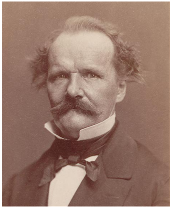
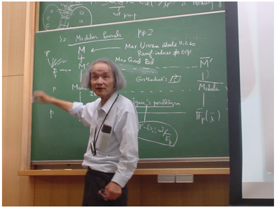
Euler-Kronecker constants and the log log log devil

Pieter Moree (MPIM, Bonn)



Ernst E. Kummer
(1810-1893)



Yasutaka Ihara
b. 1938

RIKEN, Wako Campus, Saitama
March 23, 2019
ZetaValue meeting

(Partly) joint work with

Florian Luca (Wits, Johannesburg)

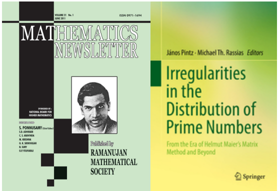
Kevin Ford (Urbana-Champaign, Illinois)

Values of the Euler phi-function not divisible by a given odd prime, and the distribution of Euler-Kronecker constants for cyclotomic fields, *Math. Comp.* **83** (2014), 1447–1476.

Overview of the talk

- Euler-Kronecker constants in general

P. Moree Counting numbers in multiplicative sets: Landau versus Ramanujan, *Mathematics Newsletter* **21**, no. 3 (2011), 73–81.



- Euler-Kronecker constants for cyclotomic number fields
- Similarities with Kummer's conjecture

P. Moree Irregular behaviour of class numbers and Euler-Kronecker constants of cyclotomic fields: the log log log devil at play, in (see picture), Springer, 2018, 143–163.

Euler-Mascheroni constant

The Euler-Mascheroni constant γ is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721566490153286\ldots$$

Some generalizations

Generalization: Stieltjes constants

$$\gamma_r = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\log^r k}{k} - \frac{\log^{r+1} n}{r+1} \right)$$

Arise as Laurent series coefficients of Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \gamma_r (s-1)^r$$

In particular,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

Definition of Euler-Kronecker constant

Let K be a number field, define its Dedekind-zeta function as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s}, \quad \mathrm{Re}(s) > 1.$$

Laurent series:

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1).$$

Euler-Kronecker constant of K: $\mathcal{EK}_K := \frac{c_0}{c_{-1}}$

$$\lim_{s\to 1} \left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1}\right) = \mathcal{EK}_K,$$

$\mathcal{EK}_K$ is constant in logarithmic derivative of $\zeta_K(s)$ at $s=1$.

Example. $\mathcal{EK}_{\mathbb{Q}} = \gamma/1 = \gamma = 0.577\ldots$

Historical background

Sums of two squares

Landau (1908) proved:

$$B(x) = \sum_{n \le x,\ n = a^2 + b^2} 1 \sim K \frac{x}{\sqrt{\log x}}.$$

Ramanujan (1913) claimed:

$$B(x) = K \int_2^x \frac{dt}{\sqrt{\log t}} + O\Big(\frac{x}{\log^r x}\Big),$$

where $r > 0$ is arbitrary.

$K = 0.764223653\ldots$: Landau-Ramanujan constant.

Shanks (1964): Ramanujan's claim is false for every $r > 3/2$.

Non-divisibility of Ramanujan's τ

$$\Delta := \eta^{24} = q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

After setting $q = e^{2\pi i z}$, the function $\Delta(z)$ is the unique normalized cusp form of weight 12 for the full modular group $SL_2(\mathbb{Z})$.

Fix a prime $q \in \{3, 5, 7, 23, 691\}$.

For these primes $\tau(n)$ satisfies an easy congruence, e.g.:

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}.$$

Put $t_n = 1$ if $q \nmid \tau(n)$ and $t_n = 0$ otherwise.

Some further claims of Ramanujan

Ramanujan in last letter to Hardy (1920):

"It is easy to prove by quite elementary methods that $\sum_{k=1}^{n} t_k = o(n)$.

It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{C_q n}{\log^{\delta_q} n}; \qquad (1)$$

and

$$\sum_{k=1}^{n} t_k = C_q \int_2^n \frac{dx}{\log^{\delta_q} x} + O\Big(\frac{n}{\log^r n}\Big), \qquad (2)$$

where r is any positive number".

