We review basic definitions regarding quasi-categories. Quasi-categories are the most common model of higher categories, highly popularized by the works of Joyal ([Jo08] and [Jo09]) and Lurie ([Lu09]). Because they are based on simplicial sets, we can effectively use combinatorial arguments to prove important results. The goal of this talk is to give a broad overview of quasi-categories and illustrate how the definition allows us to do category theory. The talk will follow mostly along the lines of the lecture notes written by Charles Rezk ([Re16]).

**Review on Simplicial Sets**

**Definition 1.1.** Let $\Delta$ be the category of finite ordinals and order preserving maps. Concretely the objects are ordinals $[0], [1], [2], ...$

**Remark 1.2.** There are maps $d^i : [n - 1] \to [n]$, which skips the $i$-th element and $s^i : [n + 1] \to [n]$, which maps $i$ and $i + 1$ to the same element. It is a fact about ordinals that every maps is generated by these maps.

**Definition 1.3.** A simplicial set $X$ is a functor $X : \Delta^{op} \to \text{Set}$. A simplicial map is just a natural transformation of functors. This gives us the category of simplicial sets. Essentially this category is just the category of presheaves on the category $\Delta$, i.e. Fun$(\Delta^{op}, \text{Set})$.

**Notation 1.4.** We usually denote the image of $n$ under the simplicial set $X$ as $X_n$, instead of $X([n])$.

Thus, a simplicial set can be thought of as a collection of sets $X_0, X_1, X_2, ...$ which has boundary maps and face maps between them, which is often depicted as follows:

$$
\vcenter{\hbox{\begin{array}{cccccccc}
X_0 & \xrightarrow{d^0} & X_1 & \xrightarrow{d^1} & \cdots & X_{n-1} & \xrightarrow{s^{n-1}} & X_n
\end{array}}} 
$$

**Definition 1.5.** For each object $[n] \in \Delta$, there is a simplicial set, denoted by $\Delta^n$, which is defined as

$$(\Delta^n)_k = \text{Hom}([k], [n])$$

i.e. the presheaf representing the object $[n]$.  


Remark 1.6. By the Yoneda Lemma we get:
\[ \text{Hom}(\Delta^n, X) \cong X_n \]
Thus \( \Delta^n \) generates the category of simplicial sets.

Example 1.7. Intuitively we think of \( \Delta^n \) as having the data of a \( n \)-dimensional simplex. This makes many arguments more understandable. Thus for the first three levels we have following cases:

\[
\begin{array}{ccc}
\Delta^0 & \Delta^1 & \Delta^2 \\
0 & 0 & 0 \\
& 1 & 1 \\
& & 2
\end{array}
\]

Before we move on we note following very important fact about simplicial sets. Recall that for two categories \( C \) and \( D \), we can form the category \( \text{Fun}(C, D) \). The same is true for simplicial sets.

Definition 1.8. For two simplicial sets \( X, Y \) we have a mapping simplicial set, \( \text{Map}(X, Y) \) defined as:
\[ \text{Map}(X, Y)_n = \text{Hom}_{sSet}(X \times \Delta^n, Y) \]
Note that in particular \( \text{Map}(X, Y)_0 = \text{Hom}(X \times \Delta^0, Y) = \text{Hom}(X, Y) \) so we can always recover the original \( \text{Hom} \) set.

Notation 1.9. In order to simplify notation, we will often denote the simplicial set \( \text{Map}(X, Y) \) as \( Y^X \).

For a more detailed discussion of simplicial sets, it would definitely help to look at Chapter I of [GJ99].

Definition of Quasi-Categories

The simplicial sets \( \Delta^n \) have certain important subobjects called \( \text{Horns} \) and denoted by \( \Lambda^n_k \) where \( n \geq 0 \) and \( 0 \leq k \leq n \). It is the largest subobject that does not include the face opposing the \( k \)-th vertex.

We say a horn is an inner horn if \( 0 < k < n \).

Example 2.1. If \( n = 2 \) then there are three horns
Note that only one of these is an inner horn, namely $\Lambda^2_1$. The inner horn $\Lambda^2_1$ has the appearance of two arrows that want to be composed. This example already indicates how inner horns are trying to capture the idea of composition. This motivates us to think of higher inner horns as capturing higher composition information.

