Review of “Unit Equations in Diophantine Number Theory”

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1 Introduction

The book concerns unit equations and their applications (mostly in algebraic number theory and arithmetic geometry). Unit equations are linear equations

$$a_1x_1 + \ldots + a_nx_n = 1,$$

with unknowns $x_1, \ldots, x_n$ taken from a multiplicative group of finite rank in the complex numbers. An important special case is when the units are taken from the unit group of the ring of integers of a number field. Since many Diophantine problems can be reduced to solving a system of unit equations, these equations are far more basic than one might naively think.

Basic problems are whether there are finitely many solutions, and if yes, how many and whether there is an upper bound for their size (in this setting typically the notion of height is used to measure size).

The authors work since the 1970s in this area and are well-known experts who contributed many interesting and important results. The book clearly reflects their command and overview of the subject. Thus one might hope that this book will become a standard work. In my review I will give some arguments why I strongly expect that this will (and should!) indeed happen.

Understanding the book requires only basic knowledge in algebra (groups, commutative rings, fields, Galois theory and elementary algebraic number theory). In particular the concepts of height, places and valuations play an important role. However, results in this area tend to be quite technical and this makes the reading not as easy as one might expect. Indeed, the subject is technical by nature also since the existing methods (e.g, linear forms in logarithms, more on this later) lead to bounds that are not very “clean” and very likely far from best possible. It is perhaps helpful to first have a look at the 1988 survey article by Evertse, Györy, Stewart and Tijdeman, which was at the basis of this book, in order to get accustomed to the subject. It is easily found online by googling for “S-units and their applications”.

The first three chapters of the book deal with preliminaries. They contain few proofs, but lots of pointers to the extensive literature. They are written in a well-structured and systematic way as indeed is the whole book.

In the third, more specialized, chapter the authors collect some fundamental and deep results from Diophantine approximation and transcendence theory. A basic result here is Roth's celebrated theorem on the approximation of algebraic numbers by rationals (hence the term "Diophantine approximation"). It states that if α is a real algebraic number that is not a rational number and $\epsilon > 0$, then there are finitely many integer pairs (x, y) with $y > 0$ such that $|\alpha - x/y| \leq \max(|x|, |y|)^{-2-\epsilon}$. Roth was awarded a Fields Medal in 1958 on the strength of this result. Schmidt's Subspace Theorem generalizes Roth's theorem to simultaneous approximation. Roughly speaking we want to have solutions making a bunch of linear forms all small in an appropriate sense (not just one, as in Roth's theorem).

The transcendence result mentioned is also associated with a Fields Medal! Namely, that of Baker (1970), who used it to derive effective bounds for the solutions of some Diophantine equations, and to solve the class number problem of finding all imaginary quadratic fields with class number 1. His result gives a non-trivial lower bound for how small say $a \log 2 + b \log 3$ (a linear form with a and b as variables with logarithms as coefficients) can be with $-N \leq a \leq N$ and $-N \leq b \leq N$. Indeed, it works likewise for linear forms in logarithms with an arbitrary number of variables. Certainly $a \log 2 + b \log 3 \neq 0$ for any $(a, b) \neq (0, 0)$, as equality would lead to the impossible identity $2^a = 3^{-b}$. With the help of Baker's theorem lots of Diophantine problems can be attacked. For example, Tijdeman in 1974 used it to show that the Catalan equation $x^p - y^q = 1$ has only finitely many solutions with $x, p, y, q > 1$. However, the drawback of this method is that it typically leads to huge bounds on the size of the solutions. For example, it does not allow the Catalan conjecture that $x^p - y^q = 1$ with $x, p, y, q > 1$ has only $(x, p, y, q) = (3, 2, 2, 3)$ as solution to be established. (That was only achieved in 2004 by Mihailescu using very different methods.)

After the recapitulation of important material in the first three chapters, the stage is set for the heart of the book consisting of Chapters 4-10.

Chapter 4 deals with effective results for unit equations in two unknowns over number fields (an example being S-unit equations with S a finite set of places including all the archimedean ones). Here the solutions are shown to be bounded in height. By Northcott's theorem this implies that there are only finitely many solutions. The authors derive the best upper bounds to date for the heights of its solutions by means of the best known effective estimates, due to Matveev (2004) and Yu (2007), for linear forms in algorithms. Aside from the material from the earlier chapters some geometry of numbers is being used. Further, the celebrated abc-conjecture is briefly discussed in this chapter.

