Euler-Kronecker constants and cusp forms

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– Partly joint work with A. Ciolan and A. Languasco –

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– A. Ciolan, A. Languasco and P. Moree, Landau and Ramanujan approximations for divisor sums and coefficients of cusp forms, https://arxiv.org/abs/2109.03288.
– B. Berndt and P. Moree, Sums of two squares and the tau function: Ramanujan's trail, in progress.

Also webpage available

Programs and numerical results for the paper
``Landau and Ramanujan approximations for divisor sums and coefficients of cusp forms''
by A. Ciolan, A. Languasco and P. Moree

In this page we include the programs (Pari/Gp and Python scripts) developed to obtain the numerical results described in the paper [1].
For the definition of the quantities $\gamma_{q,k}, \gamma'_{q,k}, \gamma_{K_r}, \gamma_{K_{2r}}$ and $S(r,q)$ we refer to [1].
In the following the acronym ``LvR'' stands for the Landau versus Ramanujan problem as stated in Section 1.2.2 of [1].

Pari/Gp and Python scripts

gammaK.gp: Pari/Gp script. It can be used via gp2c. The function to be run is: gamma_K(q1, q2, prec).
Input: q1, q2, prec: three positive integers.
Output: it computes the Euler-Kronecker constants γ_{K_r} and $\gamma_{K_{2r}}$ of the cyclotomic subfields K_r and K_{2r} for every $r \mid (q-1)/2$, $1 \le r \le 6$; where q is an odd prime running between q1 and q2.
The computation is performed with an accuracy of prec decimal digits. It uses the algorithm developed in [2]-[3] for computing the Euler-Kronecker constant of a cyclotomic field modified as in Section 9 of [1] to be able to handle the case of such cyclotomic subfields.
The output is saved in one file for each $1 \le r \le 6$ for further elaborations needed to study the LvR problem, see the Python script below.
In the folder gammaK-results you'll find the result of a computation performed with q1 = 3; q2 = 3000 and prec = 30. Each file contains the results according the values of r, $1 \le r \le 6$.

Srall-v2.gp: Pari/Gp script. It can be used via gp2c. The function to be run is: Srall(r1, r2, q1, q2, Pbound, prec).
Input: r1, r2, q1, q2, Pbound, prec: six positive integers.
Output: it computes $-S(r,q)$ (please remark the change of sign), with $1 \le r1 \le r \le r2 \le 6$; $q1 \le q \le q2$; q is an odd prime, by truncating up to Pbound the sums in its definition; prec is the internal decimal precision used.
The output is saved in one file for each $1 \le r \le 6$ for further elaborations needed to study the LvR problem, see the Python script below.
In the folder Sr-values-results you'll find the result of a computation of $S(r,q)$ performed with $1 \le r \le 6$. Pbound can be 10^8, 10^9 or 10^{10} (with prec = 19); see Section 9 of [1]. The results were first computed with Pbound = 10^8; the ones not having a sufficiently good accuracy were recomputed with Pbound = 10^9; the ones not having (yet) a sufficiently good accuracy were recomputed with Pbound = 10^{10}. All these results were then merged in the files mentioned before.

analysis-gen-gammakq.py: Python script; it uses pandas and numpy. It computes lower and upper bounds for $\gamma_{k,q}$ and $\gamma'_{k,q}$; then it decides on the LvR problem, see Theorem 4 and

Historical background

In his first letter (Jan. 16, 1913) to Hardy, Ramanujan made several claims, one of which reads:

$1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, \ldots$ *are numbers which are either themselves squares or which can be expressed as the sum of two squares. The number of such numbers greater than* A *and less than* B *equals*

$$K \int_A^B \frac{dx}{\sqrt{\log x}} + \theta(x),$$

where $K = 0.764\ldots$ *and* $\theta(x)$ *is very small compared with the previous integral.* K *and* $\theta(x)$ *have been exactly found, though complicated...*

Note: $\theta(x) = \theta(B)$

Sums of two squares

Let $B(x) = \#\{n \leq x : n = a^2 + b^2,\ a, b \in \mathbb{N}\}$.

Landau (1908) proved:

$$B(x) = \sum_{n \leq x,\ n = a^2 + b^2} 1 \sim K \frac{x}{\sqrt{\log x}}.$$

Ramanujan (1913) claimed:

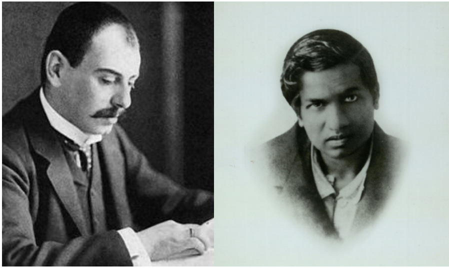
$$B(x) = K \int_2^x \frac{dt}{\sqrt{\log t}} + O\Big(\frac{x}{\log^r x}\Big),$$

where $r > 0$ is arbitrary.

Recall: Gauss' approximation $\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$ is a much better estimate for $\pi(x)$, the number of primes up to x, than is $\frac{x}{\log x}$.

Landau vs. Ramanujan

Is the Landau $\frac{Kx}{\sqrt{\log x}}$ or the Ramanujan approximation $K \int_2^x \frac{dt}{\sqrt{\log t}}$ better?



Hardy had his PhD student Gertrude Stanley (1928) work this out. Her conclusion: Landau approximation is better.

Shanks (1964): Ramanujan approximation is better.

More precise...

It can be further shown that $B(x)$ has a Poincaré asymptotic expansion:

$$B(x) = \frac{Kx}{\sqrt{\log x}}\left(1 + \sum_{j=2}^{r} \frac{C_j}{\log^{j-1} x} + O\left(\frac{1}{\log^r x}\right)\right)$$

for every $r \geq 2$, with $C_2, \ldots, C_r$ constants.

If correct, Ramanujan's integral approximation claim would imply, by partial integration,

$$B(x) = \frac{Kx}{\sqrt{\log x}}\left(1 + \sum_{j=2}^{r} \frac{C_j'}{\log^{j-1} x} + O\left(\frac{1}{\log^r x}\right)\right) \text{ with } C_2' = 1/2.$$

Shanks (1964) computed $C_2 = 0.5819486\ldots$, disproving the claim. In addition he computed $K = 0.764223654\ldots$.
Now known with 30 000 (Languasco), respectively 125 000 decimals.

