The noncommutative geometry of symplectic singularities

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1 Introduction

In electromagnetism, point charges are treated as singularities with infinite field strength at their respective positions. The hypothetical 2-dimensional magnetic monopole is described by the following symplectic form:

$$\omega = dq^i \wedge dp_i + \frac{1}{r^2} dq^1 \wedge dq^2.$$

Following [Blo17] we show how to remove this singularity yielding a Lie algebroid L_ω over $T^*\mathbb{R}^2$. The key results of this thesis are the explicit construction of an integrating Lie groupoid (Proposition 17) and the discussion of the convolution algebra associated to it (sections 5.3 and 6.1). The resulting noncommutative C*-algebra is interpreted as the noncommutative geometry of the singularity.
In section 2 we introduce Lie groupoids and discuss some properties and examples. Section 3 deals with their infinitesimal counterpart, the Lie algebroid and the passage from Lie groupoids to algebroids. We will also briefly discuss integrability and uniqueness of the t-simply connected integrating groupoid.
Section 4 discusses C*-algebras of discrete and topological groups and in 4.2 we describe with the passage from a Lie group*oid* with a Haar system to its convolution C*-algebra.
In 5 we will apply the techniques that we developed to our example. Section 5.1 introduces Dirac structures as the tool needed to remove the singularity yielding a Lie algebroid. In 5.2 we compute explicitly the magnetic Lie algebroid L_ω and its unique t-simply connected integrating Lie groupoid G. In 5.3 we construct a Haar system on G and discuss the C*-algebras of the regular and singular part.
Section 6 reviews Morita equivalence of C*-algebras and Lie groupoids and suggests to view convolution algebras modulo Morita equivalence in the absence of a distinguished Haar system. In 6.1 we then discuss properties of C*-algebras preserved under Morita equivalence that we can specify for our example.

Deutsche Zusammenfassung Im Elektromagnetismus behandelt man Punktladungen als Singularitäten mit einer an diesem Punkt unendlichen Feldstärke. Der hypothetische 2-dimensionale magnetische Monopol wird durch die folgende symplektische Form beschrieben:

$$\omega = dq^i \wedge dp_i + \frac{1}{r^2} dq^1 \wedge dq^2.$$

Basierend auf [Blo17] werden wir diese Singularität durch ein Lie Algebroid L_ω über $T^*\mathbb{R}^2$ heben. Schlüsselresultate dieser Arbeit sind die Konstruktion eines integrierenden Lie Gruppoids (Proposition 17) und die Diskussion der

assoziierten Faltungsalgebra (5.3,6.1). Diese nichtkommutative C*-Algebra kann als nichtkommutative Geometrie der Singularität interpretiert werden.
In Kapitel 2 führen wir Lie Gruppoide ein und diskutieren Beispiele und erste Eigenschaften. Kapitel 3 beschäftigt sich mit dem infinitesimalen Gegenstück, dem Lie Algebroid, und dem Übergang von Gruppoid zu Algebroid. Wir berühren auch das Thema der Integrierbarkeit und beweisen Eindeutigkeit des t-einfach zusammenhängenden integrierenden Gruppoids.
Kapitel 4 beschreibt C*-Algebren für diskrete und topologische Gruppen und im Anschluss 4.2 den Übergang von Lie Gruppoiden mit Haarsystem zu ihrer Faltungsalgebra.
In 5 werden wir diese Techniken auf unser Beispiel anwenden. Kapitel 5.1 führt zur Hebung der Singularität Dirac-Strukturen ein, wodurch wir ein Lie Algebroid erhalten. In 5.2 werden das magnetische Lie Algebroid L_ω und sein eindeutiges t-einfach zusammenhängendes integrierendes Lie Gruppoid G explizit berechnet. In 5.3 konstruieren wir ein Haarsystem auf G und diskutieren die C*-Algebren des singulären und regulären Teils.
Kapitel 6 behandelt Morita-Äquivalenz für C*-Algebren und Lie Gruppoide und begründet, dass Faltungalgebren modulo Morita-Äquivalenz betrachtet werden sollten in Abwesenheit eines ausgezeichneten Haarsystems. In 5.3 diskutieren wir Eigenschaften von C*-Algebren, die unter Morita-Äquivalenz stabil sind und die wir für unser Beispiel präzisieren können.

2 Lie Groupoids

One of the main objects of our study are Lie groupoids, a manifold version of the groupoid.

Definition 1. A **groupoid** is a category in which every morphism is an isomorphism.

Unraveling this definition, we will mostly think of a groupoid as the space of morphisms G over the space of objects M called the base. Associated to each morphism $g \in G$ is a source and a target object, i.e. maps $s, t : G \to M$. Composition of morphisms $g, h \in G$ gives the morphism $gh \in G$ which is defined whenever $s(g) = t(h)$.
This is equivalently an associative product $\circ : G^{(2)} \to G$, where $G^{(2)} = \{(g, h) \in G : s(g) = t(h)\}$ is the set of composable pairs.
Each object $x \in M$ has an identity $\epsilon(x) := \mathrm{id}_x \in G$ associated to it and since every morphism is invertible, there exists an inverse map ${}^{-1} : G \to G$.
Thus a groupoid will be denoted $G \rightrightarrows M$ with the arrows representing the maps s, t.

Example 1. (a) Every group G can be viewed as a groupoid over a point $M = \{pt\}$. Since s, t must be trivial maps, all elements are composable as usual.

(b) Similarly, group bundles or disjoint unions of groups form a groupoid with $s = t$ mapping $G \ni g \mapsto G$. The composition then restricts to the individual groups.

(c) For any set M we can form the **pair groupoid** $M \times M \rightrightarrows M$ with

$$s((x, y)) := y \qquad t((x, y)) := x \quad \text{and} \quad (z, y) \circ (y, x) = (z, x)$$

This is the prime example of a transitive groupoid. Subgroupoids of the pair groupoid are just symmetric and transitive relations on M.

(d) The fundamental groupoid $\Pi(X) \rightrightarrows X$ for a topological space X provides a great motivation for the study of groupoids, eliminating the necessity of choosing a basepoint. The maps s, t assign to each homotopy class of paths its fixed endpoints.

(e) Let G be a group acting on a space M.The **action groupoid** associated to it is $G \times M \rightrightarrows M$ with space of arrows $G \times M$ and the following maps:

$$s((g, m)) := m \qquad t((g, m)) := g.m \quad \text{and} \quad (g, h.m) \circ (h, m) = (gh, m)$$

(f) To a principal G-bundle $\pi : P \to M$ we can associate the **Atiyah groupoid** (or Gauge groupoid) $Gau(P)$.

G acts on $P \times P$ diagonally via $g.(p,q) = (g.p, g.q)$ which preserves the source and target maps $t((p,q)) = \pi(p), s((p,q)) = \pi(q)$ because we are dealing with a principal G-action. We obtain the quotient $Gau(P) = (P \times P)/G \rightrightarrows M$. Every element $[p,q] \in Gau(P)$ corresponds to a G-equivariant map

$$\phi : \pi^{-1}(\pi(p)) \to \pi^{-1}(\pi(q))$$

which is uniquely determined by $\phi(p) = q$ since G acts transitively on the fibers. The composition of these maps gives the natural composition on the Atiyah groupoid.

Adding more structure to the groupoids, we can consider topological groupoids, where the product and inverse are required to be continous and the identity section ϵ is required to be an embedding. In the smooth category we get the following definition:

Definition 2. A **Lie groupoid** $G \rightrightarrows M$ is a groupoid where G, M are smooth manifolds and the s, t are required to be smooth submersions. The identity section ϵ is required to be a smooth embedding and multiplication and inverse are required to be smooth maps.

In dealing with integrability questions it is often necessary to not require Hausdorffness of G and only that of M.
The submersion requirement ensures the smoothness of $G^{(2)}$ and that all fibers $s^{-1}(m), t^{-1}(m)$ are smooth manifolds. By the inverse function theorem smoothness of multiplication already implies the smoothness of the globally defined inverse.

