Preliminary Exam: What are Spaces ?

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Modern Homotopy Theorist: ∞-Groupoid

Example:

Let us see where this question has been answered successfully:
R -Modules

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Fundamental Question

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Fundamental Question:

Which collection is worthy of being called a collection of spaces ?
What conditions should this collection satisfy ?

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Precise Fundamental Question:

Which conditions should we impose on a higher category so that we can reasonably call it a higher category of spaces?

This is the question I want to answer.

Fundamental Question: Precise Version (Explicit Form)

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Throughout this talk we will see recent advancements which will help us phrase other conditions which we will try to combine.
In order to achieve this goal, the first step will be to define a more general higher category, which we call a *Higher Elementary Topos* (for reasons which will be mentioned). Then, we will try to figure out the right restrictions to make it a category of spaces.

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In order to achieve this goal, the first step will be to define a more general higher category, which we call a *Higher Elementary Topos* (for reasons which will be mentioned). Then, we will try to figure out the right restrictions to make it a category of spaces.
So, let's concentrate on the first step for now.

Set Theory

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The notion of set was axiomatized in 1908 by Ernst Zermelo [Ze08] and completed by Abraham Fraenkel in 1922 [Fr22] and along with the axiom of choice is now known as **ZFC**.

Categories

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Fundamental Question, Set-Theoretic Version:

Which conditions do we have to impose on a category in order to be a category of sets in the sense of ZFC.

Elementary Topos

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Elementary Topos Definition (b) (Def Page 167 [MM92]):

Alternatively, accepting the notion of a set, we can define it as a cartesian closed category with equalizer and a subobject classifier.

Subobject classifier (a)

Subobject classifier Definition (a) (Def Page 163 [MM92])

In a category $\mathcal{C}$ an object Ω and a map $u : 1 \to \Omega$ is called a subobject classifier if for every object C and mono $i : D \to C$ there exists a map $f : C \to \Omega$ such that the following square is a pullback square:

$$\begin{tikzcd} D \arrow[r, "!_D"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, hook, "u"] \\ C \arrow[r, dashed, "\exists!\ f"] & \Omega \end{tikzcd}$$

Subobject classifier (b)

Subobject classifier Definition (b) (Page 161 [MM92])

Alternatively, accepting the notion of set, let us define Sub(X) to be equivalence classes of subobjects of X. Then $1 \to \Omega$ *is a subobject classifier, if the natural map*

$$Hom(X, \Omega) \to Sub(X)$$

coming from the pullback is an isomorphism.

Power object (a)

Power object Definition (a) (Def Page 163 [MM92])

Let $\mathcal{C}$ be a category and A and object. The power object is a an object PA and map $e : A \times PA \to \Omega$ such that it satisfies following condition:
For every object B in $\mathcal{C}$ and every map $f : B \to PA$, there exists a unique map $\hat{f} : A \times B \to \Omega$, such that the following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[d, "1_A \times f"'] \arrow[dr, dashed, "\exists ! \hat{f}"] & \\ A \times PA \arrow[r, "e"] & \Omega \end{tikzcd}$$

Power object (b)

Power object Definition (b) (Def Page 161 [MM92])

Alternatively, accepting the notion of a set, we can say that a power object of A is an object PA plus a map $e : A \times PA \to \Omega$ such that for every B, we get an adjunction:

$$Hom(A \times B, \Omega) \cong Hom(B, PA)$$

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Classic Example: Category of Sets

The classic category of sets is an elementary topos. In this category $\Omega = \{0, 1\}$. Note that there is a bijection between subsets of a set A and characteristic maps $\chi : A \to \Omega$. Also, the power object is the just the power set i.e. the set of all subsets. The map $e : A \times PA \to \Omega = \{0, 1\}$ is the map which takes $a \in A$ and $B \subset A$ to 0 if $a \notin B$ and 1 if $a \in B$

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Classification of Set Theories (Sec. VI.10 [MM92]):

A category satisfies ZFC (i.e. the axioms of ZFC can be interpreted in the category in a way such that its objects and morphisms satisfy those interpretations) if and only if the category is a well-pointed elementary topos with a natural number object and satisfying the choice condition.