Rushforth, Rankin: Estimate (1) holds true.

Moree (2004): All estimates (2) are false for $r > 1 + \delta_q$.

Euler-Kronecker constants of a multiplicative set

We say that S is multiplicative if m and n are coprime integers then mn is in S iff both m and n are in S

Common example is where S is a multiplicative semigroup generated by q_i, $i = 1, 2, \ldots,$ with every q_i a prime power and $(q_i, q_j) = 1$

Example I $n = X^2 + Y^2$
Example II If q is a prime and f a multiplicative function, then

$$\{n : q \nmid f(n)\}$$

is multiplicative

If $(m, n) = 1$, then

$$q \nmid f(mn) \iff q \nmid f(m)f(n) \iff q \nmid f(n) \text{ and } q \nmid f(m)$$

Euler-Kronecker constant of a multiplicative set

Assumption. There are some fixed $\delta, \rho > 0$ such that asymptotically

$$\pi_S(x) = \delta\pi(x) + O\Big(\frac{x}{\log^{2+\rho} x}\Big).$$

We put

$$L_S(s) := \sum_{n=1,\ n\in S}^{\infty} n^{-s}.$$

Can show that, Euler-Kronecker constant

$$\gamma_S := \lim_{s\to 1+0}\Big(\frac{L_S'(s)}{L_S(s)} + \frac{\delta}{s-1}\Big)$$

exists.

Counting the elements in S

If the assumption holds, then

$$S(x) \sim C_0(S) x \log^{\delta-1} x$$

We say that the Landau approximation is better than the Ramanujan approximation if for every x sufficiently large we have

$$\left| S(x) - C_0(S) x \log^{\delta-1} x \right| < \left| S(x) - C_0(S) \int_2^x \log^{\delta-1} t dt \right|.$$

Question: Given S, is the Landau or the Ramanujan approximation better?

The second order term and γ_S

We have

$$S(x) = C_0(S)x\log^{\delta-1}x\Big(1+(1+o(1))\frac{C_1(S)}{\log x}\Big), \qquad as \quad x\to\infty,$$

where $C_1(S) = (1-\delta)(1-\gamma_S)$.

Theorem. *Suppose that $\delta < 1$. If $\gamma_S < 1/2$, the Ramanujan approximation is asymptotically better than the Landau one. If $\gamma_S > 1/2$ it is the other way around.*

Follows on noting that by partial integration we have

$$\int_2^x \log^{\delta-1} dt = x\log^{\delta-1}x\Big(1+\frac{1-\delta}{\log x}+O\Big(\frac{1}{\log^2 x}\Big)\Big).$$

A Ramanujan type claim, if true, implies $\gamma_S = 0$.

Landau versus Ramanujan for $q \nmid \varphi$

Put $S_q := \{n : q \nmid \varphi(n)\}$ and $\gamma_{\varphi,q} = \gamma_{S_q}$.

Theorem. (Moree, 2006, unpublished). *Assume* ERH. *For* $q \le 67$ *we have* $\gamma_{\varphi;q} < 1/2$ *and Ramanujan's approximation is better.*
For $q > 67$ *we have* $\gamma_{\varphi;q} > 1/2$.
Further, we have $\lim_{q\to\infty} \gamma_{\varphi;q} = \gamma$.

Theorem. (Ford-Luca-Moree, 2014). Unconditionally true!

Theorem. *We have*

- $\gamma_{\varphi;q} = \gamma + O(\frac{\log^2 q}{\sqrt{q}})$, effective constant.
- $\gamma_{\varphi;q} = \gamma + O_\epsilon(q^{\epsilon-1})$, ineffective constant.
- $\gamma_{\varphi;q} = \gamma + O(\frac{\log^2 q}{q})$, no Siegel zero.
- $\gamma_{\varphi;q} = \gamma + O(\frac{\log q(\log\log q)}{q})$, on ERH for L-functions mod q.

