POISSON STRUCTURES ON MODULI SPACES OF HIGGS BUNDLES OVER STACKY CURVES

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Abstract. We demonstrate the construction of Poisson structures via Lie algebroids on moduli spaces of stable Higgs bundles over stacky curves. This construction provides in particular new examples of holomorphic Poisson manifolds. Special attention is given to moduli spaces of parabolic Higgs bundles and of Higgs bundles over root stacks.

1. Introduction

Symplectic or Poisson structures on moduli spaces over a complex algebraic curve have been obtained in a variety of cases. A fundamental example involves the $G$-character varieties, as moduli spaces of surface group representations into a connected complex reductive group $G$. These are finite-dimensional complex symplectic manifolds when the surface $X$ is compact, while when $X$ has boundary components, they are Poisson foliated by symplectic leaves obtained by fixing the conjugacy class of the monodromy around each boundary component. The symplectic structure in the case when $X$ is compact was first conceived analytically using the method of symplectic reduction from infinite dimensional spaces in the seminal work of M. Atiyah and R. Bott [3]. At the same time, a more topological approach to the natural symplectic structure on such spaces of fundamental group representations was proposed by W. Goldman [27] interpreting these structures in terms of the intersection pairing on the underlying topological surface. The method of symplectic reduction was next further developed by N. Hitchin in [30] to produce Kähler and hyperkähler structures on the moduli space of stable Higgs bundles via the non-abelian Hodge correspondence and their counterparts to moduli spaces of solutions to the self-duality equations.

For a smooth complex algebraic curve $X$ and a reductive group $G$, the moduli space of $G$-local systems carries a Poisson structure (see for instance [25, 29]), while T. Pantev and B. Toën [42] explored such moduli on higher-dimensional smooth open varieties. Moreover, A. Kuznetsov and D. Markushevich [32] proposed a way to construct a closed holomorphic 2-form on the moduli space of sheaves over an arbitrary smooth complex projective variety via the Yoneda coupling composed with the exterior power of an Atiyah class, thus suggesting a general reason for these moduli spaces to be symplectic.

When $X$ has nonempty boundary, the symplectic structure generalizes to a Poisson structure. The symplectic leaves consist of equivalence classes of connections with fixed conjugacy class of local holonomy around each boundary component. A primary description of this theory appeared in the book of M. Atiyah [2], while in the article of M. Audin [4] a review of several approaches is presented. In particular, in [4] a detailed description is presented of the finite-dimensional discretization of this theory due to V. Fock and A. Rosly [25], viewing the compact surface as a fat graph and using it to describe the moduli space of flat connections and its Poisson structure in purely finite-dimensional terms. This is closely related to an approach using symplectic groupoids developed in [28] (see also [9, 10]).

The punctured surface case, seen as the tame case, naturally generalizes when one includes the extra topological data (beyond the $\pi_1$ representation) that classify connections/Higgs fields with poles.

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of order greater than one; this pertains to wild character varieties, the topological description of moduli spaces of meromorphic connections on principal $G$-bundles over a compact Riemann surface that can be built of pieces coming from the classification of meromorphic connections on principal $G$-bundles over a disk. We refer to the articles of P. Boalch, [14] and [19], for an intrinsic description of this extra topological data generalizing from the tame to the wild case. These spaces and their natural symplectic-Poisson structures have been extensively studied by P. Boalch (see for instance the survey article [16] as well as [17, 18]), where the Atiyah-Bott infinite-dimensional approach to moduli of flat connections was extended to allow certain singularities for the connections and involved explicit computation of the quasi-Hamiltonian structures.

Nonetheless, the hyperkähler structure on the moduli space of stable Higgs bundles from [30] has a vast generalization in this wild character variety context proved in [6]; the moduli spaces of integrable connections with fixed equivalence classes of polar parts at each irregular singularity are hyperkähler. The algebro-geometric compatibility condition between the meromorphic connection/Higgs field and the parabolic structure on a holomorphic bundle is described in [15] generalizing that of C. Simpson in the simple pole case [43].

Inspired by the aforementioned works, we seek in this article to exhibit a wider class of Poisson varieties over stacky curves, demonstrating an intrinsic property that justifies the existence of the Poisson structure.

Focusing on moduli spaces of Higgs bundles in the particular situation of parabolic vector bundles on punctured Riemann surfaces $X$ with trivial flag in the parabolic structure, Poisson structures on those were obtained independently by F. Bottacin [21] and E. Markman [36]. In the sequel, M. Logares and J. Martens in [35] considered moduli spaces $P_\alpha$ of $\alpha$-semistable parabolic Higgs bundles on $X$. In fact, the open subset $P^0_\alpha \subset P_\alpha$ of pairs involving a stable underlying parabolic bundle is a vector bundle over the moduli space $N_\alpha$ of stable parabolic bundles. M. Logares and J. Martens showed that the dual of this vector bundle in an Atiyah algebroid associated to a principal bundle over the space $N_\alpha$, thus having a complex algebraic Poisson structure that extends to a Poisson structure on the whole $P_\alpha$. It is important here the fact that the Atiyah sequence of the algebroid naturally follows from the deformation theory of parabolic vector bundles and a Serre duality map in hypercohomology plays a pivotal role in the definition of the Poisson bracket. These ideas provided important motivation for the development of the present work.

Let $X$ be a smooth and projective Deligne-Mumford stack over $\mathbb{C}$ and let $X$ be the coarse moduli space of $X$ with trivial flag in the parabolic structure, Poisson structures on those were obtained independently by F. Bottacin [21] and E. Markman [36]. The moduli problem of Higgs bundles over Deligne-Mumford stacks has been more generally studied in [47] and [48] by the second author.

Suppose that $X$ is a stacky curve, which is a smooth projective Deligne-Mumford stack of dimension one. We are thus considering the moduli space $M_H(X,\alpha)$ of stable Higgs bundles over $X$ with a fixed parabolic structure $\alpha$, as well as $M_H(X,G)$ and $M_H(X,G,\alpha)$, the relative moduli spaces of pairs where the structure group of the underlying bundles is determined by a reductive complex algebraic group $G$.

The method followed in order to obtain a Poisson structure on these moduli spaces is via the dual of Lie algebroids, as pioneered in [35]. We show that the moduli space $M_H(X,\alpha)$ is a Lie algebroid over the tangent space of the moduli space of stable bundles over the coarse moduli space, thus implying the main theorem of this article:

**Theorem 1.1** (Theorem 4.4). Let $X$ be a stacky curve over $\mathbb{C}$. The moduli space $M_H(X,\alpha)$ of stable Higgs bundles over $X$ with fixed parabolic structure $\alpha$ has a Poisson structure.

In the course of developing the proof of this result, we highlight the importance of certain short exact sequences (Atiyah sequences) that arise. This opens the way for generalizing the above theorem in several directions. We first show similarly that the moduli space $M_H(X,G,\alpha)$ for a fixed faithful representation $G \hookrightarrow \text{GL}(V)$ is also equipped with a Poisson structure:
Theorem 1.2 (Theorem 4.5). The moduli space $\mathcal{M}_H(\mathcal{X}, G, \alpha)$ of stable $G$-Higgs bundles over a stacky curve $\mathcal{X}$ with fixed parabolic structure $\alpha$ has a Poisson structure.

The Poisson structure extends on the semistable moduli space, since the locus of stable $G$-Higgs bundles is dense in the moduli space of semistable $G$-Higgs bundles. However, the notion of stability we consider here does depend on the choice of a faithful representation $G \to \text{GL}(V)$. In the particular case when the group $G$ is a semisimple linear algebraic group over $\mathbb{C}$, the stability condition does not depend on the faithful representation (see [5, 39]). An alternative notion of (semi)stability for parabolic principal $G$-bundles and parabolic $G$-Higgs bundles is considered in the more recent work of O. Biquard, O. García-Prada and I. Mundet i Riera [7]. We believe that the arguments in the proof of Theorem 4.5 can be used for constructing a Poisson structure on the moduli space with respect to the stability condition from [7] and hope to return to this question in a future article.

Note that in the special case of a root stack and a parabolic structure when all parabolic weights are rational, there is an alternative description of parabolic bundles as orbifold bundles (see [8, 26, 33, 37]). Therefore, our theorems provide an orbifold version of the result of F. Bottacin [21] and E. Markman [36] in the case of simple pole divisors, as was first conjectured by M. Logares and J. Martens [35, Section 5.2]; moreover, this result is generalized here even further to the case when $\mathcal{X}$ is a Deligne-Mumford stack and the Poisson structure is considered not only on the moduli space, but also on the moduli stack.

Let $\mathcal{M}$ be a moduli problem (as a stack), and suppose that $\mathcal{M}$ is a fine moduli space of $\mathcal{X}$. We have $\mathcal{M} \cong \text{Hom}(\cdot, \mathcal{M})$, therefore, a Poisson structure on $\mathcal{M}$ will induce a Poisson structure on $\mathcal{X}$. A theory of Poisson varieties over stacks in needed here, and this was introduced in the dissertation of J. Waldron [49]. Based on the theory of Poisson structures on stacks, we show that the moduli problem $\mathcal{M}_H(\mathcal{X}, \alpha)$ has a Poisson structure as a stack (Corollary 4.7).

Next, we show that the moduli space $\mathcal{M}_H(\mathcal{X}, \mathcal{L}, \alpha)$ of $\mathcal{L}$-twisted stable Higgs bundles over $\mathcal{X}$ with fixed parabolic structure $\alpha$ is Poisson, subject to the existence of a certain short exact sequence for any stable bundle $\mathcal{F} \in \mathcal{M}(\mathcal{X}, \alpha)$; the precise statement is the following:

**Theorem 1.3** (Theorem 5.1). Let $\mathcal{X}$ be a stacky curve over $\mathbb{C}$ and let $X$ be the coarse moduli space of $\mathcal{X}$. Let $\alpha$ be a parabolic structure on $X$ and denote by $D = p_1 + \cdots + p_k \in X$ the divisor with respect to $\alpha$. Let $\mathcal{D} = q_1 + \cdots + q_l \in \mathcal{X}$ be the corresponding divisor on $\mathcal{X}$, where $q_i$ is the point corresponding to $p_i$. If there exists a short exact sequence

$$0 \to \text{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_X) \to \mathcal{E}\text{nd}(\mathcal{F}) \to \mathcal{n} \to 0$$

for any stable bundle $\mathcal{F} \in \mathcal{M}(\mathcal{X}, \alpha)$, such that

1. the morphism $\text{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_X) \to \mathcal{E}\text{nd}(\mathcal{F})$ is not surjective,
2. $\mathcal{n}$ is a sheaf of Lie algebras supported on $\mathcal{D}$,

then the moduli space $\mathcal{M}_H(\mathcal{X}, \mathcal{L}, \alpha)$ has a Poisson structure.

For $X$ a smooth projective curve, $\widetilde{D} = p_1 + \cdots + p_k$ a reduced effective divisor on $X$ and $\mathcal{D} = (r_1, \ldots, r_k)$ a $k$-tuple of positive integers, let $X_{\widetilde{D}, \mathcal{D}}$ denote the corresponding root stack and consider the natural map $\pi : \mathcal{X} \to X$ from the root stack $\mathcal{X}$ to its coarse moduli space $X$. Denote by $\mathcal{D}$ the reduced divisor of $\pi^{-1}(\mathcal{D})$. The correspondence between parabolic bundles on $(X, \mathcal{D})$ and bundles on $\mathcal{X} := X_{\mathcal{D}, \mathcal{D}}$ implies the correspondence in the stability conditions as in [20, Remarque 10] and, more precisely, in moduli spaces $\mathcal{M}(\mathcal{X}, \alpha) \cong \mathcal{M}^{\text{par}}(X, \alpha)$. It is natural to extend this correspondence to Higgs bundles [11, 12, 37], thus having $\mathcal{M}_H(\mathcal{X}, \alpha) \cong \mathcal{M}^{\text{par}}_H(X, \alpha)$. Now let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ and denote by $L$ the corresponding parabolic line bundle of $\mathcal{L}$ on $(X, \mathcal{D})$. There is a one-to-one correspondence between $\mathcal{L}$-twisted Higgs bundles on $\mathcal{X}$ and $\mathcal{L}$-twisted parabolic bundles on $(X, \mathcal{D})$ (see [34, §5]). We have the following proposition:

**Proposition 1.4** (Proposition 6.2). Let $\mathcal{X} = X_{\mathcal{D}, \mathcal{D}}$ be a root stack. Denote by $X$ the coarse moduli space of $\mathcal{X}$. The following statements hold:
There is an isomorphism $\mathcal{M}_H^{\text{par}}(X,\alpha) \cong \mathcal{M}_H^{\text{par}}(X,\alpha)$, where $\mathcal{M}_H^{\text{par}}(X,\alpha)$ is the moduli space of strongly parabolic Higgs bundles over $X$ with parabolic structure $\alpha$.

