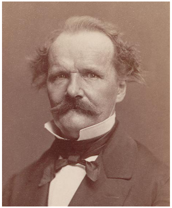
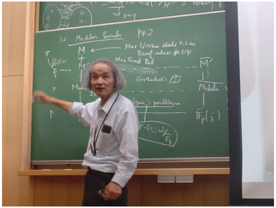
Irregular behaviour of class numbers and Euler-Kronecker constants of cyclotomic fields: the log log log devil at play

Pieter Moree (MPIM, Bonn)



Ernst E. Kummer
(1810-1893)



Yasutaka Ihara
b. 1938

Rome
April 11, 2019
RNTA meeting

Euler-Kronecker constants for cyclotomic fields

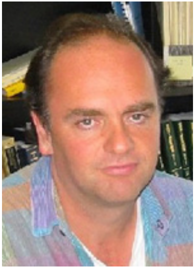
Florian Luca (Wits University, Johannesburg)

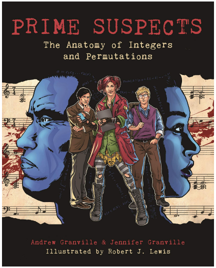
Kevin Ford (Urbana-Champaign, Illinois)

Values of the Euler phi-function not divisible by a given odd prime, and the distribution of Euler-Kronecker constants for cyclotomic fields, *Math. Comp.* **83** (2014), 1447–1476.

Follow-up in progress with Alessandro Languasco (Pisa), Sumaia Saad Eddin (Linz) and Alisa Sedunova (soon St. Petersburg).

Kummer's class number conjecture





A. Granville, On the size of the first factor of the class number of a cyclotomic field, *Invent. Math.* (1990), 321–338.

Euler-Mascheroni constant

The Euler-Mascheroni constant γ is defined as

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.57721566490153286\ldots$$



Some generalizations

Generalization: Stieltjes constants

$$\gamma_r = \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{\log^r k}{k} - \frac{\log^{r+1} n}{r+1} \Big)$$

Arise as Laurent series coefficients of Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \gamma_r (s-1)^r$$

In particular,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

Definition of Euler-Kronecker constant

Let K be a number field, define its Dedekind-zeta function as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s}, \quad \mathrm{Re}(s) > 1.$$

Laurent series:

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1).$$

Euler-Kronecker constant of K: $\mathcal{EK}_K := \frac{c_0}{c_{-1}}$

$$\lim_{s\to 1}\left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1}\right) = \mathcal{EK}_K,$$

$\mathcal{EK}_K$ is constant in logarithmic derivative of $\zeta_K(s)$ at $s = 1$.

Example. $\mathcal{EK}_{\mathbb{Q}} = \gamma/1 = \gamma = 0.577\ldots$

$\gamma_q := \mathcal{EK}_{\mathbb{Q}(\zeta_q)}$

We have

$$\mathcal{EK}_K = \lim_{x\to\infty}\Big(\log x - \sum_{N\mathfrak{p}\le x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}-1}\Big),$$

resulting in

$$\gamma_q = \lim_{x\to\infty}\Big(\log x - (q-1)\sum_{\substack{p\le x\\ p\equiv 1 \pmod{q}}}\frac{\log p}{p-1}\Big) + \text{smaller order terms}$$

On ERH we have (Ihara, FLM)

$$\gamma_q = \log(q^2) - q\sum_{\substack{p\le q^2\\ p\equiv 1 \pmod{q}}}\frac{\log p}{p-1} + O(\log\log q)$$

Unconditionally this estimate holds for all $C > 0$ and for all but $O(\pi(u)/(\log u)^C)$ primes $q \le u$.

On further assuming Elliott-Halberstam conjecture we can replace q^2 by $q^{1+\epsilon}$.

Two standard conjectures

Elliott-Halberstam Conjecture. *For every* $\epsilon > 0$ *and* $A > 0$ *we have*

$$\sum_{q \le x^{1-\epsilon}} \left| \pi(x; q, a) - \frac{\mathrm{li}(x)}{\varphi(q)} \right| \ll_{A,\epsilon} \frac{x}{\log^A x}$$

Let $\{b_1, \ldots, b_k\}$ be a set of positive integers. We say the set is admissible if $n \prod_{i=1}^{k} (b_i n + 1) \equiv 0 \pmod{p}$ has $< p$ solutions for every prime p.