Example 2. a) Lie groups can be regarded as groupoids over a point $\{pt\}$ and similarly Lie group bundles over a manifold are groupoids. A special case of this are vector bundles over manifolds.

b) Most of the groupoids above can be given a natural smooth structure: the pair groupoid, the action groupoid of a smooth action, the Atiyah groupoid and even the fundamental groupoid in some cases.

Definition 3. A **morphism of groupoids** $G \rightrightarrows M, H \rightrightarrows N$ is a functor, i.e. a map $F : G \to H$ that satisfies $F(gh) = F(g)F(h)$ for composable $g, h \in G$ inducing a base map $f : M \to N$ that makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{F} & H \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & \xrightarrow{f} & N \end{array}$$

A **Lie groupoid morphism** is a smooth morphism of Lie groupoids.

With this notion of morphism we get the categories **Grpd** and **LieGrpd**. For any groupoid $G \rightrightarrows M$ we get a unique morphism $(t, s) : G \to M \times M$ to the pair groupoid. In the restricted category of groupoids over a specific base we thus get the pair groupoid as a terminal object. For groups, regarded as groupoids over a point, the terminal object is thus $(\{pt\} \times \{pt\} \rightrightarrows \{pt\}) \cong \{0\}$.

Definition 4. A groupoid $G \rightrightarrows M$ provides an equivalence relation on points in the base M by $p \sim q :\Leftrightarrow p = s(g), q = t(g)$ for some $g \in G$. The equivalence classes are the **orbits** of G and G is called **transitive** if it has only one orbit. The orbit of $m \in M$ will be denoted $G.m$.[1]
The **isotropy groups** are $G_m = s^{-1}(m) \cap t^{-1}(m)$ for $m \in M$.

For the action groupoid associated to a group action the original notion of an orbit and the groupoid notion coincide. The fundamental groupoid is transitive if and only if the underlying space is connected and its isotropy groups are the fundamental groups associated to different basepoints.

Proposition 1. *The isotropy groups of a Lie groupoid are Lie groups and the orbits are immersed submanifolds.*

Proof. We first sketch a conceptual proof that the target map t has constant rank along the s-fibers. Let $m \in M$ and $g, g' \in s^{-1}(m)$ and let $S \subset G$ be a local bisection containing $g'g^{-1}$, i.e. a submanifold for which the restriction of s, t provides diffeomorphisms to open subsets of M. Their existence is a local question and uses that s, t are submersions.
By left multiplication a local bisection induces an s-equivariant local diffeomorphism $\mathcal{A}_S$ mapping $g \mapsto (g'g^{-1})g = g'$. By equivariance, A_S provides a local diffeomorphism between neighbourhoods of g and g' in $s^{-1}(m)$. The equivariant rank theorem implies that the rank of t is constant on the s-fibers. Level sets of $t|_{s^{-1}(m)}$ are therefore submanifolds making the isotropy group G_m a Lie group. Furthermore the image of a constant rank map is an immersed submanifold, so $G.m = t(s^{-1}(m))$ is immersed. □

Proposition 2. *Every transitive Lie groupoid $G \rightrightarrows M$ is isomorphic to an Atiyah groupoid of any of the (left) principal G_m-bundles $t^{-1}(m) \to M$.*

Proof. Let $m \in M$ be arbitrary. And let $P = t^{-1}(m)$ with bundle map $s : P \to M$. Then G_m is a Lie group acting smoothly on P via left multiplication. This action preserves s-fibers. It is free since multiplication by $g \in G_m$ is an isomorphism and it is transitive on the s-fibers. Now $s : P \to M$ is surjective because G is transitive by assumption. As seen above s has constant rank on P. Therefore s is a submersion and its normal form gives us local trivializations

[1] This is an example of a *groupoid action* on a manifold N. The difference to ordinary actions is that we need a moment map $\mu : N \to M$ and $g.n$ is defined only if $s(g) = \mu(n)$.

of P. This proves that P is a principal G_m-bundle.
Define a map

$$\begin{aligned} P \times P &\longrightarrow G \\ (x, y) &\mapsto y^{-1}x. \end{aligned}$$

This is surjective: for any $g \in G$ we find h connecting $t(g)$ to m. Then $h^{-1}(hg) = g$. Additionally, the map factors through the diagonal action of G_m on $P \times P$ giving us a bijection $\Phi : (P \times P)/G_m \cong G$ which is smooth as a quotient map. Recalling the groupoid structure of $Gau(P)$ we see that Φ is a groupoid morphism compatible with the source and target maps.
Finally, Φ is also an isomorphism of Lie groupoids. For this we only need to verify smoothness of Φ^{-1} which follows from the inverse function theorem when we show that Φ has constant rank (it must then be a submersion). However, this is easily verified since the kernel of the differential $T(P \times P) \to TG$ is just the trivial G_m orbit that is factored out. □

Corollary 3. *Let M be connected. The fundamental groupoid of* $\Pi_1(M)$ *is a Lie groupoid.*

Proof. The isotropy groups of $\Pi_1(M) \rightrightarrows M$ are just copies of the fundamental group. For manifolds the fundamental group is a countable 0-dimensional Lie group. The fiber $s^{-1}(m)$ is the set of homotopy classes of paths in M starting at m which is the standard description of the universal cover $\widetilde{M}$ of the connected manifold M. Thus $\Pi_1(M) \cong Gau(\widetilde{M})$ is a Lie groupoid. □

3 Lie Algebroids

In the case of Lie groups it is extremely fruitful to look at the infinitesimal version, the associated Lie algebra. One way to construct the Lie algebra Lie(G) of a Lie group G is by left-invariant vector fields. $X \in \mathfrak{X}(G)$ is called left-invariant if $X_g = (dL_g)_e X_e, \forall g \in G$ where $L_g : G \to G$ is left multiplication by $g \in G$.
It thus easily follows that as a vector space $Lie(G) \cong T_eG$. The bracket is the commutator bracket inherited from $\mathfrak{X}(G)$.
We will in the following examine the infinitesimal analogue of a Lie groupoid, first in the abstract algebraic setting and then the construction from a given Lie groupoid.

Definition 5. A **Lie algebroid** is a vector bundle $A \to M$ with a Lie bracket $[\cdot, \cdot]$ on the sections $\Gamma(A)$ and an **anchor map** bundle homomorphism $\rho : A \to TM$ that satisfies the following Leibniz rule:

$$[\sigma, f\tau] = f[\sigma, \tau] + \rho(\sigma)f \cdot \tau \qquad \text{where } \sigma, \tau \in \Gamma(A), f \in C^\infty(M)$$

Proposition 4. *The anchor map ρ is a Lie algebra homomorphism. It also is the unique anchor making A a Lie algebroid.*

Proof. By abuse of notation denote $\rho(\sigma) := \rho \circ \sigma \in \mathfrak{X}(M)$.
First we verify uniqueness. Let ρ, ρ' be different anchors making A a Lie algebroid. The Leibniz rule gives:

$$f[\sigma, \tau] + \rho(\sigma)f \cdot \tau = f[\sigma, \tau] + \rho'(\sigma)f \cdot \tau$$

Therefore by varying f, σ, τ we get $\rho = \rho'$.
Let $f \in C^\infty(M), \tau, \eta, \sigma \in \Gamma(A)$. Using the Leibniz rule, bilinearity and anticommutativity of the bracket we compute:

$$\begin{aligned}[[\tau, f\eta], \sigma] &= [f[\tau, \eta] + \rho(\tau)f \cdot \eta, \sigma] \\ &= f[[\tau, \eta], \sigma] - \rho(\sigma)f \cdot [\tau, \eta] + \rho(\tau)f \cdot [\eta, \sigma] - \rho(\sigma)(\rho(\tau)f) \cdot \eta \end{aligned} \tag{1}$$

$$\begin{aligned}[[f\eta, \sigma], \tau] &= [f[\eta, \sigma] - \rho(\sigma)f \cdot \eta, \tau] \\ &= f[[\eta, \sigma], \tau] - \rho(\tau)f \cdot [\eta, \sigma] - \rho(\sigma)f \cdot [\eta, \tau] + \rho(\tau)(\rho(\sigma)f) \cdot \eta \end{aligned} \tag{2}$$

$$[[\sigma, \tau], f\eta] = f[[\sigma, \tau], \eta] + \rho([\sigma, \tau])f \cdot \eta \tag{3}$$

Summing everything, by the Jacobi identity and anticommutativity of the bracket we get:

$$\begin{aligned} 0 &= [[\tau, f\eta], \sigma] + [[f\eta, \sigma], \tau] + [[\sigma, \tau], f\eta] \\ &= f \cdot 0 + \rho([\sigma, \tau])f \cdot \eta + \rho(\sigma)f \cdot 0 + \rho(\tau)f \cdot 0 \\ &\qquad - \rho(\sigma)(\rho(\tau)f) \cdot \eta + \rho(\tau)(\rho(\sigma)f) \cdot \eta \\ &= (\rho([\sigma, \tau]) - [\rho(\sigma), \rho(\tau)])f \cdot \eta \end{aligned}$$

Since this holds for arbitrary sections η and functions f, we conclude that $\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)]$ is a Lie algebra homomorphism.

