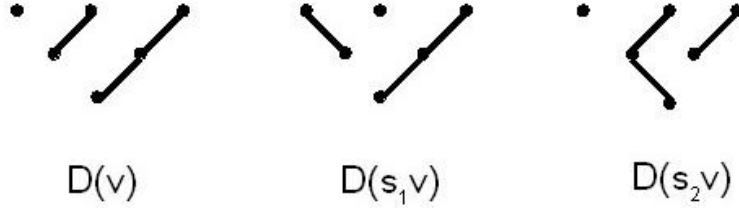


FIGURE 2

**5.1. Simple vertices.** It is easy to show that  $v$  is a simple vertex of the Gelfand-Zetlin polytope if and only if the corresponding diagram  $D(v)$  has exactly  $n - i$  edges starting at the  $i$ -th row and ending at the  $(i + 1)$ -st row (for all positive  $i < n$ ) and two such edges never start or end at the same point. In other words, the graph  $D(v)$  is the disjoint union of  $n$  simple trees  $T_1(v), \dots, T_n(v)$ . Each tree  $T_i(v)$  starts at the first row of  $D(v)$  and ends at the  $i$ -th row (that is, each  $T_i$  looks like the Dynkin diagram  $A_i$ ). The vertex of  $T_i(v)$  in the first row will be called the *starting point* of  $T_i(v)$ . Note that the coordinates  $x_{i,j}$  and  $x_{k,l}$  of the vertex  $v$  are equal if and only if the vertices  $p_{i+1,j}$  and  $p_{k+1,l}$  belong to the same tree. The diagram  $D(v)$  can also be thought of as an *RC-graph* or a *pipe dream* (see [11, 12] for details on connection between pipe-dreams and faces of the Gelfand-Zetlin polytope). Let us call the diagram of a simple vertex also *simple*. There is a different way to characterize simple diagrams (see [11, 2.2.2]). Namely, the diagram  $D(v)$  is simple if for all  $i = 2, \dots, n$  exactly  $n - i + 1$  edges end at the  $i$ -th row of  $D(v)$ , and the edges  $e_{i,j}^L$  are strictly to the left of the edges  $e_{i,j}^R$ .

Each simple diagram  $D(v)$  defines a permutation  $\sigma_v$  of elements  $1, \dots, n$  as follows: the vertex  $p_{1,i}$  is the starting point of the tree  $T_{\sigma_v(i)}$ . It is easy to check that this gives a bijective correspondence between simple vertices of  $Q_\lambda$  and elements of the symmetric group  $S_n$ , which is isomorphic to the Weyl group of  $G$  (we choose the isomorphism which sends the elementary transposition  $(i \ i + 1)$  to the simple reflection  $s_{\alpha_i}$ ). This bijection is compatible with the bijection between the vertices of the weight polytope  $P_\lambda$  and elements of the Weyl group, that is,  $p(v) = \sigma_v \lambda$ . Indeed, using the formula for the projection  $p : Q_\lambda \rightarrow P_\lambda$  from Section 2.1 we get that if  $p(u) = s_{\alpha_i} p(v)$  (and thus  $p(u) = p(v) - (p(v), \alpha_i) \alpha_i$ ), then the sums of coordinates  $\sum_{k=1}^{n-j} x_{j,k}$  for the vertices  $v$  and  $u$  only differ for  $j = i$ . This is only possible if the trees  $T_j(v)$  and  $T_j(u)$  have the same starting points for all  $j \neq i, (i + 1)$ , which implies  $\sigma_v = s_{\alpha_i} \sigma_u$ .

**5.2. Edges.** We now describe the edges of the Gelfand-Zetlin polytope. Let  $u$  and  $v$  be two vertices of the Gelfand-Zetlin polytope. We say that the diagram  $D(u)$  is obtained from the diagram  $D(v)$  by *switching* the edge  $e_{i,j}^L$  if the diagrams have the same set of edges with one exception: instead of the edge  $e_{i,j}^L$  the diagram  $D(v)$  has the edge  $e_{i,j}^R$ . Switching of  $e_{i,j}^R$  is defined in the same way. E.g. the diagrams  $D(s_1 v)$  and  $D(s_2 v)$  on Figure 2 are obtained from the diagram  $D(v)$  by switching

the edges  $e_{2,1}^R$  and  $e_{3,1}^R$ , respectively. It is easy to see that two vertices  $u$  and  $v$  are connected by an edge of the Gelfand-Zetlin polytope only if their diagrams can be obtained from each other by switching the edge  $e_{i,j}^L$  or  $e_{i,j}^R$  for some  $i$  and  $j$ . If  $D(u)$  is obtained from  $D(v)$  by switching the edge  $e_{i,j}^L$  then the diagram  $D([u, v])$  of the edge of the Gelfand-Zetlin polytope connecting  $u$  and  $v$  is obtained from  $D(v)$  by deleting the edge  $e_{i,j}^L$ .