Digression: binary cyclotomic forms

We consider the homogenized cyclotomic polynomial

$$\Phi_n(X,Y) = Y^{\varphi(n)}\Phi_n(X/Y)$$

Let

$$A(x) = \{m \le x : m = \Phi_n(a,b),\ n \ge 3,\ \max\{|a|,|b|\} \ge 2\}.$$

We have

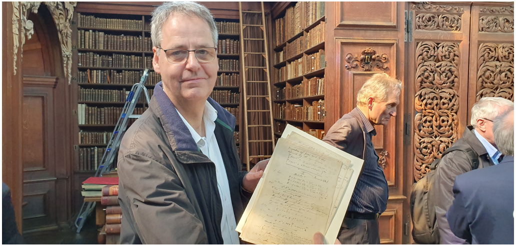
$$A(x) = \frac{x}{\sqrt{\log x}}\left(\alpha_0 - \frac{\beta_0}{\log^{1/4} x} + \frac{1}{\log x}\left(\alpha_1 - \frac{\beta_1}{\log^{1/4} x}\right) + \ldots + O\left(\frac{1}{\log^r x}\right)\right),$$

with α_0, β_0 of Landau-Ramanujan constant type.

1. E. Fouvry, C. Levesque and M. Waldschmidt, Representation of integers by cyclotomic binary forms, *Acta Arith.* **184** (2018), no. 1, 67–86.

Ramanujan's "unpublished" manuscript

Along with his final letter (Jan. 12, 1920) to Hardy, Ramanujan seems to have included a manuscript on congruence properties of $\tau(n)$ and $p(n)$.



B. Berndt and K. Ono, Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary. The Andrews Festschrift (Maratea, 1998). *Sém. Lothar. Combin.* **42** (1999), 63 pp.

Ramanujan's τ function

- If $z \in \mathbb{H}$ and $q_1 = e^{2\pi i z}$,

$$\Delta(z) := \eta^{24} = q_1 \prod_{n=1}^{\infty}(1 - q_1^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q_1^n$$

  is the modular discriminant.
- It is a cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$.
- Its Fourier coefficients $\tau(n)$ are the values of the Ramanujan τ function.
- Ramanujan realized that $\tau(n)$ has interesting arithmetic properties.
- In his "unpublished" manuscript, he discovered a few congruences for τ modulo q^e for $q \in \{2, 3, 5, 7, 23, 691\}$.

Congruences for τ

- Ramanujan showed that $2 \nmid \tau(n) \Leftrightarrow n$ is an odd square.
- Wilton's congruences:

$$\tau(p) \equiv \begin{cases} 1 \pmod{23} & \text{if } p = 23, \\ 0 \pmod{23} & \text{if } \left(\frac{p}{23}\right) = -1, \\ 2 \pmod{23} & \text{if } p = U^2 + 23V^2 \text{ with } U \neq 0, \\ -1 \pmod{23} & \text{for all other } p. \end{cases}$$

- Further, he stated the following congruences:

$$\begin{aligned} \tau(n) &\equiv n\sigma_1(n) \pmod{3}, \\ \tau(n) &\equiv n\sigma_1(n) \pmod{5}, \\ \tau(n) &\equiv n\sigma_3(n) \pmod{7}, \\ \tau(n) &\equiv \sigma_{11}(n) \pmod{691}. \end{aligned}$$

Some claims of Ramanujan

For the primes q above, namely $(3, 5, 7, 23, 691)$, Ramanujan made claims of $B(x)$-type for

$$S_{\tau,q}(x) = \#\{n \le x : q \nmid \tau(n)\}.$$

Defining $t_n = 1$ if $q \nmid \tau(n)$ and $t_n = 0$ otherwise, he typically writes:

It is easy to prove by quite elementary methods that $\sum_{k=1}^{n} t_k = o(n)$.
It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{C_q n}{\log^{\delta_q} n}$$

and

$$\sum_{k=1}^{n} t_k = C_q \int_1^n \frac{dx}{\log^{\delta_q} x} + O\left(\frac{n}{\log^{\rho} n}\right),$$

where ρ *is any positive number.*

...in handwritten form...

p being a prime of

$$\Pi_B = \prod_p \frac{1-p^{-4s}}{(1-p^{-s})(1-p^{-5s})}$$

p being a prime of the form $5k+1$.

It is easy to prove from (2.5) that

(2.6) $$t_1+t_2+t_3+\dots+t_n = o(n).$$

It can be shown by transcendental methods that

(2.7) $$t_1+t_2+t_3+\dots+t_n \sim \frac{Cn}{(\log n)^{\frac{1}{4}}},$$

and

(2.8) $$t_1+t_2+t_3+\dots+t_n = C\int_1^n \frac{dx}{(\log x)^{\frac{1}{4}}} + O\frac{n}{(\log n)^r},$$

where C is a constant and r is any positive number.

Some claims of Ramanujan

Serre (1981) proved that Frobenian multiplicative functions admit Poincaré expansions like $B(x)$. In particular,

$$S_{\tau,q}(x) = \frac{C_q x}{\log^{\delta_q} x}\left(1 + \frac{c_0}{\log x} + \frac{c_1}{\log^2 x} + \cdots + \frac{c_{j-1}}{\log^j x} + O\left(\frac{1}{\log^{j+1} x}\right)\right).$$

The integral claim implies particular values for the c_i.

Classical Theorem 1

Rankin (1988): For $q \in \{3, 5, 7, 23, 691\}$ the asymptotic claim is true, with

q	3	5	7	23	691
δ_q	1/2	1/4	1/2	1/2	1/690

Moree (2004): Computed c_0 for these q and showed they differ from Ramanujan's prediction.

The Euler-Mascheroni constant

The Euler-Mascheroni constant γ is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721566490153286\ldots$$

The Euler-Kronecker constant of a number field

- Attached to a number field K we have the Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s} \quad (\mathrm{Re}(s) > 1),$$

where $\mathfrak{a}$ runs over the non-zero ideals in the ring of integers $\mathcal{O}_K$.

- If $K = \mathbb{Q}$ we get $\zeta_{\mathbb{Q}}(s) = \sum_{n \geq 1} n^{-s}$, the Riemann zeta function.
- $\zeta_K(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, simple pole at $s = 1$.
- Unique factorization over $\mathcal{O}_K$ gives the Euler product identity

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}} \quad (\mathrm{Re}(s) > 1),$$

where $\mathfrak{p}$ runs over the prime ideals in $\mathcal{O}_K$.