□

Example 3. a) A Lie algebra is a Lie algebroid over a point $\{pt\}$ because then $C^\infty(\{pt\}) \cong \mathbb{R}$ and the Leibniz rule is plain linearity with trivial anchor. Similarly bundles of Lie algebras with trivial anchor constitute a Lie algebroid.

b) The tangent bundle TM to a manifold M with $\rho = \mathrm{id}$ forms the prime example of a Lie algebroid. The Leibniz rule is inherent in the characterisation of $\Gamma(TM)$ as derivations on M.

c) Let $\Theta : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action. Recall that a Lie algebra action is by definition a homomorphism of Lie algebras. We can associate an **action algebroid** $\mathfrak{g} \times M \to M$ which is a trivial bundle over M with anchor $\rho(X, m) = \Theta(X)_m$. The bracket is given by

$$[X, Y] = [X, Y]_{\mathfrak{g}} + \mathcal{L}_{\rho(X)} Y - \mathcal{L}_{\rho(Y)} X$$

with the identification $\Gamma(\mathfrak{g} \times M) \cong C^\infty(M, \mathfrak{g})$ and the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ evaluated pointwise. It is the unique bracket determined by the requirement that $[C_1, C_2] = [C_1, C_2]_{\mathfrak{g}}$ on constant sections C_1, C_2 which span $\Gamma(\mathfrak{g} \times M)$ as a $C^\infty(M)$-module.

d) Let ω be a closed 2-form on M. Then we can associate to it a Lie algebroid $A_\omega = TM \oplus \mathbb{L}$ where $\mathbb{L}$ is a line bundle. Then $\Gamma(A_\omega) \cong \mathfrak{X}(M) \oplus C^\infty(M)$ and bracket given by:

$$[(X, f), (Y, g)] = ([X, Y], \mathcal{L}_X g - \mathcal{L}_Y f + \omega(X, Y))$$

The closedness of ω is equivalent to this bracket satisfying the Leibniz rule.

e) Let $\pi : P \to M$ be a principal left G-bundle. Then the differential of the action induces a free and proper left G-action on TP. The quotient TP/G is called the **Atiyah algebroid** of P. π factors through this quotient making $TP/G \to M$ a vector bundle. Its differential $d\pi$ provides the anchor map.
Sections of TP/G can be identified with G-invariant vector fields on TP which are invariant under the lifted action. They form a Lie subalgebra inducing a natural bracket on TP/G.

3.1 The Lie algebroid of a Lie groupoid

Let $G \rightrightarrows M$ be a Lie groupoid. Elements $g \in G$ act on specific fibers of G when left multiplication is allowed. Namely we have:

$$\begin{aligned} L_g : t^{-1}(s(g)) &\longrightarrow t^{-1}(t(g)) \\ h &\mapsto gh \end{aligned}$$

This is a diffeomorphism between smooth manifolds with inverse $L_{g^{-1}}$ and a differential that is an isomorphism:

$$(dL_g)_h : T_h(t^{-1}(s(g))) \longrightarrow T_{gh}(t^{-1}(t(g)))$$

Again, since t is a submersion, the level sets' tangent spaces are just the kernel of dt: $T_h(t^{-1}(t(h))) = (\ker(dt))_h$. Also, by constant rank, $T^t G := ker(dt)$ is a subbundle of TG, consisting of all the fibers' tangent spaces.
To get a notion of left-invariance of vector fields on G similar to that of a Lie

group, we need to restrict our attention to sections of T^tG for which some kind of left multiplication is defined.
A vector field $X \in \mathfrak{X}(G)$ is called **left-invariant** if it is a section of T^tG and satisfies:

$$(dL_g)_h X_h = X_{gh} \qquad \forall (g,h) \in G^{(2)}$$

Lemma 5. *The set of left-invariant vector fields forms a Lie subalgebra of $\mathfrak{X}(G)$. A left-invariant vector field is uniquely determined by its values along the identity bisection $\epsilon(M) \subset G$.*

Proof. Let $X, Y \in \Gamma(T^tG)$. Then X, Y are tangent to all submanifolds $t^{-1}(m)$. By basic properties of Lie brackets we can compute their Lie bracket inside $\mathfrak{X}(t^{-1}(m))$ and the result will be another vector field tangent to $t^{-1}(m), \forall m$. This shows that $[X,Y] \in \Gamma(T^tG)$.
For left-invariance we can evaluate the bracket on the fiber $t^{-1}(t(g))$ with $f \in C^\infty(t^{-1}(t(g)))$. Then for composable g, h we have $f \circ L_g \in C^\infty(t^{-1}(t(h)))$ and we compute (with a slight abuse of notation):

$$\begin{aligned} dL_g[X,Y]_h f &= X_h(Y(f \circ L_g)) - Y_h(X(f \circ L_g)) \\ &= X_h((dL_g Y)f) - Y_h((dL_g X)f) \\ &= X_h(Yf \circ L_g) - Y_h(Xf \circ L_g) \\ &= dL_g X_h(Yf) - dL_g Y_h(Xf) \\ &= X_{gh} Y f - Y_{gh} X f = [X,Y]_{gh} f \end{aligned}$$

It is easy to see that for left-invariant vector fields:

$$X_g = dL_g X_{s(g)}$$

where we identify $M \cong \epsilon(M)$. This shows that X is already determined by its values along ϵ. In fact this is true for any bisection (a simultaneous section of s and t). $\square$

Lemma 6. *Sections of $\epsilon^* T^tG = T^tG|_M$ can uniquely be extended to left-invariant vector fields on G.*

Proof. Let $X \in \Gamma(\epsilon^* T^tG)$. Define $\widehat{X}_g := dL_g X_{s(g)}$ where we regard X as a partially defined section along a submanifold. Then $\widehat{X}$ is left-invariant: It takes values in T^tG since $\mathrm{im}(dL_g) \subset T^tG$ and we have $dL_g \widehat{X}_h = dL_g(dL_h X_{s(h)}) = dL_{gh} X_{s(h)} = \widehat{X}_{gh}$. What is left to verify is smoothness of $\widehat{X}$.
For that let $f \in C^\infty(G)$ be arbitrary. Then

$$\begin{aligned} \widehat{X} f(g) &= dL_g X_{s(g)} f \\ &= X_{s(g)}(f \circ \iota_g \circ L_g) \\ &= \frac{\partial}{\partial t}\bigg|_{t=0, h=g} f(h\Theta(s(h), t)) \end{aligned}$$

where $\iota_g : t^{-1}(t(g)) \hookrightarrow G$ is the inclusion and Θ is some locally defined flow of X. This obviously depends smoothly on g. □

Lemma 7. *$Lie(G) := T^tG|_M$ has the structure of a Lie algebroid with anchor $ds : T^tG|_M \to TM$ and the bracket induced from left-invariant vector fields.*

Proof. We have seen above the identification of $\Gamma(Lie(G))$ with left-invariant vector fields on G which endows $\Gamma(Lie(G))$ with a Lie algebra structure. To verify the Leibniz rule we first observe that multiplication of $Y \in \Gamma(Lie(G))$ by $f \in C^\infty(M)$ yields the left-invariant vector field $\widehat{fY}_g = f(s(g))\widehat{Y}_g$. We compute:

$$\begin{aligned}[X, fY] &= [\widehat{X}, \widehat{fY}]|_M = [\widehat{X}, (f \circ s)\widehat{Y}]|_M \\ &= (f \circ s)[\widehat{X}, \widehat{Y}]|_M + \widehat{X}(f \circ s)\widehat{Y}|_M \\ &= f[X, Y] + ds(X)f \cdot Y\end{aligned}$$

□

It is further possible to construct a functor **LieGrpd** → **LieAlgbd** under some suitable notion of Lie algebroid morphism which will not be necessary for our purposes. [HM90] We will now turn to a couple of examples of Lie algebroids derived in this fashion.