There is an isomorphism $\mathcal{M}_H(X,\omega_X(D),\alpha) \cong \mathcal{M}_H^{\text{par}}(X,\alpha)$, where $\omega_X(D)$ is the canonical line bundle.

Let $L$ be an invertible sheaf on $X$ and let $\pi : \mathcal{X} \to X$ be the map from the root stack to its coarse moduli space. Denote by $L$ the corresponding parabolic line bundle of $L$ on $(X,\bar{D})$.

Then there is an isomorphism $\mathcal{M}_H(X,L,\alpha) \cong \mathcal{M}_H^{\text{par}}(X,L,\alpha)$.

Under these considerations for the stack and the parabolic structure, we find that the short exact sequence

$$0 \to \text{Hom}(F \otimes L, F \otimes \omega_X) \to \text{End}(F) \to n \to 0$$

in Theorem 5.1 can be translated in the language of parabolic bundles by taking $L = \omega_X(D)$ to

$$0 \to \text{SParEnd}(F) \to \text{ParEnd}(F) \to n \to 0,$$

where $F$ is the corresponding parabolic bundle of $F$. With respect to the above observation, Theorem 5.1 gives an alternative proof to [35] on the existence of a Poisson structure on $\mathcal{M}_H^{\text{par}}(X,\alpha)$, the moduli space of stable parabolic Higgs bundles with parabolic structure $\alpha$ on $(X,\bar{D})$, where $X$ is an irreducible smooth curve and $D$ is a reduced effective divisor.

2. Preliminaries

In this preliminary section, we collect the necessary background on stacks that we shall need for our purposes and we consider parabolic structures of locally free sheaves on root stacks. The interested reader may refer to [12, 40] for further background on the material covered in §2.1 - 2.3. A good reference for §2.4 is [44, §4, §5]. In §2.5, we give the definition of the parabolic structure of a principal $G$-bundle on Deligne-Mumford stacks, which depends on a fixed faithful representation $G \hookrightarrow \text{GL}(V)$.

2.1. Deligne-Mumford Stacks. In this paper, we work over a smooth projective Deligne-Mumford stack $\mathcal{X}$ over $\mathbb{C}$; we review accordingly:

**Definition 2.1.** An algebraic space is a functor $\mathcal{X} : (\text{Sch}/\mathbb{C})^{\text{op}} \to \text{Set}$ such that

1. $\mathcal{X}$ is a sheaf with respect to the (big) étale topology;
2. the diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is representable by schemes;
3. there is a surjective étale morphism $Y \to \mathcal{X}$, where $Y$ is a $\mathbb{C}$-scheme.

**Definition 2.2.** An algebraic stack is a category fibered in groupoids $\mathcal{X} : (\text{Sch}/\mathbb{C})^{\text{op}} \to \text{Set}$ such that

1. it satisfies the effective descent condition;
2. the diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is representable by algebraic spaces;
3. there is a surjective smooth morphism $Y \to \mathcal{X}$, where $Y$ is a $\mathbb{C}$-scheme.

Note that a stack is defined as a category fibered in groupoids satisfying the effective descent condition.

**Definition 2.3.** A Deligne-Mumford stack is an algebraic stack such that there is a surjective étale morphism $Y \to \mathcal{X}$, where $Y$ is a $\mathbb{C}$-scheme.

Let $\mathcal{X}$ be a Deligne-Mumford stack over $\mathbb{C}$. A local chart of $\mathcal{X}$ is a pair $(U,u)$, where $U$ is a scheme over $\mathbb{C}$ and $u : U \to \mathcal{X}$ is an étale morphism. Let $(U,u)$ and $(V,v)$ be two local charts of $\mathcal{X}$. A morphism of local charts $(U,u)$ and $(V,v)$ is a morphism $f_{uv} : (U,u) \to (V,v)$ of schemes such that the following diagram commutes

$$
\begin{array}{ccc}
U & \xrightarrow{f_{uv}} & V \\
\downarrow{u} & \searrow{v} & \\
\mathcal{X} & \xrightarrow{u} & \mathcal{X} \\
\end{array}
$$
Let $P$ be a property of $\mathbb{C}$-schemes. We say a Deligne-Mumford stack is $P$, if there is a surjective étale morphism $Y \to X$ such that the scheme $Y$ is $P$. The property $P$ can involve notions of being separated, proper, smooth, projective, locally of finite type, etc.

If $X$ is locally of finite type and with finite diagonal, then there exists a coarse moduli space $X$ of $X$ [40, Theorem 11.1.2]. Furthermore, if we assume that $X$ is smooth and projective (or proper), then there exists a surjective étale morphism $Y \to X$ such that $Y$ is a smooth projective variety [44, Theorem 5.4]. A smooth projective Deligne-Mumford stack $X$ in this paper is a Deligne-Mumford stack, which is smooth and projective, has a coarse moduli space $X$ and can be written as a global quotient.

**Definition 2.4.** A stacky curve in this paper is defined as a smooth projective geometrically connected Deligne-Mumford stack of dimension one.

**Definition 2.5.** Let $X$ be a smooth projective Deligne-Mumford stack. A Cartier divisor $D$ on $X$ is defined on each local chart such that for each local chart $(U, u)$, there is a Cartier divisor $D_u$ on $U$ such that if $f_{uv} : (U, u) \to (V, v)$ is a morphism of local charts, then $f_{uv}^*(D_v) = D_u$. We say a Cartier divisor $D$ has normal crossings if for each local chart $(U, u)$, the divisor $D_u$ has normal crossings.

**Definition 2.6.** A geometric point of $X$ is a morphism $x : \text{Spec}(\mathbb{C}) \to X$. An étale neighborhood of a geometric point $x$ is a commutative diagram

$$
\text{Spec}(\mathbb{C}) \longrightarrow U \xrightarrow{u} X
$$

where $u : U \to X$ is étale.

2.2. Sheaves.

**Sheaves over Stacks.** Let $X$ be a Deligne-Mumford stack. A coherent (resp. quasi-coherent) sheaf $F$ over $X$ is defined on the local charts of $X$. More precisely, on each local chart $(U, u)$ of $X$, let $F_u$ be a coherent (resp. quasi-coherent) sheaf on $U$. Let $\alpha_{uv} : F_u \to f_{uv}^*F_v$ be an isomorphism of coherent (resp. quasi-coherent) sheaves. The coherent (resp. quasi-coherent) sheaf $F$ is defined by the data $(F_u, \alpha_{uv})_{u, v \in \mathcal{I}}$. Sometimes we use the notation $F = (F_u, \alpha_{uv})_{u, v \in \mathcal{I}}$ to work on the local charts of the coherent (resp. quasi-coherent) sheaf $F$. Given a local chart $(U, u)$ and a coherent sheaf $F$ over $X$, if there is no ambiguity, we will prefer to use the notation

$$
F(U) = F_U := F_u
$$

for the coherent sheaf over $U$. A coherent sheaf $F$ is locally free if for each local chart $(U, u)$, the coherent sheaf $F_u$ is locally free. In this paper, a vector bundle is considered as a locally free sheaf.

There is an equivalent way to define a coherent sheaf on $X$. Let $(U, u)$ be a local chart of $X$. Note that $u : U \to X$ is an étale surjective morphism. A coherent sheaf $F$ is equivalent to a pair $(F, \sigma)$, where $F$ is a coherent sheaf on $U$ and $\sigma : s^*F \cong t^*F$ is an isomorphism, where $s, t : U \times_X U \to U$ are the source and target maps. Let $\text{Coh}(X)$ be the category of coherent sheaves on $X$, and let $\text{Coh}(U \times_X U \cong U)$ be the category of pairs $(F, \sigma)$. Then, these two categories are equivalent (see [40, Chap 4]).

We next provide some examples of coherent sheaves on $X$. For simplicity, we omit the isomorphisms $\alpha_{uv}$. The structure sheaf $O_X$ is defined as $O_X(U) := O_U$ on each local chart $u : U \to X$. A coherent sheaf on $X$ is also called an $O_X$-module. For each local chart $u : U \to X$, we define $\Omega^1_X(U) := O^1_{U/C}$ to be the sheaf of differentials on $U$. These local charts give a well-defined sheaf on $X$, and we use the notation $\Omega^1_X$ for this sheaf. The sheaf $\Omega^1_X$ is known as the sheaf of differentials on $X$.

**Higgs Bundles over stacks.** Let $X$ be a Deligne-Mumford stack. A Higgs bundle over $X$ is a pair $(F, \Phi)$, where $F$ is a locally free sheaf on $X$ and $\Phi : F \to F \otimes \Omega^1_X$ is a morphism called the Higgs field. More precisely, on each local chart $u : U \to X$, the morphism $\Phi$ is given by $\Phi_u : F_u \to F_u \otimes \Omega^1_u$ such that the following diagram commutes:
Stability Condition. Suppose that \( \mathcal{X} \) has a coarse moduli space \( X \), which is a projective scheme. Denote by \( \pi : \mathcal{X} \to X \) the natural map, and by \( q : \mathcal{X} \to \mathbb{C} \) the structure morphism. Let \( \mathcal{F} \) be a locally free sheaf over \( \mathcal{X} \) and \( \Phi : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L} \) is a morphism called the \( \mathcal{L} \)-twisted Higgs field.

For a line bundle \( \mathcal{L} \) over \( \mathcal{X} \), an \( \mathcal{L} \)-twisted Higgs bundle over \( \mathcal{X} \) is a pair \((\mathcal{F}, \Phi)\), where \( \mathcal{F} \) is a locally free sheaf over \( \mathcal{X} \) and \( \Phi : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L} \) is a morphism called the \( \mathcal{L} \)-twisted Higgs field.

Stability Condition. Suppose that \( \mathcal{X} \) has a coarse moduli space \( X \), which is a projective scheme. Denote by \( \pi : \mathcal{X} \to X \) the natural map, and by \( q : \mathcal{X} \to \mathbb{C} \) the structure morphism. Let \( \mathcal{F} \) be a locally free sheaf over \( \mathcal{X} \). The degree of \( \mathcal{F} \) over \( \mathcal{X} \) is defined (see [20]) as

\[
\deg(\mathcal{F}) := q_*(c_1(\mathcal{F})\pi^*(\mathcal{O}_X(1))^{n-1}).
\]

A locally free sheaf \( \mathcal{F} \) is called semistable (resp. stable), if for any subsheaf \( \mathcal{F}' \) with \( \text{rk}(\mathcal{F}') < \text{rk}(\mathcal{F}) \), it is

\[
\frac{\deg(\mathcal{F}')}{\text{rk}(\mathcal{F}')} \leq \frac{\deg(\mathcal{F})}{\text{rk}(\mathcal{F})} \quad \text{(resp. <)}.
\]

Similarly, a Higgs bundle \((\mathcal{F}, \Phi)\) is called semistable (resp. stable), if for any \( \Phi \)-invariant subsheaf \( \mathcal{F}' \) with \( \text{rk}(\mathcal{F}') < \text{rk}(\mathcal{F}) \), it is

\[
\frac{\deg(\mathcal{F}')}{\text{rk}(\mathcal{F}')} \leq \frac{\deg(\mathcal{F})}{\text{rk}(\mathcal{F})} \quad \text{(resp. <)}.
\]

2.3. Principal \( G \)-bundles over Deligne-Mumford Stacks.

Principal \( G \)-bundles. Let \( G \) be a reductive complex linear algebraic group. Denote by \( \mathfrak{g} \) the Lie algebra of \( G \). A principal \( G \)-bundle \( \mathcal{E} \) over \( \mathcal{X} \) consists of a principal \( G \)-bundle \( \mathcal{E}_u \) on each local chart \((U, u)\) of \( X \) together with isomorphisms \( \alpha_{uv}^{\mathcal{E}} : \mathcal{E}_u \to f_{uv}^*\mathcal{E}_v \) between local charts \((U, u)\) and \((V, v)\). The following equivalence [12, Proposition 1.2] is well-known:

\[
\text{Bun}_G \mathcal{X} \cong \text{Hom}_C(\mathcal{X}, BG),
\]

where \( \text{Bun}_G \mathcal{X} \) is the category of principal \( G \)-bundles on \( \mathcal{X} \) and \( BG \) is the classifying space of \( G \).