Hardy-Littlewood Conjecture. If $\{b_1, \ldots, b_k\}$ is admissible, then the number of primes $n \le x$ for which the numbers $b_i n + 1$ are all prime, is

$$\gg \frac{x}{\log^{k+1} x}$$

Ihara's conjectures

For a density 1 set of primes q there exists $c > 0$ such that

$$-c \log\log q \leq \frac{\gamma_q}{\log q} \leq 2 + \epsilon$$

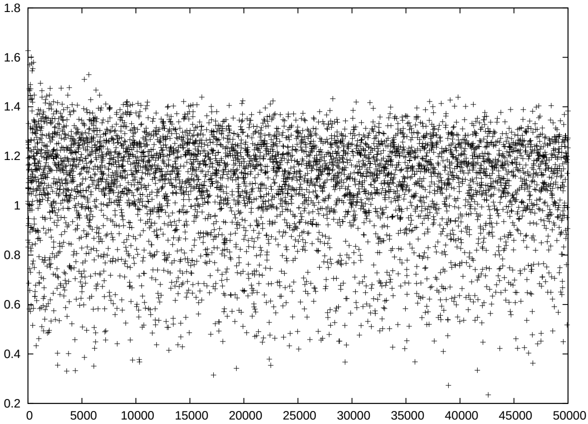
On ERH true for all q sufficiently large.

Ihara (2009).
(i) $\gamma_q > 0$ ('very likely')
(ii) Conjectures that

$$\frac{1}{2} - \epsilon \leq \frac{\gamma_q}{\log q} \leq \frac{3}{2} + \epsilon$$

for q sufficiently large

$\frac{\gamma_q}{\log q}$ for $q \le 50000$



A. Languasco (March 2019): Extended to $q \le 100000$

Ihara's conjectures. II

(FLM, 2014): We have $\gamma_{964477901} = -0.1823\ldots$

Theorem (FLM, 2014). On a quantitative version of the HL conjecture we have

$$\liminf_{q\to\infty} \frac{\gamma_q}{\log q} = -\infty$$

Theorem (FLM, 2014). Under the EH conjecture we have for density 1 sequence of primes

$$1-\epsilon < \frac{\gamma_q}{\log q} < 1+\epsilon$$

(That is, γ_q has normal order $\log q$)

Fouvry (2013) Dyadic average of γ_q is $\log q$:

$$\frac{1}{Q}\sum_{Q<q\leq 2Q} \gamma_q = \log Q + O(\log\log Q).$$

Sketch that $\gamma_q < 0$ infinitely often und ERH,HL

$$\gamma_q = 2\log q - q \sum_{\substack{p \le q^2 \\ p \equiv 1 \pmod q}} \frac{\log p}{p-1} + O(\log\log q)$$

Find b_i, $i = 1, 2, \ldots s$ such that $n, 1 + b_1 n, 1 + b_2 n, \ldots$ satisfies conditions of the HL conjecture AND

$$\sum_{i=1}^{s} \frac{1}{b_i} > 2$$

Take $\{b_i\} = \{2, 6, 8, 12, 18, 20, 26, \ldots\}$ sequence of greedy prime offsets and $s = 2088$ so that sum is > 2.

By HL conjecture $q, 1 + b_1 q, 1 + b_2 q, \ldots, 1 + b_s q$ are infinitely often ALL prime with $1 + b_s q \le q^2$ and so

$$q \sum_{\substack{p \le q^2 \\ p \equiv 1 \pmod q}} \frac{\log p}{p-1} > q \log q \sum_{i=1}^{s} \frac{1}{b_i q} > \log q \sum_{i=1}^{s} \frac{1}{b_i} > (2 + \epsilon_0) \log q$$

Admissible sets

The measure of an admissible set S is defined as

$$m(S) = \sum_{s \in S} \frac{1}{s}.$$

The theorem is a consequence of the fact that $m(S) \to \infty$.

FLM gave a 5 line proof of this based on a 1961 result of Erdős. However... The divergence result is due to Granville and answered a 1988 conjecture of ... Erdős in the positive.