Example 4. a) The pair groupoid $M \times M \rightrightarrows M$ has Lie algebroid TM with its standard anchor and bracket.

b) The Lie algebroid of the action groupoid $G \ltimes M \rightrightarrows M$ is the action algebroid $\mathfrak{g} \times M$. We can see this in the following way:
First, note that $t^{-1}(m) = \{(h, h^{-1}.m) | h \in G\}$. Now we can observe that any left-invariant field X must be tangent to this foliation and thus that its flow $\Theta^{(g,m)}(t) = (\Theta_G^{(g,m)}(t), \Theta_M^{(g,m)}(t)) \in G \times M$ must stay inside the t-fibers. Therefore $\Theta^{(g,m)}(t) = (\Theta_G^{(g,m)}(t), \Theta_G^{(g,m)}(t)^{-1}.m)$. We know that such a vector field is determined uniquely by its values along the identity section $(e, m), m \in M$ where X takes values

$$X_m = \frac{d}{dt}\bigg|_{t=0} (\Theta_G^{(e,m)}(t), \Theta_M^{(e,m)}(t)^{-1}.m) \in T_{(e,m)}(G \times M) \cong \mathfrak{g} \times T_mM$$

Recalling that for a left G-action τ the canonical Lie algebra homomorphism $\widehat{\tau} : \mathfrak{g} \to \mathfrak{X}(M)$ is given by $\gamma'(0) \mapsto \frac{d}{dt}\big|_{t=0} \gamma(t)^{-1}.m$, we see that the vector field X is uniquely determined by a section Y of the action algebroid $\mathfrak{g} \times M$ via $X_m = (Y_m, \widehat{\tau}_m(Y_m))$. From this representation we easily see that the anchor also coincides with the anchor of the action algebroid.
As noted above, $\Gamma(\mathfrak{g} \times M)$ is a $C^\infty(M)$-module spanned by the constant sections. The bracket on constant sections then determines the whole

bracket by the Leibniz rule. So consider $Y, Z \in \mathfrak{g}$ as constant sections. They determine left-invariant vector fields on G (in the usual Lie *group* sense) and vector fields on M via $\hat{\tau}$. It is a straightforward computation to see that their lift to a left-invariant vector field on the Lie *groupoid* $G \ltimes M$ is given by $\widehat{Y}_{(g,m)} = (Y_g, \hat{\tau}_m(Y))$ and similarly for $\widehat{Z}$.
The bracket can then be computed componentwise to yield:

$$[\widehat{Y}, \widehat{Z}] = ([Y, Z], [\hat{\tau}(Y), \hat{\tau}(Z)]) = ([Y, Z], \hat{\tau}([Y, Z]))$$

Therefore the bracket of these constant sections is just their ordinary Lie bracket regarded as a constant section. This concludes the example.

c) As one might expect, the Lie algebroid of the Atiyah (or Gauge) groupoid $(P \times P)/G$ is the Atiyah algebroid TP/G. A simple consequence is example a: For the trivial bundle $M \to M$, the associated Atiyah groupoid is the pair groupoid with algebroid TM.

As we see from the construction of $Lie(G)$ is essentially determined by a small open neighbourhood of the identity section in G. Denote $G^O \subset G$ the set of $g \in G$ that belong to the connected component of $t^{-1}(t(g))$ containing $\epsilon(t(g))$. This G^O is obviously ***t*-connected**, i.e. has connected t-fibers. Furthermore the following theorem holds:

Theorem 8. *G^O is a wide, open and t-connected subgroupoid of G with the same Lie algebroid.*

Proof. The maps $L_g : t^{-1}(s(g)) \to t^{-1}(t(g))$ are homeomorphisms. If g has $s(g) = x, t(g) = y$ and lies in the same connected component of $t^{-1}(y)$ as id_y, then $L_g(id_x) = g$ and thus the homeomorphism L_g maps the connected component of id_x to the connected component of id_y. This means that multiplication on G^O is well-defined. Also inversion restricts to G^O: $g^{-1} = (L_g)^{-1}(id_x)$ so that g^{-1} must have been in the connected component of id_y. We conclude by showing that G^O is open in G giving it a unique smooth Lie groupoid structure. It is evident that they must then yield the same Lie algebroid whose construction depends only on a neighbourhood of the identity section.
Since t is a submersion, t is locally a projection and for each $m \in M$ we can find a neighbourhood U_m of id_m that in some local coordinates is a product neighbourhood so that $U_m \cap t^{-1}(n)$ is connected $\forall n \in M$. Denote $U = \bigcup_{m \in M} U_m$ the resulting neighbourhood of the identity section. Then $G^O = G^O \cdot U$ is open as a union of open sets. $\square$

3.2 Integrability of Lie algebroids

It is now natural to ask whether all Lie algebroids can be integrated to a (possibly non-Hausdorff) Lie groupoid, i.e. if every Lie algebroid A is isomorphic to $Lie(G)$ for some Lie groupoid G. For the case of Lie groups Lie's third theorem provides an affirmative answer: There is a unique simply connected Lie group integrating any Lie algebra. It is also possible to extend this to Lie algebra bundles [CdSW99]. However, not every Lie algebroid is integrable. An important observation is the following:

Theorem 9. *If G is a t-connected Lie groupoid, there exists a unique t-simply connected Lie groupoid $\widetilde{G}$ and a homomorphism $\mathcal{F} : \widetilde{G} \to G$ such that $\mathcal{F}$ is a local diffeomorphism and $\widetilde{G}$ and G have the same Lie algebroid.*

Proof. The only possible construction is to piece together the universal covers of the t-fibers of G. So define $\widetilde{t}^{-1}(m)$ to be the universal cover of $t^{-1}(m)$ consiting of homotopy classes of paths ending at m. $\widetilde{s}$ will then be the starting point and multiplication will be multiplication of homotopy classes. This obviously defines a groupoid $\widetilde{G}$.
We find a smooth structure on $\widetilde{G}$ by the following trick: G is foliated by $\mathcal{F}_t$, the fibers of the submersion t. The associated fundamental groupoid $\Pi_1(\mathcal{F}_t)$ is a (possibly non Hausdorff) Lie groupoid (see Proposition 10). If we denote its target map by $p : \Pi_1(\mathcal{F}_t) \to G$, then $\widetilde{G} = p^{-1}(M)$ since $\Pi_1(\mathcal{F}_t)$ consists of homotopy classes of paths inside the t-fibers which by construction belong to $\widetilde{G}$ precisely when their endpoint is some $id_m \in M$. Thus $\widetilde{G}$ is an embedded submanifold as a preimage of an embedded submanifold by a submersion. To show that the structure maps are submersions we just note that they have constant rank (as shown above) and are surjective. Multiplication and inversion as well as the identity section are smooth by restriction. ☐

Proposition 10. *If $\mathcal{F}$ is a regular smooth foliation of M then the groupoid $\Pi_1(\mathcal{F})$ consisting of homotopy classes of paths that stay inside leafs is a Lie groupoid. It is not Hausdorff in general. [CF11]*

Proof. We will sketch a construction of charts. Let $[\gamma] \in \Pi_1(\mathcal{F})$ be a path connecting $p, q \in M$. Let W_0, W_1 be foliation charts around p, q with $W_i \cong U_i \times V_i \subset \mathbb{R}^p \times \mathbb{R}^q$. p will be the dimension of the leafs and V_i the transverse space to the leafs. We can assume U_i to be simply-connected.
Shrinking V_0 and V_1 we get a transverse 'holonomy lift' of γ to each of the leafs starting in V_0 and ending in V_1. We get a chart by the injection $U_0 \times V_0 \times U_1 \to \Pi_1(\mathcal{F})$ that sends a point (x_1, x_2, y) to the following concatenation: $[\gamma]$ lifts to the leaf of x_2 and connects this to V_2. We precompose this with the unique homotopy class connecting x_1 with the start of this lift and follow it by connecting the endpoint to y.
The source and target maps in these charts are just projections onto U_0, U_1 and thus are smooth submersions. ☐