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Choice Condition (Page 342 [MM92])

Satisfying following senctence:

$\forall y(\forall x \in y(x \neq \emptyset) \to \exists f : y \to \cup y(\forall x \in y(f(x) \in x)))$

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The Quest to find Higher Elementary Topos

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1. Higher Topos Theory
2. Homotopy Type theory

Grothendieck Topos

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Presentable Category (Def. 1.17 [AR94])

A presentable category is a category with small colimits and a set of λ-presentable objects such that every object is a λ-directed colimit of those presentable objects (for some regular cardinal λ). More concretely, a presentable category is a localization of $\mathcal{PSh}(\mathcal{C})$ of some small category $\mathcal{C}$.

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Grothendieck Topos (Def III.4.3 and Prop App.4 [MM92])

A Grothendieck Topos is a presentable elementary topos. More concretely, a Grothendieck topos is a left-exact localization of $\mathcal{P}Sh(\mathcal{C})$ of some small category $\mathcal{C}$. Alternatively, it is of the form $\mathcal{S}hv(\mathcal{C}, J)$, where J is a Grothendieck site on some small category $\mathcal{C}$.

∞-Categories

Several people have generalized this definition from classical category theory to higher category theory. I will mention three of those in the context of ∞-categories, in the sense of Joyal and Lurie. Let me first give the definition:

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∞-Category (Def 1.2.2.4 [Lu09]):

An ∞-category is a simplicial set S such that for every $n \geq 2$ and $0 < i < n$ the following diagram has a lift.

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, tail, "i"'] & S \\ \Delta^n \arrow[ur, dashed, "\exists \hat{f}"'] & \end{tikzcd}$$

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An ∞-topos is a left-exact accessible localization of $\mathcal{P}Sh(\mathcal{C})$, the category of presheaves into spaces, where $\mathcal{C}$ is a small ∞-category.

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The definition of Lurie is highly dependent on the notion of presentability and as such is not very useful from my point of view. Rezk's definition might be much more suitable in this regard.

∞-Topos (Rezk-Descent Version):

∞-Topos (Theorem 6.9 [Re05]):

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Descent (informal definition) (6.5 [Re05]):

Let $\mathcal{C}$ be a complete ∞-category and let $\mathcal{C}at_\infty$ be the (large) ∞-category of ∞-categories. Then, we have a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}at_\infty$, defined by $\mathcal{F}(U) = \mathcal{C}_{/U}$. Now, $\mathcal{C}$ satisfies descent if $\mathcal{F}$ carries colimits to limits.

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∞-Topos (Theorem 6.1.6.8 [Lu09]):

A presentable ∞-category $\mathcal{C}$ is an ∞-topos if colimits in $\mathcal{C}$ are universal and for a sufficiently large cardinal κ there exists a object classifier for all relatively κ-presentable object i.e. there exists Ω_∞^κ such that for every κ-presentable object $C \in \mathcal{C}$ there is an equivalence of ∞-categories $\mathcal{C}^\kappa_{/C} \simeq Hom_{\mathcal{C}}(C, \Omega^\kappa_\infty)$

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Universal Colimit (Page 527 [Lu09]):

A complete ∞-category $\mathcal{C}$ has universal colimits if for every morphism $f : T \to S$ the pullback functor $f^* : \mathcal{C}_{/S} \to \mathcal{C}_{/T}$ preserves colimits.

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Bounded Local Classes (Conditions in Prop 6.1.6.3 [Lu09]):

The mentioned collections of suitable objects (of κ-presentable objects) are called bounded local classes. The collection of all of them form a poset with inclusion.