The Euler-Kronecker constant of a number field

- Laurent series:

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1)$$

- Euler-Kronecker constant of K:

$$\gamma_K := \frac{c_0}{c_{-1}}$$

- Around $s=1$ we have

$$\frac{\zeta_K'(s)}{\zeta_K(s)} = -\frac{1}{s-1} + \gamma_K + O(s-1),$$

  which can be continued in terms of Stieltjes constants etc.
- Equivalently,

$$\gamma_K = \lim_{s\to 1^+}\Big(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1}\Big).$$

- Example: $\gamma_{\mathbb{Q}} = \gamma/1 = \gamma = 0.577\ldots$

The Euler-Kronecker constant of a number field

- Alternative formula given by:

$$\gamma_K = \lim_{x\to\infty}\Big(\log x - \sum_{N\mathfrak{p}\le x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}-1}\Big).$$

- The existence of (many) prime ideals in $\mathcal{O}_K$ of small norm has a decreasing effect on γ_K.
- For $K=\mathbb{Q}$ we obtain the well-known formula

$$\gamma_{\mathbb{Q}} = \gamma = \lim_{x\to\infty}\Big(\log x - \sum_{p\le x}\frac{\log p}{p-1}\Big).$$

The Euler-Kronecker constant of a multiplicative set

We say that $S \subset \mathbb{N}$ is multiplicative if the following holds:

$$\text{If } (m,n)=1, \text{ then } mn \in S \Leftrightarrow m,n \in S.$$

Alternatively: The indicator function 1_S is multiplicative.

Example 1: $\{n \in \mathbb{N} : n = a^2 + b^2,\ a,b \in \mathbb{N}\}$

Example 2: $\{n : q \nmid f(n)\}$ with q prime and f multiplicative

Let

$$L_S(s) := \sum_{n \in S} n^{-s}.$$

If the limit

$$\gamma_S := \lim_{s \to 1^+} \left(\frac{L_S'(s)}{L_S(s)} + \frac{\alpha}{s-1} \right)$$

exists for some $\alpha \neq 0$, the set S admits an Euler-Kronecker constant γ_S.

The Euler-Kronecker constant of a multiplicative set

The second order behavior of $S(x)$ is determined by the Euler-Kronecker constant γ_S.

Classical Theorem 2

Let S be a multiplicative set. If there are $\rho > 0$ and $0 < \delta < 1$ such that

$$\text{Condition A:} \quad \sum_{p \le x,\ p \in S} 1 = (1-\delta) \sum_{p \le x} 1 + O\left(\frac{x}{\log^{2+\rho} x}\right)$$

holds true, then γ_S exists and, as $x \to \infty$,

$$S(x) = \sum_{n \le x,\ n \in S} 1 = \frac{c_0\, x}{\log^{\delta} x}\left(1 + \frac{(1-\gamma_S)\delta}{\log x}(1 + o(1))\right),$$

with $c_0 > 0$.

The Euler-Kronecker constant of a multiplicative set

Classical Theorem 2 (continued)

If the primes in S are, with finitely many exceptions, precisely those in a finite union of arithmetic progressions, for any $j \geq 1$ we have

$$S(x) = \frac{c_0\, x}{\log^{\delta} x}\left(1 + \frac{c_1}{\log x} + \frac{c_2}{\log^2 x} + \cdots + \frac{c_j}{\log^j x} + O_{j,S}\left(\frac{1}{\log^{j+1} x}\right)\right),$$

with $c_0, \ldots, c_j$ constants, $c_0 > 0$ and $c_1 = (1 - \gamma_S)\delta$.

Landau vs. Ramanujan

We call

$$c_0 \frac{x}{\log^\delta x} \quad \text{and} \quad c_0 \int_2^x \frac{dt}{\log^\delta t}$$

the Landau, respectively the Ramanujan approximation to $S(x)$.

If for every x sufficiently large we have

$$\left| S(x) - c_0 \frac{x}{\log^\delta x} \right| < \left| S(x) - c_0 \int_2^x \frac{dt}{\log^\delta t} \right|,$$

we say that the Landau approximation is better than the Ramanujan one (and the other way around if the reverse inequality holds).

Partial integration gives us

$$c_0 \int_2^x \frac{dt}{\log^\delta t} = \frac{c_0\, x}{\log^\delta x} \left(1 + \frac{\delta}{\log x} + O\left(\frac{1}{\log^2 x} \right) \right).$$

Landau vs. Ramanujan

Suppose that condition A is satisfied.

Ramanujan: $c_1 = \delta$.
Landau: $c_1 = 0$.
Truth: $c_1 = (1 - \gamma_S)\delta$.

Criterion 1

If S is a multiplicative set satisfying Condition A, the associated Euler-Kronecker constant γ_S exists.

If $\gamma_S < 1/2$, then Ramanujan's approximation is better than Landau's (and the other way around if $\gamma_S > 1/2$).

Note: A Ramanujan-type claim, if true, implies $\gamma_S = 0$.

Landau vs. Ramanujan for $\varphi(n)$

Ford, Luca & Moree (2014) were the first to tackle the "Landau vs. Ramanujan problem" for infinitely many primes q by studying $\varphi(n)$.

Theorem 1 (Ford-Luca-Moree, 2014)

Put $S_q := \{n : q \nmid \varphi(n)\}$.
For $q \le 67$ *we have* $\gamma_{S_q} < 1/2$ *and Ramanujan's approximation is better.*
For $q > 67$ *we have* $\gamma_{S_q} > 1/2$ *and Landau's approximation is better.*
Furthermore, $\lim_{q\to\infty} \gamma_{S_q} = \gamma$ *and*

a) $\gamma_{S_q} = \gamma + O(\frac{\log^2 q}{\sqrt{q}})$, *effective constant.*

b) $\gamma_{S_q} = \gamma + O_\epsilon(q^{\epsilon-1})$, *ineffective constant.*

A crucial role in the analysis is played by the Euler-Kronecker constant γ_q of the cyclotomic field $\mathbb{Q}(\zeta_q)$.