\( G \)-Higgs bundles. Let \( \mathcal{X} \) be Deligne-Mumford stack. A \( G \)-Higgs bundle over \( \mathcal{X} \) is a pair \((\mathcal{E}, \Phi)\) such that \( \mathcal{E} \) is a principal \( G \)-bundle on \( \mathcal{X} \) and \( \Phi \in H^0(\mathcal{X}, \mathcal{E}(\mathfrak{g}) \otimes \Omega^1_{\mathcal{X}}) \) is a section, where \( \mathcal{E}(\mathfrak{g}) := \mathcal{E} \times_G \mathfrak{g} \) is the adjoint bundle. The Higgs field \( \Phi \) is defined on local charts with respect to isomorphisms between the local charts; here is the definition: Let \( u : U \to \mathcal{X} \) be a local chart of \( \mathcal{X} \) and let \( \Phi_u \) be an element in \( H^0(U, \mathcal{E}_u(\mathfrak{g}) \otimes \Omega^1_{U/\mathbb{C}}) \), which is an \( \mathcal{E}_u(\mathfrak{g}) \)-valued 1-form and \( \mathcal{E}_u(\mathfrak{g}) := \mathcal{E}_u \times_G \mathfrak{g} \). Now given two local charts \((U, u)\) and \((V, v)\) of \( \mathcal{X} \), let \( f_{uv} : (U, u) \to (V, v) \) be a morphism between them. We then have the following pullback diagram

\[
f_{uv}^*\mathcal{E}_v \longrightarrow \mathcal{E}_v \\
\downarrow \quad \downarrow \\
U \quad f_{uv} \quad V
\]

thus we get an element \( f_{uv}^*(\Phi_v) \in H^0(U, f_{uv}^*\mathcal{E}_v(\mathfrak{g}) \otimes \Omega^1_{U/\mathbb{C}}) \). The condition we want is that the isomorphism \( \alpha_{uv}^{\mathcal{E}} : \mathcal{E}_u \to f_{uv}^*\mathcal{E}_v \) also implies that

\[
\Phi_u = (\alpha_{uv}^{\mathcal{E}})^* f_{uv}^* \Phi_v.
\]

Therefore, a Higgs field \( \Phi \) is given by the data \((\Phi_u, (\alpha_{uv}^{\mathcal{E}})^*)_{u,v \in \mathcal{I}}\).
**Stability Condition.** Suppose that $\mathcal{X} = [U/\Gamma]$ is a global quotient, where $\Gamma$ is a finite group. By definition, a $G$-Higgs bundle $E$ on $\mathcal{X}$ is equivalent to a $\Gamma$-linearized $G$-Higgs bundle on $U$ (see [5, §2]). Let $G \hookrightarrow \text{GL}(V)$ be a faithful representation. Denote by $\mathcal{E}(V) := E \times_G V$ the associated bundle. In this case, a Higgs field $\Phi \in H^0(\mathcal{X}, \mathcal{E}(g))$ corresponds to an element in $H^0(\mathcal{X}, \mathcal{End}(\mathcal{E}(V)) \otimes \Omega^1_{\mathcal{X}})$. A $G$-Higgs bundle $(\mathcal{E}, \Phi)$ is called semistable (resp. stable), if the associated Higgs bundle $(\mathcal{E}(V), \Phi)$ is semistable (resp. stable) under the faithful representation $G \hookrightarrow \text{GL}(V)$. The semistability condition of $G$-Higgs bundles is called semiharmonic in C. Simpson’s papers [46, page 49] and [44, §7].

**Remark 2.7.** Indeed, there are several ways to define the semistability of principal $G$-bundles. In this paper, we shall use the above definition for the semistability, since it has been proved that the moduli space of $G$-Higgs bundles over a projective Deligne-Mumford stack exists in this case [44].

**2.4. Root Stacks.** Root stacks are a highly significant case of Deligne-Mumford stacks. It is known that a smooth projective Deligne-Mumford stack is locally isomorphic to a root stack (see [44]). We shall review this result in this subsection.

Let $X$ be a smooth projective scheme, and let $L$ be a line bundle on $X$. Note that the following categories are equivalent

$$\{ \text{invertible sheaves on } X \} \longleftrightarrow \{ \text{morphisms: } X \to BG_m \}.$$

Let $s \in \Gamma(X, L)$ be a section of $L$. The pair $(L, s)$ defines a morphism $X \to [\mathbb{A}^1/G_m]$. The category of pairs $(L, s)$, where $L$ is a line bundle on $X$ and $s \in \Gamma(X, L)$, and the category of morphisms $X \to [\mathbb{A}^1/G_m]$ are equivalent. This equivalence can be generalized to $n$ line bundles and $n$ sections. More precisely, the category of morphisms $X \to [\mathbb{A}^n/G_m^n]$ is equivalent to the category of $n$-tuples $(L_i, s_i)_{i=1}^n$, where $L_i$ is a line bundle on $X$ and $s_i \in \Gamma(X, L_i)$ (see [22, Lemma 2.1.1]).

Let $\theta_i : [\mathbb{A}^1/G_m] \to [\mathbb{A}^1/G_m]$ be the morphism induced by $r$-th power maps on both $\mathbb{A}^1$ and $G_m$. Let $X_{(L,s,r)}$ be the fiber product $X \times_{[\mathbb{A}^1/G_m], \theta_i} [\mathbb{A}^1/G_m]$. The stack $X_{(L,s,r)}$ is then called the $r$-th root stack.

Let $\bar{D} = (D_1, \ldots, D_k)$ be a $k$-tuple of effective Cartier divisors $D_i \subseteq X$. Let $\bar{r} = (r_1, \ldots, r_k)$ be a $k$-tuple of positive integers. We define $\theta_{\bar{r}} : [\mathbb{A}^k/G_m^k] \to [\mathbb{A}^k/G_m^k]$ to be the morphism $\theta_{r_1} \times \cdots \times \theta_{r_k}$. The **Cadman-Vistoli root stack** $X_{\bar{D}, \bar{r}}$ is defined as the fiber product $X \times_{[\mathbb{A}^k/G_m^k], \theta_{\bar{r}}} [\mathbb{A}^k/G_m^k]$, where the morphism $X \to [\mathbb{A}^k/G_m^k]$ is defined by $(\mathcal{O}(D_i), s_{D_i})_{i=1}^k$ (see [22, Definition 2.2.4]).

Let $\mathcal{X}$ be a Deligne-Mumford stack locally of finite presentation with finite diagonal. Denote by $X$ its coarse moduli space. Let $x$ be a geometric point of $\mathcal{X}$. Denote by $G_x$ the automorphism group of $x$. Since $\mathcal{X}$ is a Deligne-Mumford stack, $G_x$ is a finite group. Let $\bar{x}$ be the corresponding point of $x$ in the coarse moduli space. There exists (see [40, Theorem 11.3.1]) a neighborhood $(U, u)$ of $\bar{x}$ and a finite morphism $V \to U$ such that

$$\mathcal{X} \times_X U \cong [V/T_x].$$

This property tells us that a Deligne-Mumford stack is locally a quotient stack. Furthermore, if we assume that $\mathcal{X}$ is smooth and projective, the local picture of $\mathcal{X}$ is equivalent to look at a Cadman-Vistoli root stack. More precisely, let $x \in \mathcal{X}$ be a geometric point, and let $\bar{x}$ be its corresponding point in $X$. There is an étale neighborhood $(U, u)$ of $\bar{x} \in X$ such that

$$\mathcal{X} \times_X U \cong U_{\bar{D}, \bar{r}}$$

for some $\bar{D}$ and $\bar{r}$ (see [44, §4 and §5]).

**2.5. Parabolic Bundles.** Now we consider a special case of a root stack. Let $X$ be a smooth projective curve over $\mathbb{C}$ and let $\bar{D} = (D_1, \ldots, D_k)$ be a $k$-tuple of divisors $D_i \subseteq X$ such that $D_i = p_i$ is a single point. Let $\bar{r} = (r_1, \ldots, r_k)$ be a $k$-tuple of positive integers. The notation $r(p)$ shall refer to the integer in $\bar{r}$ corresponding to the point $p \in \bar{D}$. 
Parabolic Structures of Parabolic Bundles. We assume that $X_{D, \bar{r}}$ can be written as a global quotient $[U/I] \cong X_{D, \bar{r}}$, where $U$ is a projective scheme. There is a natural map $\pi : U \to X$. A locally free sheaf $\mathcal{F}$ of rank $n$ over $X_{D, \bar{r}}$ is equivalent to a locally free sheaf $F$ on $U$ together with a local trivialization $\Theta_p : F_p \to U_p \times \mathbb{C}^n$ for each point $p \in \bar{D}$, where $U_p$ is a neighborhood of $\pi^{-1}(p)$ and $F_p := F|_{U_p}$, such that $\Theta_p$ is $\mathbb{Z}_{r(p)}$-equivariant with respect to the following action around the point $p \in \bar{D}$

$$t(z; z_1, z_2, ..., z_n) = (t_2; t^{\alpha_1(p)}z_1, t^{\alpha_2(p)}z_2, ..., t^{\alpha_n(p)}z_n),$$

where $\alpha_1(p), ..., \alpha_n(p)$ are integers such that $0 \leq \alpha_1(p) \leq \alpha_2(p) \leq ... \leq \alpha_n(p) \leq r(p)$. We can take local holomorphic sections $f_1, ..., f_n$ of $F$ such that $\{f_1(p), ..., f_n(p)\}$ is a basis of $F_p$ consisting of eigenvectors. Then, we can set

$$\Theta = (t^{-\alpha_1(p)}(t \cdot f_1), ..., t^{-\alpha_n(p)}(t \cdot f_n)),$$

where $t \cdot f_i(x) = t^{\alpha_i(p)}f_i(x)$. This action defines a weighted filtration of $F$ on $\pi^{-1}(p)$,

$$F|_{\pi^{-1}(p)} = F_1(p) \supseteq \cdots \supseteq F_n(p) \supseteq F_{n+1}(p) = 0,$$

$$\frac{\alpha_1(p)}{r(p)} \leq \cdots \leq \frac{\alpha_n(p)}{r(p)},$$

which shall be called a parabolic structure.

A parabolic bundle over $(X, D)$ is a pair $(F, \alpha)$, where $F$ is a locally free sheaf and $\alpha$ is a parabolic structure over each of the points in $\bar{D}$. A locally free sheaf $\mathcal{F}$ on $X_{D, \bar{r}}$ is equivalent to a locally free sheaf $F$ on $X$ together with a parabolic structure $\alpha$, which is defined on each point $p \in \bar{D}$. We refer the reader to [8, 26, 37] for more details on this correspondence. Furthermore, the parabolic structure itself is an important topological invariant of a locally free sheaf over $X_{D, \bar{r}}$, which can be used in describing the connected components of the moduli space of locally free sheaves on $X$ [33]. With respect to this correspondence, we say that a locally free sheaf $\mathcal{F}$ on $X_{D, \bar{r}}$ has parabolic structure $\alpha$ if the corresponding parabolic bundle on $X$ is of parabolic type $\alpha$.

A parabolic structure of a locally free sheaf over a point $p$ corresponds to a parabolic group. A parabolic structure over a point is given by integers $\alpha_1(p) \leq \cdots \leq \alpha_n(p)$, which defines a type. For example, the sequence of integers $(1, 1, 3, 3, 3, 3, 4, 4, 4)$ gives us the type $(2, 4, 3)$. This type in turn uniquely determines a parabolic subgroup of $GL_n$: $\left( \begin{array}{ccc} A_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & A_3 \end{array} \right)$, where $A_1$ is a 2 by 2 matrix, $A_2$ is a 4 by 4 matrix and $A_3$ is a 3 by 3 matrix.

The correspondence between parabolic structures and parabolic groups can be also understood from Higgs fields. Let now $\Phi$ be a Higgs field over $\mathcal{F}$. With respect to the above setup, $\Phi$ can be written as follows around $p \in \bar{D}$:

$$\Phi = (\Phi_{ij})_{1 \leq i, j \leq n},$$

where

$$\Phi_{ij} = \begin{cases} z^{\alpha_i(p)-\alpha_j(p)} \hat{\Phi}_{ij}(z)r(p) \frac{dz}{z} & \text{if } \alpha_i \geq \alpha_j \\ 0 & \text{if } \alpha_i < \alpha_j, \end{cases}$$

and $\hat{\Phi}_{ij}$ are holomorphic functions on $F$. Note that the morphism $\Phi$ is also an element in the parabolic group. In fact, the calculation also works for any endomorphism of $\mathcal{F}$ (see [33, 37]).

Parabolic Degree and Stability Condition. The parabolic degree of a parabolic vector bundle $F$ is given by

$$\text{pardeg}(F) = \text{deg}(F) + \sum_{p \in \bar{D}} \sum_{i=1}^{n} \alpha_i(p).$$
We call a parabolic vector bundle $F$ semistable (resp. stable) if for all parabolic subbundles $F'$, it is
\[
\frac{\text{pardeg}(F')}{\text{rk}(F')} \leq \frac{\text{pardeg}(F)}{\text{rk}(F)} \quad \text{(resp. <)}.
\]
From the correspondence between parabolic bundles $F$ over $X$ and bundles $F$ over $X_{\bar{D}}$, we have
\[
\text{deg}(F) = \text{pardeg}(F).
\]
Note that this property provides that if $F$ is semistable (resp. stable), then $F$ is also semistable (resp. stable).

