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Irregular behaviour of class numbers and Euler-Kronecker constants of cyclotomic fields: the $\log\log\log$ devil at play

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We study two invariants for cyclotomic number fields $\mathbb{Q}(\zeta_q)$, where q is a prime, namely the first factor of the class number and the Euler-Kronecker constant. In particular, we consider the connection between a conjecture by Kummer on the asymptotic behaviour of the former and a conjecture by Ihara on the positivity of the latter.

1 The Euler-Kronecker constant

The *Euler-Mascheroni constant* γ is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577...$$

and in general we define the *Stieltjes constants* as

$$\gamma_r = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\log^r k}{k} - \frac{\log^{r+1} n}{r+1}\right)$$

for $r \geq 0$, which arise as the coefficients of the Laurent series expansion of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \gamma_r (s-1)^r \,.$$

In particular, we have

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)\,.$$

Recall that the *Dedekind-zeta function* of a number field K is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s}\,, \qquad \mathrm{Re}(s) > 1\,,$$

where $\mathfrak{a}$ runs over all integral ideals of K. The Laurent series of ζ_K is such that

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1)\,.$$

The *Euler-Kronecker constant* of K, introduced by Ihara, is then defined as $\mathcal{EK}_K := c_0/c_{-1}$, which is the constant term in the logarithmic derivative of $\zeta_K(s)$ at $s = 1$:

$$\lim_{s\to 1} \left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1} \right) = \mathcal{EK}_K \,.$$

For example, we have $\mathcal{EK}_{\mathbb{Q}} = \gamma$. The Euler-Kronecker constant satisfies

$$\mathcal{EK}_K = \lim_{x\to\infty} \left(\log x - \sum_{N\mathfrak{p} \leq x} \frac{\log N\mathfrak{p}}{N\mathfrak{p} - 1} \right),$$

where $\mathfrak{p}$ runs over the primes of K, so that for cyclotomic fields $\mathbb{Q}(\zeta_q)$, setting $\gamma_q := \mathcal{EK}_{\mathbb{Q}(\zeta_q)}$, the main contribution is given by the rational primes p which split completely in $\mathbb{Q}(\zeta_q)$:

$$\gamma_q = \lim_{x\to\infty} \left(\log x - (q-1) \cdot \sum_{\substack{p \leq x \\ p \equiv 1 \bmod q}} \frac{\log p}{p-1} \right) + \text{smaller order terms}\,.$$

Under the assumption of the Extended Riemann Hypothesis (ERH), Ihara, and by different methods, Ford, Luca and Moree showed the following approximation:

$$\gamma_q = \log(q^2) - q \cdot \sum_{\substack{p \le q^2 \\ p \equiv 1 \bmod q}} \frac{\log p}{p-1} + O(\log\log q)\,. \qquad (1)$$

Unconditionally this estimate holds for all $C > 0$ and for all but $O(\pi(u)/(\log u)^C)$ primes $q \le u$. Assuming the Elliot-Halberstam conjecture (Conj. 1) we may replace q^2 by $q^{1+\epsilon}$ in (1).

2 Ihara's conjectures

We first introduce two standard conjectures.

Conjecture 1 (Elliot-Halberstam (EH))**.** *For every $\epsilon > 0$ and $A > 0$ we have*

$$\sum_{q \le x^{1-\epsilon}} \left|\pi(x;q,a) - \frac{\mathrm{li}(x)}{\varphi(q)}\right| \ll_{A,\epsilon} \frac{x}{\log^A x}\,,$$

where $\pi(x;q,a)$ denotes the number of primes p less than x with $p \equiv a \bmod q$, and φ is Euler's totient function.

We say that a set $\{b_1, \ldots, b_k\}$ of positive integers is *admissible* if the congruence $n\prod_{i=1}^{k}(b_i n + 1) \equiv 0 \bmod p$ has $< p$ solutions for every prime p.