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Homotopy type theory is a type theory which was introduced by Vladimir Voevodsky [Vo10] and separately by Steve Awodey and Michael Warren [AW09] as a foundation more suitable to homotopy theory, because it treats equality and homotopy in a similar way, which is exactly what homotopy theorists would expect.

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Introducing homotopy type theory will take us too far astray from our current discussion. The important part I am concerned with, is the notion of univalence introduced by Voevodsky which seems to have unexpected connections to higher topoi.

Univalent Fibrations

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Voevodsky, Kapulkin and Lumsdaine introduced the notion of a univalent fibration (Def 3.2.10 [KLV12]).
This notion has lead other mathematician to work on stronger relations between higher topos theory and homotopy type theory.
We will focus on Gepner and Kock's work in [GK12].

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4. Combined Approach

First Approach: Higher Topos Theory

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Higher Topos Theory Conjecture:

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This conjecture suggests the following approach:

∞-Topos Approach:

Study ∞-categories which satisfy descent and/or the object classifier condition.

Concrete Questions:

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HT Question 1 (∞-category object classifier)

Propose a definition for an object-classifying ∞-category object in an ∞-category $\mathcal{C}$. This should be simplicial object $\Omega_\bullet \in \mathcal{C}_\Delta$ satisfying the right lifting conditions in the category, such that for every object X, $\mathcal{C}_{/X} \simeq Map_{\mathcal{C}}(X, \Omega_\bullet)$. Then prove that a presentable ∞-category is an ∞-topos if and only if it has object-classifying ∞-category objects for suitable classes of maps.

Generalization

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Elementary Topos Theory Conjecture:

For any reasonable definition of a higher elementary topos, the 0-truncation of a higher elementary topos is an elementary topos.

This conjecture suggests the following approach:

Elementary Topos Approach:

Find a homotopical definition of an elementary topos and then try to generalize it.

Concrete Question:

Remember that an elementary topos involves three components: limits, subobject classifier and power objects. Limits already generalize and for subobject classifier we will see that they might have a connection to univalent maps, which leaves power objects.

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ET Question 1: Power Objects from the homotopical perspective

It is easy to see that for any topos $\mathcal{E}$, we have following isomorphism of posets:

$$\tau_{-1}(\mathcal{E}_{/C}) \cong Hom_{\mathcal{E}}(1, PC)$$

However, we cannot define power objects this way as the adjunction of power sets does not follow. Add a precise, correct condition to get the other side.

Univalent Maps

We go to the third approach.
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Univalent Maps (2.4 [GK12])

Let $\mathcal{C}$ be an ∞-category. Then for every $p : X \to S$ we have the following two sheaves of spaces on the ∞-category $\mathcal{C}_{/S\times S}$. The first takes $(f,g) : C \to S \times S$ to $Hom_{\mathcal{C}_{/S\times S}}((f,g),(1_S,1_S))$ and the second takes $(f,g) : C \to S \times S$ to $Eq_{/C}(f^*X, g^*X)$ (the subspace of equivalences), where f^*X and g^*X are the pullbacks. p is univalent if the natural map from $Hom_{\mathcal{C}_{/S\times S}}((f,g),(1_S,1_S)) \to Eq_{/C}(f^*X, g^*X)$ is a weak equivalence.

Univalent Maps (Explanation)

This means that this map has to be an equivalence:

$$Hom_{\mathcal{C}_{/S \times S}}((f, g), (1_S, 1_S)) \longrightarrow Eq_{/C}(f^*X, g^*X)$$

Univalent Maps (Explanation)

This means that this map has to be an equivalence:

$$Hom_{\mathcal{C}_{/S\times S}}((f,g),(1_S,1_S)) \longrightarrow Eq_{/C}(f^*X, g^*X)$$

$$\begin{tikzcd} C \arrow[rr, dashed, "h"] \arrow[dr, "{(f,g)}"'] & & S \arrow[dl, "{(1_S,1_S)=\Delta}"] \\ & S \times S & \end{tikzcd} \longrightarrow \begin{tikzcd} f^*X \arrow[rr, dashed, "\cong", "\hat{h}"'] \arrow[dr] & & g^*X \arrow[dl] \\ & C & \end{tikzcd}$$

Univalent maps and Bounded Local Classes

Using this definition, Gepner and Kock proved the following:

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Univalent Maps and Bounded Local Classes (Prop 2.9 [GK12])

In a presentable ∞-category with universal colimits, there is an isomorphism of posets between the poset of univalent maps and the poset of bounded local classes.