**Parabolic Higgs Bundles.** We may also define the parabolic, strongly parabolic homomorphisms and tensor products, and thus the category of parabolic vector bundles forms a tensor category. The precise definitions can be found in [51].

We denote the sheaf of parabolic homomorphisms between two parabolic vector bundles $E$ and $F$ by $\text{ParHom}(E, F)$, the sheaf of strongly parabolic homomorphisms by $\text{SParHom}(E, F)$ and the tensor product by $E \otimes F$. In addition, we denote $\text{ParEnd}(E) = \text{ParHom}(E, E)$ and $\text{SParEnd}(E) = \text{SParHom}(E, E)$. We now define:

1. A parabolic Higgs bundle over $(X, \bar{D})$ is a pair $(E, \Phi)$, where $E$ is a parabolic vector bundle and $\Phi \in H^0(X, \text{ParEnd}(E) \otimes \omega_X(\bar{D}))$, where $\omega_X(\bar{D})$ is the canonical line bundle of $X$ over the divisor $\bar{D}$. We call it stable (resp. semistable) if it is slope stable (resp. semistable) with respect to the $\Phi$-fixed sub-bundles.

2. A strongly parabolic Higgs bundle $(X, \bar{D})$ is a pair $(E, \Phi)$ where $E$ is a parabolic vector bundle and $\Phi \in H^0(X, \text{SParEnd}(E) \otimes \omega_X(\bar{D}))$.

3. Let $L$ be a parabolic line bundle on $X$. We call $L$-twisted parabolic Higgs bundle over $(X, \bar{D})$ a pair $(E, \Phi)$, where $E$ is a parabolic vector bundle and $\Phi \in H^0(X, \text{ParEnd}(E) \otimes L)$.

**Parabolic Structure of $G$-Principal Bundles.** Closing this subsection, we give the definition of the parabolic structure of a principal $G$-bundle. Fix a faithful representation of $G$, $G \hookrightarrow \text{GL}(V)$. Let $E$ be a principal $G$-bundle on $X$ and denote by $E(V)$ the associated bundle. We say that the parabolic structure of $E$ is $\alpha$, if the parabolic structure of the corresponding associated bundle $E(V)$ is $\alpha$. Note that the parabolic structure of a principal $G$-bundle depends on the choice of the faithful representation.

3. Deformation Theory on Moduli Spaces of Higgs Bundles over Deligne-Mumford Stacks

In this section, we review some results on the deformation theory of moduli spaces of Higgs bundles over Deligne-Mumford stacks, which will help us to calculate the tangent space of the moduli spaces. In §3.1, we define all the moduli spaces we consider in this paper and in §3.2, we review the deformation theory on those moduli spaces. In §3.3, we review the Grothendieck duality of coherent sheaves over Deligne-Mumford stacks, while in §3.4, we restrict to stacky curves and apply the results from §3.2 and §3.3 to construct a morphism $T^*(\mathcal{M}_H(\mathcal{X})) \to T(\mathcal{M}_H(\mathcal{X}))$, which we shall use to construct a Poisson structure on $\mathcal{M}_H(\mathcal{X})$ later on in §4.

3.1. Moduli Space of Higgs Bundles on Smooth Projective Deligne-Mumford Stacks. Let $\mathcal{X}$ be a smooth and projective Deligne-Mumford stack. We first review the process of constructing the moduli space $\mathcal{M}_H(\mathcal{X})$ (see [44] for more details).

There exists a surjective étale morphism $Y \to \mathcal{X}$ such that $Y$ is a smooth projective variety [44, Theorem 5.4]. For a smooth projective (or proper) Deligne-Mumford stack $\mathcal{X}$, this admits a proper hyper-covering by smooth projective varieties [44, Theorem 5.8]. In other words, there is a simplicial resolution of $\mathcal{X}$ by smooth projective varieties. We briefly review next the construction of a simplicial resolution of $\mathcal{X}$. The first step of this construction is given by the existence of a surjective étale morphism $Y_0 \to \mathcal{X}$. Then we look at $Y_0 \times_{\mathcal{X}} Y_0$. By resolving singularities, we get a smooth projective variety $Y_1$. This gives us the starting point of a simplicial resolution $Y_1 \Rightarrow Y_0 \to \mathcal{X}$. Iterating the
process, we get the simplicial resolution $Y_*$ of $\mathcal{X}$. Since each $Y_k$ is a smooth projective variety over $\mathbb{C}$, the moduli space $\mathcal{M}_H(Y_k)$ of stable Higgs bundles over $Y_k$ exists [45, Theorem 4.7]. Thus, there is a natural way to construct the moduli space $\mathcal{M}_H(Y_*)$ of semistable Higgs bundles over $Y_*$ [44, §6]. Indeed, the moduli space of stable (resp. semistable) Higgs bundles over $Y_*$ is isomorphic to the moduli space of stable (resp. semistable) Higgs bundles over $\mathcal{X}$ [44, §9]:

\[ \mathcal{M}_H(Y_*) \cong \mathcal{M}_H(\mathcal{X}). \]

The moduli space of Higgs bundles over $\mathcal{X}$ is proved to be a quasi-projective scheme [44, §6].

In this paper, we prefer to consider the stable locus of the moduli space, but some of our results can be extended to the semistable case. We use the following notations for the moduli spaces we consider:

- $\mathcal{M}(\mathcal{X}, \bullet_1, \bullet_3)$: the moduli space of stable bundles on $\mathcal{X}$,
- $\mathcal{M}_H(\mathcal{X}, \bullet_1, \bullet_2, \bullet_3)$: the moduli space of stable Higgs bundles on $\mathcal{X}$,
- $\mathcal{M}^{\text{par}}(\mathcal{X}, \bullet_1, \bullet_3)$: the moduli space of stable parabolic bundles on $\mathcal{X}$,
- $\mathcal{M}^{\text{par}}(\mathcal{X}, \bullet_1, \bullet_2, \bullet_3)$: the moduli space of stable parabolic Higgs bundles on $\mathcal{X}$,

where $\bullet_1$ is the position for the structure group $G$, $\bullet_2$ is for the line bundle $L$ (as the twisting bundle) and $\bullet_3$ is for the parabolic structure $\alpha$. For example, $\mathcal{M}_H(\mathcal{X}, G, \alpha)$ denotes the moduli space of stable $G$-Higgs bundles on $\mathcal{X}$ with parabolic structure $\alpha$. The moduli spaces over Deligne-Mumford stacks are constructed in [44], while their parabolic analogs were constructed in [50].

### 3.2. Deformation Theory

The goal of this subsection is to calculate the tangent space of $\mathcal{M}_H(\mathcal{X})$, which is the moduli space of Higgs bundles on $\mathcal{X}$.

The moduli space $\mathcal{M}_H(\mathcal{X})$ represents the following moduli problem

\[ \widetilde{\mathcal{M}}_H(\mathcal{X}) : (\text{Sch}/\mathbb{C})^{\text{op}} \to \text{Set} \]

such that for each $T \in (\text{Sch}/\mathbb{C})$, $\widetilde{\mathcal{M}}_H(\mathcal{X})(T)$ is the set of isomorphism classes of pairs $(\mathcal{F}_T, \Phi_T)$ satisfying the following conditions

1. $\mathcal{F}_T$ is a locally free sheaf on $\mathcal{X} \times T$ and $\Phi_T : \mathcal{F}_T \to \mathcal{F}_T \otimes p_\mathcal{X}^* \Omega^1_{\mathcal{X}}$ is a morphism, where $p_\mathcal{X} : \mathcal{X} \times T \to \mathcal{X}$ is the natural projection.
2. $\mathcal{F}_T$ is flat over $T$.
3. The support of $\mathcal{F}_T$ is proper over $T$.
4. $(\mathcal{F}_T, \Phi_T)$ is stable.
5. Two pairs $(\mathcal{F}_T, \Phi_T)$ and $(\mathcal{E}_T, \Psi_T)$ are isomorphic, if there exists a line bundle $L$ on $T$ such that $\mathcal{F}_T \otimes p_\mathcal{T}^* L = \mathcal{E}_T$ and $\Phi_T \otimes 1_{p_\mathcal{T}^* L} = \Psi_T$.

Similarly, we can also define a moduli problem of semistable Higgs bundles. However, the moduli space of semistable Higgs bundles does not represent such a moduli problem, and it universally co-represents the moduli problem of semistable Higgs bundles (see [45, Theorem 4.7]).

A moduli problem is defined as a contravariant functor. Moreover, the moduli problem is also understood as a category fibered over groupoids satisfying the descent condition. In other words, a moduli problem is a stack [23]. In this section, we shall be using the notation $\widetilde{\mathcal{M}}$ for the moduli problem $\mathcal{M}_H(\mathcal{X})$ and $\mathcal{M}$ for the moduli space of stable Higgs bundles.

Let Spec$(A)$ be an affine scheme in Sch$/\mathbb{C}$, and let $M$ be an $A$-module. Let $\xi = (\mathcal{F}, \Phi)$ be an element in $\mathcal{M}(A)$, where $\mathcal{M}(A) := \mathcal{M}(\text{Spec}(A))$. There is a natural map

\[ \widetilde{\mathcal{M}}(A[M]) \to \mathcal{M}(A). \]

Denote by $\widetilde{\mathcal{M}}\xi(A[M])$ the pre-image of the element $\xi \in \mathcal{M}(A)$. In other words, $\widetilde{\mathcal{M}}\xi(A[M])$ is the set of elements whose restriction to $\mathcal{X}_A$ is $\xi$. The set $\widetilde{\mathcal{M}}\xi(A[M])$ is known as the set of deformations of $\xi$ with respect to the extension

\[ 0 \to M \to A[M] \to A \to 0. \]
Now let $A = \mathbb{C}$ and let $M$ be the free rank one $A$-module generated by $\varepsilon$. We consider the short exact sequence

$$0 \to (\varepsilon) \to \mathbb{C}[\varepsilon] \to \mathbb{C} \to 0,$$

where $\mathbb{C}[\varepsilon]$ is the ring $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ and we abuse the notation here.

Given an element $\xi \in \mathcal{M}(\mathbb{C})$, an infinitesimal deformation of $\xi$ is an element in $\widetilde{\mathcal{M}}_\xi(\mathbb{C}[\varepsilon])$. It is well-known that the set of all infinitesimal deformations of $\xi$ is the tangent space of the moduli space $\mathcal{M}$ at the point $\xi$.

Let $\xi = (\mathcal{F}, \Phi)$ be an element in $\widetilde{\mathcal{M}}(A)$. The deformation complex $C^*_M(\mathcal{F}, \Phi)$ is defined as

$$C^*_M(\mathcal{F}, \Phi) : C^0_M(\mathcal{F}) = \mathcal{E}nd(\mathcal{F}) \otimes M \xrightarrow{\varepsilon(\Phi)} C^1_M(\mathcal{F}) = \mathcal{E}nd(\mathcal{F}) \otimes p_X^* \Omega^1_X \otimes M,$$

where the map $\varepsilon(\Phi)$ is given by

$$\varepsilon(\Phi)(s) = -\rho(s)(\Phi)$$

and $p_X : X \times_A \mathbb{C} \to X$ is the natural projection. If there is no ambiguity, we omit the symbols $M$, $\mathcal{F}$, $\Phi$ and use the notation

$$C^* : C^0 = \mathcal{E}nd(\mathcal{F}) \otimes M \xrightarrow{\varepsilon(\Phi)} C^1 = \mathcal{E}nd(\mathcal{F}) \otimes \Omega^1_X \otimes M,$$

for the deformation complex. I. Biswas and S. Ramanan in [13] first described the set of infinitesimal deformations of a Higgs bundle over a smooth projective scheme and proved that the set of deformations is isomorphic to the first hypercohomology of a two-term complex. This approach was generalized to the case of Higgs bundles over a Deligne-Mumford stack in [47]. Therefore, we have the following proposition:

**Proposition 3.1** (Proposition 3.3 in [47]). Let $\xi = (\mathcal{F}, \Phi)$ be a Higgs bundle in $\widetilde{\mathcal{M}}(A)$. The set of deformations $\mathcal{M}_\xi(A[M])$ is isomorphic to the hypercohomology group $\mathbb{H}^1(C^*)$, where $C^*$ is the complex

$$C^* : C^0 = \mathcal{E}nd(\mathcal{F}) \otimes M \xrightarrow{\varepsilon(\Phi)} C^1 = \mathcal{E}nd(\mathcal{F}) \otimes \Omega^1_X \otimes M,$$

where $\varepsilon(\Phi)(s) = -\rho(s)(\Phi)$ is defined as above.

