Preliminary Exam Syllabus

Nima Rasekh

March 3rd 2015

What is a *space* ? The answer to this question might seem fairly obvious to any mathematician, however, it might not be as trivial as it first seems. To an analyst, a space will probably be a topological space. To an algebraic geometer, it might be a locale. To a category theorist it will probably be a compactly generated Hausdorff topological space (as a matter of fact look at page 188 of [ML98]). To an algebraic topologist or homotopist it could be CW-complexes, simplicial sets or even ∞-groupoids. All of these are acceptable models of what we would intuitively think of as a space. Although their categories are all Quillen equivalent, with the obvious model structure, every one of them has their own unique traits which make them suitable to specific situations. This leads us to two fundamental and very imprecise questions:

- What is a good axiomatization of a space ?
- Which collection of objects should be called a collection of spaces ?

Before I continue, let me first find a precise framing for the second question. The recent years have shown that the right notion of a collection of spaces is the notion of a higher category (quasi-categories, model categories, ...). So, now we can phrase the second question thusly:

Which conditions should we impose on a higher category so that we can reasonably call it a higher category of spaces ?

My main objective for the coming years is to try to answer this question. One way to approach this question is to see how it has been answered for another very foundational concept: *Sets*. The correct notion of collection of sets is a category and so the analogous question was phrased as follows:

Which category is suitable to be called a category of sets in the sense of *ZFC* (Zermelo-Frankel set theory with the axiom of choice)? This question was answered in 1969 by *William Lawvere* and *Myles Tierney*, introducing the notion of an *elementary topos*: A category (not enriched over sets), which has finite limits, a sub-object classifier and power objects. Using this category, Lawvere showed that we have the following theorem: a category satisfies bounded ZFC if and only if it is a well-pointed elementary topos which has a natural number object and satisfies the choice condition. This answer is quite satisfactory and later we will see how it reflects on the question I have posed above.

In addition to the results mentioned in the previous paragraph there are two recent advancements which could help answering this question. First, *Charles Rezk* [Re05], *Jacob Lurie* [Lu09], and *Bertrand Toen* and *Gabriele Vezzosi* [TV05] managed to generalize the notion of a *Grothendieck topos* (which is by definition a presentable elementary topos) to the higher categorical setting. Second, based on the work of *Per Martin-Löf*, *Steve Awodey* and *Michael Warren* [AW09], and *Vladimir Voevodsky* [Vo10] introduced a specific version of type theory, *homotopy type theory*, which is a logical foundation closely tied to homotopical intuition.

In light of these explanations, from now on, I am going to call the higher category of spaces I am trying to understand and define a *higher elementary topos*, although it might be a misnomer. Based on two previous paragraphs, I am considering these three different approaches to this question:

1. **Higher Topos Theory**: As we mentioned *higher topoi* are the higher categorical analogues of Grothendieck topoi. But Grothendieck topoi are just presentable elementary topoi and so it seems suggestive that a presentable higher elementary topos should be the same as a higher topos. Therefore, one goal should be to study higher topoi and its properties and realize which one are more essential. Some preliminary studies suggest that the notions of *descent* and *object-classifier*, introduced by Charles Rezk in his definition of a higher topos play a key role. So, the first approach is to study (not necessarily presentable) higher categories which satisfy those conditions. As a first step, I propose to solve the following question:

   Question 1. Propose a definition for an object-classifying ∞-category object in an ∞-category $\mathcal{C}$ i.e an ∞-category object $\Omega_\bullet \in \mathcal{C}_\Delta$ such that for every object X, $\mathcal{C}_{/X} \simeq Map_{\mathcal{C}}(X, \Omega_\bullet)$. Then prove that a presentable

∞-category is an ∞-topos if and only if it has object-classifying ∞-category objects for suitable classes of maps.

Possible Approach 1. There is already a partial result in this regard, as can be seen in Theorem 6.1.6.8 of [Lu09]. The problem is that Lurie assumes that colimits are universal and that the classifying object is an merely an object and not a simplicial object. So, the goal is to strengthen the object classifier condition and then prove that colimits are universal.