Conjecture 2 (Hardy-Littlewood (HL))**.** *If $\{b_1, \ldots, b_k\}$ is admissible, then the number of primes $n \le x$ for which the integers $b_i n + 1$ are all prime is*

$$\gg \frac{x}{\log^{k+1} x}\,.$$

Ihara's conjecture concerns the positivity of the constants γ_q, and it gives bounds for the ratio $\gamma_q/\log q$. In fact, it is known unconditionally that for a density 1 set of primes q there exists a constant $c > 0$ such that

$$-c \log\log q \leq \frac{\gamma_q}{\log q} \leq 2 + \epsilon \,.$$

Assuming ERH, this property is true for all sufficiently large primes q.

Conjecture 3 (Ihara, 2009)**.** *Let $q \geq 3$ be a prime. We have:*

(i) $\gamma_q > 0$ *('very likely');*

(ii) for fixed $\epsilon > 0$ and q sufficiently large

$$\frac{1}{2} - \epsilon \leq \frac{\gamma_q}{\log q} \leq \frac{3}{2} + \epsilon \,.$$

However γ_q can be negative [1]:

$$\gamma_{964477901} = -0.1823\ldots,$$

and furthermore, assuming HL, one can prove that this happens infinitely often:

Theorem 1. *On a quantitative version of the HL conjecture we have*

$$\liminf_{q\to\infty} \frac{\gamma_q}{\log q} = -\infty \,.$$

In favour of Ihara's conjecture we have:

Theorem 2. *Under the EH conjecture, for a density 1 sequence of primes q we have*

$$1 - \epsilon < \frac{\gamma_q}{\log q} < 1 + \epsilon$$

(that is, γ_q has normal order $\log q$*).*

Sketch of proof of Theorem 1. Assume ERH and the HL conjecture. We need to find $b_1, \ldots, b_s$ such that the integers $n, 1 + b_1 n, 1 + b_2 n, \ldots$ satisfy the conditions of the HL conjecture and

$$\sum_{i=1}^{s} \frac{1}{b_i} > 2\,.$$

We may take $\{b_i\}$ to be the sequence of *greedy prime offsets*, namely $\{2, 6, 8, 12, 18, 20, 26, \ldots\}$, and $s = 2088$. Then by the HL conjecture $q, 1+b_1 q, 1+b_2 q, \ldots, 1+b_s q$ are infinitely often all prime with $1+b_s q \leq q^2$, and so we have

$$q \sum_{\substack{p \leq q^2 \\ p \equiv 1 \bmod q}} \frac{\log p}{p-1} > q \log q \sum_{i=1}^{s} \frac{1}{b_i q} > \log q \sum_{i=1}^{s} \frac{1}{b_i} > (2 + \epsilon_0) \log q\,.$$

The proof is now concluded on invoking estimate (1). □

The *measure* of an admissible set S is defined as

$$m(S) = \sum_{s \in S} \frac{1}{s}\,.$$

Theorem 1 is a consequence of the fact that there exists an admissible set S with $m(S) > 2$. Ford, Luca and Moree gave a short proof of this fact based on a result by Erdös from 1961. However, the divergence result is due to Granville and it confirmed a conjecture of Erdös from 1988:

Theorem 3 (Granville [2]). *There is a sequence of admissible sets* $S_1, S_2, \ldots$ *such that* $\lim_{i\to\infty} m(S_i) = \infty$.