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Univalent Maps and Bounded Local Classes (Prop 2.9 [GK12])

In a presentable ∞-category with universal colimits, there is an isomorphism of posets between the poset of univalent maps and the poset of bounded local classes.

Note that for $p : C \to D$ and $q : A \to B$ univalent, $q \leq p$ if we have following diagram:

$$\begin{tikzcd} A \arrow[r, dashed, "\exists ! f"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "p"] \\ B \arrow[r, dashed, "\exists ! g"] & D \end{tikzcd}$$

Univalent Maps and Topoi

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This suggests an interesting possible connection between topoi and univalent maps, as we already showed that ∞-topoi classify all bounded local classes. So, it seems worthwhile to understand univalent maps in different context and maybe even classify them.

Univalent Maps and Topoi

This suggests an interesting possible connection between topoi and univalent maps, as we already showed that ∞-topoi classify all bounded local classes. So, it seems worthwhile to understand univalent maps in different context and maybe even classify them. Based on this observation, I started to study univalent maps in a framework more familiar and with many known results:

$$1 - \mathit{Categories}$$

Univalence in 1-Categories

Univalence in 1-Categories

Let restrict our attention to 1-Categories. From the following diagram:

$$\begin{tikzcd} B \arrow[rr, dashed, "h"] \arrow[dr, "(f,g)"'] & & C \arrow[dl, "(1_S,1_S)=\Delta"] \\ & A & \end{tikzcd}$$

We have the following:

Univalence in 1-Categories

Let restrict our attention to 1-Categories. From the following diagram:

$$\begin{tikzcd} B \arrow[rr, dashed, "h"] \arrow[dr, "{(f,g)}"'] & & C \arrow[dl, "{(1_S,1_S)=\Delta}"] \\ & A & \end{tikzcd}$$

We have the following:

$$Hom_{\mathcal{C}_{/S\times S}}((f,g),(1_S,1_S)) = \emptyset \text{ if and only if } f \neq g$$

and

$$Hom_{\mathcal{C}_{/S\times S}}((f,g),(1_S,1_S)) = \{*\} \text{ if and only if } f = g$$

Univalence in 1-Catgories

Univalence in 1-Catgories

If p is univalent this implies the following (for $f \neq g$):

$$Eq_{/C}(f^*X, g^*X) = \emptyset$$

and

$$Eq_{/C}(f^*X, f^*X) = \{1_{f^*X}\}$$

Simple Proposition

Based on this result we can prove the following:

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Basic Proposition

In a complete 1-category with subobject-classifier Ω *, there is a bijection between the poset of mono univalent maps and the poset of subobjects of the subobject-classifier* $(Hom(\Omega, \Omega) \cong P(\Omega))$.

Simple Proposition

Based on this result we can prove the following:

Basic Proposition

In a complete 1-category with subobject-classifier Ω , there is a bijection between the poset of mono univalent maps and the poset of subobjects of the subobject-classifier $(Hom(\Omega, \Omega) \cong P(\Omega))$.

Proof.

Proof of the Proposition. $\square$

Concrete Questions 1

Based on this we can phrase the following concrete questions:

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HoTT Question 1

We know that in any topos there is a correspondence between left-exact localizations and a specific subset of Hom(Ω,Ω) (see Chapter VII Cor. 7 of [MM92]). In light of the previous result, characterize univalent maps which correspond to left-exact localizations. Then generalize that characterization to the higher categorical setting. The hope is that it could help understand left-exact localizations of higher topoi.