Overview of Euler-Kronecker constants discussed so far

set	γ_{set}	winner	author
$n = a^2 + b^2$	$-0.1638\ldots$	Ramanujan	Shanks
$3 \nmid \tau$	$+0.5349\ldots$	Landau	Moree
$5 \nmid \tau$	$+0.3995\ldots$	Ramanujan	Moree
$7 \nmid \tau$	$+0.2316\ldots$	Ramanujan	Moree
$23 \nmid \tau$	$+0.2166\ldots$	Ramanujan	Moree
$691 \nmid \tau$	$+0.5717\ldots$	Landau	Moree
$q \nmid \varphi$, $q \leq 67$	< 0.4977	Ramanujan	FLM
$q \nmid \varphi$, $q \geq 71$	> 0.5023	Landau	FLM

Intermezzo: The Euler-Kronecker constant of $\mathbb{Q}(\zeta_q)$

Ihara (2009) conjectured that $\gamma_q > 0$ and that, for q sufficiently large,

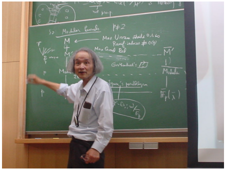
$$\frac{1}{2} - \epsilon \leq \frac{\gamma_q}{\log q} \leq \frac{3}{2} + \epsilon.$$

Badzyan (2010): Under GRH we have $|\gamma_q| = O((\log q) \log\log q)$.

Fouvry (2013): The dyadic average of γ_q is $\log q$:

$$\frac{1}{Q} \sum_{Q<q\leq 2Q} \gamma_q = \log Q + O(\log\log Q).$$

Intermezzo: The Euler-Kronecker constant of $\mathbb{Q}(\zeta_q)$



Recall

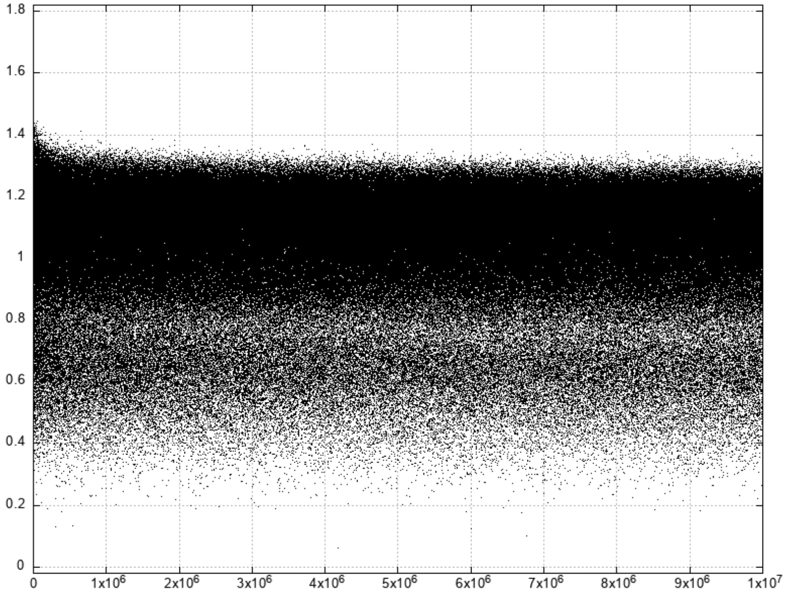
$$\gamma_K = \lim_{x\to\infty}\left(\log x - \sum_{N\mathfrak{p}\le x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}-1}\right).$$

Ihara conjecture (2009): $\gamma_q > 0$.
Ford-Luca-Moree (2014): $\gamma_{964477901} = -0.1823\ldots$
Languasco (2020): $\gamma_{9109334831} = -0.2487\ldots$; $\gamma_{9854964401} = -0.0964\ldots$
Languasco-Righi (2021): $\gamma_{50040955631} = -0.1659\ldots$

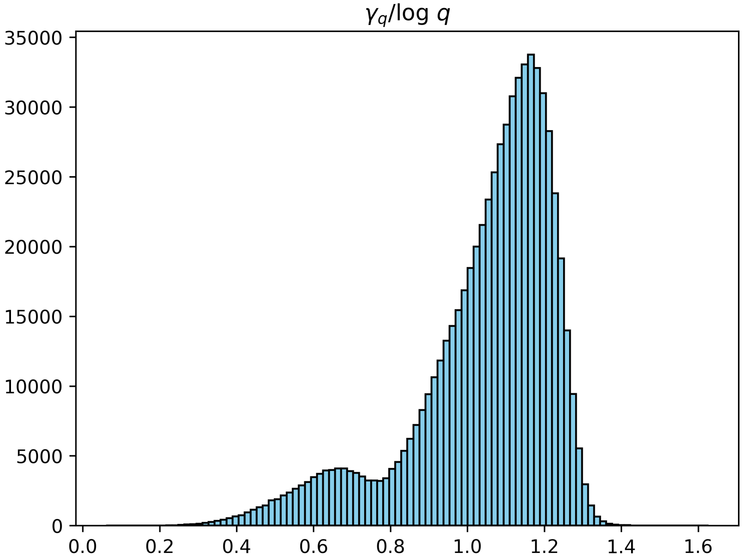
Ford-Luca-Moree (2014): On Hardy-Littlewood conjecture we have

$$\lim\inf_{q\to\infty}\frac{\gamma_q}{\log q} = -\infty.$$

$\frac{\gamma_q}{\log q}$ for $q \leq 10^7$



$\frac{\gamma_q}{\log q}$ for $q \le 10^7$ - histogram



Analogy with Kummer's Conjecture

Kummer conjectured in 1851 that

$$h_1(q) \sim 2q\left(\frac{q}{4\pi^2}\right)^{\frac{q-1}{4}},$$

with $h_1(q)$ the ratio of the class number of $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$
Put $r(q) = h_1(q)/RHS$. Conjecture thus states that

$$r(q) \sim 1.$$

Masley and Montgomery (1976):

$$|\log r(q)| < 7 \log q, \qquad q > 200.$$

Used this to determine all cyclotomic fields of class number 1.