Proposition 1 (Granville [2]). *There is an admissible set S with elements $\leq x$, such that $m(S) \geq (1 + o(1)) \log\log x$. For any admissible set we have $m(S) \leq 2 \log\log x$.*

3 Analogy with Kummer's Conjecture

Kummer conjectured in 1851 that

$$h_1(q) = \frac{h(q)}{h_2(q)} \sim G(q) := 2q\left(\frac{q}{4\pi^2}\right)^{\frac{q-1}{4}},$$

where $h(q)$ and $h_2(q)$ are the class numbers of $\mathbb{Q}(\zeta_q)$ and of its maximal real subfield $\mathbb{Q}(\zeta_q)^+ := \mathbb{Q}(\zeta_q + \zeta_q^{-1})$, respectively. Define the *Kummer's ratio* as $r(q) := h_1(q)/G(q)$. Then the conjecture amounts to

$$r(q) \sim 1\,.$$

Masley and Montgomery (1976) showed that $|\log r(q)| < 7\log q$ for $q > 200$ and used this result to determine all cyclotomic fields of class number 1. Ram Murty and Petridis (2001) showed that there exists a constant $c > 1$ such that for a density 1 set of primes q we have $1/c \le r(q) \le c$.

Both γ_q and $h_1(q)$ are related to special values of Dirichlet L-series. Hasse (1952) showed that

$$r(q) = \prod_{\chi(-1)=-1} L(1,\chi),$$

where χ runs over all the odd characters modulo q. Furthermore, using the definition of the Euler-Kronecker constant, one can find the Taylor series expansion around $s = 1$:

$$\frac{\zeta_{\mathbb{Q}(\zeta_q)}(s)}{\zeta_{\mathbb{Q}(\zeta_q)^+}(s)} = r(q)\Big(1 + (\gamma_q - \gamma_q^+)(s-1) + O_q((s-1)^2)\Big),$$

where $\gamma_q^+ := \mathcal{EK}_{\mathbb{Q}(\zeta_q)^+}$, which involves both γ_q and $h_1(q)$.

Both quantities $\log r(q)$ and $(\gamma_q - \gamma_q^+)/\log q$ are related to the distribution of primes $p \equiv \pm 1 \bmod q$. In fact, they are analytically similar in the following way

$$\frac{\gamma_q - \gamma_q^+}{\log q} \approx \frac{(q-1)}{2}\left(\sum_{\substack{p \le q(\log q)^A \\ p \equiv 1 \bmod q}} \frac{1}{p} - \sum_{\substack{p \le q(\log q)^A \\ p \equiv -1 \bmod q}} \frac{1}{p}\right) \approx \log r(q)\,. \qquad (2)$$

If we assume HL and EH, then Kummer's conjecture is false. We have the following result:

Theorem 4 (Granville [2]). *Assume both the HL and the EH conjecture. Then $r(q)$ has $[0, \infty]$ as set of limit points.*

Similarly, in view of (2), we have that, assuming both HL and EH, the sequence $(\gamma_q - \gamma_q^+)/\log q$ can be shown to be dense in $(-\infty, \infty)$ (see [3]). In the same way, exploiting the analytic similarity of $\gamma_q/\log q$ with $1 - 2|\log r(q)|$, the sequence $\gamma_q/\log q$ is dense in $(-\infty, 1]$ (see [1]).

Exploiting these results, we obtain the following speculations, where the log log log 'devil' appears:

1. (Granville [2]) the Kummer's ratio $r(q)$ asymptotically satisfies

$$(-1 + o(1)) \log\log\log q \leq 2 \log r(q) \leq (1 + o(1)) \log\log\log q \,;$$

2. (Languasco, Moree, Saad Eddin, Sedunova [3])

$$(-1 + o(1)) \log\log\log q \leq 2 \frac{(\gamma_q - \gamma_q^+)}{\log q} \leq (1 + o(1)) \log\log\log q \,;$$

3. (Ford, Luca, Moree [1])

$$\frac{\gamma_q}{\log q} \geq (-1 + o(1)) \log\log\log q \,.$$

These bounds are best possible in the sense that there exist infinite sequences of primes q for which all the indicated bounds are attained.

References

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[2] A. Granville, *On the size of the first factor of the class number of a cyclotomic field*. Inv. Math. **100** (1990), 321-338.

[3] A. Languasco, P. Moree, S. Saad Eddin, A. Sedunova, working paper.

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