It turns out that reconstructing this unique integrating groupoid purely from paths in the Lie algebroid is possible. [CF03] The only obstruction to integrability then is the existence of a smooth structure on this *Weinstein groupoid.*

4 C*-algebras

Definition 6. A ***-algebra** is an algebra over $\mathbb{C}$ with a conjugate linear involution $a \mapsto a^*$ satisfying $(ab)^* = b^*a^*$. A ***-homomorphism** is an algebra homomorphism compatible with the involutions.
A **C*-algebra** is a *-algebra with a submultiplicative norm $\|\cdot\|$ making it a Banach space. The last additional constraint is $\|a^*a\| = \|a\|^2$.
A **representation** of a C*-algebra is a pair (π, H) of a Hilbert space H and a *-homomorphism $\pi : A \to B(H)$. Two representations $(\pi_1, H_1), (\pi_2, H_2)$ are **unitarily equivalent** if there is a unitary operator $U : H_1 \to H_2$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for all $a \in A$.

It turns out that any *-homomorphism ϕ between C*-algebras is already continuous with $\|\phi\| \leqslant 1$ and ϕ is injective if and only if ϕ is isometric.
The standard example of C*-algebras are the bounded linear operators $B(H)$ on a (complex) Hilbert space H with the operator norm and hermititan adjoint as involution.
If X is a locally compact Hausdorff space, $C_0(X)$ becomes a commutative C*-algebra with the supremum norm and complex conjugation. For $X = \{pt\}$ this just yields $\mathbb{C}$.
However, note that it is explicitly not required that C*-algebras are unital (or even commutative).
As a result of the Gelfand-Naimark-Segal construction every C*-algebra is *-isomorphic to a to a C*-subalgebra of $B(H)$ for some Hilbert space H (c.f. [Put19]) If A is a commutative C*-algebra we get an even stronger result. Namely, the Gelfand representation shows that A is isometrically *-isomorphic to $C_0(X)$ where X is locally compact and Hausdorff. Explicitly, X is the set of (multiplicative, complex valued) characters of A with the weak-*-topology denoted $Spec(A)$. Furthermore $Spec(A)$ is compact if and only if A is unital.
The Gelfand duality is a contravariant equivalence of categories between commutative C*-algebras and locally compact Hausdorff spaces. This suggests that noncommutative C*-algebras arise from "noncommutative spaces". The following will develop a way of noncommutative convolution on Lie groupoids giving rise to such algebras.

4.1 Group C*-algebras

Before diving into the case of Lie groupoids we will present briefly the usual constructions of C*-algebras of discrete and topological groups.

Discrete Groups. Let G be a finite group. The complex valued functions on G form a $|G|$-dimensional vector space $\mathbb{C}G$ with standard basis $\{\delta_g | g \in G\}$ called the **group algebra**. We will define a product $*$ on $\mathbb{C}G$ by defining $\delta_g * \delta_h := \delta_{gh}$ and extending it by bilinearity to all of $\mathbb{C}G$. This yields the following convolution formula:

$$\phi * \psi = \sum_{h \in G} \sum_{g \in G} \phi(g)\psi(g^{-1}h)\delta_h \qquad \phi, \psi \in \mathbb{C}G$$

We thereby have extended the group structure of G to a "linear object" $\mathbb{C}G$. We further get an involution on $\mathbb{C}G$ by $\phi^*(g) := \overline{\phi(g^{-1})}$ making $\mathbb{C}G$ a *-algebra.

The deep relationship between G and $\mathbb{C}G$ is reflected in the bijective correspondence of unitary representations of G with nondegenerate *-representations of $\mathbb{C}G$ induced by linear continuation (see Theorem 11 below). We conclude by remarking that $\mathbb{C}G$ is a C*-algebra if we equip it with the norm induced by the injective left regular representation.

Topological Groups. Let G be a locally compact, Hausdorff group. Then there exists a (left) Haar measure μ on G that is unique up to scalar multiples, i.e. a Radon measure that is left-invariant: $\mu(gE) = \mu(E)$ for all $g \in G$ and $E \subset G$ measurable. Similarly one defines right Haar measures. Any left Haar measure induces a right Haar measure via $\mu^{-1}(E) = \mu(E^{-1})$ and vice versa. On $\mathbb{R}^n$ the Haar measure is just the Lebesgue measure, for discrete groups it is the counting measure. We will see below a proof for general Lie groups.

Measure-theoretic induction shows that μ is a left Haar measure if and only if

$$\int_G \phi(gh) d\mu(h) = \int_G \phi(h) d\mu(h) \qquad \text{for } \phi \in L^1(G).$$

Right-shift by any $g \in G$ gives us a new left Haar measure $\mu_g(E) := \mu(Eg)$. By uniqueness we must have $\mu_g = \Delta(g)\mu$. This assignment defines the **modular function** $\Delta : G \to \mathbb{R}_+$ which can be easily seen to be a continuous group homomorphism.

Generalizing the case of discrete groups, we are now going to construct a convolution algebra structure on $C_c(G)$ using the unique Haar measure μ. Define the convolution product and involution by:

$$\phi*\psi(g) := \int_G \phi(h)\psi(h^{-1}g) d\mu(h) \qquad \phi^*(g) := \Delta(g)^{-1}\overline{\phi(g^{-1})} \qquad \phi, \psi \in C_c(G)$$

This makes $C_c(G)$ into a *-algebra. Convolution is commutative if and only if G is abelian. By Young's inequality all this extends to $L^1(G)$. Using the modular function, one can verify that $*$ is an isometric involution and convolution is submultiplicative. However, L^1 may still fail the C*-identity. The solution that will also be described later is a completion with respect to another norm.

Theorem 11. *Let G be a locally compact, Hausdorff group. If $\pi : G \to U(H)$ is a unitary representation of G, then*

$$\begin{aligned} \widetilde{\pi} : L^1(G) &\longrightarrow B(H) \\ \phi &\mapsto \int_G \phi(g)\pi(g)dg \end{aligned}$$

*is a nondegenerate *-representation of $L^1(G)$. Every unitary-representation is uniquely determined in this way by a representation of $L^1(G)$.*

Remark 1. The operator $\widetilde{\pi}(\phi)$ needs to be understood in this way:

$$(\eta, \xi) \mapsto \int_G \langle \phi(g)\pi(g)\eta, \xi \rangle dg \qquad \eta, \xi \in H$$

is bilinear and thus, by Lax-Milgram theorem, induced by $\widetilde{\pi}(\phi) \in B(H)$ in such a way that this bilinear map takes the form $\langle \widetilde{\pi}(\phi)\eta, \xi \rangle$.

Remark 2. Any group admits the **left regular representation** $\lambda : L^1(G) \to U(L^2(G))$ by $[\lambda(g)\phi](h) := \phi(g^{-1}h)$. This representation induces the convolution representation $\widetilde{\lambda}(\phi)\eta = \phi * \eta$ for $\phi \in L^1, \eta \in L^2$. $\widetilde{\lambda}$ is faithful as can be seen by approximate units in $L^2(G)$.

For a proof we refer to [Fol16]. The Renault disintegration theorem provides a generalization of the above for groupoids, but requires representations of groupoids on Borel Hilbert bundles. (c.f. [RW98], [Ren80])

4.2 The C*-algebra of a Lie groupoid

Much as in the case for topological (Lie) groups our aim is to construct an algebraic invariant of the underlying groupoid. The appropriate analogue and generalization of the Haar measure on topological groupoids is the notion of a Haar system. [Ren80]

Definition 7. Let $G \rightrightarrows M$ be a topological locally compact and Hausdorff groupoid (or a Lie groupoid). A **left Haar system** on G is a family of Radon measures $\{\mu_m\}, m \in M$ indexed by the base manifold such that:

(i) $\mathrm{supp}(\mu_m) = t^{-1}(m)$

(ii) for any $f \in C_c(M)$ the map

$$m \mapsto \int_{t^{-1}(m)} f d\mu_m$$

is continuous (respectively smooth).