Any model of higher category should capture the idea of ”higher composition” and should find a way to make this precise. The intuition we stated in the example above about inner horns, thus motivates us to introduce following definition.

**Definition 2.2.** A simplicial set $X$ is called a quasi-category, if the following diagram lifts:

$$
\begin{array}{ccc}
\Lambda^n_k & \longrightarrow & X \\
\uparrow & & \downarrow \\
\Delta^n & \longrightarrow & \Delta^0
\end{array}
$$

for every $n \geq 0$ and every $0 < k < n$.

**Remark 2.3.** Note that $\Delta^0$ is the final object and so the maps into $\Delta^0$ are actually redundant i.e. we could have removed it and still preserved the same condition. Here we added it to motivate the idea that eventually quasi-categories will be fibrant objects in some model structure (more on that in the last section).

**Remark 2.4.** We have the following special cases of the definition above:

- We say $X$ is a quasi-groupoid, if this condition is true for all $0 \leq k \leq n$.
- We say $X$ is a category, if the lifts in the diagram exists uniquely.
- We say $X$ is a groupoid, if the lifts exists for all $0 \leq k \leq n$ and is unique.

**Remark 2.5.** Note that when are calling the simplicial set $X$ a ”category” we are identifying a category with it’s underlying nerve. For any category $C$, we define the simplicial set $N(C)$ called the nerve of $C$ as

$$
N(C)_n = Hom_{Cat}([n], C) = Fun([n], C)
$$
This is gives us a functor from categories to simplicial sets, which is fully faithful. Thus we can always identify a category with its nerve. Thinking about it from this perspective, the condition stated above will exactly classify all simplicial sets that are the nerve of a category. As the condition is more restrictive than the condition for a quasi-category, we realize that every category is itself a quasi-category.

**Remark 2.6.** At a closer look we notice that our definition of a quasi-groupoid exactly coincides with the definition of a Kan complex, that we have given before. We will come back to that when we have discussed isomorphisms.

There is the following very useful recognition principle for quasi-categories:

**Theorem 2.7.** A simplicial set $X$ is a quasi-category if and only if
\[ \text{Map}(\Delta^2, X) \to \text{Map}(\Lambda^2_1, X) \]
is a trivial Kan fibration.

The proof relies on some combinatorial lifting conditions the map $\Lambda^2_1 \to \Delta^2$ satisfies and can be found in Proposition 16.7 and Corollary 16.8 in [Re16]

**Categorical Constructions in Quasi-Categories**

In this section we will cover basic categorical notions in the context of quasi-categories.

**Definition 3.1.** An object in a quasi-category $X$ is just any element in the set $X_0$.

**Definition 3.2.** A functor of quasi-categories is just a map of simplicial sets.

**Definition 3.3.** A product of quasi-categories is just the product of the underlying simplicial sets. The lifting conditions guarantees that this product is also a quasi-category.

We also have following relevant fact about the mapping simplicial set.

**Theorem 3.4.** If $Y$ is a quasi-category, then the mapping simplicial set $\text{Map}(X, Y)$ is actually a quasi-category.

This is also one of those proof which heavily relies on the lifting conditions that inner horns satisfy. For more please see Proposition 16.2 in [Re16]

**Example 3.5.** For every quasi-category $X$, the simplicial set $\text{Map}(\Delta^1, X) = X^{\Delta^1}$ is thus a quasi-category, which is commonly known as the quasi-category of arrows.

One of the great features of higher categories is that for any two objects, we are able to construct a mapping space.
Definition 3.6. For any two objects \( x, y \) in the quasi-category \( X \), we define the mapping space \( \text{Map}(x, y) \) by the following pullback diagram:

\[
\begin{array}{ccc}
\text{Map}(x, y) & \rightarrow & X^\Delta^1 \\
\downarrow & & \downarrow \left\langle s, t \right\rangle \\
\Delta^0 & \rightarrow & X \times X \\
\end{array}
\]

By using the quasi-category condition appropriately, we can prove that \( \text{Map}(x, y) \) is actually a Kan complex. This is one of the many statements follows by applying the "Joyal Extension Theorem" (Theorem 27.2 in [Re16]) and "Joyal Lifting Theorem" (Theorem 27.13 in [Re16]) and which is proven in detail in Proposition 29.2 of [Re16].