Possible Way of Solving HoTT Question 1

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Proposed Solution to HoTT Question 1

One reasonable way to solve this question is to phrase the three conditions for a subobject of Ω to correspond to a left-exact localization purely in the language of the poset structure. Then we translate those conditions back to the world of univalent maps.

Possible Way of Solving HoTT Question 1

Proposed Solution to HoTT Question 1

One reasonable way to solve this question is to phrase the three conditions for a subobject of Ω to correspond to a left-exact localization purely in the language of the poset structure. Then we translate those conditions back to the world of univalent maps.

The actually interesting step would be to see whether a univalent map in a higher category satisfying those conditions has a connection to left-exact localizations of higher categories or not.

Concrete Question 2

HoTT Question 2

Prove that all univalent maps in 1-categories are mono.
In case this conjecture is wrong, try to find a characterization of non-mono univalent maps. This would require more than the subobject classifier and could help construct new invariants of topoi.

Possible Way of Solving HoTT Question 2

Proposed Solution to HoTT Question 2

It is not hard to show that if u is univalent and $u=ip$ is any factorization such that i is mono then i is univalent as well.

Proof.

Proof of the Lemma □

Possible Way of Solving HoTT Question 2

Proposed Solution to HoTT Question 2

Combining the lemma above with the fact that every topos has a epi-mono factorization we get the following (for a map $u : A \to B$ with image I):

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Proposed Solution to HoTT Question 2

Combining the lemma above with the fact that every topos has a epi-mono factorization we get the following (for a map $u : A \to B$ with image I):

$$\begin{tikzcd} A \arrow[d, two heads, "p"] & \\ I \arrow[r, hook, "!"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, hook, "t"] \\ B \arrow[r, hook, "f"] & \Omega \end{tikzcd}$$

Possible Way of Solving HoTT Question 2

Proposed Solution to HoTT Question 2

The hope is that we can prove p is iso, which seems to be true in some easy cases, but turns out to be rather tricky for the general case.

Concrete Question 3

HoTT Question 3

One special special corollary of mentioned proposition is that in any category with subobject classifier the map $1 \to \Omega$ is univalent. Find the right conditions on a univalent map of the form $1 \to A$ in a category, such that A is a subobject classifier.

Possible Way of Solving HoTT Question 3

Proposed Solution to HoTT Question 3

This goes back to the poset structure. The map $1 \to \Omega$ *is the maximal mono univalent map as every other such map is its pullback. So, this suggests asking this question: Assume that the poset of univalent maps in a category has a maximum element. Prove this is the subobject-classifier.*

Combined Questions

The three separate approaches mentioned lead to some combined questions:

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The first question goes back to part 1 and 3. Assuming we can prove *HT Question 1 (∞-category object classifier)*, this would open the way towards a better connection with univalent maps. Concretely, we can pose the following question:

Combined Question 1

Combined Question 1

The result of Gepner and Kock could be improved in two ways:

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1. *Gepner and Kock assume colimits are universal. A positive answer to HT question 1 would allow us to remove this condition and prove the result for more general categories.*

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The result of Gepner and Kock could be improved in two ways:

1. *Gepner and Kock assume colimits are universal. A positive answer to HT question 1 would allow us to remove this condition and prove the result for more general categories.*
2. *Gepner and Kock construct an isomorphism of posets. However, now our objects are simplicial (the collection of object-classifying ∞-category objects) and so it is very reasonable to assume that it has higher structure. The next step would then be to understand this higher structure and extend the isomorphism of posets to an isomorphism of simplicial sets.*

Combined Question 2

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Combined Question 2

With the newfound definitions of power objects and subobject classifier, we can define topoi in a completely homotopical language. Now, generalize this definition to the higher categorical level and study the resulting object. In particular, show that if we add presentability we get an ∞-topos.