(iii) for any $f \in C_c(M)$ the following left invariance holds:

$$\int f(h) d\mu_{t(g)}(h) = \int f(gh) d\mu_{s(g)}(h)$$

Note that the last formula makes sense since $\text{supp}(\mu_{s(g)}) = t^{-1}(s(g))$ and thus the composition of g, h in the integrand is well-defined on the support of $\mu_{s(g)}$.

Proposition 12. *Any Lie groupoid admits a smooth (left) Haar system.*

Proof. Let $G \rightrightarrows M$ be a Lie groupoid with associated Lie algebroid $A \to M$. Then $k := \text{rank}A$ is also the dimension of the t-fibers. Let $\rho \in |\Lambda^k A^*|$ be a top degree non-zero section.
We can extend ρ smoothly to G by setting $\rho_g = L^*_{g^{-1}} \rho_{s(g)}$. Then ρ satisfies $L^*_g \rho = \rho$, because:

$$\begin{aligned}(L^*_g \rho)_h &= L^*_g \rho_{gh} = L^*_g (L^*_{h^{-1}g^{-1}}) \rho_{s(gh)} \\ &= L^*_{h^{-1}g^{-1}g} \rho_{s(h)} = \rho_h.\end{aligned}$$

We get positive linear functionals on $C^\infty_c(G)$ by

$$f \mapsto \int_{t^{-1}(m)} f\rho$$

which by continuity extend to $C_c(G)$. The Riesz representation theorem now yields a family of Radon measures $\{\mu_m\}, m \in M$ on G with support in $t^{-1}(m)$ such that $\int f d\mu_m = \int_{t^{-1}(m)} f\rho$.
The assignment $m \mapsto \int f d\mu_m$ is easily seen to be smooth and we furthermore compute:

$$\begin{aligned}\int f(gh) d\mu_{s(g)}(h) &= \int_{t^{-1}(s(g))} (f \circ L_g)\rho = \int_{t^{-1}(s(g))} L^*_g (f\rho) \\ &= \int_{t^{-1}(t(g))} f\rho = \int f(h) d\mu_{t(g)}(h)\end{aligned}$$

where we used that $L_g : t^{-1}(s(g)) \to t^{-1}(t(g))$ is a diffeomorphism and that ρ is invariant under pullback by L_g. Thus we have found a Haar system on G. In the case of Lie groups we have in fact proven existence of a unique Haar measure up to a multiplicative constant. □

We can now describe the construction of the C*-algebra of a groupoid. For this purpose, let $G \rightrightarrows M$ be a locally compact Hausdorff groupoid with a fixed Haar system $\{\mu_m\}_{m \in M}$. First, $C^\infty_c(G)$ becomes a $*$-algebra by the **convolution**

$$\phi * \psi(g) := \int \phi(h) \psi(h^{-1}g) d\mu_{t(g)}(h) \qquad \phi^*(g) := \overline{\phi(g^{-1})} \qquad \phi, \psi \in C^\infty_c(G).$$

The convolution is well-defined. It is smooth (as can most easily be checked in local coordinates by applying dominated convergence) and is compactly supported inside $\mathrm{supp}\phi \cdot \mathrm{supp}\psi$. It is associative and the verification of the axioms is a computation using basic properties of the Haar system. The isomorphism of group, viewed as a groupoid, from the groupoid convolution algebra to the group version is given by $f \mapsto \Delta^{\frac{1}{2}} f$. There is no canonical analogue of a modular function on groupoids as there is no uniqueness of the Haar system.

We are now constructing representations of $C_c^\infty(G)$ following [Muh97].

Definition 8. Let λ be a measure on M. Denote the **induced measure** on G by $\nu = \lambda \circ \mu$ given by successive integration:

$$\int f d\nu = \int_M \int f d\mu_m d\lambda(m).$$

Denote ν^{-1} the measure determined by precomposition of ν with the inversion homeomorphism. To integrate with respect to ν^{-1} we can use $\int f d\nu^{-1} = \int f(g^{-1}) d\nu(g)$.

Define the norm $\|f\|_I := \max(\|f\|_{I,t}, \|f\|_{I,s})$ where

$$\|f\|_{I,t} = \sup_{m \in M} \int |f| d\mu_m \qquad \|f\|_{I,s} = \sup_{m \in M} \int |f(g^{-1})| d\mu_m(g)$$

We define the **induced representation** of λ on $L^2(\nu^{-1})$ to be

$$[\mathrm{Ind}\lambda(\phi)\xi](g) := \phi * \xi(g) \qquad \phi \in C_c^\infty(G), \xi \in L^2(\nu^{-1})$$

A $*$-representation π is called **bounded** if $\|\pi(f)\| \leqslant \|f\|_I$.

Lemma 13. *$\mathrm{Ind}\lambda$ is indeed a bounded $*$-representation of $C_c^\infty(G)$.*

Proof. Let $\phi, \psi \in C_c^\infty(G)$ and $\xi, \eta \in L^2(\nu^{-1})$. We will first show that indeed $\phi * \xi \in L^2(\nu^{-1})$ and that the norm is bounded. First note that by Cauchy-Schwarz inequality

$$\begin{aligned} |\phi * \xi(g)| &\leqslant \int |\phi(h)||\xi(h^{-1}g)| d\mu_{t(g)}(h) \\ &\leqslant \left[\int |\phi(h)||\xi(h^{-1}g)|^2 d\mu_{t(g)}(h)\right]^{\frac{1}{2}} \left[\int |\phi| d\mu_{t(g)}\right]^{\frac{1}{2}}. \end{aligned}$$

The second factor is always smaller than $\|\phi\|_I^{\frac{1}{2}}$. We use this to compute:

$$\begin{aligned} \|\phi * \xi\|^2_{L^2(\nu^{-1})} &= \int \int |\phi * \xi(g^{-1})|^2 d\mu_m(g) d\lambda(m) \\ &\leqslant \int d\lambda \|\phi\|_I \int_{t^{-1}(m)} \int_{t^{-1}(m)} |\phi(g^{-1}h)||\xi(h^{-1})| d\mu_m(h) d\mu_m(g) \\ &\leqslant \|\phi\|_I \int d\lambda \int |\xi(h^{-1})| \int |\phi(g^{-1}h)| d\mu_m(g) d\mu_m(h) \\ &\leqslant \|\phi\|_I^2 \|\xi\|^2_{L^2(\nu^{-1})} \end{aligned}$$

Therefore Indλ is bounded. It is a homomorphism since Ind$\lambda(\phi * \psi)\xi = (\phi * \psi) * \xi = \phi * (\psi * \xi) = \text{Ind}\lambda(\phi)\text{Ind}\lambda(\psi)\xi$ by associativity of convolution. Lastly, Indλ is also a $*$-homomorphism:

$$\begin{aligned}\langle \text{Ind}\lambda(\phi^*)\xi, \eta\rangle &= \int (\phi^* * \xi)(g^{-1})\overline{\eta(g^{-1})}d\nu \\ &= \int d\lambda \int d\mu_m(g) \left[\int \overline{\phi(h^{-1})}\xi(h^{-1}g^{-1})d\mu_{t(g^{-1})}\overline{\eta(g^{-1})}\right] \\ &= \int d\lambda \int d\mu_m(g) \int d\mu_m(h)\overline{\phi(h^{-1}g)}\xi(h^{-1})\overline{\eta(g^{-1}h)} \\ &= \int d\lambda \int d\mu_m(h)\xi(h^{-1})\overline{\int \phi(h^{-1}g)\eta(g^{-1})d\mu_m(g)} \\ &= \int d\lambda \int d\mu_m(h)\xi(h^{-1})\overline{\phi * \eta(h^{-1})} \\ &= \langle \xi, \text{Ind}\lambda(\phi)\eta\rangle = \langle \text{Ind}\lambda(\phi)^*\xi, \eta\rangle\end{aligned}$$

And thus Ind$\lambda(\phi^*) = \text{Ind}\lambda(\phi)^*$. □

For the Dirac measure ϵ_m concentrated at $m \in M$ we thus get representations Indϵ_m on $L^2(s^{-1}(m))$ equipped with the inverse Haar measure. We refer to them as **left regular representations**.
For $g \in G$ the two representations Ind$\epsilon_{s(g)}$ and Ind$\epsilon_{t(g)}$ are unitarily equivalent by $U : L^2(s^{-1}(s(g))) \to L^2(s^{-1}(t(g)))$ with $Uf(h) = f(hg)$.

