In 1927, in a conversation with Helmut Hasse, Emil Artin made the following conjecture (now known as Artin's primitive root conjecture): for any integer α, not equal to ± 1 or a square, the natural density

$$\lim_{x\to\infty} \frac{\#\{p \text{ prime} : p \le x \text{ and } \langle\alpha\rangle = \mathbf{F}_p^*\}}{\#\{p \text{ prime} : p \le x\}} \tag{1}$$

of the set of primes p for which α is a primitive root modulo p inside the set of all prime numbers exists and is positive. Artin based this conjecture on the observation that α is a primitive root modulo p if and only if for all primes l, the prime p does not split completely in $\mathbf{Q}(\zeta_l, \sqrt[l]{\alpha})/\mathbf{Q}$. Here ζ_l denotes a primitive l-th root of unity. By Chebotarev's density theorem, the density of the set of primes that do split completely in $\mathbf{Q}(\zeta_l, \sqrt[l]{\alpha})/\mathbf{Q}$ is equal to $[\mathbf{Q}(\zeta_l, \sqrt[l]{\alpha}) : \mathbf{Q}]^{-1}$. Using the principle of inclusion and exclusion, one expects the limit (1) to be equal to

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{[\mathbf{Q}(\zeta_k, \sqrt[k]{\alpha}) : \mathbf{Q}]}. \tag{2}$$

It is not clear from this expression whether the density is positive. However, one can show that the infinite sum is equal to a positive rational multiple $c_\alpha \cdot A$ of Artin's constant

$$A = \sum_{n=1}^{\infty} \frac{\mu(n)}{n\varphi(n)} = \prod_{l \text{ prime}} \Big(1 - \frac{1}{l(l-1)}\Big) \approx 0.3739558136.$$

In 1967 Hooley [J. Reine Angew. Math. 225 (1967), 209–220; MR 34 #7445] proved that the density (1) is indeed equal to the sum (2), if the Generalized Riemann Hypothesis (GRH) holds true. Furthermore, he explicitly determined c_α in terms of the prime factorization of α.

Over arbitrary number fields, there are two ways in which Artin's conjecture can be generalized. Fix a number field K with ring of integers $\mathcal{O}$ and a non-zero element $\alpha \in \mathcal{O}$ which is not a root of unity. If we replace $\mathbf{Q}$ by K in the above discussion, we expect the following generalization to hold: the set of primes $\mathfrak{p}$ of K for which α generates the cyclic group $(\mathcal{O}/\mathfrak{p}\mathcal{O})^*$ has a density inside the set of all primes of K. Moreover, the situation is highly similar to the rational case, as the set of primes $\mathfrak{p}$ of K for which $(\mathcal{O}/\mathfrak{p}\mathcal{O})^*$ is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^*$ has density 1. In 1975 Cooke and Weinberger [Comm. Algebra 3 (1975), 481–524. MR 52 #8094] proved that if GRH holds, this generalization of Artin's conjecture is indeed true and the density is given by (2) with $\mathbf{Q}$ replaced by K.

The author considers a generalization of Artin's conjecture in a 'rational' direction: the set of rational primes p for which the order of α in $(\mathcal{O}/p\mathcal{O})^*$ is equal to the exponent of this group has a density inside the set of all rational primes. He proves that, assuming GRH, this conjecture holds for quadratic fields K. To state his results we fix a quadratic field K with ring of integers $\mathcal{O}$ and $\alpha \in K^*$ not a root of unity with conjugate $\bar{\alpha}$. As the exponent of $(\mathcal{O}/p\mathcal{O})^*$ depends on the splitting behaviour of p in $K/\mathbf{Q}$, he defines the sets

$$S^- = \{p : p \text{ is inert in } K/\mathbf{Q} \text{ and } \langle\alpha\rangle = (\mathcal{O}/p\mathcal{O})^*\}$$
$$S^+ = \{p : p \text{ splits in } K/\mathbf{Q} \text{ and } \alpha \text{ has order } p-1 \text{ in } (\mathcal{O}/p\mathcal{O})^*\}.$$