Proposition 3.3 in [47] actually proves the statement for an $\mathcal{L}$-twisted Higgs bundle, and we will use this to calculate the tangent space of $\mathcal{M}(X, \mathcal{L})$ (see Corollary 3.3). By taking the complex

$$0 \to (\varepsilon) \to \mathbb{C}[\varepsilon] \to \mathbb{C} \to 0,$$

the above proposition implies the following corollary:

**Corollary 3.2.** The tangent space of $\mathcal{M}_H(X)$ at the point $\xi = (\mathcal{F}, \Phi)$ is isomorphic to $\mathbb{H}^1(C^*_\varepsilon)$, where $C^*_\varepsilon$ is the complex

$$C^*_\varepsilon : C^0_\varepsilon = \mathcal{E}nd(\mathcal{F}) \longrightarrow C^1_\varepsilon = \mathcal{E}nd(\mathcal{F}) \otimes \Omega^1_X.$$

The result can be generalized to the $\mathcal{L}$-twisted case:

**Corollary 3.3.** The tangent space of $\mathcal{M}_H(X, \mathcal{L})$ at the point $\xi = (\mathcal{F}, \Phi)$ is isomorphic to $\mathbb{H}^1(C^*_\varepsilon)$, where $C^*_\varepsilon$ is the complex

$$C^*_\varepsilon : C^0_\varepsilon = \mathcal{E}nd(\mathcal{F}) \longrightarrow C^1_\varepsilon = \mathcal{E}nd(\mathcal{F}) \otimes \mathcal{L}.$$

Let $\mathcal{E}$ be a principal $G$-bundle. We fix a faithful representation $G \hookrightarrow \text{GL}(V)$, and consider the associated bundle $\mathcal{E}(V)$. Then, we can use the same argument as in Proposition 3.1 to calculate the tangent space of $\mathcal{M}_H(X, G)$.

**Corollary 3.4.** The tangent space of $\mathcal{M}_H(X, G)$ at the point $\xi = (\mathcal{E}, \Phi)$ is isomorphic to $\mathbb{H}^1(C^*_{\varepsilon,G})$, where $C^*_{\varepsilon,G}$ is the complex

$$C^*_{\varepsilon,G} : C^0_{\varepsilon,G} = \mathcal{E}(g) \longrightarrow C^1_{\varepsilon,G} = \mathcal{E}(g) \otimes \Omega^1_X.$$
3.3. Grothendieck Duality. Let $\mathcal{X}$ and $\mathcal{Y}$ be separated and finite type Deligne-Mumford stacks over $\mathbb{C}$. Denote by $D^b(\mathcal{X})$ the derived category of complexes of coherent sheaves over $\mathcal{X}$, where # represents here either of $b, +, -$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of stacks. The morphism induces the following functors of the categories of coherent sheaves

$$f_* : \text{Coh}(\mathcal{X}) \to \text{Coh}(\mathcal{Y}), \quad f^* : \text{Coh}(\mathcal{Y}) \to \text{Coh}(\mathcal{X}),$$

such that $f_*$ is right adjoint to $f^*$. From these two functors, we can define the derived functors

$$Rf_* : D^b(\mathcal{X}) \to D^b(\mathcal{Y}), \quad Lf^* : D^b(\mathcal{Y}) \to D^b(\mathcal{X}).$$

Note that the functor $Rf_*$ is still right adjoint to $Lf^*$:

$$\text{Hom}(\mathcal{E}, Rf_* \mathcal{F}) \cong \text{Hom}(Lf^* \mathcal{E}, \mathcal{F}).$$

In fact, we have another functor $f^! : D^b(\mathcal{Y}) \to D^b(\mathcal{X})$, which is right adjoint to $Rf_*$, satisfying

$$Rf_* R\text{Hom}(\mathcal{E}, f^! \mathcal{F}) \cong R\text{Hom}(Rf_* \mathcal{E}, \mathcal{F}),$$

where $\mathcal{E}^\bullet \in D^b(\mathcal{X})$ and $\mathcal{F}^\bullet \in D^b(\mathcal{Y})$.

**Theorem 3.5** (Theorem 2.22 in [38]). Let $\sigma : \mathcal{X} \to \text{Spec}(\mathbb{C})$ be a smooth proper Deligne-Mumford stack of dimension $n$ over $\mathbb{C}$. Then $\sigma^! \mathcal{O}$ is canonically isomorphic to the complex $\omega_{\mathcal{X}}[n]$.

Considering the morphism $f : \mathcal{X} \to \text{Spec}(\mathbb{C})$, we have

$$\text{Hom}_{D^b(\mathcal{X})}(\mathcal{E}^\bullet, \omega_{\mathcal{X}}[n]) \cong \text{Hom}(\mathcal{R}(\mathcal{E}^\bullet), \mathbb{C}).$$

If $\mathcal{E}^\bullet$ is a coherent sheaf $\mathcal{E}$ over $\mathcal{X}$, then $\text{Ext}^i(\mathcal{E}, \omega_{\mathcal{X}}) \cong H^{n-i}(\mathcal{X}, \mathcal{E})^\bullet$.

Let $\mathcal{E}^\bullet = 0 \to E_0 \to \cdots \to E_m \to 0$ be an element in $D^b(\mathcal{X})$. We have

$$H^i(\mathcal{X}, \mathcal{E}^\bullet) \cong H^{1-i+n}(\mathcal{E}^\bullet) \otimes \omega_{\mathcal{X}}$$

by Grothendieck duality, where $\mathcal{E}^{\bullet, *}$ is the “dual” complex

$$0 \to (E_m)^* \to \cdots \to (E_0)^* \to 0.$$

3.4. Application. For $\mathcal{X}$ a stacky curve, we then have $\Omega^1_{\mathcal{X}} \cong \omega_{\mathcal{X}}$. In §3.2, we have seen that the tangent space $T_{\xi}(\mathcal{M}_H(\mathcal{X}))$ of the moduli space $\mathcal{M}_H(\mathcal{X})$ at the point $\xi = (\mathcal{F}, \Phi)$ is isomorphic to $\mathbb{H}^1(\mathcal{C}_{\xi}^\bullet)$, where

$$C_{\xi}^\bullet : C_{\xi}^0 = \mathcal{E}nd(\mathcal{F}) \longrightarrow C_{\xi}^1 = \mathcal{E}nd(\mathcal{F}) \otimes \omega_{\mathcal{X}}$$

is a two term complex. Thus the cotangent space $T_{\xi}^*(\mathcal{M}_H(\mathcal{X}))$ is isomorphic to $\mathbb{H}^1(\mathcal{C}_{\xi}^\bullet)$, where $C_{\xi}^{\bullet,*}$ is the dual of the complex $C_{\xi}^\bullet$. Tensoring the complex $C_{\xi}^{\bullet,*}$ by $\omega_{\mathcal{X}}$, the complex $C_{\xi}^\bullet$ is dual to $C_{\xi}^{\bullet,*} \otimes \omega_{\mathcal{X}}$ in the derived category by the Grothendieck duality. Note that there is a natural morphism $C_{\xi}^{\bullet,*} \to C_{\xi}^{\bullet,*} \otimes \omega_{\mathcal{X}}$. This induces the morphism

$$C_{\xi}^{\bullet,*} \to C_{\xi}^{\bullet,*} \otimes \omega_{\mathcal{X}} \cong C_{\xi}^\bullet$$

and so

$$T_{\xi}^*(\mathcal{M}_H(\mathcal{X})) \cong \mathbb{H}^1(C_{\xi}^{\bullet,*}) \longrightarrow \mathbb{H}^1(C_{\xi}^\bullet) \cong T_{\xi}(\mathcal{M}_H(\mathcal{X})).$$

Similarly, we also have the morphism $T_{\xi}^*(\mathcal{M}_H(\mathcal{X}, G)) \to T_{\xi}(\mathcal{M}_H(\mathcal{X}, G))$ induced by the morphism $C_{\xi,G}^{\bullet,*} \to C_{\xi,G}^\bullet$. The significance of this map will be demonstrated in the next section, where we will show that this induces a Poisson structure on the moduli spaces $\mathcal{M}_H(\mathcal{X}, \alpha)$ and $\mathcal{M}_H(\mathcal{X}, G, \alpha)$.
4. Lie Algebroids and Poisson Structures

The principal method by which we shall obtain Poisson structures on our moduli spaces of interest is via the duals of Lie algebroids. Poisson structures are usually defined on a projective scheme. Nonetheless, Poisson structures over stacks were first conceived in the dissertation of J. Waldron [49]. In §4.1, we shall review the definitions and some properties of Lie algebroids and Poisson structures over projective schemes and stacks; we refer to [49, §7.2] for a complete overview. In §4.2 and §4.3, we prove the main theorems of this section that the moduli spaces $M_H(X, \alpha)$ and $M_H(X, G, \alpha)$ have a Poisson structure (Theorems 4.4 and 4.5). In §4.4, we consider the moduli problem $\tilde{M}_H(X, \alpha)$, which is a stack. As an application of Theorem 4.4, we next show that the moduli problem also has a Poisson structure.

4.1. Lie algebroids and Poisson structures. We start by recalling the basic notions for Lie algebroids and Poisson structures on projective schemes over $\mathbb{C}$ and then pass to a reasonable generalisation of these notions over stacks.

**Projective Schemes.** A Lie algebroid over a projective scheme $X$ over $\mathbb{C}$ is a vector bundle $E \to X$ together with a Lie bracket $[\cdot, \cdot]$ on the space of global sections of the bundle $\Gamma(E)$ and a morphism $a : E \to TX$, called the anchor map of $E$, which induces a Lie algebra morphism $\Gamma(E) \to \Gamma (TX)$ and satisfies the Leibniz rule

$$[\xi, f\nu] = a(\xi)(f)\nu + f[\xi, \nu],$$

for all $\xi, \nu \in \Gamma(E)$ and $f \in C^\infty(X)$. A morphism of Lie algebroids over $X$ is a morphism of vector bundles inducing a Lie algebra morphism between spaces of sections and commuting with the anchor maps. The category of Lie algebroids over $X$ will be denoted by $\mathcal{LA}$. A Poisson bracket on the projective scheme $X$ is a Lie bracket $\{\cdot, \cdot\} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ satisfying the Leibniz rule $\{f, gh\} = \{f, g\} h + g \{f, h\}$, for all smooth functions $f, g, h$ on $X$. Poisson brackets bijectively correspond to bi-vector fields $\Pi \in \Gamma (\wedge^2 TX)$ for which the Schouten-Nijenhuis bracket on $\Gamma (\wedge TX)$ vanishes

$$[\Pi, \Pi] = 0.$$

The Poisson bracket $\{\cdot, \cdot\}_\Pi$ that corresponds to such a bi-vector field $\Pi$ is given by $\{f, g\}_\Pi = \Pi (df, dg)$. Associated to a Poisson structure $\Pi$ on $X$, there is a Lie algebroid structure $T^*_\Pi X$ on $T^*X$ (see [49, §7.2.1]).

The class of examples of Poisson manifolds that we are interested in is the one that arises from Lie algebroids through the following theorem, which in the case of differentiable manifolds is due to T. J. Courant [24]:

**Theorem 4.1.** If $E$ is a Lie algebroid on $X$, then the total space of the dual vector bundle $E^*$ has a natural Poisson structure.

The Poisson structure on the total space of the dual vector bundle $E^*$ is determined by the following relations

$$\left\{\xi, \hat{\nu}\right\} \equiv [\xi, \hat{\nu}],$$

$$\{\pi^* f, \pi^* g\} \equiv 0,$$

$$\left\{\xi, \pi^* f\right\} \equiv \pi^* (\xi(f)),$$

where $\xi, \nu$ are global sections of $E$, $\pi$ is the vector bundle projection, $f, g$ are smooth functions on $X$, while for any section $\xi \in \Gamma (E)$, $\hat{\xi}$ denotes the corresponding fiber-wise linear function on $E^*$. 
We now consider an important example of a Lie algebroid, the Atiyah algebroid. Let \( G \) be a group acting freely and properly on a projective scheme \( X \) over \( \mathbb{C} \). We have a natural projection \( \pi : X \to X/G \). We have the following exact sequence

\[
0 \to T_{\text{orbits}}X \to TX \to \pi^*T(X/G) \to 0.
\]

The group action \( G \) on \( X \) induces a natural free action on \( TX \). We have a natural surjective morphism \( TX/G \to T(X/G) \), which is considered as an anchor map. Therefore, we have the following exact sequence

\[
0 \to \text{Ad}(X) \to TX/G \xrightarrow{\alpha} T(X/G) \to 0,
\]

where \( \text{Ad}(X) := X \times_G \mathfrak{g} \). This exact sequence is called the Atiyah sequence. It is easy to check that the Atiyah sequence gives a Lie algebroid structure on \( TX/G \). By Theorem 4.1, the total space of \( (TX/G)^* \) has a Poisson structure; note here that we are viewing \( TX/G \) as a vector bundle over \( X/G \).