Table: Overview of Euler-Kronecker constants discussed

set	γ_{set}	winner	author
$\mathbb{Z}_{\geq 1}$	$+0.5772\ldots$		Euler
$n = a^2 + b^2$	$-0.1638\ldots$	Ramanujan	Shanks
$3 \nmid \tau$	$+0.5349\ldots$	Landau	M.
$5 \nmid \tau$	$+0.3995\ldots$	Ramanujan	M.
$7 \nmid \tau$	$+0.2316\ldots$	Ramanujan	M.
$23 \nmid \tau$	$+0.2166\ldots$	Ramanujan	M.
$691 \nmid \tau$	$+0.5717\ldots$	Landau	M.
$q \nmid \varphi$, $q \leq 67$	< 0.4977	Ramanujan	FLM
$q \nmid \varphi$, $q \geq 71$	> 0.5023	Landau	FLM

Connection with $\gamma_q := \mathcal{EK}_{\mathbb{Q}(\zeta_q)}$

Put $f_p = |\langle p \pmod{q} \rangle|$ and

$$S(q) := \sum_{p \neq q,\ f_p \geq 2} \frac{\log p}{p^{f_p} - 1},$$

We have

$$\gamma_{\varphi;q} = \gamma - \frac{(3-q)\log q}{(q-1)^2(q+1)} - S(q) - \frac{\mathcal{EK}_{\mathbb{Q}(\zeta_q)}}{q-1}.$$

Given $\epsilon > 0$ we have $S(q) < \epsilon/q$ for a subset of primes of natural density 1, and $S(q) < 45/q$ for every q.

Conclusion:

$$\gamma_{\varphi;q} \approx \gamma - \frac{\mathcal{EK}_{\mathbb{Q}(\zeta_q)}}{q-1}$$

$\mathcal{EK}_{\mathbb{Q}(\zeta_q)} = \gamma_q$

$$\mathcal{EK}_K = \lim_{x\to\infty}\Big(\log x - \sum_{N\mathfrak{p}\le x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}-1}\Big)$$

gives

$$\frac{\gamma_q}{q-1} = -\frac{\log q}{(q-1)^2} - S(q) - \lim_{x\to\infty}\Big(\frac{\log x}{q-1} - \sum_{\substack{p\le x\\ p\equiv 1 \pmod q}}\frac{\log p}{p-1}\Big)$$

On ERH we have (Ihara, FLM)

$$\gamma_q = 2\log q - q\sum_{\substack{p\le q^2\\ p\equiv 1 \pmod q}}\frac{\log p}{p-1} + O(\log\log q)$$

Unconditionally also true with error $O_C(\log\log q)$ for all but at most $O(\pi(u)/(\log u)^C)$ primes $q\le u$.

On further assuming the Elliott-Halberstam conjecture we can replace 2 by $1+\epsilon$.

Two standard conjectures

Elliott-Halberstam Conjecture. *For every* $\epsilon > 0$ *and* $A > 0$ *we have*

$$\sum_{q \leq x^{1-\epsilon}} \left| \pi(x; q, a) - \frac{\mathrm{li}(x)}{\varphi(q)} \right| \ll_{A,\epsilon} \frac{x}{\log^A x}$$

Let $\{b_1, \ldots, b_k\}$ be a set of positive integers. We say it is admissible if the collection of forms n and $b_i n + 1$, $1 \leq i \leq k$, has no fixed prime factor.

Hardy-Littlewood Conjecture. If $\{b_1, \ldots, b_k\}$ is admissible, then the number of primes $n \leq x$ for which the numbers $b_i n + 1$ are all prime, is

$$\gg \frac{x}{\log^{k+1} x}$$

Ihara's conjectures

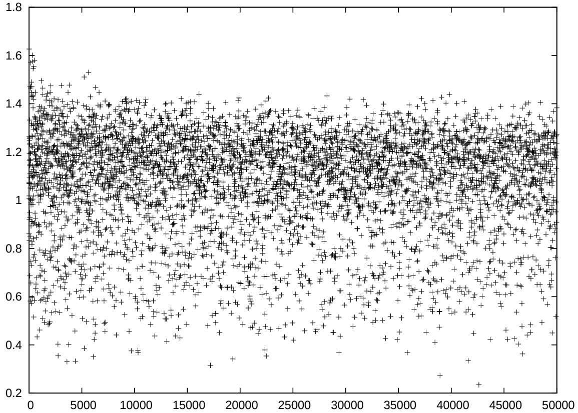
Badzyan (2010). On GRH, we have $|\gamma_q| = O((\log q)\log\log q)$