The author proves that if GRH holds, then there exist rational numbers c_α^+ and c_α^- such that these sets have densities

$$\delta(S^+) = c_\alpha^+ \cdot \prod_{l \text{ prime}} \Big(1 - \frac{1}{l^r(l-1)}\Big), \qquad \delta(S^-) = c_\alpha^- \cdot \prod_{\substack{l \text{ odd} \\ \text{prime}}} \Big(1 - \frac{2}{l(l-1)}\Big),$$

where r is the rank of the multiplicative subgroup $\langle \alpha, \bar{\alpha} \rangle \subset K^*$.

The above theorem for the set S^- and a restrictive set of imaginary quadratic integers α has been proved independently by Chen, Kitaoka and Yu [Nagoya Math. J. 158 (2000), 167–184, MR 2001k:11230].

Note that the generalization of Artin's conjecture that was proved by Cooke and Weinberger does not imply the existence of the density of S^+; it is possible that p is in S^+ while α does not generate $(\mathcal{O}/\mathfrak{p}\mathcal{O})^*$ for either of the primes $\mathfrak{p}$ above p.

The proof of the theorem is along the lines of Hooley's proof of Artin's conjecture. The primes in S^+ and S^- are characterised in terms of splitting conditions in the fields $L_l = K(\zeta_l, \sqrt[l]{\alpha}, \sqrt[l]{\bar{\alpha}})$, with l ranging over the prime numbers. These fields are the analogues of the fields $\mathbf{Q}(\zeta_l, \sqrt[l]{\alpha})$ occurring in the proof of Artin's original conjecture. To express the densities as infinite sums similar to (2), The author adapts Hooley's analytic arguments. This is straightforward for the set S^+; for the inert primes additional arguments are needed. In order to prove that these infinite sums are equal to the Euler products in the above theorem, one needs to know the obstruction for the fields $\{L_l\}_{l \text{ prime}}$ to be linearly disjoint over K. It turns out that there exists a finite set of primes I, depending on α, such that the fields $\{L_l\}_{l \notin I}$ are linearly disjoint over K. The rational numbers c_α^+ and c_α^- take care of the remaining primes, and can be seen as 'correction factors'.

The author does not give explicit formulas for c_α^+ and c_α^- for general α. He does treat several examples which give an idea of the problems involved in computing these constants. Moreover he proves the equalities $c_\alpha^+ = \frac{1}{2}$ and $c_\alpha^- = \frac{1}{4}$ for 'generic' α. In addition he classifies all α for which either S^+ or S^- has zero density.

Two non-obvious examples of this occurrence are $\alpha = (\frac{15+3\sqrt{5}}{2})^{15}$, for which $S^+ \cup S^-$ has zero density, and $\alpha = -150 + 57\sqrt{7}$, for which S^- has zero density.

The main reason for restricting to quadratic fields is the fact that the analytic part of his proof of the conjecture in the quadratic case does not readily generalize to higher degree fields. The author remarks that such a generalization to fields of degree $k \geq 3$ implies that the set of primes dividing a 'generic' k-th order linear recurrent sequence has a positive lower density (something he showed in [J. Théor. Nombres Bordeaux 13 (2001), 303–314, MR 2002c:11013]).

In an earlier paper the author [J. Number Theory 81 (2000), 93–109, 2000k:11128, erratum J. Number Theory 85 (2000), 108; MR 2001m:11202] addressed the related (but easier) problem of determining the density of rational primes p for which the fundamental unit ε of a real quadratic number field K has maximal order (namely $p \pm 1$ or $2(p+1)$ depending on the norm of ε and the decomposition of p in K.

The author's present paper and the two other papers quoted above together form the, well-written, 'primitive root' part of his PhD thesis entitled 'Primitive Roots and Singular Moduli' (Leiden University, 2003).