ϵ-δ Proofs

See `http://www.maths.dur.ac.uk/users/steven.charlton/analysis1_1314` for updates.

When you first meet them, ϵ-δ proofs are conceptually quite difficult. It usually takes a lot of time and effort thinking over the ideas before the concept finally 'clicks', makes sense and seems natural.

Firstly recall the definition:

Definition (ϵ-δ limit definition)**.** We say $\lim_{x\to c} f(x) = L$ if:

$$\text{Given any real } \epsilon > 0 \text{, we can find some real } \delta > 0 \text{ such that}$$
$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

⟦ In this definition L and c are real numbers (or complex numbers), in particular L and c are finite. Limits as $x \to \infty$, or limits which equal infinity require different definitions. ⟧

So let's say that you claim some function $f(x)$ has limit L at c. Then if I pick any ϵ, you should be able to give me the corresponding δ: I tell you how close to L I want the output of f to be (within ϵ), then you me how close to c I need to look (within δ).

1 Numerical Example

Say we're looking at $f(x) = 3x - 1$. You claim $f(x)$ has limit $L = 5$ at $c = 2$.

I want $\epsilon = 1$. We find $|3x - 1 - 5| < 1 \iff |3(x-2)| < 1 \iff |x - 2| < 1/3$. So you can tell me to take $\delta = 1/3$. That is, if I plug in any x from the interval $(2 - 1/3, 2 + 1/3)$, the output I get from $f(x)$ is in the interval $(5 - 1, 5 + 1)$.

Now I want $\epsilon = 1/7$. Similarly we find $|3x - 1 - 5| < 1/7 \iff |3(x - 2)| < 1/7 \iff |x - 2| < 1/21$. So you will tell me to take $\delta = 1/21$.

We can do this over and over again (and not just with rational numbers), and make a table like

Given ϵ	Found δ
1	1/3
1/7	1/21
6/101	2/101
0.60335...	0.20111...
0.035516...	0.011838...

When dealing with a particular function, the δ you give back to me can only possibly depend on the ϵ I give to you. That is, δ is a function of ϵ only. The expression you finally write down for $\delta = \cdots$ can only contain ϵ, it can't contain x or anything else.

2 Example ϵ-δ Proofs

In this section don't worry (too much) about why I choose a particular values for δ in the proof. Just concentrate on the structure of the proof itself, unencumbered by the working which you would have to do beforehand to find the value to use.

This will make it clearer what is the ϵ-δ proof, and what is the working out we need to do alongside.

Problem: Give an ϵ-δ proof that $\lim_{x\to 2}(3x-1)=5$.

Solution: Given $\epsilon>0$, let $\delta=\epsilon/3$. If $|x-2|<\delta$ then

$$|3x-1-5|=|3(x-2)|=3\underbrace{|x-2|}_{<\delta}<3\delta=3\cdot\epsilon/3=\epsilon\,.$$

So the ϵ-δ definition holds, and $\lim_{x\to 2}(3x-1)=5$, as required. $\square$

Problem: Give an ϵ-δ proof that $\lim_{x\to 4}x^2=16$.

Solution: Given $\epsilon>0$, let $\delta=\min\{1,\epsilon/9\}$. Note this means $\delta<1$ and $\delta<\epsilon/9$.
If $|x-4|<\delta$ then

$$|x-4|<1 \implies -1<x-4<1 \implies 7<x+4<9 \implies |x+4|<9\,,$$

and

$$\left|x^2-16\right|=\underbrace{|x+4|}_{<9}|x-4|<9\underbrace{|x-4|}_{<\delta}<9\delta<9\cdot\epsilon/9=\epsilon\,.$$

So the ϵ-δ definition holds, and $\lim_{x\to 4}x^2=16$, as required. $\square$

Problem: Given an ϵ-δ proof that $\lim_{x\to 3}\frac{x+3}{(x+1)(x-2)}=\frac{3}{2}$.

Solution: Given $\epsilon>0$, let $\delta=\min\{\frac{1}{2},\frac{\epsilon}{5}\}$. Note this means $\delta<\frac{1}{2}$ and $\delta<\frac{\epsilon}{5}$.
Firstly if $|x-3|<\frac{1}{2}$, then

$$-\tfrac{1}{2}<x-3<\tfrac{1}{2} \implies \tfrac{5}{2}<x<\tfrac{7}{2}\,.$$

So

$$\tfrac{7}{2}<x+1<\tfrac{9}{2} \implies \tfrac{1}{|x+1|}=\tfrac{1}{x+1}<\tfrac{2}{7}\,,$$

and

$$\tfrac{1}{2}<x-2<\tfrac{3}{2} \implies \tfrac{1}{|x-2|}=\tfrac{1}{x-2}<2\,,$$

and

$$\tfrac{23}{2}<3x+4<\tfrac{29}{2} \implies |3x+4|<\tfrac{29}{2}$$

Now if $|x-3|<\delta$, then $|x-3|<\frac{1}{2}$, and

$$\left|\frac{x+3}{(x+1)(x-2)}-\frac{3}{2}\right|=\frac{|3x+4|\,|x-3|}{2\,|x+1|\,|x-2|}<\frac{29}{2}\cdot\delta\cdot\frac{1}{2}\cdot\frac{2}{7}\cdot 2=\frac{29}{7}\delta<\frac{29}{7}\frac{\epsilon}{5}=\frac{29}{35}\epsilon<\epsilon$$

So the ϵ-δ definition holds, and $\lim_{x\to 3}\frac{x+3}{(x+1)(x-2)}=\frac{3}{2}$, as required. $\square$

3 Writing Your Own ϵ-δ Proofs

Now you should be wondering how I came up with the particular values for δ in the above proofs? This time I'll include the working out needed beforehand.