2. **Elementary Topos Theory**: Most topologists would agree that the notion of set and 0-types should be equivalent. Therefore, it's seems reasonable to say that the full subcategory of 0-truncated objects of a higher elementary topos should be an elementary topos. This is particularly useful as it puts a reasonable restriction on any proposed definition. In addition to that, it might be possible to find a definition of an elementary topos in a homotopic language and then try to generalize that to the higher setting. Concretely, we can already form one question:

   Question 2. One of the central notions of topos theory (besides the subobject-classifier) is the power object. The notion of subobject classifier has shown interesting connections to univalent maps (see 3.), so it might be that there are higher categorical notions which can be related to the power object. What could those be ?

   Possible Approach 2. One first approach to this question is to look at truncations. It is very straightforward to show that for any object X in a topos $\mathcal{C}$ there is an isomorphism of posets $\tau_{-1}(\mathcal{C}_{/X}) \cong Hom_{\mathcal{C}}(1, PX)$, where τ_{-1} is the -1-truncation of the category, however the definition of a power object involves a certain adjunction which cannot be deduced easily. The next step is to actually carve out a definition of a power object from this initial observation and try to understand its ramifications on higher categories.

3. **Homotopy Type Theory**: Vladimir Voevodsky introduced the notion of *univalence* in his study of homotopy type theory. In [GK12] David Gepner and Joachim Kock used the notion of univalence to define *univalent families* in ∞-categories (a particular model of a higher category). Based on that, they proved that in presentable ∞-categories

with universal colimits the posets of bounded local classes and univalent families are isomorphic. On the other side, a presentable ∞-category with universal colimits is an ∞-topos if and only if every bounded local class has an object classifier. The combination of these two results suggests a connection between ∞-topoi and univalence, as it gives us a connection between univalent families and object classifiers. Therefore, the final approach is to study the notion of univalence in higher category theory and see how it relates to object classifiers and other notions related to higher topos theory. Concretely, we can show the following:

Lemma. *In any 1-category $\mathcal{C}$ with subobject-classifier Ω, there is an isomorphism of posets between the poset of mono univalent maps and subobjects of Ω i.e. $Sub(\Omega) \cong Hom(\Omega, \Omega)$.*

This leads us to three questions:

Question 3. We know that in any topos there is a correspondence between left-exact localizations and a specific subset of Hom(Ω,Ω) (see Chapter VII Cor. 7 of [MM92]). In light of the previous result, characterize univalent maps which correspond to left-exact localizations. Then generalize that characterization to the higher categorical setting. The hope is that it could help understand left-exact localizations of higher topoi.

Possible Approach 3. Considering the poset isomorphism the main idea in solving this question should be to phrase the conditions for left-exact localizations purely in terms of poset structure and then use the isomorphism to transfer them to the side of univalent maps.

Question 4. Characterize non-mono univalent maps in a topos (the hope is that all univalent maps are mono as it would complete the lemma result and is true for basic example, however it might not be true).

Possible Approach 4. In order to prove it we can use the epi-mono factorization. We can show that if u is univalent and $u = ip$ is the epi-mono factorization, then i (the mono part) is also univalent and so we get a lot of information about i using the characterization above. The next step is to understand the epi part using the information we got from the mono part.

Question 5. One special special corollary of mentioned result is that the map $1 \to \Omega$ is univalent. Now, find the right conditions on a univalent map of the form $1 \to A$ in a category, such that A is a subobject classifier.

Possible Approach 5. We can show that in a topos the map $1 \to \Omega$ is the maximal mono univalent map (using the standard poset structure). So, the most reasonable step is to show the opposite is true: If the poset of mono univalent maps has a maximum, then this map is the subobject-classifier.

Note that a negative answer to the second question will subsequently lead to the following question:

Question 6. Try to find a characterization of non-mono univalent maps (this would then require more than the subobject classifier). This question would be interesting from the opposite point of view as it might help construct new invariants of topoi.

The hope is that each approach will lead to individual results which can be compared to each other and used to make the correct definitions. Based on the combination of the approaches the following questions can be proposed:

Question 7. Let us assume we managed to answer question 1 successfully. Then this would open the possibility to a more profound and precise connection between univalent maps and object-classifiers. The correspondence of Gepner and Kock could be improved in two ways:

(i) Gepner and Kock's result work only for higher categories with universal colimits. We can remove this condition and study the correspondence for general higher categories, because the the answer to question 1 tells us that this assumption is not necessary for our definition of a higher topos.