Ihara (2009).
(i) $\gamma_q > 0$ ('very likely')
(ii) Conjectures that

$$\frac{1}{2} - \epsilon \leq \frac{\gamma_q}{\log q} \leq \frac{3}{2} + \epsilon$$

for q sufficiently large

$\frac{\gamma_q}{\log q}$ for $q \leq 50.000$



A. Languasco (March 2019): Extended to $q \leq 100.000$

Results of Ford-Luca-M. on γ_q

We have $\gamma_{964477901} = -0.1823\ldots$

Theorem. On a quantitative version of the HL conjecture we have

$$\liminf_{q\to\infty} \frac{\gamma_q}{\log q} = -\infty$$

Conjecture. For density 1 sequence of primes we have

$$1-\epsilon < \frac{\gamma_q}{\log q} < 1+\epsilon$$

(That is, γ_q has normal order $\log q$)

Fouvry (2013) Dyadic average of γ_q is $\log q$:

$$\frac{1}{Q}\sum_{Q<q\leq 2Q} \gamma_q = \log Q + O(\log\log Q).$$

Sketch of proof of theorem

$$\gamma_q = 2\log q - q \sum_{\substack{p \le q^2 \\ p \equiv 1 \pmod{q}}} \frac{\log p}{p-1} + O(\log\log q)$$

Construct infinite sequence b_i, $i = 1, 2, \ldots$ such that $n, 1 + 2b_1 n, 1 + 2b_2 n, \ldots$ satisfies conditions of the HL conjecture AND

$$\sum_{i=1}^{s} \frac{1}{b_i} \to \infty$$

Take $\{b_i\} = \{2, 6, 8, 12, 18, 20, 26, \ldots\}$ sequence of greedy prime offsets and $s = 2088$ so that sum is > 4.

By HL conjecture $q, 1 + 2b_1 q, 1 + 2b_2 q, \ldots, 1 + 2b_s q$ are infinitely often ALL prime with $1 + 2b_s q \le q^2$. Then

$$q \sum_{\substack{p \le q^2 \\ p \equiv 1 \pmod{q}}} \frac{\log p}{p-1} > q \log q \sum_{i=1}^{s} \frac{1}{2b_i q} > (2 + \epsilon_0)\log q$$

Analogy with Kummer's Conjecture

Kummer conjectured in 1851 that

$$h_1(q) = \frac{h(q)}{h_2(q)} \sim G(q) := 2q(\frac{q}{4\pi^2})^{\frac{q-1}{4}}$$

Ratio of the class number of $\mathbb{Q}(\zeta_q)$, respectively $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$

Put $r(q) = h_1(q)/G(q)$

Ankeny and Chowla (1949):

$$\log r(q) = o(\log q) \quad \Rightarrow \quad \log h_1(q) \sim q(\log q)/4$$

Masley and Montgomery (1976):

$$|\log r(q)| < 7 \log q, \quad q > 200.$$

Used this to determine all cyclotomic fields of class number 1.

Connection with $L(1,\chi)$ and $L'(1,\chi)$

$$\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s,\chi) = \zeta_{\mathbb{Q}(\zeta_q)^+}(s) \prod_{\chi(-1)=-1} L(s,\chi)$$

$$\gamma_q = \gamma + \sum_{\chi \neq \chi_0} \frac{L'(1,\chi)}{L(1,\chi)} = \gamma_q^+ + \sum_{\chi(-1)=-1} \frac{L'(1,\chi)}{L(1,\chi)}$$

Hasse (1952): $\quad r(q) = \prod_{\chi(-1)=-1} L(1,\chi).$

$$\frac{\zeta_{\mathbb{Q}(\zeta_q)}(s)}{\zeta_{\mathbb{Q}(\zeta_q)^+}(s)} = r(q)(1 + (\gamma_q - \gamma_q^+)(s-1) + O_q((s-1)^2)).$$

$$\frac{\gamma_q}{\log q}$$ analytically similar to $1 - 2|\log r(q)|$.

