Geometry I – Very detailed syllabus

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1. Smooth manifolds

Topological manifold; coordinate charts, examples: vector spaces, spheres, projective spaces; connected \( \iff \) path connected; smooth structures: atlas, transition function, smooth maps, diffeomorphisms, equivalence of atlases vs. maximal atlases; more examples: Grassmannians, matrix groups; manifolds with boundary

2. Tangent vectors

Tangent vector as an equivalence class of smooth paths; tangent vector as derivation at a point; tangent space; equivalence of two definitions using Hadamard’s lemma; tangent vector in local coordinates

3. The differential of a smooth map

The differential as maps of paths; submersion, immersions, local diffeomorphisms (\(=\) étale maps); chain rule for differential; normal form of immersions and submersions; immersions as maps with local retractions, submersions as maps with local sections; normal form of regular map; embeddings, examples and counterexamples: figure 8, Kronecker foliation of torus; embedded and immersed submanifold; fibers of submersion are embedded submanifolds

4. The tangent bundle

Definition of fiber bundle, structure group, reduction of structure group, example: Moebius strip, sections of fiber bundle; submersions are not fiber bundles; definition of vector bundle; construction of tangent bundle, parallelizability; morphisms of fiber bundles and vector bundles; the tangent functor; products, sums, tensor products, duals, exterior and symmetric products of vector bundles; canonical line bundle, orientability of manifold

5. Vector fields

Vector fields as sections of tangent bundle; vector fields as derivations; dictionary geometry–algebra; definition of Lie algebra; the Lie algebra of derivations, flows; [aside on multilinear algebra (upon student request): symmetric and antisymmetric
tensor products as $S_n$-coinvariants, equivalence of coinvariants and invariants, tensor algebra, symmetric algebra, exterior algebra; vector field of a flow; integration of flow, reminder on integration of ODEs/Picard-Lindelöf; maximal integral curves, local flows, maximal local flows; complete flows; lemma: incomplete integral curve escapes any compact subset, vector fields on compact manifolds are complete; Lie derivative of functions and vector fields

6. Lie groups and Lie algebras

Definition of Lie group; Examples: real line, real and complex units, circle, tori, the standard matrix groups; representation and action of Lie group, equivariant maps; an equivariant map from $G$-manifold has constant rank; invariant vector fields; Lie algebra of a Lie group; examples: Lie algebras of matrix groups; invariant vector fields are complete, exponential map, 1-parameter subgroups; vector fields related by smooth map, homomorphisms of Lie groups induce homomorphisms of Lie algebras

7. Foliations and distributions

Definition of foliation, examples; fibers of surjective submersion are foliation; definition of distribution, integrability, involutivity, integral submanifolds; Frobenius theorem; examples for integrable and non-integrable distributions; application to Lie groups: every Lie subalgebra comes from a Lie subgroup, Lie’s third theorem, equivalence of category of simply connected Lie groups and Lie algebras, Lie group is abelian iff Lie algebra is abelian.

8. Differential forms, Cartan calculus

Cotangent bundle, covariant and contravariant tensor bundles, upper and lower indices; differential forms, multivector fields; functoriality of tangent bundle (not vector fields) and differential forms (not cotangent bundle); minicourse on graded algebra: graded vector space, graded tensor product, graded algebras, sign rules, braid relations, Koszul sign, graded symmetric/antisymmetric algebras, internal homomorphisms, graded derivations, graded Lie algebras, graded commutator algebra; Cartan calculus: inner derivative, de Rham differential, and Lie derivative as graded derivations on graded commutative ring of differential forms, derivation of all commutator relations, Cartan magic formula

9. Riemannian metrics

Definition of Riemannian metric; metric in local coordinates, inverse metric, raising and lowering indices; gradient; examples: Euclidean spaces, sphere, hyperboloid; every manifold has Riemannian metric (including reminder on partitions of unity); length and energy functional of paths; distance is a metric; metric topology is manifold topology; relation between length and energy; derivation of geodesic equation by variation of energy, Christoffel symbols, advantages of energy vs. length; exponential map, injectivity radius; Riemann normal coordinates, vanishing of first derivatives of metric in normal coordinates, geodesics in normal coordinates, polar normal coordinates; local existence of unique length minimizing geodesic; existence
of shortest path in homotopy class on compact manifold; geodesic completeness; Hopf-Rinow theorem

10. Connections
Ehresmann connection, horizontal sections, flat Ehresmann connection; horizontal lift of vectors and vector fields; Ehresmann connection is flat iff horizontal lift of vector fields is homomorphism of Lie algebras; linear connection; covariant derivative, equivalence with linear connections, connection coefficients; parallel transport; curvature, tensoriality of curvature;