We now focus on the edges going out of a given simple vertex  $v$ . Their diagrams are obtained by deleting one of the edges of the diagram  $D(v)$ . Denote by  $e$  the  $i$ -th edge of the tree  $T_j$  for  $i = 1, \dots, j - 1$  (that is the edge of the tree  $T_j(v)$  starting at the  $i$ -th row of the diagram  $D(v)$  and ending at the  $(i + 1)$ -st row). Recall that we denoted by  $e(v, \alpha)$  the edge of the Gelfand-Zetlin polytope whose projection  $p(e(v, \alpha))$  is parallel to the root  $\alpha$ . It is easy to check using again the formula for the projection  $p : Q_\lambda \rightarrow P_\lambda$  (see Section 2.1) that if we delete the edge  $e$  from the diagram  $D(v)$  we get the diagram of the edge  $e(v, \alpha)$ , where  $\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$  if  $e$  is of type  $L$  and  $\alpha = -\alpha_i - \alpha_{i+1} - \dots - \alpha_{j-1}$  if  $e$  is of type  $R$ . Indeed, let  $p_{i,s}$  and  $p_{i+1,s}$  be the vertices of the edge  $e$ . Then switching  $e$  only changes coordinates of  $v$  corresponding to the vertices of the tree  $T_j(v)$  lying strictly below  $p_{i,s}$ . This coordinates increase by the same number  $x_{i-1,s+1}(v) - x_{i-1,s}(v)$ . Hence, the sums of coordinates  $\sum_{k=1}^{n-r} x_{r,k}$  increase by the same number for  $r = i, \dots, j - 1$ , and stay the same for all other  $r$ . In particular, for each simple root  $\alpha_i$  the diagram of the edge  $e(v, \pm\alpha_i)$  is obtained from  $D(v)$  by deleting the lowest edge (that is, the  $i$ -th edge) of the tree  $T_{i+1}(v)$ , and the sign in  $\pm\alpha$  is determined by the slope of the lowest edge. Thus we get an explicit one-to-one correspondence between the edges  $e(v, \alpha)$  of the Gelfand-Zetlin polytope and the edges of the diagram  $D(v)$ .

**5.3. Faces  $\Gamma(v, B)$  and proof of Proposition 3.3.** It is now easy to describe the diagrams of the faces  $\Gamma(v, B)$  in terms of the diagram for  $v$ . Namely, we should delete all edges in  $D(v)$  that correspond to the roots in  $R(v, B)$  under the above correspondence. E.g. when  $B = B^-$  is lower-triangular, the diagram of  $\Gamma(v, B^-)$  is obtained from the diagram  $D(v)$  by deleting all edges of type  $R$ .

**Remark 5.1.** *The faces  $\Gamma(v, B^-)$  are exactly the so-called Gelfand-Zetlin faces considered in [11, Subsection 2.2.1]. Note that notation in [11] is different: my  $x_{i,j}$  is his  $\lambda_{i+j,i}$  and my  $\sigma_v$  is his  $w_v^{-1}$ .*

We now determine under which conditions two simple vertices  $u$  and  $v$  of the Gelfand-Zetlin polytope are connected by an edge. The necessary condition  $p(u) = s_\alpha p(v)$  for some root  $\alpha$  is obviously not sufficient (e.g. the vertices  $s_2v$  and  $s_1s_2v$  on Figure 1 are not connected by the edge though  $p(s_1s_2v) = s_1p(s_2v)$ ).

**Lemma 5.2.** *Let  $u$  and  $v$  be two simple vertices of the Gelfand-Zetlin polytope such that the weights  $p(u)$  and  $p(v)$  can be obtained from each other by the reflection  $s_\alpha$  with respect to some root  $\alpha$ . Then  $u$  and  $v$  are connected by the edge if and only if the diagram  $D(u)$  can be obtained from the diagram  $D(v)$  by switching the edge of  $D(u)$  corresponding to the root  $\alpha$ .*

*Proof.* Choose  $\alpha$  so that  $(p(v), \alpha) < (p(u), \alpha)$ . Then the vertices  $v$  and  $u$  can only be connected by the edge  $e(v, \alpha)$  (which will then coincide with the edge  $e(u, -\alpha)$ ), and the lemma immediately follows from the description of edges in the Gelfand-Zetlin polytope.  $\square$

To prove Proposition 3.3 we will need the following two lemmas.