Problem: Give an ϵ-δ proof that $\lim_{x\to 3} x^2 + 2x - 1 = 14$.

Solution: Firstly:

$$|f(x) - L| = \left|x^2 + 2x - 1 - 14\right| = \left|x^2 + 2x - 15\right| = |x-3|\,|x+5|$$

Now we have a term $|x-3|$ which we know will be $< \delta$ in the proof. But how can we deal with the other term $|x+5|$? Can we say anything about it? Well, we have: $|x-3| < \delta \implies -\delta < x - 3 < \delta \implies 8 - \delta < x + 5 < 8 + \delta$, so $|x+5| < 8 + \delta$.

We could make this work, but why deal with arbitrary δ, and quadratic equations in δ? The limit only cares what goes on close to $x = 3$, so we can take (arbitrarily) $\delta \leq 1$. If $\delta = 1$ works for $\epsilon = 1/2$, then it also works for $\epsilon = 50$; if it gives $|f(x) - 14| < 1/2$, then obviously we also have $|f(x) - 14| < 50$ since $1/2 < 50$.

If we assume $\delta \leq 1$, the we have $|x+5| < 9$, so we get:

$$|x-3|\,|x+5| < 9\delta\,.$$

We can make $|x-3|\,|x+5| < \epsilon$ by making $9\delta \leq \epsilon$, so by taking $\delta \leq \epsilon/9$.

So we must have $\delta \leq 1$, and $\leq \epsilon/9$, i.e. $\delta \leq \min\{1, \epsilon/9\}$. Therefore let's take $\delta = \min\{1, \epsilon/9\}$. We've now finished the working out, and so we can start the proof proper. The above does work as a proof, as long as you make sure all the implications are in the right direction, but it's good to see directly this works:

Given $\epsilon > 0$, let $\delta = \min\{1, \epsilon/9\}$. Note $\delta \leq 1$ and $\leq \epsilon/9$. Then:

$$|x-3| < \delta \implies |x-3| < 1 \implies |x+5| < 9\,.$$

So, if $|x-3| < \delta$, then:

$$|f(x) - L| = \underbrace{|x+5|}_{<9}\,\underbrace{|x-3|}_{<\delta} < 9\delta < 9 \cdot \epsilon/9 = \epsilon\,.$$

So the ϵ-δ definition holds, and $\lim_{x\to 3} \frac{x+3}{(x+1)(x-2)} = \frac{3}{2}$, as required. $\square$

Problem: Give an ϵ-δ proof that $\lim_{x\to 4} \frac{1}{x-3} = 1$.

Solution: Firstly:

$$|f(x) - L| = \left|\frac{1}{x-3} - 1\right| = \frac{|x-4|}{|x-3|}\,.$$

We have the $|x-4|$ bit which we know is $< \delta$. Can we find an upper bound on $\frac{1}{|x-3|}$ after restricting δ a bit? Let's assume $\delta \leq 1$. Then $|x-4| < \delta \implies |x-4| < 1 \implies -1 < x - 4 < 1 \implies 0 < x - 3 < 2$. And all this says is $\frac{1}{|x-3|} = \frac{1}{|x-3|} > 1/2$, which is no good.

To fix this we need to restrict δ even more. The problem is the vertical asymptote $x = 3$ which is only 1 unit away from the limit point at $x = 4$. Taking anything < 1 means we don't reach all the way to the asymptote, and so there are no problems.

So assume $\delta \leq 1/2$. Then $|x-4| < 1/2 \implies 1/2 < x-3 < 3/2 \implies \frac{1}{|x-3|} < 2$. So we get

$$|f(x) - L| = \frac{|x-4|}{|x-3|} < 2\delta\,,$$

and we can make it $< \epsilon$ by taking $\delta \leq \epsilon/2$.

So taking $\delta = \min\{1/2, \epsilon/2\}$ gives us the ϵ-δ proof. □

4 General Strategy for Rational Functions

Suppose we have to give an ϵ-δ proof that $\lim_{x\to c} \frac{a(x)}{b(x)} = L$. Do the following:

Step 1: Compute $\left|\frac{a(x)}{b(x)} - L\right|$

Step 2: Write it in the form $|x-c| \frac{|p(x)|}{|q(x)|}$, where $p(x)$ and $q(x)$ are polynomials, and $q(c) \neq 0$. You *will* be able to do this. ⟦ First cancel all factors of $(x-c)$ on top and bottom, so you can evaluate the limit by substituting in $x = c$. There can't be a factor of $(x-c)$ on the bottom since the limit exits, so $q(c) \neq 0$. And there has to be a factor of $(x-c)$ on the top since at $x = c$, we get L. ⟧

Step 3: Find the nearest asymptote (root of $q(x)$) to the point $x = c$. Pick any D less than this distance. Assume $\delta \leq D$.

Step 4: Use $|x-c| < D$ to say $c - D < x < c + D$, and use this to find an upper bound P on $p(x)$. So get $|x-c| < \delta \implies |p(x)| < P$.

Step 5: Also use this to find a lower bound Q on $q(x)$. So get $|x-c| < \delta \implies \frac{1}{|q(x)|} < \frac{1}{Q}$.

Step 6: Using these we get

$$\left|\frac{a(x)}{b(x)} - L\right| = |x-c| \frac{|p(x)|}{|q(x)|} < \delta \frac{P}{Q}\,,$$

so we can make it $< \epsilon$, by taking $\delta < \frac{Q}{P}\epsilon$ as well.

Take $\delta = \min\{D, \frac{Q}{P}\epsilon\}$, then $|x-c| < \delta \implies \left|\frac{a(x)}{b(x)} - L\right| < \epsilon$. Hence we get an ϵ-δ proof that $\lim_{x\to c} \frac{a(x)}{b(x)} = L$.