(ii) The correspondence of Gepner and Kock is an isomorphism of posets of objects (object classifiers). However, if we now take a collection of simplicial objects (the collection of object-classifying ∞-category objects) then it is very reasonable to assume that it has higher structure. The next step would then be to understand this higher structure and extend the isomorphism of posets to an isomorphism of simplicial sets.

Proving the existence of such a strong correspondence would be an excellent step in showing how fundamental the notion of univalence is in topos theory, as it would mean that a higher topos can be defined purely in terms of its univalent maps.

Question 8. Let us assume we have found a satisfying answer to questions 4 and 5, which means we can characterize power objects in terms of truncations and the subobject classifier in terms of univalent maps. This would directly result in a definition of elementary topoi in the language of higher categories. Now impose these conditions on a higher category and study its properties. In particular, show that, if presentable, it is a higher topos.

Finally, if all these lead to results then I can start to tackle the following question:

Question 9. Propose a definition of a higher elementary topos and show it a has desirable properties. Any proposed definition should satisfy at least the following conditions:

1. It is finitely complete and cocomplete
2. Any higher topos should be an elementary higher topos
3. On the other side any presentable higher elementary topos is a higher topos
4. The zero truncation of every higher elementary topos is an elementary topos
5. Any model of the category of spaces should be a higher elementary topos generated by the final object (which exists by 1)

Given that definition, find necessary conditions which makes it intuitively equivalent to a category of spaces.

If we can actually find such a definition then we can try to understand the higher category of elementary higher topoi and what properties it has inside the higher category of higher categories. We could also try to understand its connections to homotopy type theory. For example, we could check if every elemementary higher topos is a model of the axioms of homotopy type theory.

One problem in any such approach is that many definitions of higher categories themselves depend on some model of spaces. For example, the definition of quasi-categories and complete Segal spaces depend on simplicial sets. And so, defining spaces using such categories would be circular. However, there are definitions of higher categories which can be phrased independent of any notion of set or space, for example model categories, but it would be helpful to have other and maybe stronger definitions. So, one other question (a side question) which arises from this explanation is to find other notions of higher categories equivalent to the existent ones, which can phrased in a language independent of sets. This gives us following questions, which could be helpful in our quest:

Question 10. Propose new definitions for higher categories independent of the notion of spaces, which give us different approaches to the notion of higher categories.

In my approaches, I will mainly rely on the following sources:

- Higher Topos Theory, Jacob Lurie ([Lu09])
- Toposes and homotopy toposes, Charles Rezk ([Re05])
- Sheaves in Geometry and Logic, Saunder MacLane and Ieke Moerdijk ([MM92])
- Univalence in locally cartesian closed ∞-categories, David Gepner and Joachim Kock ([GK12])

Finally, I wanted to point to some of the main applications of this idea. Certainly a suitable definition of the category of spaces would be a very satisfying result from a foundational point of view. However, it might also have some more concrete benefits. As we saw in the begining there are several models of spaces, each one of them constructed in a very particular manner in order to answer questions the previous model of spaces could not answer. Now, a theory of spaces could provide us with a much more direct way to construct new models which can help us in the study of homotopy theory. It might even reveal some aspects of the nature of spaces which previously went unnoticed, similar to the development of set theory in the late nineteenth century. At last, in light of the recent advancements of homotopy type theory, elementary higher topoi might contribute to advancements in the foundations of mathematics as a whole and serve as an application of homotopy theory to mathematical logic.

References

[AW09] Awodey, Steven and Warren, Michael A.. *Homotopy theoretic models of identity types.* Math. Proc. Camb. Phil. Soc., 146:45-55, 2009.

[GK12] Gepner, David and Kock, Joachim. *Univalence in locally cartesian closed ∞-categories.* arXiv:1208.1749, 2012.

[Lu09] Lurie, Jacob. *Higher Topos Theory.* Princeton University Press, 2009.

[ML98] MacLane, Saunder. *Categories for the working mathematician.* Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York.

[MM92] MacLane, Saunder and Moerdijk, Ieke *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer-Verlag, Berlin and New York, 1992.

[Re05] Rezk, Charles. *Toposes and homotopy toposes.* 2005.

[TV05] Toen, Bertrand and Vezzosi, Gabriele. *Homotopical algebraic geometry i: topos theory.* Advances in Mathematics, 193(2):257 - 372, 2005.

[Vo10] Voevodsky, Vladimir. *Univalent foundations project.* 2010.