Remark 3.7. One of the very important fact about categories is that for three objects \( x, y, z \) we have a map:

\[
\text{Hom}(x, y) \times \text{Hom}(y, z) \rightarrow \text{Hom}(x, z)
\]

Despite of the definition we introduced above, this does not hold for quasi-categories. Instead, we have the space of compositions. For three objects \( x, y, z \), we define \( \text{Map}(x, y, z) \) as

\[
\begin{array}{ccc}
\text{Map}(x, y, z) & \rightarrow & X^\Delta^2 \\
\downarrow & & \downarrow \left\langle d_0, d_1, d_2 \right\rangle \\
\Delta^0 & \rightarrow & X \times X \times X \\
\end{array}
\]

Using the inner horn lifting condition we get following equivalence:

\[
\text{Map}(x, y, z) \xrightarrow{\sim} \text{Map}(x, y) \times \text{Map}(y, z)
\]

Now using the natural map \( \text{Map}(x, y, z) \rightarrow \text{Map}(x, z) \) we get an "indirect composition":

\[
\text{Map}(x, y) \times \text{Map}(y, z) \xleftarrow{\sim} \text{Map}(x, y, z) \rightarrow \text{Map}(x, z)
\]

Thus we do not have a direct composition map. It is only if we go to the homotopy category where we get a real composition map.

Isomorphisms in Quasi-Categories

In a regular category an isomorphism is a map \( f : x \rightarrow y \) such that there exists \( g : y \rightarrow x \) which satisfies \( fg = \text{id}_y \) and \( gf = \text{id}_x \). This does not happen in the context of quasi-categories. Similar to the case of compositions where we need to consider the "higher data", we have to be more careful about isomorphisms.
**Definition 4.1.** We say a map $f \in X_1$ is an isomorphism if there exists a map $g \in X_1$, a two 2-cells $\alpha, \beta \in X_2$ such that we can fill out following diagram:

$$
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
| & \downarrow{\beta} & | \\
| & \searrow{\alpha} & | \\
| & \swarrow{g} & | \\
x & \xleftarrow{id_x} & y
\end{array}
$$

Now that we have a definition of an isomorphism, we might wonder what happens if every map is an isomorphism. In the classical context, we call those categories *groupoids*. That suggests that the name *quasi-groupoid* for a quasi-category in which every map is an isomorphism. However, we have already used that terminology before and thus it might seem unreasonable. The following theorem will resolve that issue:

**Theorem 4.2.** Let $X$ be a quasi-category. The following are equivalent

1. Every map in $X$ is an isomorphism.
2. $X$ is a Kan complex.

One side, $(2) \Rightarrow (1)$, is not difficult at all. However, the other side is quite tricky and again utilizes the Joyal extension theorem. The proof of $(1) \Rightarrow (2)$ can be found in Proposition 28.2 of [Re16]. Thus in the context of quasi-categories, we are justified in calling a quasi-category that satisfies either one of these conditions a quasi-groupoid.

**Natural Transformations and Natural Isomorphisms**

**Definition 5.1.** *Natural transformation* of quasi-categories between functors $S, T : X \to Y$ is a map of quasi-categories $H : X \times \Delta^1 \to Y$ such that $H$ restricted to $X \times \{0\}$ gives us $S$ and $H$ restricted to $X \times \{1\}$ gives us $T$.

Recall that $\text{Map}(X,Y)_1 = \text{Hom}(X \times \Delta^1, Y)$, so we can think of the maps in the mapping quasi-category as natural transformations.

**Notation 5.2.** Note that this implies that for every object $x \in X_0$ we have chosen a map $H_x : S(x) \to T(x)$ such that the following diagram can be filled out with two-cells $\alpha$ and $\beta$:

$$
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
| & \downarrow{\beta} & | \\
| & \searrow{\alpha} & | \\
| & \swarrow{g} & | \\
x & \xleftarrow{id_x} & y
\end{array}
$$
In the last section we showed how to define and isomorphism in a quasi-category. Given that the maps in the quasi-category $Y^X$ are just natural transformations, we have following definition:

**Definition 5.3.** Global condition for natural isomorphisms: A natural transformation between quasi-categories $X$ and $Y$ is called a global natural isomorphism if it is an isomorphism in the quasi-category $Y^X$.