Connection with $L(1,\chi)$ and $L'(1,\chi)$

We have

$$\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s,\chi),$$

$$\zeta_{\mathbb{Q}(\zeta_q)^+}(s) = \zeta(s) \prod_{\chi(-1)=-1} L(s,\chi)$$

$$\gamma_q = \gamma + \sum_{\chi \neq \chi_0} \frac{L'(1,\chi)}{L(1,\chi)}, \ \gamma_q^+ + \sum_{\chi(-1)=-1} \frac{L'(1,\chi)}{L(1,\chi)}$$

Hasse (1952): $$r(q) = \prod_{\chi(-1)=-1} L(1,\chi).$$

$$\frac{\zeta_{\mathbb{Q}(\zeta_q)}(s)}{\zeta_{\mathbb{Q}(\zeta_q)^+}(s)} = r(q)(1 + (\gamma_q - \gamma_q^+)(s-1) + O_q((s-1)^2)).$$

$$\frac{\gamma_q}{\log q} \text{ analytically similar to } 1-2|\log r(q)|.$$

Granville (1990): If Kummer's conjecture is true then

$$\sum_{\substack{p\le q^{1+\delta}\\ p\equiv 1 \pmod q}} \frac{1}{p} - \sum_{\substack{p\le q^{1+\delta}\\ p\equiv -1 \pmod q}} \frac{1}{p} = o(\frac{1}{q}),$$

for every $\delta>0$, for all but at most $2x/\log^3 x$ exceptions $q\le x$. We have

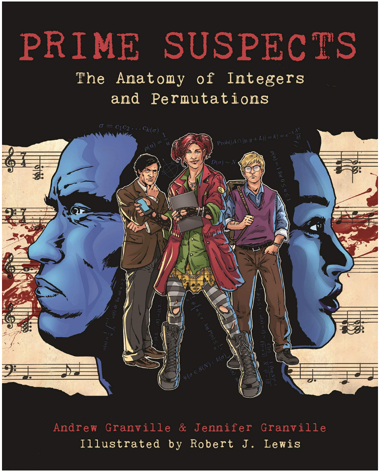
$$\gamma_q-\gamma_q^+ = \frac{(q-1)}{2}\lim_{x\to\infty}\Big(\sum_{\substack{p\le x\\ p\equiv 1 \pmod q}}\frac{\log p}{p-1} - \sum_{\substack{p\le x\\ p\equiv -1 \pmod q}}\frac{\log p}{p-1}\Big).$$

Assume Hardy-Littlewood conjecture and Elliott-Halberstam conjecture.

Granville: $r(q)$ has $[0,\infty]$ as set of limit points.

FLM: $\gamma_q/\log q$ has $(-\infty,1]$ as set of limit points.

So who are the exceptional prime suspects?



Exceptional primes q

Due to the work of Deligne, Serre and Swinnerton-Dyer we now know that the primes $q \in \{2, 3, 5, 7, 23, 691\}$ for which Ramanujan proved congruences are part of a larger (finite) list of exceptional primes modulo which congruences hold for the coefficients $\tau_w(n)$ of the six cusp forms:

Weight w	12	16	18	20	22	26
Form	Δ	$Q\Delta$	$R\Delta$	$Q^2\Delta$	$QR\Delta$	$Q^2R\Delta$

with

$$Q = 1 + 240\sum_{n\geq 1}\sigma_3(n)q_1^n, \quad R = 1 - 504\sum_{n\geq 1}\sigma_5(n)q_1^n, \quad \Delta = \frac{Q^3 - R^2}{1728}.$$

Remark 1: $Q = E_4$ and $R = E_6$.

Remark 2: the weights $w \in \{12, 16, 18, 20, 22, 26\}$ are precisely those for which the associated spaces of cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ are 1-dimensional.

Classical properties

For $w \in \{12, 16, 18, 20, 22, 26\}$ the following properties hold:

Classical Theorem 2

1) τ_w is integer valued.
2) τ_w is multiplicative: $\tau_w(mn) = \tau_w(m)\tau_w(n)$ whenever $(m, n) = 1$.
3) $\tau_w(p^{e+1}) = \tau_w(p)\tau_w(p^e) - p^{w-1}\tau_w(p^{e-1})$ for any prime p and $e \geq 2$.
4) We have
$$\sum_{n=1}^{\infty} \frac{\tau_w(n)}{n^s} = \prod_p \frac{1}{1 - \tau_w(p)p^{-s} + p^{w-1-s}}.$$
5) $|\tau_w(p)| \leq 2p^{(w-1)/2}$.

Far reaching consequences in number theory!!
M.R. Murty and V.K. Murty, *The mathematical legacy of Srinivasa Ramanujan*, Springer, New Delhi, 2013.

Special congruences mod q

Deligne, Haberland, Serre and Swinnerton-Dyer classified all primes q modulo which certain congruences hold for τ_w :

(i) $\tau_w(n) \equiv n^v \sigma_{w-1-2v}(n) \pmod{q}$ for all $(n,q)=1$, with $v \in \{0,1,2\}$.

(ii) $\tau_w(n) \equiv 0 \pmod{q}$ if and only if $\left(\frac{n}{q}\right) = -1$.

(iii) $p^{1-w}\tau_w^2(p) \equiv 0, 1, 2$ or $4 \pmod{q}$ for all primes $p \neq q$.

Type (i) congruences $\Longrightarrow$ $q \nmid n^a \sigma_\ell(n)$

Remark: $q \nmid \sigma_\ell(n) \Leftrightarrow q \nmid \sigma_{(\ell, q-1)}(n)$

For simplicity: If $q \mid a(n) \Leftrightarrow q \mid b(n)$, write $a(n) \cong b(n) \pmod{q}$

Example 1: If $a \geq 1$, then $n^a \sigma_\ell(n) \cong n\sigma_\ell(n) \pmod{q}$
Example 2: $\sigma_\ell(n) \cong \sigma_{(\ell, q-1)}(n) \pmod{q}$

Type (i): Exceptional primes with $q > w$

It turns out that $v = 0$ and

$$\tau_w(n) \equiv \sigma_{w-1}(n) \cong \sigma_r(n) \pmod{q},$$

with $r = (w-1, q-1)$ as in the table:

w	12	16	18	20	22	26
Form	Δ	$Q\Delta$	$R\Delta$	$Q^2\Delta$	$QR\Delta$	$Q^2R\Delta$
q	691	3617	43867	283, 617	131, 593	657931
r	1	1	1	1, 1	1, 1	5

Computational fact: $\tau_w(q) \equiv 1 \pmod{q}$
(and so $\tau_w(q^e) \equiv 1 \pmod{q}$)

Type (i): Exceptional primes with $q < w$

If $q < w$ is exceptional, then $\tau_w(n) \cong n\sigma_r(n) \pmod{q}$ with r given in the table:

Form	w	$q = 2$	3	5	7	11	13	17	19	23
Δ	12	1	1	1	3	No				
$Q\Delta$	16	1	1	1	1	1	No			
$R\Delta$	18	1	1	1	3	5	3	No		
$Q^2\Delta$	20	1	1	1	3	1	1	No	No	
$QR\Delta$	22	1	1	1	1	No	1	1	No	
$Q^2R\Delta$	26	1	1	1	3	1	No	1	1	No

Computational fact: $q \mid \tau_w(q)$
(and so $q \mid \tau_w(q^e)$)

Goal

- study how often for $q \nmid n^a\sigma_\ell(n)$ with $a \in \{0,1\}$ and $\ell \mid q-1$
- apply the results to all the exceptional primes q and the coefficients τ_w of the associated weight w cusp forms
- put the work of Moree (2004) into a general framework
- solve the "Landau versus Ramanujan problem" for fixed ℓ and all primes $q \equiv 1 \pmod{\ell}$

P. Moree, On some claims in Ramanujan's "unpublished" manuscript on the partition and tau functions, *Ramanujan J.* **8** (2004), 317–330.