Conditions on an Elementary Higher Topos

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1. It is finitely complete and cocomplete

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Conditions on an Elementary Higher Topos

1. It is finitely complete and cocomplete
2. Any higher topos should be an elementary higher topos
3. On the other side any presentable higher elementary topos is a higher topos
4. The zero truncation of every higher elementary topos is an elementary topos
5. Any model of the category of spaces should be a higher elementary topos generated by the final object (which exists by 1)

What to do after that ?

The next step would be to study the newfound definition and its connection to other already known results.

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1. First and foremost, having a definition, we can start finding the right condition to get a reasonable definition for a category of spaces, which was the main aim of the talk. One conditions should be generation by the final object. However, there are probably more and the next step would be to find those in a precise manner.

What to do after that ?

The next step would be to study the newfound definition and its connection to other already known results.

1. First and foremost, having a definition, we can start finding the right condition to get a reasonable definition for a category of spaces, which was the main aim of the talk. One conditions should be generation by the final object. However, there are probably more and the next step would be to find those in a precise manner.
2. We can also study the higher category of higher elementary topoi and study its different properties, in particular if it is (co)complete. We can also study embeddings and surjections as they already have been very nicely characterized in the category of elementary topoi (Page 366 [MM92]).

What to do after that ?

3. There has been considerable effort in showing that every ∞ -topos is a model for the axioms of homotopy type theory (see [AW09], [GK12] and [KLV12]). We can try to build on that by showing that every elementary higher topos is a model for homotopy type theory.

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3. There has been considerable effort in showing that every ∞ -topos is a model for the axioms of homotopy type theory (see [AW09], [GK12] and [KLV12]). We can try to build on that by showing that every elementary higher topos is a model for homotopy type theory.
4. More interestingly, we can work on the opposite question: Is every model of homotopy type theory an elementary higher topos and, if not, how different are they. In that regard we can again draw an analogy to elementary topoi. John Bell constructs a type theory which he names Local Set Theory and proves that every model of this type theory is an elementary topos (Theorem 3.37 [Be88]) so our effort would be a generalization of that.

Applications

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1. This result would be theoretically very satisfying as it gives a classification of all models of spaces. It would settle questions about what can be proven in homotopy theory and what the main object of study in homotopy theory is.

Applications

Finally, we should see why it is a good idea to think about higher elementary topoi:

1. This result would be theoretically very satisfying as it gives a classification of all models of spaces. It would settle questions about what can be proven in homotopy theory and what the main object of study in homotopy theory is.
2. It might help us construct new models for the category of spaces, which could help us get a better understanding of homotopy theory. It might also lead to the construction of non-standard models which have unexpected properties, as suggested by Shulman ([Sh12]).

Applications

3. One other step to go from here is to get a more axiomatic approach to higher category theory. Currently, there are several models of higher category theory relying on some notion of space. A theory of spaces could help us get a clear theory of higher categories.

Thank you note

$\mathfrak{T}$hank $\mathfrak{Y}$ou for your time.

I also thank $\mathfrak{C}$harles $\mathfrak{R}$ezk and $\mathfrak{M}$att $\mathfrak{A}$ndo for all their help and support.

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Proof of Proposition

Basic Proposition

In a complete 1-category with subobject-classifier Ω , there is a bijection between the poset of mono univalent maps and the poset of subobjects of the subobject-classifier ($Hom(\Omega, \Omega) \cong P(\Omega)$).

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In order to prove we first have to prove two lemmas:

Lemma 1

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Let us have to following pullback square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "p"] \\ B \arrow[r, "i"] & D \end{tikzcd}$$

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Then we have the following two facts:

1. *If q is univalent then i is mono*
2. *If p is univalent and i is mono then q is univalent.*

Proof of Lemma 1 (Part 1)

Proof

1. *Let q be univalent. Also, let $f, g : E \to B$ such that if=ig. This gives us the following diagrams:*

Proof of Lemma 1 (Part 1)

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1. *Let q be univalent. Also, let $f, g : E \to B$ such that $if{=}ig$. This gives us the following diagrams:*

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "p"] \\ E \arrow[r, "f"] & B \arrow[r, "i"] & D \end{tikzcd}$$

and

$$\begin{tikzcd} g^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "p"] \\ E \arrow[r, "g"] & B \arrow[r, "i"] & D \end{tikzcd}$$

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But if=ig and so $f^*A \cong g^*A$.