**Lemma 5.3.** *If  $\Gamma(u, B)$  is a facet of  $\Gamma(v, B)$ , then the vertices  $v$  and  $u$  are connected by the edge.*

*Proof.* First, note that the assumptions of the lemma imply that  $\mathcal{O}(u, B)$  precedes  $\mathcal{O}(v, B)$  with respect to the Bruhat order. Hence,  $p(u) = s_\alpha p(v)$  for some root  $\alpha \in R(v, B)$ . Let  $(i\ j)$  be the transposition corresponding to  $s_\alpha$ , and  $e_{i+1,s}^R$  the edge of the diagram  $D(v)$  corresponding to the root  $\alpha$  (we assume that this edge is of type  $R$ ; type  $L$  case is completely analogous). We now compare the edges starting at the  $i$ -th rows of the diagrams  $D(v)$  and  $D(u)$ . Let  $e_{i+1,s-k}^L$  be the last edge of type  $L$  (when going from left to right) starting at the  $i$ -th row of the diagram  $D(v)$ . We want to show that  $k = 1$ , so that  $e_{i+1,s}^R$  can be switched and the resulting diagram remains simple. Consider all edges of  $D(v)$  between  $e_{i+1,s-k}^L$  and  $e_{i+1,s}^R$ , that is, the edges  $e_{i+1,s-k+1}^R, \dots, e_{i+1,s-1}^R$ . The above explicit correspondence between simple vertices  $v$  and permutations  $\sigma_v$  implies that the trees  $T_l(v)$  and  $T_l(u)$  have the same starting points unless  $l = i, j$ . From this it is easy to deduce that the diagram  $D(u)$  contains the edges  $e_{i+1,s-k+1}^L, \dots, e_{i+1,s}^L$ . Moreover, if  $e_{i+1,s-k+l}^R$  in  $D(v)$  for  $l = 1, \dots, k-1$  corresponds to a root  $\beta$ , then  $e_{i+1,s-k+l+1}^L$  in  $D(u)$  corresponds to  $-\beta$ . Finally,  $e_{i+1,s-k+1}^L$  corresponds to the root  $-\alpha$ . Hence, the diagrams  $D(\Gamma(v, B))$  and  $D(\Gamma(u, B))$  will differ in at least  $k$  edges. Indeed, whenever the diagram  $D(\Gamma(v, B))$  contains (or does not contain) the edge  $e_{i+1,s-k+l}^R$ , the diagram  $D(\Gamma(u, B))$  does not contain (or contains) the edge  $e_{i+1,s-k+l+1}^L$  for  $l = 1, \dots, k-1$ . Also  $D(\Gamma(v, B))$  does not contain the edge  $e_{i+1,s-k+1}^L$ , while  $D(\Gamma(u, B))$  does. It remains to note that the diagram of  $\Gamma(v, B)$  is obtained from the diagram of  $\Gamma(u, n)$  by deleting exactly one edge (since  $\Gamma(u, B)$  is a facet in  $\Gamma(v, n)$ ). Hence,  $k = 1$ .  $\square$

**Lemma 5.4.** *If  $v$  and  $u$  are two simple vertices of the Gelfand-Zetlin polytope such that  $p(v) = s_\alpha p(u)$  for the root  $\alpha = \alpha_i + \dots + \alpha_{j-1}$ , then*

$$|(p(v), \alpha)| = |\lambda_r - \lambda_s|,$$

where  $s = \sigma_v^{-1}(i)$  and  $r = \sigma_v^{-1}(j)$  (that is,  $p_{1,s}$  and  $p_{1,r}$  are the starting points of the trees  $T_i(v)$  and  $T_j(v)$ , respectively).

*Proof.* Since  $p(v) = \sigma_v \lambda$ , we have  $(p(v), \alpha) = (\sigma_v \lambda, \alpha) = (\lambda, \sigma_v^{-1} \alpha)$ . Note that the reflection defined by the root  $\sigma_v^{-1} \alpha$  corresponds to the transposition  $(\sigma_v^{-1}(i) \ \sigma_v^{-1}(j)) = (s \ r)$ . Hence,  $|(\lambda, \sigma_v^{-1} \alpha)| = |\lambda_r - \lambda_s|$ .  $\square$