Non-divisibility of $\sigma_k(n)$ – Set up

- Given a divisor m of $q-1$, let K_m be the unique subfield of $K=\mathbb{Q}(\zeta_q)$ of degree $[K:K_m]=(q-1)/m$.

  Examples: $K_1 = K = \mathbb{Q}(\zeta_q), \quad K_{q-1}=\mathbb{Q}$
  $K_2 = \mathbb{Q}(\zeta_q+\zeta_q^{-1}) = \mathbb{Q}(\cos(2\pi/q))$ is the maximal real subfield of K
- Put $r=(k,q-1)$ and assume that $h=(q-1)/r$ is even.
- Let $f_p = \text{ord}_q(p)$ and $g_p=\text{ord}_q(p^r)$
  ($p^{f_p}\equiv 1 \pmod q$, $p^{rg_p}\equiv 1 \pmod q$)
- Let $S_{k,q}=\{n\in\mathbb{N} : q\nmid \sigma_k(n)\}$ and $S'_{k,q}=\{n\in\mathbb{N} : q\nmid n\sigma_k(n)\}$, with $\gamma_{k,q}$ and $\gamma'_{k,q}$ the associated Euler-Kronecker constants.
- The associated counting functions are

$$S_{k,q}(x)=\sum_{n\le x,\ q\nmid\sigma_k(n)} 1, \quad S'_{k,q}(x)=\sum_{n\le x,\ q\nmid n\sigma_k(n)} 1.$$

Non-divisibility of $\sigma_k(n)$ – Set up

- Define

$$S(r,q) := -\sum_{g_p=1} \frac{(q-1)\log p}{p^{q-1}-1} + \sum_{g_p=1} \frac{q \log p}{p^q - 1} - \sum_{g_p \geq 3} \frac{(g_p-1)\log p}{p^{g_p-1}-1} + \sum_{g_p\geq 3} \frac{g_p \log p}{p^{g_p}-1} + \sum_{g_p=2} \frac{\log p}{p^2-1} + \sum_{\substack{2 \mid g_p \\ g_p > 2}} \frac{\log p}{p^{g_p/2} - p^{-g_p/2}}.$$

- Compare with

$$S(q) = \sum_{f_p \geq 2} \frac{\log p}{p^{f_p} - 1}$$

Non-divisibility of $\sigma_k(n)$ – Main result

- Rankin (1961) determined the asymptotic behavior of $S_{k,q}(x)$ for general k and primes q.
- Scourfield (1964) established asymptotics in the case where a prescribed prime power is required to exactly divide $\sigma_k(n)$.

Theorem 2 (Ciolan–Languasco–M., 2021)

For any odd prime q, there is a Poincaré asymptotic expansion for $S_{k,q}(x)$ with $\delta_q = 1/h$. In particular, there is a constant $C_{k,q} > 0$ such that

$$S_{k,q}(x) = \frac{C_{k,q}\, x}{\log^{1/h} x}\left(1 + \frac{1-\gamma_{k,q}}{h\log x} + O_{k,q}\left(\frac{1}{\log^2 x}\right)\right),$$

with

$$\gamma_{k,q} = \gamma - \frac{1}{h}(2\gamma_{K_{2r}} - \gamma_{K_r}) - \frac{\log q}{h(q-1)} - S(r,q).$$

Non-divisibility of $\sigma_k(n)$ – Main result

Theorem 2 (continued)

Similarly, we have

$$S'_{k,q}(x) = \frac{C'_{k,q}\ x}{\log^{1/h} x}\left(1 + \frac{1-\gamma'_{k,q}}{h\log x} + O_{k,q}\left(\frac{1}{\log^2 x}\right)\right),$$

with

$$C'_{k,q} = \left(1 - \frac{1}{q}\right) C_{k,q} \quad \textit{and} \quad \gamma'_{k,q} = \gamma_{k,q} + \frac{\log q}{q-1}.$$

- Recall: If $\gamma_{k,q} < 1/2$, Ramanujan's approximation is better than Landau's (the other way around if $\gamma_{k,q} > 1/2$). The same for $\gamma'_{k,q}$.
- A Ramanujan-type claim would imply $\gamma_{k,q} = 0$.
- Suffices to study $\gamma_{r,q}$.

Landau vs. Ramanujan for $S_{k,q}$ and $S'_{k,q}$

Theorem 3 (Ciolan–Languasco–M., 2021)

There exists an absolute constant c_1 such that for every positive integer r, every prime $q \geq e^{2r(\log r + \log\log r + c_1)}$ satisfying $q \equiv 1 \pmod{2r}$ and every positive integer k satisfying $(k, q-1) = r$, the Landau approximation is better than the Ramanujan approximation for both $S_{k,q}(x)$ and $S'_{k,q}(x)$.

Theorem 4 (Ciolan–Languasco–M., 2021)

Let $k \geq 1$ be an integer and q an odd prime such that $(k, q-1) = 1$. The Landau approximation for $S_{k,q}(x)$ is better than the Ramanujan one for all primes q other than $q \in \{3, 5, 7, 11, 13, 17, 23, 29, 37, 41, 43, 47, 53, 59, 73\}$, in which cases the Ramanujan approximation is better. The Landau approximation for $S'_{k,q}(x)$ is better than the Ramanujan one for all primes q other than $q = 5$.

Ramanujan's claims repeated

Ramanujan, in the unpublished manuscript:

It is easy to prove by quite elementary methods that $\sum_{k=1}^n t_k = o(n)$.