While this definition is conceptually helpful, it is often very difficult to use this condition as mapping quasi-categories can be quite intricate. Thus it would be nice to have an easier condition. It turns out there is such condition! Generalizing from the case of natural isomorphism of categories, we have following definition:

**Definition 5.4.** Local definition of natural isomorphism. For two functors $S, T$ from a quasi-category $X$ to $Y$, a natural transformation from $S$ to $T$ is called a local natural isomorphism if for every object $x \in X$ the map $H_x : Sx \to Tx$ is an isomorphism in the quasi-category $Y$.

We have following theorem which makes these definitions useful.

**Theorem 5.5.** For $Y$ a quasi-category, a map in the quasi-category $Y^X$ is a global natural isomorphism if and only if it is a local natural isomorphism. Thus we have a sort of "local-to-global"-recognition principle to identify natural isomorphisms.

Again this might seem quite easy to accept but it is difficult to prove in the context of quasi-categories and uses Joyal extension theorem. There is a detailed proof in [Re16] (Theorem 28.10).

**Fundamental Theorem of Quasi-Categories**

Whenever we have a functor $F : C \to D$ we might ask ourselves whether this map is an equivalence. One thing to do is to simply find an inverse i.e. a functor $G : D \to C$ such that $FG \simeq id_D$ and $GF \simeq id_C$. This definition generalizes to the world of quasi-categories:

**Definition 6.1.** A map of quasi-categories $T : C \to D$ is a categorical equivalence if there exists another map $S : D \to C$, such that $S \circ T \simeq 1_C$ and $T \circ S \simeq 1_D$. 

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However, from the world of category theory we already know that this definition is very often not that useful, as it might be quite difficult to construct an inverse. The good thing is there is a way to avoid it. Namely, we have following fact:

*A functor is an equivalence iff it is fully faithful and essentially surjective.*

This generalizes to the world of quasi-categories thusly:

**Lemma 6.2.** *A map of quasi-categories* \( T : C \to D \) *is a categorical equivalence if it satisfies the following two conditions:

- **Fully Faithful:** For all two objects \( c, c' \in C \), the map of Kan complexes
  \[
  \operatorname{Map}(c, c') \xrightarrow{\cong} \operatorname{Map}(Tc, Tc')
  \]
  is a Kan equivalence.

- **Essentially Surjective:** For every object \( d \in D \) there exists an object \( c \in C \) such that \( Tc \) is equivalent to \( d \).

Because of its utmost importance this theorem has been named the *Fundamental Theorem of Quasi-Categories* by Rezk in [Re16] (Section 30) and the main proof can be found in Section 38.

**Joyal Model Structure**

The category of simplicial sets has a model structure in which the fibrant objects are exactly the quasi-categories. It is known as the *Joyal model structure*.

It has following specifications. A map \( X \to Y \) is a ...

**W** Weak equivalence if for any quasi-category \( W \) the induced map

\[
\operatorname{Map}(Y, W) \to \operatorname{Map}(X, W)
\]

is a categorical equivalence.

**C** Cofibration if it is an inclusion.

**F** Fibration if it satisfies the right lifting property with respect to trivial cofibrations.

[Re16] gives an explicit proof that these three collections of maps satisfy all conditions of a model structure (Theorem 40.6), by heavily relying on lifting techniques that have been developed before.

The Joyal model structure does (and does not) satisfy many interesting properties which are definitely worth mentioning:
1. By definition of a quasi-category it follows then directly that the fibrant objects are just the quasi-categories.

2. Every object is cofibrant and thus the model structure is left proper.

3. The model structure is NOT right proper however (for example we can look at following pullback diagram

\[
\begin{array}{ccc}
\Delta^0 \amalg \Delta^0 & \rightarrow & \Delta^1 \\
\downarrow & & \downarrow \\
\Lambda_1^2 & \rightarrow & \Delta^2 \\
\end{array}
\]

where the right map maps 0 to 0 and 1 to 2 which is a fibration and the bottom is the standard inner horn inclusion, which by definition is an equivalence.

4. Every categorical equivalence is a Kan equivalence. Thus the Kan model structure on simplicial sets is a localization of the Joyal model structure. In particular, every Kan fibration is a fibration the Joyal model structure.

5. The Joyal model structure is NOT simplicial. It is however enriched over itself as a model structure, as can be clearly seen from the weak equivalence condition we stated above.

6. The Joyal model structure is Cartesian closed.

References