We now prove Proposition 3.3. Let  $\Gamma(u, B)$  be a facet of  $\Gamma(v, B)$ . By Lemma 5.3 the vertices  $v$  and  $u$  are connected by the edge. We have  $p(u) = s_\alpha p(v)$  for some

root  $\alpha$ . Suppose that  $\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$  where  $0 < i < j < n$ . By Lemma 5.4 we have that  $|(p(v), \alpha)| = |\lambda_r - \lambda_s|$ , where  $p_{1,s}$  and  $p_{1,r}$  are the starting points of the trees  $T_i(v)$  and  $T_j(v)$ , respectively. We now show that  $|\lambda_r - \lambda_s| = d(v, \Gamma(u, B))$ . Denote by  $e$  the  $i$ -th edge of the tree  $T_j(v)$ . Since  $u$  and  $v$  are connected by the edge we get by Lemma 5.2 that the diagram  $D(u)$  is obtained from  $D(v)$  by switching the edge  $e$ . We again consider the case where  $e$  is of type  $R$ , since the proof for the other case is completely the same. Let  $p_{i,l+1}$  and  $p_{i+1,l}$  be the vertices of the edge  $e$ . Denote by  $F$  the facet of the Gelfand-Zetlin polytope given by the equation  $x_{i-1,l} = x_{i,l}$ . It is easy to check that  $\Gamma(u, B) = F \cap \Gamma(v, B)$ . Hence, the integral distance  $d(v, \Gamma(u, B))$  is by definition equal to the distance  $d(v, F)$ . To compute the latter we note that the equation  $x_{i-1,l} = x_{i,l}$  defining  $F$  is already primitive. Since  $p_{i,l}$  belongs to  $T_i(v)$  and  $p_{i+1,l}$  to  $T_j(v)$  we get that the  $x_{i-1,l}$ -coordinate of  $v$  is equal to  $\lambda_s$  and the  $x_{i,l}$ -coordinate to  $\lambda_r$ . Hence,  $d(v, F) = \lambda_r - \lambda_s$ .

**5.4. Chevalley formula for arbitrary faces  $\Gamma(v, B)$ .** The same arguments as in the proof of Proposition 3.3 allow us to prove a more general Chevalley type formula for the faces of the Gelfand-Zetlin polytope. Let  $\Gamma(v, B)$  be any (not-necessarily) admissible face, and  $\Gamma(u, B)$  a face such that  $\mathcal{O}(u, B)$  precedes  $\mathcal{O}(v, B)$  (but we no longer require that  $\Gamma(u, B) \subset \Gamma(v, B)$ ). Let  $\alpha = \alpha_i + \dots + \alpha_{j-1}$  be the root such that  $p(v) = s_\alpha p(u)$ . Consider those edges  $e_1, \dots, e_k$  ending at the  $(i+1)$ -st row of the diagram  $D(u)$  that differ by the slope from the corresponding edges at the  $(i+1)$ -st row of  $D(v)$ . Each such edge considered alone gives the diagram of a facet in the Gelfand-Zetlin polytope. Denote by  $F_i$  the facet defined by the edge  $e_i$ . Put  $d(v, u) := d(v, F_1) + \dots + d(v, F_k)$ .

**Theorem 5.5.** *Let  $\Gamma(v, B)$  be any (not-necessarily) admissible face. Then*

$$H_\lambda \overline{\mathcal{O}(v, B)} = \sum d(v, u) \overline{\mathcal{O}(u, B)},$$

where the sum is taken over all Schubert cells  $\mathcal{O}(u, B)$  that precede  $\mathcal{O}(v, B)$ .

Note that for admissible faces Theorem 5.5 reduces to Theorem 3.4 (since we have  $k = 1$  by Lemma 5.3 and  $\Gamma(u, n) = F_1 \cap \Gamma(v, n)$ ). Theorem 5.5 is important for a realization of Schubert cycles by unions of faces of the Gelfand-Zetlin polytope [9].

*Proof.* The proof is almost the same as for admissible faces. We have  $|(p(v), \alpha)| = |\lambda_r - \lambda_s|$  by Lemma 5.4. Assume that  $r > s$ . We can also write  $\lambda_r - \lambda_s$  as  $(\lambda_r - \lambda_{i_{k-1}}) + (\lambda_{i_{k-1}} - \lambda_{i_{k-2}}) + \dots + (\lambda_{i_1} - \lambda_s)$ , where  $\lambda_{i_1}, \dots, \lambda_{i_{k-1}}$  correspond to the starting points of the trees in  $D(u)$  containing the edges  $e_1, \dots, e_{k-1}$ , respectively. It is easy to check that  $(\lambda_{i_i} - \lambda_{i_{i-1}}) = d(v, F_i)$  using the same argument as in the proof of Proposition 3.3.  $\square$

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