It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{C_q n}{\log^{\delta_q} n}; \quad (1)$$

and

$$\sum_{k=1}^{n} t_k = C_q \int_2^n \frac{dx}{\log^{\delta_q} x} + O\Big(\frac{n}{\log^{\rho} n}\Big), \quad (2)$$

where ρ *is any positive number.*

Main cusp form result

Theorem 5 (Ciolan–Languasco–M., 2021)

Let $f = \sum_{n\geq 1} \tau_w(n) q_1^n$ be any of the six cusp forms and let q be any odd exceptional prime of type (i) *or* (ii). *If*

$$t_n = \begin{cases} 0 & \text{if } q \mid \tau_w(n), \\ 1 & \text{if } q \nmid \tau_w(n), \end{cases}$$

then the claim (1) *holds for some positive numbers C_q and δ_q.However, the Ramanujan-type claim* (2) *is false for any $\rho > 1 + \delta_q$.Ramanujan's approximation is better than Landau's if one of the following is satisfied:*

a) $q = 5$;

b) $q = 7$ *and* $f \in \{\Delta, Q^2\Delta, Q^2 R\Delta\}$;

c) $f = R\Delta$ *and* $q > 5$.

In all remaining cases, Landau's approximation is better. For primes of type (i) *we have $\delta_q = r/(q-1)$. For type* (ii) *we have $\delta_q = 1/2$.*

Euler-Kronecker constants for $q > w$

form	w	r	q	$\gamma_{r,q}$
Δ	12	1	691	$0.571714\ldots$
$Q\Delta$	16	1	3617	$0.574566\ldots$
$R\Delta$	18	1	43867	$0.57669\ldots.$
$Q^2\Delta$	20	1	283	$0.552571\ldots$
$Q^2\Delta$	20	1	617	$0.567565\ldots$
$QR\Delta$	22	1	131	$0.532695\ldots$
$QR\Delta$	22	1	593	$0.568078\ldots$
$Q^2R\Delta$	26	5	657931	$0.57701\ldots.$

- Computation of final entry took 6.5 days!
- E-K constants of involved fields very fast; bottle neck $S(r, q)$

Euler-Kronecker constants for $q < w$

r	q	$\gamma'_{r,q}$
1	2	$-0.677823\ldots$
1	3	$0.534921\ldots$
1	5	$0.399547\ldots$
1	7	$0.712434\ldots$
3	7	$0.231640\ldots$
1	11	$0.522413\ldots$
5	11	$0.044497\ldots$
1	13	$0.614357\ldots$
3	13	$0.194544\ldots$
1	17	$0.518971\ldots$
1	19	$0.720414\ldots$

- Moree (2004) values (in red) were confirmed and computed with higher precision

Proof ingredients

Computation of $\gamma_{k,q}$

- Determining the associated Dirichlet series $T(s)$
- Splitting of primes in K_r and K_{2r}
- L-series factorization of $T(s)$ via $\zeta_{K_r}(s)$ and $\zeta_{K_{2r}}(s)$
- Algorithms for numerical evaluation of L'/L

Behavior of $\gamma_{k,q}$ for fixed k and large q

- Upper estimates of the form $S(r,q) < c\,q^{-1/r}$
- Explicit zero free regions for Dirichlet L-series

Proof – preliminaries

- By the multiplicativity of

$$t_n = \begin{cases} 0 & \text{if } q \mid \sigma_k(n), \\ 1 & \text{if } q \nmid \sigma_k(n), \end{cases}$$

the associated Dirichlet series $T(s)$ admits an Euler product:

$$T(s) = \sum_{n=1}^{\infty} \frac{t_n}{n^s} = \prod_p \sum_{j=0}^{\infty} \frac{t_{p^j}}{p^{js}}.$$

- The problem comes down to studying

$$\sigma_k(p^a) \equiv 0 \pmod{q}.$$

- Assume $p \neq q$, since $\sigma_k(q^a) \equiv 1 \pmod{q}$.

Proof preliminaries

- We have

$$\sigma_k(p^a) = \frac{p^{k(a+1)} - 1}{p^k - 1} \iff a \equiv -1 \pmod{\mu_p}$$

with

$$\mu_p = \begin{cases} q & \text{if } g_p = 1, \\ g_p & \text{if } g_p > 1. \end{cases}$$

- $q \mid \sigma_k(p) \Leftrightarrow p$ splits completely in K_{2r}, but not in the larger field K_r.
- We compute

$$T(s) = \frac{1}{1-q^{-s}} \prod_{p \neq q} \frac{1 - p^{-(\mu_p - 1)s}}{(1-p^{-s})(1-p^{-\mu_p s})}$$

$$= \frac{1}{1-q^{-s}} \prod_{g_p = 2} \frac{1}{1-p^{-2s}} \prod_{g_p \neq 2} \frac{1 - p^{-(\mu_p - 1)s}}{(1-p^{-s})(1-p^{-\mu_p s})}.$$

Challenge: Express $T(s)$ in terms of Dirichlet L-series.

Dedekind zeta function factorizations of K_m

Given a divisor m of $q-1$, let K_m be the unique subfield of $K=\mathbb{Q}(\zeta_q)$ of degree $[K:K_m]=(q-1)/m$.
Recall the Euler product identity

$$\zeta_K(s)=\prod_{\mathfrak{p}}\frac{1}{1-N\mathfrak{p}^{-s}}\quad (\mathrm{Re}(s)>1).$$

If $m \mid (q-1)/2$, then

$$\zeta_{K_m}(s)=\frac{1}{1-q^{-s}}\prod_{p\neq q}\left(\frac{1}{1-p^{-sg_p}}\right)^{\frac{q-1}{g_p}}=\zeta(s)\prod_{\chi\in X_m\setminus\{\chi_0\}}L(s,\chi).$$

Logarithmic differentiation yields

$$\gamma_{K_m}=\gamma+\sum_{\chi\in X_m\setminus\{\chi_0\}}\frac{L'(1,\chi)}{L(1,\chi)}.$$

The Euler-Kronecker constant $\gamma_{k,q}$

- In terms of Dedekind zeta functions,

$$T(s)^h = (1 - q^{-s})^{-1}\zeta(s)^h H(s)^{h/2}\zeta_{K_r}(s)\zeta_{K_{2r}}(s)^{-2},$$

  where $H(s)$ is a regular function as $s \to 1^+$ and

$$\frac{H'(1)}{2H(1)} = -S(r,q).$$

- Some L-series manipulation and logarithmic differentiation yield

$$\gamma_{k,q} = \gamma - \frac{r}{q-1}\Big(2\gamma_{K_{2r}} - \gamma_{K_r} + \frac{\log q}{q-1}\Big) - S(r,q), \;\; r = (k, q-1).$$

This expression is highly suitable for numerical evaluation, not for determining its asymptotic behavior for $q \to \infty$

$\gamma(k, q)$ for large q

- For our application we mostly have $r = 1$.
- For fixed r it can be shown that $\gamma_{r,q} \to \gamma$.
- There thus exists $q_0(r)$ such that $\gamma_{r,q} > 1/2$ for $q \geq q_0$ and Landau wins.
- In particular, we can hope to determine all q for which Ramanujan wins for small r.