**Stacks.** There is a natural way to define a Lie algebroid over a stack \( \mathcal{M} \). A Lie algebroid \( F \) over \( \mathcal{M} \) is defined on local charts of \( X \). On each local chart \((U,u)\), there is a Lie algebroid \((F_u,[\cdot],a_u)\) such that the data \((F_u)_{u \in \mathcal{I}}\) defines a locally free sheaf \( F \) on \( X \) and the following diagram commutes

\[
\begin{array}{ccc}
F_u & \xrightarrow{a_u} & TU \\
\downarrow{\alpha^X_{uv}} & & \downarrow{\alpha^X_{uv}} \\
F_v & \xrightarrow{f^*_uv} & f^*_uvTV
\end{array}
\]

for each morphism \( f_{uv} : (U,u) \to (V,v) \).

Now we assume that \( M \) is an algebraic stack, and it has a smooth covering \( M \to \mathcal{M} \), where \( M \) is a scheme. We consider the simplicial resolution of \( M \)

\[\begin{array}{c}
M \\
\downarrow{\sigma} \\
M \times_M M \xrightarrow{s} M \to \mathcal{M}.
\end{array}\]

Let \((F,\sigma)\) be a locally free sheaf on \( M \times_M M \). In this case, a Lie algebroid on \( M \) is equivalent to a Lie algebroid \((F,\sigma,[\cdot],a)\) on \( M \), where \([\cdot]\) is a Lie bracket on \( \mathcal{O}(F) \) and \( a : F \to TM \) is a morphism, such that \( \sigma \) induces an isomorphism

\[s^*[\cdot] \cong t^*[\cdot]\]

on the Lie bracket. In fact, this Lie algebroid is defined on the simplicial resolution \( M \times_M M \).

Denote by \( \mathcal{LA}(M \times_M M \to M) \) the category of Lie algebroids over the resolution. The category \( \mathcal{LA}(\mathcal{M}) \) of Lie algebroids over \( \mathcal{M} \) is equivalent to \( \mathcal{LA}(M \times_M M \to M) \), of which the objects are pairs \((F,\sigma)\) (see [49, Chap 7]).

Denote by \( \mathcal{M}_{\text{lis-ét}} \) the smooth site of the stack \( \mathcal{M} \). One can define a Poisson structure on \( \mathcal{M} \) as follows. There is a pre-sheaf

\[\textbf{Poiss} : \mathcal{M}_{\text{lis-ét}} \to \textbf{Set} \]

\[(U,u) \mapsto \text{Poiss}_U,
\]

where \( \text{Poiss}_U \) is the set of all Poisson brackets on \( C^\infty(U) \). Let \( f_{uv} : (U,u) \to (V,v) \) be a morphism between two charts. This is actually a sheaf. In fact, this also implies that \( \text{Poiss} \) has a stack structure. A Poisson structure \( \Pi_v \) on \( (V,v) \) induces a Poisson structure \( f^*_uv\Pi_v \) on \( (U,u) \). Moreover, the natural map

\[T_{f^*_uv\Pi_v}U \to f^*_uv(T^*_vV)\]
is an isomorphism of Lie algebroids. The above indicates that there is a morphism of sheaves over \( \mathcal{M}_{\text{lis-ét}} \):

\[
T^*: \text{Poiss} \to \mathcal{L}A
\]

\[
\Pi \mapsto T^*_\Pi \mathcal{M}
\]

We may now define:

**Definition 4.2.** A Poisson structure \( \Pi \) on a stack \( \mathcal{M} \) is a morphism

\[
\Pi: \mathcal{M}_{\text{lis-ét}} \to \text{Poiss}
\]

of stacks.

Associated to such a Poisson structure \( \Pi \) is a Lie algebroid \( T^*_\Pi \mathcal{M} \) defined by the composition

\[
\mathcal{M}_{\text{lis-ét}} \xrightarrow{\Pi} \text{Poiss} \xrightarrow{T^*} \mathcal{L}A.
\]

To describe the basic example of Poisson structures that we shall be considering, let \( \mathcal{F} \) be a Lie algebroid over an algebraic stack \( \mathcal{M} \). Suppose that \( u: U \to \mathcal{M} \) is a smooth atlas, which is a surjective morphism, with \( F_u \) the induced algebroid over \( (U,u) \). Denote \( U_1 := U \times \mathcal{M} U \), and let \( s,t: U_1 \to U \) be the source and target maps. In this case, a Lie algebroid \( \mathcal{F} \) on \( \mathcal{M} \) is exactly a Lie algebroid \( F_u \) on \( U \) such that \( s^*F_u \cong t^*F_u \). The same argument holds for Poisson structures. Thus, we have the desired generalization of Theorem 4.1:

**Proposition 4.3.** [49, §7.2.5] Let \( \mathcal{F} \) be a Lie algebroid over \( \mathcal{M}_{\text{lis-ét}} \). The total space of \( \mathcal{F}^* \) has a natural Poisson structure.

We will see in §4.4 that the moduli problems we are studying in this article fall in the case described in the above proposition.

4.2. **Poisson Structure on** \( \mathcal{M}_H(\mathcal{X},\alpha) \). Let \( \mathcal{X} = [U/\Gamma] \) be a stacky curve over \( \mathbb{C} \) and let \( \mathcal{X} \) be the coarse moduli space of \( \mathcal{X} \). In this section, we will prove the main theorem of this paper.

**Theorem 4.4.** The moduli space \( \mathcal{M}_H(\mathcal{X},\alpha) \) of stable Higgs bundles over \( \mathcal{X} \) with fixed parabolic structure \( \alpha \) has a Poisson structure.

*Proof.* The idea in the proof is to show that \( \mathcal{M}_H(\mathcal{X},\alpha) \) is a Lie algebroid over \( TM(\mathcal{X}) \), the tangent space of the moduli space of stable bundles on \( \mathcal{X} \), by constructing a map \( \mathcal{M}_H(\mathcal{X},\alpha) \to TM(\mathcal{X}) \). Thus, \( \mathcal{M}_H(\mathcal{X},\alpha) \) shall have a Poisson structure by Theorem 4.1. Instead of working on \( \mathcal{M}_H(\mathcal{X},\alpha) \) directly, we first restrict to \( \mathcal{M}_H^0(\mathcal{X},\alpha) \), the moduli space of stable Higgs bundles \( (\mathcal{F},\Phi) \), of which the underlying bundles \( \mathcal{F} \) are also stable as bundles. This moduli space is a dense open set of \( \mathcal{M}_H(\mathcal{X},\alpha) \), hence a Poisson structure on \( \mathcal{M}_H^0(\mathcal{X},\alpha) \) can be extended to \( \mathcal{M}_H(\mathcal{X},\alpha) \) naturally. We thus only have to construct a Poisson structure on \( \mathcal{M}_H^0(\mathcal{X},\alpha) \).

By the discussion in §2.4, we know that \( \mathcal{X} \) can be considered as a root stack locally. Therefore, we assume that \( \mathcal{X} \) is a root stack \( X_\Delta,F \). Denote by \( \pi: \mathcal{X} \to \mathcal{X} \) the natural morphism; note that the dimension of \( \mathcal{X} \) is one. The divisor \( D = p_1 + \cdots + p_n \) is a sum of distinct points and let \( q_i \) be the corresponding point of \( p_i \) in \( \mathcal{X} \). Note that \( \pi^{-1}(p_i) = r(p_i)q_i \). Let us denote by \( D = q_1 + \cdots + q_n \) the divisor in \( \mathcal{X} \).

Let \( \mathcal{F} \) be a locally free sheaf on \( \mathcal{X} \) with parabolic structure \( \alpha \). Given \( q \in D \), denote by \( \alpha(q) \) the parabolic structure of \( \pi_*\mathcal{F} \) around \( p = \pi(q) \in D \). Let \( P_q \) be the parabolic group of \( GL_n(\mathbb{C}) \) corresponding to \( \alpha(q) \) as we discussed in §2.5. Let \( P_q = L_qN_q \) be the Levi decomposition of \( P \), where \( L_q \) is the Levi factor and \( N_q \) is a unipotent group. Denote by \( l_q, n_q \) the Lie algebras of \( L_q \) and \( N_q \) respectively. Define \( \mathcal{F}' := \pi^*\pi_*\mathcal{F} \) to be the coherent sheaf on \( \mathcal{X} \). We have the following exact sequence

\[
0 \to \mathcal{E}nd(\mathcal{F}) \to \mathcal{E}nd(\mathcal{F}') \to \prod_{q \in D} n_q \otimes O_q \to 0.
\]
This induces the long exact sequence
\[ 0 \to \text{End}(\mathcal{F}) \to \text{End}(\mathcal{F}') \to H^0(\mathcal{X}, \prod_{q \in \mathbb{D}} n_q \otimes \mathcal{O}_q) \to \]
\[ \to \text{Ext}^1(\mathcal{F}, \mathcal{F}) \to \text{Ext}^1(\mathcal{F}', \mathcal{F}') \to H^1(\mathcal{X}, \prod_{q \in \mathbb{D}} n_q \otimes \mathcal{O}_q) \to 0. \]

Note that the last term \( H^1(\mathcal{X}, \prod_{q \in \mathbb{D}} n_q \otimes \mathcal{O}_q) \) is trivial, and thus we have a short exact sequence
\[ (4.2) \quad 0 \to \text{Ad} \to \text{Ext}^1(\mathcal{F}, \mathcal{F}) \to \text{Ext}^1(\mathcal{F}', \mathcal{F}') \to 0, \]
where \( \text{Ad} \) is the kernel of the surjective map \( \text{Ext}^1(\mathcal{F}, \mathcal{F}) \to \text{Ext}^1(\mathcal{F}', \mathcal{F}') \). By our assumption, \( \mathcal{F} \) is a stable bundle, and note that we are working over the field \( \mathbb{C} \). Therefore, \( \mathcal{F} \) is simple and \( \text{End}(\mathcal{F}) \cong \mathbb{C} \). The functor
\[ \pi_* : \text{Coh}(\mathcal{X}) \to \text{Coh}(X) \]
is an exact functor. The exactness of the functor \( \pi_* \) implies that \( \pi_* \mathcal{F} \) is stable, and so is \( \mathcal{F}' \). Therefore, \( \text{End}(\mathcal{F}') \cong \mathbb{C} \). Now we go back to the term \( \text{Ad} \). If \( \mathcal{F} \) is stable, then
\[ \text{Ad} \cong H^0(\mathcal{X}, \prod_{q \in \mathbb{D}} n_q \otimes \mathcal{O}_q), \]
which is supported over \( q \in \mathbb{D} \). Over each point \( q \in \mathbb{D} \), \( \text{Ad}_q \) is isomorphic to the Lie algebra \( n_q \). Therefore, \( \text{Ad}_q \) is the adjoint representation of the unipotent group \( N_q \).

In the short exact sequence \( (4.2) \), the third term \( \text{Ext}^1(\mathcal{F}', \mathcal{F}') \) is isomorphic to the tangent space of the moduli space of stable bundles over \( X \) at the point \( \mathcal{F}' \), thus
\[ \text{Ext}^1(\mathcal{F}', \mathcal{F}') \cong T_{\mathcal{F}'}(\mathcal{M}(X)). \]

With respect to this isomorphism, we have a natural map
\[ \text{Ext}^1(\mathcal{F}, \mathcal{F}) \to T_{\mathcal{F}'}(\mathcal{M}(X)). \]

Now we consider the tangent bundles on \( \mathcal{M}(\mathcal{X}, \alpha) \) and \( \mathcal{M}(X) \):

\[ \begin{array}{ccc}
\mathcal{M}(\mathcal{X}, \alpha) \times \mathcal{X} & \xrightarrow{\mu_1} & \mathcal{M}(\mathcal{X}, \alpha) \\
\mathcal{M}(X) \times X & \xleftarrow{\mu_2} & \mathcal{X}
\end{array} \]

Let \( \mathcal{F}, \mathcal{F}' \) be the universal bundles on \( \mathcal{M}(\mathcal{X}, \alpha) \times \mathcal{X} \) and \( \mathcal{M}(X) \times X \) respectively. Therefore, we have
\[ 0 \to \mathcal{E}_{\text{nd}}(\mathcal{F}) \to \mathcal{E}_{\text{nd}}(\mathcal{F}') \to \prod_{q \in \mathbb{D}} n_q \otimes \mathcal{O}_{\nu_1^{-1}(q)} \to 0 \]
and
\[ 0 \to \mathcal{A}d \to R^1(\mu_1)_* \mathcal{E}_{\text{nd}}(\mathcal{F}) \to R^1(\mu_1)_* \mathcal{E}_{\text{nd}}(\mathcal{F}') \to 0. \]

Clearly, the term
\[ R^1(\mu_1)_* \mathcal{E}_{\text{nd}}(\mathcal{F}) \cong TM(\mathcal{X}, \alpha) \]
is the tangent bundle of \( \mathcal{M}(\mathcal{X}, \alpha) \), and
\[ R^1(\mu_1)_* \mathcal{E}_{\text{nd}}(\mathcal{F}') \cong TM(\mathcal{X}) \]
is the tangent bundle of \( \mathcal{M}(X) \). For the rest of the proof, we prove the following two statements
\begin{enumerate}
  \item \( (R^1(\mu_1)_* \mathcal{E}_{\text{nd}}(\mathcal{F}))^* \) is isomorphic to \( \mathcal{M}_{\mu_1}(\mathcal{X}, \alpha) \) as bundles over \( \mathcal{M}(\mathcal{X}, \alpha) \), and
  \item \( R^1(\mu_1)_* \mathcal{E}_{\text{nd}}(\mathcal{F}) \) is a Lie algebroid over \( \mathcal{M}(X) \).
\end{enumerate}
which shall imply that the moduli space $M_H(X,\alpha)$ has a Poisson structure.