Theorem (G, 1990). There is a sequence of admissible sets $S_1, S_2, \ldots$ such that $\lim_{i \to \infty} m(S_i) = \infty$.
Proposition (G, 1990). There is an admissible set S with elements $\leq x$, such that $m(S) \geq (1 + o(1)) \log \log x$. For any admissable set $m(S) \leq 2 \log \log x$.

Analogy with Kummer's Conjecture

Kummer conjectured in 1851 that

$$h_1(q) = \frac{h(q)}{h_2(q)} \sim G(q) := 2q\Big(\frac{q}{4\pi^2}\Big)^{\frac{q-1}{4}}$$

Ratio of the class number of $\mathbb{Q}(\zeta_q)$, respectively $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$

Put $r(q) = h_1(q)/G(q)$. Conjecture thus states that

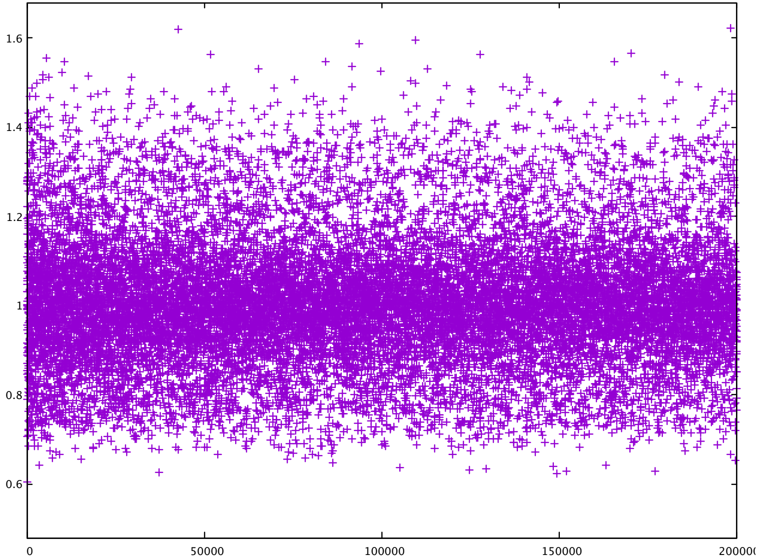
$$r(q) \sim 1.$$

Masley and Montgomery (1976): $|\log r(q)| < 7 \log q,\ \ q > 200.$

Used this to determine all cyclotomic fields of class number 1.

Ram Murty and Petridis (2001): There exists $c > 1$ such that for a density 1 set of primes q we have $1/c \le r(q) \le c$.

$r(q)$ for $q \leq 200000$



Current max: $r(6766811) = 1.709\ldots$

Connection with $L(1,\chi)$ and $L'(1,\chi)$

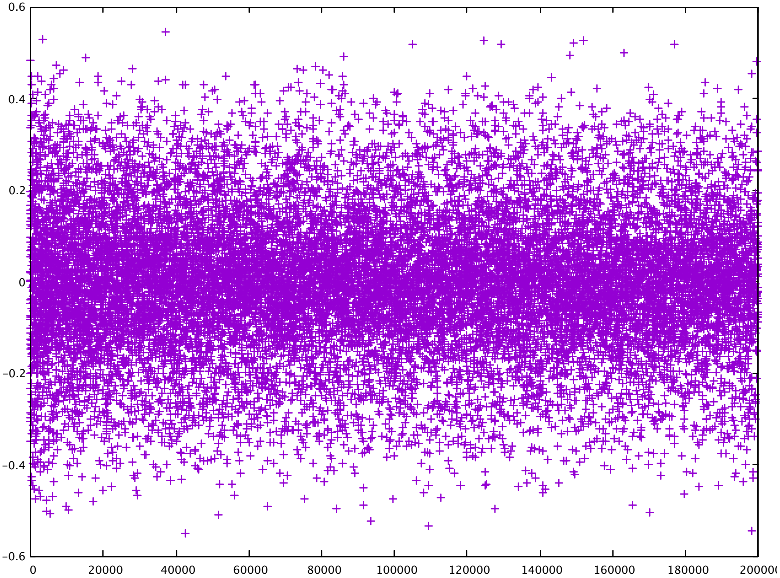
$$\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s,\chi) = \zeta_{\mathbb{Q}(\zeta_q)^+}(s) \prod_{\chi(-1)=-1} L(s,\chi)$$

$$\gamma_q = \gamma + \sum_{\chi \neq \chi_0} \frac{L'(1,\chi)}{L(1,\chi)} = \gamma_q^+ + \sum_{\chi(-1)=-1} \frac{L'(1,\chi)}{L(1,\chi)}$$