Aim: determine this $q_0(r)$

- Earlier $\gamma_{k,q}$ expression is not useful for this, look for another one.

$\gamma(k, q)$ for large q

With $r = (k, q-1)$ we have

$$\gamma_{k,q} = \gamma - \sum_{i=1}^{r} \lim_{x\to\infty}\Big(\frac{\log x}{q-1} - \sum_{\substack{n\le x\\ n\equiv a_i \pmod q}} \frac{\Lambda(n)}{n}\Big) - S(r,q),$$

with $a_1, \ldots, a_r$, with $0 < a_i < q$, the solutions of $x^r \equiv -1 \pmod q$.

- Summand in limit can be estimated using zero free region of Dirichlet L-series (technical, we skip this)
- For fixed r we have $S(r,q) \to 0$ as $q \equiv 1 \pmod r$ and tends to infinity.
- $S(r,q) \ll (\log q)^2 q^{-1/r}$.

It follows that there exist $C_1, C_2 > 0$ such that

$$\gamma_{k,q} \ge \gamma - C_1 \frac{r \log^2 q}{\sqrt{q}} - C_2 \frac{\log^2 q}{q^{1/r}} = \gamma - F_r(q).$$

$\gamma(k, q)$ for large q

- Recall that there exist C_1, C_2 such that

$$\gamma_{k,q} \geq \gamma - C_1 \frac{r \log^2 q}{\sqrt{q}} - C_2 \frac{\log^2 q}{q^{1/r}} = \gamma - F_r(q).$$

- By taking c large enough we can ensure that $F_r(q) < 0.077$ for any $q \geq e^{2r(\log r + \log\log r + c)}$ satisfying $q \equiv 1 \pmod{2r}$, hence $\gamma_{k,q} > 1/2$.
- Conclusion: For any fixed $r \geq 1$, Landau's approximation is better for any such (large enough) q.
- Using the fact that $\gamma'_{k,q} = \gamma_{k,q} + \log q/(q-1) > \gamma_{k,q}$, we obtain the same conclusion for $\gamma'_{k,q}$.
- For $r = 1$ we can determine all the (finitely many) q such that Ramanujan wins. They satisfy $q \leq 73$.

HAPPY 75 and > 75!

On the numerical computations

- Evaluation of $\gamma_{k,q}$ splits in two parts: the pair $(\gamma_{K_r}, \gamma_{K_{2r}})$ and $S(r,q)$
- $$\gamma_{K_m} = \gamma + \sum_{\chi \in X_m \setminus \{\chi_0\}} \frac{L'(1,\chi)}{L(1,\chi)}$$

  for $m = r$ and $m = 2r$ can be evaluated with the same computational cost as in the case $m = 1$
- Implemented in Pari/Gp, with a precision of 30 decimal digits for $q \leq 3000$, using an approach developed by Languasco & Righi (2020) to compute $\gamma_{K_1} (= \gamma_q)$ and γ_{K_2} for $q < 10^7$
- FFT algorithm for $q > 3000$ for $m = 1$
- The slow decay of certain summands in $S(r,q)$ prevents us from getting a good enough accuracy for $r \geq 2$

Special cases

- $q = 2$: We have $\tau_w(n) \equiv n\sigma_1(n) \pmod 2$ and

$$\sum_{2 \nmid \tau_w(n)} 1 = \frac{\sqrt{x}}{2} + O(1).$$

- Haberland (1983) proved that the case $w = 16,\ q = 59$ is the only one of type (iii) using Galois cohomological methods, establishing a conjecture of Swinnerton-Dyer.

  The relevant algebraic field is non-abelian with a non-solvable Galois group and thus a factorization of $T(s)$ solely in terms of Dirichlet L-series and a regular factor is not expected to exist.

Special cases

- Type (ii) congruences $\rightsquigarrow m = (q-1)/2,\ K_m = \mathbb{Q}(\sqrt{q^*})$ is quadratic, with $q^* = \left(\frac{-1}{q}\right) q$
- Two cases: $w = 12,\ q = 23$ and $w = 16,\ q = 31$
- If $w = (q+1)/2$, we have

$$\tau_w(p) \equiv \begin{cases} 1 \pmod{q} & \text{if } p = q, \\ 0 \pmod{q} & \text{if } \left(\frac{p}{q}\right) = -1, \\ 2 \pmod{q} & \text{if } p = U^2 + qV^2 \text{ with } U \neq 0, \\ -1 \pmod{q} & \text{for all other } p \end{cases}$$

  for any $q \in \{23, 31\}$.
- $\gamma_{(\mathrm{ii})}$ can be computed in terms of $\gamma_{\frac{q-1}{2},q}$ etc.

Landau vs. Ramanujan for $S_{k,q}$ and $S'_{k,q}$

Conjecture 1 (Ciolan–Languasco–M., 2021)

If $r = 3$, the Landau approximation for $S_{k,q}(x)$ is better than the Ramanujan one for all primes q other than

$$q \in \{7, 13, 19, 31, 37, 61, 67, 79, 97, 103, 109, 127, 181\},$$

in which cases the Ramanujan approximation is better. The Landau approximation for $S'_{k,q}(x)$ is better than the Ramanujan one for all primes q other than

$$q \in \{7, 13, 19, 31, 61, 67, 97, 109\}.$$

Landau vs. Ramanujan for $S_{k,q}$ and $S'_{k,q}$

Conjecture 2 (Ciolan–Languasco–M., 2021)

If $r = 5$, the Landau approximation for $S_{k,q}(x)$ is better than the Ramanujan one for all primes q other than

$$q \in \{11, 31, 41, 71, 101, 131, 241, 271, 311\},$$

in which cases the Ramanujan approximation is better. The Landau approximation for $S'_{k,q}(x)$ is better than the Ramanujan one for all primes q other than

$$q \in \{11, 31, 71, 131, 241, 311\}.$$

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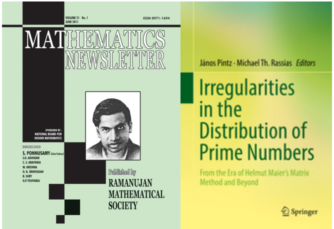

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