To prove the first statement, we work locally on a point $F \in M(X,\alpha)$. The space $\text{Ext}^1(F,F)$ is the tangent space of $M(X,\alpha)$ at the point $F$, and we have

$$H^1(X,\mathcal{E}nd(F)) \cong \text{Ext}^1(F,F) \cong T_F(M(X,\alpha)).$$

By Grothendieck duality (see §3.3), the following isomorphism holds

$$H^1(X,\mathcal{E}nd(F)) \cong H^0(X,\mathcal{E}nd(F) \otimes \omega_X)^*.$$

This isomorphism tells us that

$$T^*_F(M(X,\alpha)) \cong H^0(X,\mathcal{E}nd(F) \otimes \omega_X),$$

and let $\Phi \in H^0(X,\mathcal{E}nd(F) \otimes \omega_X)$ be a Higgs field. This finishes the proof of the first statement. Furthermore, we have

$$T_\Phi T^*_F(M(X,\alpha)) \cong \mathbb{H}^1(\mathcal{E}nd(F) \to \mathcal{E}nd(F) \otimes \omega_X) \cong T(\mathcal{M}_H(X,\alpha)).$$

For the second statement, note that in summary we have the following exact sequence

$$0 \to \mathfrak{A}d \to T\mathcal{M}(X,\alpha) \to T\mathcal{M}(X) \to 0,$$

which is an Atiyah sequence as studied in §4.1. Therefore, the second statement also holds and this finishes the proof of the theorem. \qed

4.3. Poisson Structure on $M_H(X,G,\alpha)$. Let $G \hookrightarrow \text{GL}(V)$ be a fixed faithful representation. In this subsection, we will prove that there exists a Poisson structure on $M_H(X,G,\alpha)$.

**Theorem 4.5.** The moduli space $M_H(X,G,\alpha)$ of stable $G$-Higgs bundles over a stacky curve $X$ with fixed parabolic structure $\alpha$ has a Poisson structure.

**Proof.** The proof is similar to that for the moduli space $M_H(X,\alpha)$. We work on the open dense set $M^0_H(X,G,\alpha)$, where the underlying principal $G$-bundles are also stable. Denote by $M(X,G)$ the moduli space of stable principal $G$-bundles on $X$ and consider

\[
\begin{array}{ccc}
M(X,G,\alpha) \times X & \overset{\mu_1}{\longrightarrow} & M(X,\alpha) \\
\downarrow & & \leftarrow \downarrow \\
M(X,G) \times X & \overset{\mu_2}{\longrightarrow} & X
\end{array}
\]

Let $\mathcal{E}$, $\mathcal{E}'$ be the universal bundles on $M(X,G,\alpha) \times X$ and $M(X,G) \times X$ respectively. Denote by $\mathcal{E}(V)$ and $\mathcal{E}'(V)$ the associated bundles with respect to $G \hookrightarrow \text{GL}(V)$. Around a point $q \in \mathbb{D}$, let $P_q$ be the corresponding parabolic subgroup in $\text{GL}(V)$ with respect to $\mathcal{E}(V)$. Denote by $P_q = L_q N_q$ the Levi factorization. With the same discussion as in the proof of Theorem 4.4, we have

$$0 \to \mathcal{E}nd(\mathcal{E}(V)) \to \mathcal{E}nd(\mathcal{E}'(V)) \to \prod_{q \in \mathbb{D}} n_q \otimes \mathcal{O}_{\mathcal{E}_q^{-1}(q)} \to 0.$$

Let $P'_q$, $L'_q$ and $N'_q$ be the pre-images of $P_q$, $L_q$ and $N_q$ in $G$ respectively via the fixed faithful representation. Therefore, we have

$$0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd(\mathcal{E}') \to \prod_{q \in \mathbb{D}} n'_q \otimes \mathcal{O}_{\mathcal{E}_q^{-1}(q)} \to 0,$$

where $n'_q$ is the Lie algebra of $N'_q$. This short exact sequence induces the following one

$$0 \to \mathfrak{A}d \to R^1(\mu_1)_* \mathcal{E}nd(\mathcal{E}) \to R^1(\mu_1)_* \mathcal{E}nd(\mathcal{E}') \to 0.$$

Clearly, we have

$$R^1(\mu_1)_* \mathcal{E}nd(\mathcal{E}) \cong T\mathcal{M}(X,G,\alpha)$$
and
\[ R^1(\mu_1)_* \mathcal{End}(\mathcal{E}) \cong T\mathcal{M}(X,G) \]
is the tangent bundle of \( \mathcal{M}(X,G) \). Therefore, \( R^1(\mu_1)_* \mathcal{End}(\mathcal{E}) \) is a Lie algebroid over \( \mathcal{M}(X,G) \). At the same time, the tangent space of \( (R^1(\mu_1)_* \mathcal{End}(\mathcal{E}))^* \) is isomorphic to the tangent space of \( \mathcal{M}^0_H(\mathcal{X},G,\alpha) \), in other words,
\[ T(R^1(\mu_1)_* \mathcal{End}(\mathcal{E}))^* \cong T\mathcal{M}^0_H(\mathcal{X},G,\alpha). \]

Therefore, the moduli space of \( G \)-Higgs bundles \( \mathcal{M}_H(\mathcal{X},G,\alpha) \) has a Poisson structure. \( \square \)

Remark 4.6. The above Theorem demonstrates the construction of a Poisson structure on the moduli space of stable \( G \)-Higgs bundles. Since the locus of stable \( G \)-Higgs bundles is dense in the moduli space of semistable \( G \)-Higgs bundles, the Poisson structure extends on the semistable moduli space. However, the notion of stability we considered does depend on the choice of a faithful representation \( G \hookrightarrow \text{GL}(V) \). In the particular case when the group \( G \) is a semisimple linear algebraic group over \( \mathbb{C} \), the stability condition does not depend on the faithful representation (see [5, 39]). An alternative notion of (semi)stability for parabolic principal \( G \)-bundles and parabolic \( G \)-Higgs bundles is considered in the more recent work of O. Biquard, O. García-Prada and I. Mundet i Riera [7]. From [31] and [1, Proposition 2.10] a principal \( G \)-bundle \( E_G \) is semistable if and only if the adjoint vector bundle \( \text{ad}(E_G) \) is semistable (see [39]). We believe that adding the Higgs field in the discussion, the above correspondence of the semistability conditions still holds for the \( G \)-Higgs bundles, thus providing strong motivation that the arguments in the proof of Theorem 4.5 can be used for constructing a Poisson structure on the moduli space with respect to the stability condition from [7]. This is formulated as Proposition 7.4 in [7]; we hope to return to this question in a future article.

4.4. Stacks. In the last subsection, we proved that there exists a Poisson structure on the moduli space \( \mathcal{M}_H(\mathcal{X},\alpha) \). In this subsection, we consider the existence of a Poisson structure on the moduli problem \( \mathcal{M}_H(\mathcal{X},\alpha) \), as a stack (see §3). We show that the stack \( \mathcal{M}_H(\mathcal{X},\alpha) \) has a Poisson structure, which is induced by the Poisson structure on \( \mathcal{M}_H(\mathcal{X},\alpha) \). Below we sketch the extension of the construction of a Poisson structure over a general stack; we are hoping to provide a detailed proof of this extension in a successive article.

Let \( \mathcal{X} \) be a stacky curve with coarse moduli space \( X \). It is known that \( \mathcal{M}_H(\mathcal{X},\alpha) \) is a stack [23]. Since this moduli problem is defined for the stable Higgs bundles, \( \mathcal{M}_H(\mathcal{X},\alpha) \) is a fine moduli space of \( \mathcal{M}_H(\mathcal{X},\alpha) \) [48, Theorem 1.3]. Thus,
\[ \mathcal{M}_H(\mathcal{X},\alpha)(-) \cong \text{Hom}(-,\mathcal{M}_H(\mathcal{X},\alpha)). \]

This isomorphism provides the following corollary:

Corollary 4.7. The stack \( \mathcal{M}_H(\mathcal{X},\alpha) \) has a Poisson structure.

In general, a moduli problem may not have a fine moduli space, for example \( \mathcal{M}_H^\text{ss}(\mathcal{X},\alpha) \) the moduli space of semistable Higgs bundles. To deal with the general case, we have to find another approach to construct a Poisson structure on the stack.

In the rest of this subsection, we give a brief idea about the construction of a Poisson structure on a moduli problem \( \tilde{\mathcal{M}} \). The idea is that we want to find an atlas \( \{ M_i \}_{i \in I} \) of the given moduli problem \( \tilde{\mathcal{M}} \) in the Lisse-étale site (see [40, Example 2.1.15]), where \( M_i \) are schemes. We can try to construct a Poisson structure on each \( M_i \) and check whether they can be glued together. Furthermore, if the moduli problem \( \tilde{\mathcal{M}} \) is an algebraic stack, we can assume that there exists a smooth surjective map \( M \to \tilde{\mathcal{M}} \), where \( M \) is a scheme, and construct a Poisson structure on \( M \). We also have to check that the pull-backs \( s, t : M \times_{\tilde{\mathcal{M}}} M \to M \) of the Poisson structure on \( M \) are isomorphic.

Now we take the moduli problem \( \mathcal{M}_H^\text{ss}(\mathcal{X},\alpha) \) as an example. Let \( \xi = (\mathcal{F}, \Phi) \) be a point in \( \mathcal{M}_H^\text{ss}(\mathcal{X},\alpha) \). There is a quasi-projective substack \( \tilde{M}_\xi \subseteq \text{Quot}(\mathcal{G}, P) \), where \( \mathcal{G} \) is a coherent sheaf, \( P \) is a (Hilbert)
polynomial and $\tilde{\text{Quot}}$ is the Quot-functor (see [41, §1]), such that $\tilde{M}_\xi \to \tilde{\mathcal{M}}_H^\| (\mathcal{X}, \alpha)$ is a smooth morphism (see the proof of Proposition 6.3 in [48]). Note that the Quot-functor $\text{Quot}(\mathcal{G}, P)$ is represented by a quasi-projective scheme $\text{Quot}(\mathcal{G}, P)$ [41, Theorem 4.4]. Denote by $\tilde{M}_\xi$ the subscheme of $\text{Quot}(\mathcal{G}, P)$ representing $\tilde{M}_\xi$. Then, we have a smooth morphism from a scheme $\tilde{M}_\xi$ to $\tilde{\mathcal{M}}_H^\| (\mathcal{X}, \alpha)$. Running over all points in $\tilde{\mathcal{M}}_H^\| (\mathcal{X}, \alpha)$, we get an atlas $\{M_\xi\}_{\xi \in \tilde{\mathcal{M}}_H^\| (\mathcal{X}, \alpha)}$ of $\tilde{\mathcal{M}}_H^\| (\mathcal{X}, \alpha)$. In §4.2, we construct a Poisson structure at each point $\xi = (\mathcal{F}, \Phi) \in \mathcal{M}_H(\mathcal{X}, \alpha)$, and the induced construction on the universal bundle gives a Poisson structure on the moduli space globally. The point-wise argument also works for the stack $\tilde{\mathcal{M}}_H^\| (\mathcal{X}, \alpha)$. The problem is that we have to construct the Poisson structure globally. With respect to the atlas we find, we can work on $M_\xi$ and try to construct a Poisson structure on it. The scheme $M_\xi$ is a subscheme in $\text{Quot}(\mathcal{G}, P)$. The Quot-scheme has a universal sheaf, which gives us the possibility to construct a Poisson structure globally. We thus conjecture:

**Conjecture 4.8.** There is a Poisson structure on the stack $\tilde{\mathcal{M}}_H^\| (\mathcal{X}, \alpha)$ of semistable Higgs bundles over $\mathcal{X}$ with fixed parabolic structure $\alpha$.

5. **Poisson Structure on the Moduli Space of $\mathcal{L}$-twisted Stable Higgs Bundles over Stacky Curves**

In this section, we work on $\mathcal{M}_H(\mathcal{X}, \mathcal{L}, \alpha)$, the moduli space of $\mathcal{L}$-twisted stable Higgs bundles on a stacky curve $\mathcal{X}$ with fixed parabolic structure $\alpha$. We prove that a Poisson structure exists over $\mathcal{M}_H(\mathcal{X}, \mathcal{L}, \alpha)$ under certain conditions.