Hasse (1952): $$r(q) = \prod_{\chi(-1)=-1} L(1,\chi).$$

$$L(s) := \frac{\zeta_{\mathbb{Q}(\zeta_q)}(s)}{\zeta_{\mathbb{Q}(\zeta_q)^+}(s)} = r(q)(1 + (\gamma_q - \gamma_q^+)(s-1) + O_q((s-1)^2)).$$

$\gamma_q - \gamma_q^+$ for $q \leq 200000$



$r(q)$ and distribution of primes $p \equiv \pm 1 \pmod{q}$

We have

$$r(q) = \prod_{\chi(-1)=-1} L(1,\chi).$$

Can be written using orthogonality of characters as

$$r(q) = \exp\Big(\frac{(q-1)}{2} \lim_{x\to\infty} \Big(\sum_{m\geq 1} \frac{1}{m}\Big(\sum_{\substack{p^m \leq x \\ p^m \equiv 1 \pmod{q}}} \frac{1}{p^m} - \sum_{\substack{p^m \leq x \\ p^m \equiv -1 \pmod{q}}} \frac{1}{p^m}\Big)\Big)\Big)$$

Ignoring the $m \geq 2$ contributions and taking logarithm:

$$\log r(q) \approx \frac{(q-1)}{2} \lim_{x\to\infty} \Big(\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{q}}} \frac{1}{p} - \sum_{\substack{p \leq x \\ p \equiv -1 \pmod{q}}} \frac{1}{p}\Big).$$

$(\gamma_q - \gamma_q^+)/\log q$ analytically similar to $r(q)$

$$\gamma_q - \gamma_q^+ = \frac{(q-1)}{2} \lim_{x\to\infty} \Big(\sum_{\substack{p\le x \\ p\equiv 1 \pmod q}} \frac{\log p}{p-1} - \sum_{\substack{p\le x \\ p\equiv -1 \pmod q}} \frac{\log p}{p-1} \Big).$$

$$\frac{\gamma_q - \gamma_q^+}{\log q} \approx \frac{(q-1)}{2} \Big(\sum_{\substack{p\le q(\log q)^A \\ p\equiv 1 \pmod q}} \frac{1}{p-1} - \sum_{\substack{p\le q(\log q)^A \\ p\equiv -1 \pmod q}} \frac{1}{p-1} \Big).$$

$$\frac{\gamma_q - \gamma_q^+}{\log q} \approx \frac{(q-1)}{2} \Big(\sum_{\substack{p\le q(\log q)^A \\ p\equiv 1 \pmod q}} \frac{1}{p} - \sum_{\substack{p\le q(\log q)^A \\ p\equiv -1 \pmod q}} \frac{1}{p} \Big) \approx \log r(q).$$

Exploiting the similarity

Assume Hardy-Littlewood conjecture and Elliott-Halberstam conjecture.

Granville: $r(q)$ has $[0, \infty]$ as set of limit points.

Analytic similarity of $(\gamma_q - \gamma_q^+)/\log q$ with $\log r(q)$ yields:

LMSS: $(\gamma_q - \gamma_q^+)/\log q$ is dense in $(-\infty, \infty)$.

Analytic similarity of $\gamma_q/\log q$ with $1 - 2|\log r(q)|$ yields:

FLM, 2014: $\gamma_q/\log q$ is dense in $(-\infty, 1]$.

The log log log devil makes its appearance...

Granville (1990): Kummer's ratio asymptotically satisfies

$$(-1 + o(1)) \log\log\log q \leq 2 \log r(q) \leq (1 + o(1)) \log\log\log q.$$

(LMSS, 2019) Euler-Kronecker analogue:

$$(-1 + o(1)) \log\log\log q \leq 2 \frac{(\gamma_q - \gamma_q^+)}{\log q} \leq (1 + o(1)) \log\log\log q.$$

(FLM, 2014) Euler-Kronecker analogue:

$$\frac{\gamma_q}{\log q} \geq (-1 + o(1)) \log\log\log q.$$

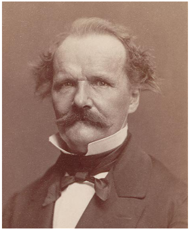
These bounds are best possible in the sense that there exist infinite sequences of primes q for which all the indicated bounds are attained.