Definition 9. The **reduced C^*-algebra** $C^*_{\text{red}}(G)$ is the completion of $C_c^\infty(G)$ in the norm

$$\|\phi\|_{\text{red}} = \sup\{\|\text{Ind}\epsilon_m(\phi)\| : m \in M\}.$$

The **full C^*-algebra** $C^*_{\text{full}}(G)$ is the completion with respect to the supremum norm that ranges over all bounded $*$-representations of $C_c^\infty(G)$.

These are obviously seminorms. We have shown that these are actually norms by the construction of the representations above so that the supremum is always positive. We remark that construction makes direct use of one particular Haar measure on G. But in fact, the resulting C^*-algebra is somewhat independent of the Haar measure used (see section 6). In what follows we are only going to consider reduced C*-algebras. By unitary equivalence of many of the representations Indϵ_m, it actually suffices to consider only one for each G-orbit.

Example 5. a) The trivial groupoid $M \rightrightarrows M$ has as Haar system the counting or Dirac measures. Convolution is just pointwise multiplication and the reduced norm is the supremum norm on $C_c^\infty(M)$. Its completion is $C^*(M) = C_0(M)$.

b) Let M be a manifold and λ a measure with full support on M. Then the pair groupoid $M \times M \rightrightarrows M$ has a Haar system given by $\epsilon_m \times \lambda$. Convolution of $f, g \in C_c^\infty(M \times M)$ is then given by convolution of Hilbert-Schmidt integral kernels:

$$f * g(m,n) = \int_M f(m,k)g(k,n)d\lambda(k).$$

All representations being unitarily equivalent, we see that $C_c^\infty(M \times M)$ is completed in the norm of the representation π on $L^2(M)$ given by:

$$\pi(f)\xi(m) = \int_M f(m,n)\xi(n)d\lambda(n).$$

Thus every f can be identified with a smooth integral kernel in the compact operators $K(L^2(M))$. To see that they lie dense, it suffices to show that they are dense in the finite rank operators. This is easily verified using that $(L^2)^* = L^2$. Therefore we conclude that $C^*(M \times M) \cong K(L^2(M))$.

5 The magnetic monopole

This section finally deals with the magnetic symplectic singularity. We will remove this singularity by means of Dirac structures. The resulting Lie algebroid is integrable to a Lie groupoid (Proposition 17). We will then treat its convolution algebra in section 5.3. This procedure might generally be interesting to classify and describe types of removable singularities.

5.1 Dirac structures

A Dirac structure is a simultaneous generalization of Poisson and Symplectic Geometry in that both provide canonical examples of such a structure.
On the bundle $TM \oplus T^*M \to M$ we have the following additional structure: A nondegenerate symmetric and fibrewise bilinear form:

$$\langle X + \alpha, Y + \beta \rangle = \beta(X) + \alpha(Y) \qquad X, Y \in T_mM \quad \alpha, \beta \in T_m^*M,$$

and the **Courant bracket** $[\![\cdot,\cdot]\!]$ on the sections of $TM \oplus T^*M$

$$[\![(X,\alpha),(Y,\beta)]\!] = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}d(\alpha(Y) - \beta(X)). \tag{4}$$

The Courant bracket is *not* a Lie bracket. A long calculation gives the following violation of the Jacobi identity. [Bur13]

$$[\![[\![a_1,a_2]\!],a_3]\!] + c.p. = \frac{1}{3}d(\langle [\![a_1,a_2]\!],a_3\rangle + c.p.) \tag{5}$$

where *c.p.* means cyclic permutations.

Definition 10. A **Dirac structure** over M is a subbundle $L \subset TM \oplus T^*M$ that satisfies

(i) $L = L^\perp$ with respect to the pairing $\langle \cdot, \cdot \rangle$

(ii) $[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$

Using equation (5) and (i) we see that (ii) is equivalent to $\langle [\![a_1, a_2]\!], a_3 \rangle = 0$ for all $a_1, a_2, a_3 \in \Gamma(L)$. By nondegeneracy, (i) is equivalent to $\mathrm{rank}(L) = \dim(M)$ and $\langle \cdot, \cdot \rangle|_L = 0$.
We also see that $[\![\cdot, \cdot]\!]|_L$ does provide a Lie bracket. Using Cartan's magic formula on (4) and the vanishing of $\langle \cdot, \cdot \rangle$ on L we immediately get a different expression for the Courant bracket: $[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - i_Y d\alpha$.

Lemma 14. *Any Dirac structure $L \to M$ provides a Lie algebroid with anchor pr_{TM} and the Courant bracket.*

Proof. We only have to check the Leibniz rule. Let $X + \alpha, Y + \beta \in \Gamma(L)$ and $f \in C^\infty(M)$. Then

$$\begin{aligned}
[\![X + \alpha, f(Y + \beta)]\!] &= [X, fY] + \mathcal{L}_X(f\beta) - \mathcal{L}_{fY}\alpha + \frac{1}{2}d(\alpha(fY) - f\beta(X)) \\
&= f[X, Y] + Xf \cdot Y + Xf \cdot \beta + f\mathcal{L}_X\beta - f\mathcal{L}_Y\alpha - \alpha(Y)df \\
&\quad + \frac{1}{2}(\alpha(Y) - \beta(X))df + \frac{1}{2}fd(\alpha(Y) - \beta(X)) \\
&= f[\![X + \alpha, Y + \beta]\!] + Xf \cdot (Y + \beta).
\end{aligned}$$

We are done because $pr_{TM}(X + \alpha) = X$. □

The canonical examples of Dirac structures are the graphs of Poisson bivectors and presymplectic forms. Since it is more relevant to our case, we will only discuss the latter.

Example 6. Let $\omega \in \Omega^2(M)$ be a closed 2-form, i.e. (M, ω) is a presymplectic manifold. The canonical map $\omega^\sharp : TM \to T^*M$ given by interior multiplication gives us a graph Dirac structure.

$$\mathrm{Graph}(\omega) = L_\omega = \{X + \omega^\sharp(X) : X \in TM\} \subset TM \oplus T^*M$$

By skew-symmetry $\langle \cdot, \cdot \rangle|_{L_\omega} = 0$ and it follows as in the remark above that $L_\omega = L_\omega^\perp$. An easy calculation yields

$$\langle [\![X + \omega^\sharp(X), Y + \omega^\sharp(Y)]\!], Z + \omega^\sharp(Z) \rangle = d\omega(X, Y, Z) = 0,$$

so that $[\![\cdot, \cdot]\!]|_{L_\omega}$ does provide a Lie bracket and $L_\omega = \mathrm{Graph}(\omega)$ is a Dirac structure. Note that $L_\omega \cap T^*M = \{0\}$. In fact, it is easy to see that this last property uniquely identifies Dirac structures that are induced by a closed 2-form.

5.2 The Dirac structure of the magnetic monopole

This section follows [Blo17]. We are going to deal with the *magnetic symplectic form* ω of a magnetic monopole in 2 dimensions, which is a 2-form on $M := T^*\mathbb{R}^2$ given by

$$\omega = \omega_0 + B = dq^i \wedge dp_i + \frac{1}{r^2} dq^1 \wedge dq^2,$$

where $\{q^i, p_i\}$ are the standard global Darboux coordinates on M and $r^2 = |q|^2 = (q^1)^2 + (q^2)^2$. ω_0 is the standard symplectic form on M.
ω has a singularity at the origin which just means that ω is not defined on and cannot be extended to $T_0^*\mathbb{R}^2$.
[Blo17] proposes a process to get rid of this singularity by introducing a Dirac structure.