For some classes of concrete Diophantine equations using the LLL lattice basis reduction algorithm upper bounds for the sizes of the solutions can be substantially reduced and the Diophantine equation then solved using a computer. Indeed, in Chapter 5 the algorithmic resolution, i.e. finding actually all solutions, of concrete unit equations in two unknowns is considered and the role of the LLL lattice basis reduction algorithm explained. In this process it is important that the basic objects are given effectively, that is in such a way that they can serve as an input for an algorithm (Chapter 1 fleshes out some details).

In Chapter 6 unit equations in several unknowns are considered. There is currently a big difference with the two-unknowns case in that for three or more unknowns still no effective results are known. However, here ineffective finiteness results can be established that allow to give an explicit upper bound for the number of non-degenerate solutions. There are two main approaches here to derive such results, one of which is based on the p-adic subspace theorem of Schmidt and Schlickewei (and followed in this book), the other being based on Faltings' product theorem. The chapter discusses also the classic result of Beukers and Schlickewei (1996), who showed that if H is a subgroup of $K^* \times K^*$ of finite rank r, with K^* the non-zero elements of a field K of characteristic zero, then the equation $x_1 + x_2 = 1$ has at most $2^{8(r+1)}$ solutions (x_1, x_2) with $(x_1, x_2) \in H$. A detailed proof of this result is given, which is not based on the deep subspace theorem, but instead uses some theory of hypergeometric functions. The latter allow one to explicitly construct certain important auxiliary polynomials in Diophantine approximation which have to vanish to high order.

Chapter 7 considers analogues of the results of Chapter 6 over function fields, with a strong emphasis on the characteristic zero case. The preliminaries for this chapter are discussed in Chapter 2. In the function field case Mason's abc-theorem for polynomials (which for number fields is a very difficult open problem) plays the role of the earlier bounded-height results.

Chapter 8 deals, among others, with equations $a_1x_1 + a_2x_2 = c$, where the unknowns x_1 and x_2 are taken from the unit group of an arbitrary finitely generated $\mathbb{Z}$-subalgebra of the complex numbers. It is shown that the finitely many solutions of such an equation can be determined in principle if the underlying algebra is explicitly given in a well-defined sense. Here, Mason's theorem, the effective results from Chapter 4 and some effective specializations play an important role.

Chapter 9 deals with decomposable form equations in an arbitrary number of unknowns. These are equations of the type $F(x_1, \ldots, x_n) = a$, where F is a decomposable form, that is the product of homogeneous linear forms in n variables, a is a non-zero constant, and the solutions $x_1, \ldots, x_n$ are taken from the integers or more generally from finitely generated $\mathbb{Z}$-subalgebras of the complex numbers. Important classes are the Thue equations, which are decomposable form equations in two unknowns, norm form equations, where the underlying decomposable form is the norm of a linear form, and discriminant form equations, which have close ties with algebraic number theory.

The authors present a finiteness criterion for decomposable form equations and show that it is equivalent with the finiteness criterion of unit equations given in Chapter 6. In some special cases they provide effective results. If the finiteness criterion is not satisfied, still the solutions can be meaningfully grouped into a finite set of infinite families.

The final chapter gives a sketchy overview of various applications of unit equations; it relies heavily on the material from Chapters 4 and 6. We mention here prime factors of sums of integers and the zero-multiplicity of linear recurrences. The latter deals with the question how often a non-trivial k-th order linear recurrence can assume the value zero. Here the subspace theorem shows that this zero multiplicity is finite (and a very difficult theorem of W. Schmidt, mentioned without proof in the book gives an upper bound which is triply exponential in k).

A number of further applications are given in a second book of the authors entitled "Discriminant equations in Diophantine number theory", which also relies considerably on the material from Chapters 4 and 6.

I think my review amply demonstrates the importance of the mathematics discussed in this book. This and the structured way in which the authors approach their subject, makes for challenging (due to the technical nature of the material), but rewarding reading. In addition to providing a survey, the authors improve both formulations and proofs of various important existing results in the literature, making their book also a valuable asset for researchers in this area.

Given how specialized this field is, the help of the authors in answering some of my mathematical questions related to it was very useful in preparing this review. I thank them for their fast and clear answers.