Granville (1990): If Kummer's conjecture is true then

$$\sum_{\substack{p \le q^{1+\delta} \\ p \equiv 1 \pmod q}} \frac{1}{p} - \sum_{\substack{p \le q^{1+\delta} \\ p \equiv -1 \pmod q}} \frac{1}{p} = o(\frac{1}{q}),$$

for every $\delta > 0$, for all but at most $2x/\log^3 x$ exceptions $q \le x$.

$$\gamma_q - \gamma_q^+ = \frac{(q-1)}{2} \lim_{x \to \infty} \Big(\sum_{\substack{p \le x \\ p \equiv 1 \pmod q}} \frac{\log p}{p-1} - \sum_{\substack{p \le x \\ p \equiv -1 \pmod q}} \frac{\log p}{p-1} \Big).$$

Assume Hardy-Littlewood conjecture and Elliott-Halberstam conjecture.

Granville: $r(q)$ has $[0, \infty]$ as set of limit points.

FLM: $\gamma_q/\log q$ has $(-\infty, 1]$ as set of limit points.

The log log log devil makes its appearance...

Granville (1990): Kummer's ratio asymptotically satisfies

$$(-1+o(1))\log\log\log q \leq 2\log r(q) \leq (1+o(1))\log\log\log q.$$

These bounds are best possible in the sense that there exist two infinite sequences of primes q for which the lower, respectively upper bound are attained.

(Moree, 2018) Euler-Kronecker analogue:

$$\frac{\gamma_q}{\log q} \geq (-1+o(1))\log\log\log q.$$

The bound is best possible in the sense that there exists an infinite sequence of primes q for which the bound is attained.

The log log log devil

$$\gamma_q \approx \log q - q \sum_{\substack{2q+1 \le p \le q(\log q)^A \\ p \equiv 1 \pmod{q}}} \frac{\log p}{p-1}$$

$$\frac{\gamma_q}{\log q} \approx 1 - q \sum_{\substack{2q+1 \le p \le q(\log q)^A \\ p \equiv 1 \pmod{q}}} \frac{1}{p}$$

Brun-Titchmarsh (with $c = 2$)

$$\pi(x; q, 1) \le c \frac{x}{(q-1)\log(x/q)}.$$

Get

$$\frac{\gamma_q}{\log q} \approx -c(\log\log(q \log^A q) - \log\log(2q+1)),$$

so

$$\frac{\gamma_q}{\log q} \approx -c \log\log\log q$$

Conjecturally: $c = 1$

log log (?) devil

The speculations imply that

$$\liminf_{q\to\infty} \frac{\gamma_q}{(\log\log\log q)\log q} = 2\liminf_{q\to\infty} \frac{\log r(q)}{\log\log\log q} = -1.$$

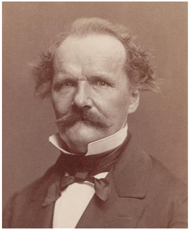
Weaker version:

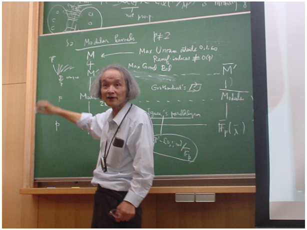
There exists a function $g(q)$ such that

$$\liminf_{q\to\infty} \frac{\gamma_q}{g(q)\log q} = 2\liminf_{q\to\infty} \frac{\log r(q)}{g(q)} < 0.$$

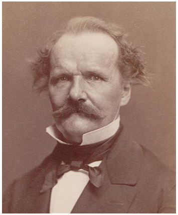
Badzyan (2010): We have $g(q) = O(\log\log q)$

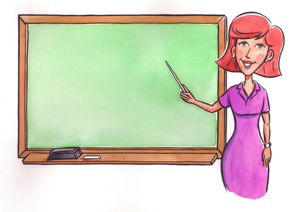
DEVIL ?

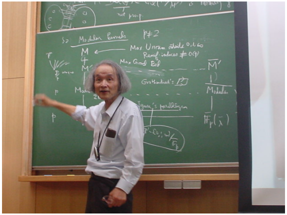




TEACHER!







THANK YOU!