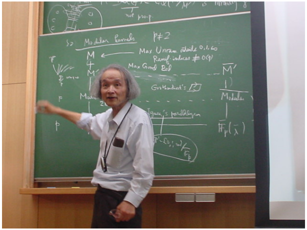
log log (?) devil

Badzyan (2010): Under GRH we have

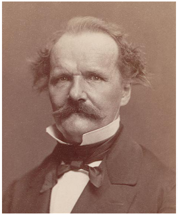
$$|\gamma_q| = O((\log q) \log \log q)$$

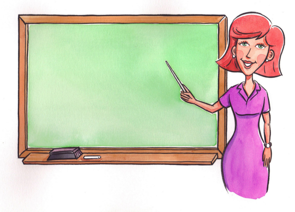
DEVIL ?

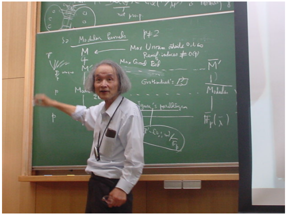




TEACHER!







THANK YOU!

Euler-Kronecker constant in general

Given an L-series $L(s)$, one can define the associated Euler-Kronecker constant as the constant in the logarithmic derivative of $L(s)$.

This can be used for example to disprove some conjectures of Ramanujan in which case $L(s)$ raised to an appropriate power is a product of Dirichlet L-series and the Euler-Kronecker constant can be rigorously computed.

Some claims of Ramanujan

Ramanujan (1913) claimed:

$$\sum_{n\le x,\ n=a^2+b^2} 1 = K\int_2^x \frac{dt}{\sqrt{\log t}} + O\Big(\frac{x}{\log^r x}\Big),$$

where $r > 0$ is arbitrary.

Further,

$$\sum_{n\le x,\ 5\nmid\tau(n)} 1 = C\int_2^x \frac{dt}{(\log t)^{1/4}} + O\Big(\frac{x}{\log^r x}\Big),$$

where $r > 0$ is arbitrary.

p being a prime of the form ...

$$\prod_{\mathfrak{B}} = \prod_{p} \frac{1-p^{-4s}}{(1-p^{-s})(1-p^{-5s})}$$

p being a prime of the form $5k+1$.

It is easy to prove from (2.5) that

(2.6) $$t_1 + t_2 + t_3 + \dots + t_n = o(n).$$

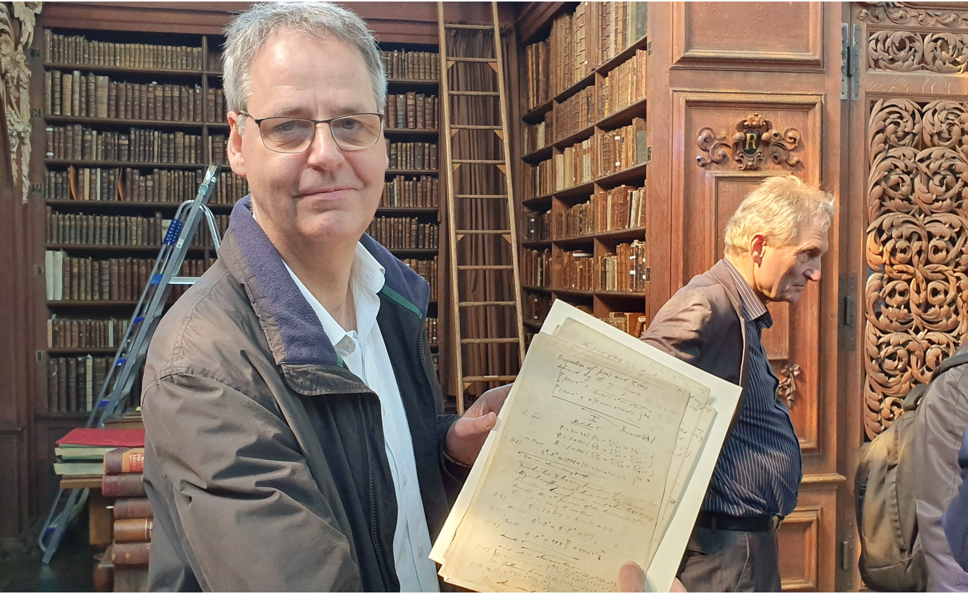
It can be shown by transcendental methods that