Let $\mathcal{X} = [U/\Gamma]$ be a stacky curve. Denote by $X$ the coarse moduli space of $\mathcal{X}$. Let $\alpha$ be a parabolic structure on $X$, and denote by $\bar{D} = p_1 + \cdots + p_k \in X$ the divisor with respect to $\alpha$. Let $D = q_1 + \cdots + q_k \in \mathcal{X}$ be the corresponding divisor on $\mathcal{X}$, where $q_i$ is the point corresponding to $p_i$.

**Theorem 5.1.** If there exists a short exact sequence

$$0 \to \text{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_{\mathcal{X}}) \to \text{End}(\mathcal{F}) \to n \to 0,$$

for any stable bundle $\mathcal{F} \in \mathcal{M}(\mathcal{X}, \alpha)$ such that

1. the morphism $\text{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_{\mathcal{X}}) \to \text{End}(\mathcal{F})$ is not surjective,
2. $n$ is a sheaf of Lie algebras supported on $D$,

then the moduli space $\mathcal{M}_H(\mathcal{X}, \mathcal{L}, \alpha)$ has a Poisson structure.

**Proof.** Analogously to the proof Theorem 4.4, we only have to work with $\mathcal{M}_H(\mathcal{X}, \mathcal{L}, \alpha)$, the moduli space of $\mathcal{L}$-twisted stable Higgs bundles, such that the underlying locally free sheaf is stable. If the short exact sequence (5.1) exists, then we have the long exact sequence

$$0 \to \text{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_{\mathcal{X}}) \to \text{End}(\mathcal{F}) \to H^0(\mathcal{X}, n) \to$$
$$\to \text{Ext}^1(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_{\mathcal{X}}) \to \text{Ext}^1(\mathcal{F}, \mathcal{F}) \to H^1(\mathcal{X}, n) \to 0.$$

By our assumption that the support of $n$ is contained in $\bar{D}$, the last term $H^1(\mathcal{X}, n)$ is trivial. This implies a short exact sequence

$$0 \to \text{Ad} \to \text{Ext}^1(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_{\mathcal{X}}) \to \text{Ext}^1(\mathcal{F}, \mathcal{F}) \to 0,$$

where $\text{Ad}$ is the kernel of the map $\text{Ext}^1(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_{\mathcal{X}}) \to \text{Ext}^1(\mathcal{F}, \mathcal{F})$. Let $\mathcal{F}$ be the universal bundle on $\mathcal{M}(\mathcal{X}, \alpha) \times \mathcal{X}$ and consider

$$\begin{array}{ccc}
\mathcal{M}(\mathcal{X}, \alpha) \times \mathcal{X} & \xleftarrow{\mu_2} & \mathcal{M}(\mathcal{X}, \alpha) \\
\downarrow{\nu_1} & & \downarrow{\nu_2} \\
\mathcal{X} & & \mathcal{X}
\end{array}$$

where $\mu_1$ is the projection $\mu_1(x, \alpha, \xi) = (\alpha, \xi)$.
The short exact sequence (5.2) induces the following sequence for universal bundles
\[(5.3) \quad 0 \to s\mathcal{A}d \to R^1(\mu_1)_*\mathcal{H}om(\mathcal{F} \otimes \nu^*_1\mathcal{L}, \mathcal{F} \otimes \nu^*_1\omega_X) \to R^1(\mu_1)_*\mathcal{E}nd(\mathcal{F}) \to 0.\]

With respect to our assumption about \(n\), the short exact sequence (5.3) gives a Lie algebroid structure on \(R^1(\mu_1)_*\mathcal{H}om(\mathcal{F} \otimes \nu^*_1\mathcal{L}, \mathcal{F} \otimes \nu^*_1\omega_X)\). Note that
\[R^1(\mu_1)_*\mathcal{E}nd(\mathcal{F}) \cong T\mathcal{M}(\mathcal{X}, \alpha).\]

Therefore, \(R^1(\mu_1)_*\mathcal{H}om(\mathcal{F} \otimes \nu^*_1\mathcal{L}, \mathcal{F} \otimes \nu^*_1\omega_X)\) is a Lie algebroid over \(T\mathcal{M}(\mathcal{X}, \alpha)\). This implies that \((R^1(\mu_1)_*\mathcal{H}om(\mathcal{F} \otimes \nu^*_1\mathcal{L}, \mathcal{F} \otimes \nu^*_1\omega_X))^*\) has a Poisson structure. We only have to prove that
\[R^1(\mu_1)_*\mathcal{E}nd(\mathcal{F} \otimes \nu^*_1\mathcal{L}, \mathcal{F} \otimes \nu^*_1\omega_X)^* \cong \mathcal{M}_H^0(\mathcal{X}, \mathcal{L}, \alpha),\]
which would imply the result.

By Grothendieck duality (see §3.3), we see that
\[\text{Ext}^1(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_X)^* \cong H^0(\mathcal{X}, \mathcal{E}nd(\mathcal{F}) \otimes \mathcal{L}).\]

Equivalently,
\[R^1(\mu_1)_*\mathcal{H}om(\mathcal{F} \otimes \nu^*_1\mathcal{L}, \mathcal{F} \otimes \nu^*_1\omega_X) \cong (\mu_1)_*\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes \nu^*_1\mathcal{L}).\]

Therefore, \(R^1(\mu_1)_*\mathcal{H}om(\mathcal{F} \otimes \nu^*_1\mathcal{L}, \mathcal{F} \otimes \nu^*_1\omega_X)^*\) is isomorphic to \(\mathcal{M}_H^0(\mathcal{X}, \mathcal{L}, \alpha)\) and this finishes the proof of the theorem. \(\square\)

**Remark 5.2.** Note that if the morphism \(\mathcal{H}om(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_X) \to \mathcal{E}nd(\mathcal{F})\) is surjective, then \(n\) is zero. Therefore, the term \(s\mathcal{A}d\) in (5.3) is zero. This means that (5.3) is not an Atiyah sequence, and the approach in Theorem 5.1 fails in this case. However, it will be in this case \(\mathcal{H}om(\mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \omega_X) \cong \mathcal{E}nd(\mathcal{F})\) and so \(\mathcal{L} \cong \omega_X\). This is precisely the case treated in Theorem 4.4.

### 6. Relation With the Work of Logares-Martens

In this section, we compare the results of Sections 4 and 5 to the main result of M. Logares and J. Martens [35]. In the special case of a root stack and a parabolic structure when all parabolic weights are rational, we show that the moduli space of stable orbifold \(G\)-Higgs bundles is Poisson.

Let \(X\) be a smooth projective curve. Let \(D = p_1 + \cdots + p_k\) be a reduced effective divisor on \(X\), and let \(\bar{r} = (r_1, \ldots, r_k)\) be a \(k\)-tuple of positive integers. We denote by \(X_{\bar{D}, \bar{r}}\) the corresponding root stack and there is a natural map \(\pi : X_{\bar{D}, \bar{r}} \to X\). Denote by \(D := \pi^{-1}(D)\). By the discussion in §2.5, we know that there is a correspondence between parabolic bundles on \((X, D)\) and bundles on \((\mathcal{X}, \mathcal{X}) := X_{\bar{D}, \bar{r}}\). Moreover, we have the following equivalence in the language of tensor categories from [20]:

**Proposition 6.1.** [20, Théorème 4, 5] There is an equivalence of tensor categories between bundles \(\mathcal{F}\) with parabolic structure \(\alpha\) on \(\mathcal{X}\) and parabolic bundles \(F\) with the same parabolic structure \(\alpha\) on \((X, \bar{D})\). In particular, this equivalence preserves the degree, that is, \(\text{pardeg}(F) = \text{deg}(\mathcal{F})\).

As discussed in §2.5, this equivalence implies the correspondence in stability as in [20, Remarque 10]. More precisely, we have \(\mathcal{M}(\mathcal{X}, \alpha) \cong \mathcal{M}^\text{par}(X, \alpha)\). It is natural to extend this correspondence to Higgs bundles [11, 12, 37], thus giving \(\mathcal{M}_H(\mathcal{X}, \alpha) \cong \mathcal{M}_H^\text{par}(X, \alpha)\). Now let \(\mathcal{L}\) be a line bundle on \(\mathcal{X}\) and let \(\pi : \mathcal{X} \to X\) be the map from the root stack \(\mathcal{X}\) to its coarse moduli space \(X\). Denote by \(L\) the corresponding parabolic line bundle of \(\mathcal{L}\) on \((X, \bar{D})\). There is a one-to-one correspondence between \(\mathcal{L}\)-twisted stable Higgs bundles on \(\mathcal{X}\) and \(L\)-twisted stable parabolic bundles on \((X, \bar{D})\) (see [34, §5]).

In conclusion, we have the following proposition.

**Proposition 6.2.** Let \(\mathcal{X} = X_{\bar{D}, \bar{r}}\) be a root stack and denote by \(X\) the coarse moduli space of \(\mathcal{X}\). The following statements hold:

1. There is an isomorphism \(\mathcal{M}_H(\mathcal{X}, \alpha) \cong \mathcal{M}_H^\text{par}(X, \alpha)\), where \(\mathcal{M}_H^\text{par}(X, \alpha)\) is the moduli space of strongly parabolic Higgs bundles over \(X\) with parabolic structure \(\alpha\).

2. There is an isomorphism \(\mathcal{M}_H(\mathcal{X}, \omega_X(\bar{D}), \alpha) \cong \mathcal{M}_H^\text{par}(X, \alpha)\), where \(\omega_X(\bar{D})\) is the canonical line bundle on \(\mathcal{X}\) over the divisor \(\bar{D}\).
Corollary 6.3. The moduli space of stable orbifold \( G \)-Higgs bundles is Poisson.

In [35], the following short exact sequence is used to construct the Poisson structure on \( \mathcal{M}_H^\text{par}(X, \alpha) \)
\[
0 \to \text{SParEnd}(F) \to \text{ParEnd}(F) \to \prod_{p \in \mathcal{D}} I_p \otimes \mathcal{O}_p \to 0,
\]
where \( F \) is a parabolic bundle on \((X, \bar{D})\), \( \text{SParEnd}(F) \) is the sheaf of strongly parabolic endomorphisms and \( \text{ParEnd}(F) \) is the sheaf of parabolic endomorphisms. Translating to the language of stacks, we have
\[
\text{Hom}(\mathcal{F}(\mathcal{D}), \mathcal{F}) \cong \text{SParEnd}(F), \quad \text{Hom}(\mathcal{F}, \mathcal{F}) \cong \text{ParEnd}(F),
\]
where \( \mathcal{F} \) is the corresponding bundle to \( F \). Therefore, the sequence (6.1) is equivalent to
\[
0 \to \text{Hom}(\mathcal{F}(\mathcal{D}), \mathcal{F}) \to \text{Hom}(\mathcal{F}, \mathcal{F}) \to \prod_{p \in \mathcal{D}} I_p \otimes \mathcal{O}_p \to 0.
\]
Note that when \( \mathcal{L} = \omega_X(\mathcal{D}) \), the short exact sequence (5.1) becomes
\[
0 \to \text{Hom}(\mathcal{F}(\mathcal{D}), \mathcal{F}) \to \text{Hom}(\mathcal{F}, \mathcal{F}) \to \mathfrak{n} \to 0.
\]
Clearly, the sequence (6.2) satisfies the conditions of Theorem 5.1. With respect to the above discussion, we can prove alternatively to [35] the following:

Corollary 6.4. The moduli space \( \mathcal{M}_H^\text{par}(X, \alpha) \) has a Poisson structure.

Proof. Given the data \((X, \bar{D}, \alpha)\), we can construct a root stack \( \mathcal{X} = X_{\bar{D}, \mathcal{F}} \) (see §2.4 and §2.5). Denote by \( \mathcal{D} \) the corresponding divisor on \( \mathcal{X} \). There is a one-to-one correspondence between parabolic Higgs bundles with parabolic structure \( \alpha \) and Higgs bundles with parabolic structure \( \alpha \) on \( \mathcal{X} \). Under this correspondence, the line bundle \( \omega_X(\mathcal{D}) \) on \( X \) corresponds to \( \omega_X(\mathcal{D}) \) on \( \mathcal{X} \). By Proposition 6.2, this induces an isomorphism between \( \mathcal{M}_H^\text{par}(X, \alpha) \) and \( \mathcal{M}_H(\mathcal{X}, \omega_X(\mathcal{D}), \alpha) \). Therefore, it is enough to prove that the moduli space \( \mathcal{M}_H(\mathcal{X}, \omega_X(\mathcal{D}), \alpha) \) has a natural Poisson structure. When \( \mathcal{L} = \omega_X(\mathcal{D}) \), the condition in Theorem 5.1 is automatically satisfied. This finishes the proof of the corollary. \( \square \)

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