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Now q is univalent and $f^*A \cong g^*A$ and so $f = g$ and we are done.

$\square$

Proof of Lemma 1 (Part 2)

Proof

2. *Let us assume p is univalent and i is mono. We show that q satisfies:*

Proof of Lemma 1 (Part 2)

Proof

2. *Let us assume p is univalent and i is mono. We show that q satisfies:*
   1. *For any $f, g : E \to B$, $f^*A \cong g^*A$ implies $f = g$*

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2. *Let us assume p is univalent and i is mono. We show that q satisfies:*
   1. For any $f, g : E \to B$, $f^*A \cong g^*A$ implies $f = g$
   2. For any $f : E \to B$

$$Eq_{/E}(f^*A, f^*A) = \{1_{f^*A}\}$$

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So, we will show both separately. Note, that for the second part we don't even need i to be mono.

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But p is univalent and so $if = ig$.

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But p is univalent and so $if = ig$.
But i is mono and so $f = g$.

Proof of Lemma 1 (Part 2)

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2. *Now we show the second part.*

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But p is univalent and so $Eq_{/E}(f^*A, f^*A) = \{1_{f^*A}\}$.
And so we are done

$\square$

Corollary

Note that we get the following corollary:

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Corollary

If p is univalent, then q is univalent if and only if i is mono.

Lemma 2

Lemma 2

If $\mathcal{C}$ is a complete category with subobject classifier $t : 1 \to \Omega$ then t is univalent

Proof of Lemma 2

Proof

Note that by definition the natural transformation from $Hom(-, \Omega) \to Sub(-)$ *taking a map* $f : X \to \Omega$ *to the mono map* $i : f^*1 \to X$ *is an natural equivalence of presheaves of sets. Note that Sub(X) is the set of equivalence classes of subobjects of X.*

Proof of Lemma 2

Proof

Note that by definition the natural transformation from $Hom(-, \Omega) \to Sub(-)$ *taking a map* $f : X \to \Omega$ *to the mono map* $i : f^*1 \to X$ *is an natural equivalence of presheaves of sets. Note that Sub(X) is the set of equivalence classes of subobjects of X. The fact that the natural equivalence is injective implies that if* $f^*1 \cong g^*1$ *then* $f = g$*, which shows one part.*

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And so we have $jh = j = j1_{f^*1}$, but j is mono and so $h = 1_{f^*1}$ and so we proved that t is univalent. □

Proposition

Finally, we can state and prove the proposition

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Basic Proposition

In a complete 1-category with subobject-classifier Ω *, there is a bijection between the poset of mono univalent maps and the poset of subobjects of the subobject-classifier* $(Hom(\Omega, \Omega) \cong P(\Omega))$.

Proof of Proposition

Proof

First, let us mention the poset structure of $P(\Omega)$ *and* $\mathcal{U}$ *(which we take to be the poset of univalent mono maps).*

Proof of Proposition

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First, let us mention the poset structure of $P(\Omega)$ and $\mathfrak{U}$ (which we take to be the poset of univalent mono maps).

If $i : A \to \Omega$ and $j : B \to \Omega$ are two subobjects of Ω then $A \leq B$ if there is a mono map $k : A \to B$ such that $jk = i : A \to B$.

$$\begin{tikzcd} A \arrow[r, hook, "k"] \arrow[dr, hook, "i"'] & B \arrow[d, hook, "j"] \\ & \Omega \end{tikzcd}$$

Proof of Proposition

Proof

If $p : C \to D$ and $q : A \to B$ are univalent then $q \leq p$ if there are maps $f : B \to D$ and $g : A \to C$ such that the following square is a pullback square:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "p"] \\ B \arrow[r, "f"] & D \end{tikzcd}$$

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Note that based on Lemma 1, f and g will be necessarily mono.

