Brizolis conjectured that for every prime $p > 3$ there exists an x and a primitive root g mod p such that $x \equiv g^x (\mathrm{mod}\ p)$. Let $N(p)$ denote the number of $1 \leq x \leq p-1$ such that x is a primitive root mod p and x and $p-1$ are coprime. It is easy to see that if $N(p) \geq 1$, then Brizolis' conjecture is true for the prime p. As usual let φ denote Euler's totient, $\omega(n) := \sum_{p|n} 1$ and $d(n) := \sum_{d|n} 1$, where the first sum is over the distinct prime divisors p of n. In the present note the authors, using charactersums and the Weil bounds, prove that

$$N(p) = \frac{\varphi(p-1)^2}{p-1} + O(\sqrt{p}\, 4^{\omega(p-1)} \log p),$$

and use this to show that Brizolis' conjecture is true for every prime $p > 10^{50}$ (actually they prove the slightly weaker bound where $4^{\omega(p-1)}$ is replaced by the square of $d(p-1)$, but their proof immediately gives the above sharpening).

This approach to Brizolis' conjecture and the estimate for $N(p)$ are not new, vide Wen Peng Zhang [Pure Appl. Math. 11 (1995), suppl. 1–3, MR 98d:11099], a paper the authors were apparently unaware of. Zhang's proof is easier, but his estimate not completely explicit due to the use of the Polya-Vinogradov inequality.

Let $M(p)$ denote the number of $1 \leq x \leq p-1$ such that there exists a primitive root g satisfying $x \equiv g^x (\mathrm{mod}\ p)$. The reviewer likes to remark that the authors method can be easily extended to prove, on noting that for a given x Brizolis conjecture is satisfied if and only if x is of residual index $\gcd(x, p-1)$, that

$$M(p) = \frac{1}{p-1} \sum_{r|p-1} \varphi\left(\frac{p-1}{r}\right)^2 + O\left(\sqrt{p}\, d((p-1)^4) \log p\right).$$