(2.7) $$t_1 + t_2 + t_3 + \dots + t_n \sim \frac{Cn}{(\log n)^{\frac{1}{2}}},$$

and

(2.8) $$t_1 + t_2 + t_3 + \dots + t_n = C\int_1^n \frac{dx}{(\log x)^{\frac{1}{2}}} + O\frac{n}{(\log n)^r},$$

where C is a constant and r is any positive number.



Kronecker limit formula

Let $E(\tau, s)$ be the real analytic Eisenstein series, given by

$$E(\tau, s) = \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m\tau + n|^{2s}}.$$

Then

$$E(\tau, s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log(2) - \log(\sqrt{y}|\eta(\tau)|^2)) + O(s-1),$$

where

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n), \ q = e^{2\pi it}$$

denotes the Dedekind eta function.
So the Eisenstein series has a pole at $s = 1$ of residue π, and the (first) Kronecker limit formula gives the constant term of the Laurent series at this pole.

$\mathcal{EK}_K$

$$\mathcal{EK}_K = \lim_{x\to\infty}\Big(\log x - \sum_{N\mathfrak{p}\le x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}-1}\Big)$$

$$\tilde{\zeta}_K(s) = \tilde{\zeta}_K(1-s)$$

$$\tilde{\zeta}_K(s) = \tilde{\zeta}_K(0)e^{\beta_K s}\prod_\rho\Big(1-\frac{s}{\rho}\Big)e^{s/\rho}$$

$$-\beta_K = \sum_\rho \frac{1}{\rho}$$

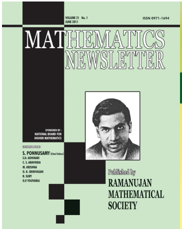
$$-\beta_K = \mathcal{EK}_K - (r_1+r_2)\log 2 + \frac{\log|D_K|}{2} - \frac{[K:\mathbb{Q}]}{2}(\gamma+\log\pi)+1$$

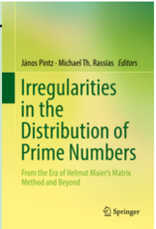
Theorem. (Ihara, 2006). *Under GRH we have*

$$-c_1\log|D_K| \le \mathcal{EK}_K \le c_2\log\log|D_K|$$

Expository accounts

P. Moree Counting numbers in multiplicative sets: Landau versus Ramanujan, *Mathematics Newsletter* **21**, no. 3 (2011), 73–81.





P. Moree Irregular behaviour of class numbers and Euler-Kronecker constants of cyclotomic fields: the log log log devil at play, in (see picture), Springer, 2018, 143–163.

Non-divisibility of Ramanujan's τ

$$\Delta := \eta^{24} = q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

After setting $q = e^{2\pi i z}$, the function $\Delta(z)$ is the unique normalized cusp form of weight 12 for the full modular group $SL_2(\mathbb{Z})$.

Fix a prime $q \in \{3, 5, 7, 23, 691\}$.

For these primes $\tau(n)$ satisfies an easy congruence, e.g.:

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}.$$

Put $t_n = 1$ if $q \nmid \tau(n)$ and $t_n = 0$ otherwise.

Some further claims of Ramanujan

Ramanujan in last letter to Hardy (1920):

"It is easy to prove by quite elementary methods that $\sum_{k=1}^{n} t_k = o(n)$.

It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{C_q n}{\log^{\delta_q} n}; \tag{1}$$

and

$$\sum_{k=1}^{n} t_k = C_q \int_2^n \frac{dx}{\log^{\delta_q} x} + O\Big(\frac{n}{\log^r n}\Big), \tag{2}$$

where r is any positive number".

Rushforth, Rankin: Estimate (1) holds true.

Moree (2004): All estimates (2) are false for $r > 1 + \delta_q$.