Proof of Proposition

Proof

Let me remind you that $\mathcal{U}$ is actually a poset. Let $f, f' : B \to D$ be two maps such that we have the following pullback squares:

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "p"] \\
B \arrow[r, "f"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "g'"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "p"] \\
B \arrow[r, "f'"] & D
\end{tikzcd}
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But p is univalent and $A \cong A$, and so $f = f'$.

Proof of Proposition

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$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "t"] \\ B \arrow[r, "f"] & \Omega \end{tikzcd}$$

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But p and t are univalent and so f is mono and so $B \to \Omega$ is a subobject.

Proof of Proposition

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B \arrow[r, "f"] & D \arrow[r, "h"] & \Omega
\end{tikzcd}$$

Proof of Proposition

Proof

This gives me the following diagram:

$$\begin{tikzcd} B \arrow[rr, hook, "f"] \arrow[dr, hook, "h \circ f"'] & & D \arrow[dl, hook, "h"] \\ & \Omega & \end{tikzcd}$$

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This gives me the following diagram:

$$\begin{tikzcd} B \arrow[rr, hook, "f"] \arrow[dr, hook, "h\circ f"'] & & D \arrow[dl, hook, "h"] \\ & \Omega & \end{tikzcd}$$

and so we get an order preserving morphism.

Proof of Proposition

Proof.

Now we show that this map has an inverse.

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Let $f : B \to \Omega$ be a subobject (i.e. f is mono) and let us have this diagram:

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Now f is mono and t is univalent and so p is univalent.

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Now f is mono and t is univalent and so p is univalent. This map is exactly the inverse of the first one described. So, we get an isormorphism of posets. $\square$

Conclusion of Proof

We effectively proved that the poset $P(\Omega)$ and $\mathcal{U}$ of univalent mono maps are the same.

Back to the Basic Proposition

Proof of Lemma

Proof

Assume $u : A \to B$ is univalent and u=ip where $i : I \to B$ is mono. Note that any pullback of i is mono. Let $f : X \to B$, if $h \in Eq_{/X}(f^*I, f^*I)$, then we have the following diagram:

Proof of Lemma

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$$\begin{tikzcd}
f^*I \arrow[rr, dashed, "h"] \arrow[dr, hook, "j"'] & & f^*I \arrow[dl, hook', "j"] \\
& X &
\end{tikzcd}$$

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f^*I \arrow[rr, dashed, "h"] \arrow[dr, hook, "j"'] & & f^*I \arrow[dl, hook', "j"] \\
& X &
\end{tikzcd}$$

And so we have $jh = j = j1_{f^*X}$, but j is mono and so $h = 1_{f^*X}$ and so we proved that i satisfies the first condition.

Proof of Lemma

Proof

For the second part, let $f, g : X \to B$ *such that* $f^*I \cong g^*I$. *We can pull it back another time to get these diagrams:*

Proof of Lemma

Proof

For the second part, let $f, g : X \to B$ such that $f^*I \cong g^*I$. We can pull it back another time to get these diagrams:

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "p"] \\
f^*I \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & I \arrow[d, "i"] \\
X \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
g^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "p"] \\
g^*I \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & I \arrow[d, "i"] \\
X \arrow[r, "g"] & B
\end{tikzcd}$$

Proof of Lemma

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From $f^*I \cong g^*I$ we get that the top squares are isomorphic and so we in the following isomorphic diagrams:

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$$\begin{tikzcd} f^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "u"] \\ X \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} g^*A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "u"] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

we have $f^*A \cong g^*A$. But u is univalent and so $f = g$ and so we are done. $\square$

Conclusion of Lemma

We proved that if u is univalent and $u = ip$ where i is mono then i is univalent.

Back to the Lemma