Proposition 15. *The Dirac structure Graph(ω) of the magnetic form $\omega \in \Omega^2(T^*\mathbb{R}^2 \backslash T_0^*\mathbb{R}^2)$ extends to a smooth Dirac structure $L := \overline{Graph(\omega)}$ on all of $T^*\mathbb{R}^2$.*
However, L is not the graph of any symplectic form.

Proof. By our example above Graph(ω) is a Dirac structure over $T^*\mathbb{R}^2_\times = T^*\mathbb{R}^2 \backslash T_0^*\mathbb{R}^2$ where we denote $\mathbb{R}^2_\times = \mathbb{R}^2 \backslash \{0\}$.
Since $\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i}$ span $T(T^*\mathbb{R}^2_\times)$, Graph($\omega$) will be spanned by

$$\left\{ \frac{\partial}{\partial q^i} + i_{\frac{\partial}{\partial q^i}} \omega, \frac{\partial}{\partial p^i} + i_{\frac{\partial}{\partial p^i}} \omega : i = 1, 2 \right\}.$$

Computing these, we get the following sections:

$$\begin{array}{ll} \widetilde{a}_1 = \frac{\partial}{\partial q^1} + dp_1 + \frac{1}{r^2} dq^2 & b_1 = \frac{\partial}{\partial p_1} - dq^1 \\ \widetilde{a}_2 = \frac{\partial}{\partial q^2} + dp_2 - \frac{1}{r^2} dq^1 & b_2 = \frac{\partial}{\partial p_2} - dq^2 \end{array}$$

To remove the singular part of $\widetilde{a}_i$, we introduce the sections

$$\begin{aligned} a_1 &= -r^2 \widetilde{a}_2 = -r^2 \frac{\partial}{\partial q^2} - r^2 dp_2 + dq^1 \\ a_2 &= r^2 \widetilde{a}_1 = r^2 \frac{\partial}{\partial q^1} + r^2 dp_1 + dq^2 \end{aligned}$$

On $\mathbb{R}^2_\times$, r^2 is a positive function, so that $\{a_i, b_i\}$ will still span Graph(ω). But we further observe that a_i, b_i can be extended to $T_0^*\mathbb{R}^2$ where they remain linearly independent. This makes the topological closure $L = \overline{\text{Graph}(\omega)}$ a smooth subbundle.
Being a subbundle already suffices to equip L with a Dirac structure: Since the pairing $\langle \cdot, \cdot \rangle$ is a smooth operation, we see that the singular fiber will again be isotropic by continuity, i.e. $L = L^\perp$. The bracket will also restrict since $\langle [\![X_1, X_2]\!], X_3 \rangle$ is smooth and will vanish on $T^*\mathbb{R}^2_\times$. It must then vanish identically on all of $T^*\mathbb{R}^2$.

□

As seen above, Dirac structures can be viewed as Lie algebroids.

Proposition 16. *The magnetic Lie algebroid $L \to T^*\mathbb{R}^2$ is a trivial vector bundle spanned by the 4 sections a_1, a_2, b_1, b_2 with anchor ρ and bracket $[\cdot,\cdot]$ given by:*

$$\boxed{\begin{array}{lll} \rho(a_1) = -r^2\frac{\partial}{\partial q^2} & \rho(b_1) = \frac{\partial}{\partial p_1} & [a_1, a_2] = -2q^1 a_1 - 2q^2 a_2 \\ \rho(a_2) = r^2\frac{\partial}{\partial q^1} & \rho(b_2) = \frac{\partial}{\partial p_2} & [a_i, b_j] = [b_i, b_j] = 0 \end{array}}$$

Proof. We are only left to compute the Lie brackets. For this we can use the trick that ρ is a Lie algebra homomorphism.
It is easy to see that $[\rho(a_i), \rho(b_j)] = [\rho(b_i), \rho(b_j)] = 0$. Away from the singular fiber, any section $c \in \Gamma(L)$ is given by $c = \rho(c) + i_{\rho(c)}\omega$. It then follows necessarily that $[a_i, b_j] = [b_i, b_j] = 0$.
For computing $[a_1, a_2]$ just note that

$$\left[-r^2\frac{\partial}{\partial q^2}, r^2\frac{\partial}{\partial q^1}\right] = 2q^1r^2\frac{\partial}{\partial q^2} - 2q^2r^2\frac{\partial}{\partial q^1}.$$

The same trick as above implies $[a_1, a_2] = -2q^i a_i$. □

The natural question that arises is whether L is integrable. Denote by $L_\times$ the restriction of L to $T^*\mathbb{R}^2_\times$ which is just the graph of ω as a Dirac structure. We see that $\rho : L_\times \to T(T^*\mathbb{R}^2_\times)$ is an isomorphism (because $r^2 \neq 0$). Hence $L_\times$ is integrable by the pair groupoid $T^*\mathbb{R}^2_\times \times T^*\mathbb{R}^2_\times \rightrightarrows T^*\mathbb{R}^2_\times$. However, this groupoid is not t-simply connected as every t-fiber is isomorphic to $T^*\mathbb{R}^2_\times$ which is homotopic to $\mathbb{C}\backslash\{0\}$. Using the universal cover $\mathbb{C} \to \mathbb{C}\backslash\{0\} : z \mapsto e^z$, we can replace the target fibers by their universal cover (cf. Theorem 9) and we get the following t-simply connected groupoid integrating $L_\times$:

$$\Gamma = T^*\mathbb{C} \times T^*\mathbb{R}^2_\times \rightrightarrows T^*\mathbb{R}^2_\times$$

$$s\left((z, v), (q, p)\right) = (q, p) \quad t((z, v), (q, p)) = (qe^z, v)$$

$$((z, v), (q, p))((z', p), (q', p')) = ((z + z', v), (q', p')) \quad \text{where } q = q'e^{z'}$$

We are identifying here $\mathbb{C} \cong \mathbb{R}^2 \cong T^*_q\mathbb{R}^2 \cong T^*_q\mathbb{C}$ by using the usual global trivialization by coordinate differentials. The isotropy at any point is isomorphic to $\mathbb{Z}$ corresponding to the Deck transformations of the universal cover. If L is integrable by a t-simply connected Lie groupoid $G \rightrightarrows T^*\mathbb{R}^2$ we must find Γ as the subgroupoid $s^{-1} \cap t^{-1}(T^*\mathbb{R}^2_\times)$ by uniqueness of the integrating groupoid. This suggests to take some closure of Γ.

Proposition 17. *$L \to T^*\mathbb{R}^2$ is integrable to a Lie groupoid $G \rightrightarrows T^*\mathbb{R}^2$ given by:*

$$G = T^*\mathbb{C} \times T^*\mathbb{R}^2 \rightrightarrows T^*\mathbb{R}^2 \tag{6}$$
$$s(z,v,q,p) = (q,p) \quad t(z,v,q,p) = (qe^{\overline{q}z}, v) \tag{7}$$
$$(z,v,q,p)(z',p,q',p') = (e^{q'\overline{z}'}z + z', v, q', p') \quad \text{where } q = q'e^{\overline{q}'z'} \tag{8}$$
$$\mathrm{id}_{(q,p)} = (0,p,q,p) \tag{9}$$
$$(z,v,q,p)^{-1} = (-ze^{-q\overline{z}}, p, qe^{\overline{q}z}, v) \tag{10}$$

Proof. We need to verify first that G is indeed a groupoid, then that it is Lie and lastly that its Lie algebroid of left-invariant vector fields is isomorphic to L.
Source and target maps are well-defined making the canonical identifications $\mathbb{C} \cong \mathbb{R}^2$ and $T^*\mathbb{C} \cong T^*\mathbb{R}^2$. It is then easy to see that the composition behaves well with respect to the source map, i.e. $s(gh) = s(h)$. For the target map we compute:

$$q'e^{\overline{q}'\left(e^{q'\overline{z}'}z+z'\right)} = \left(q'e^{\overline{q}'z'}\right)e^{\overline{q'e^{\overline{q}'z'}}z} = qe^{\overline{q}z}.$$

This proves that $t((z,v,q,p)(z',p,q',p')) = t((z,v,q,p))$. For each $(q,p) \in T^*\mathbb{R}^2$ $\mathrm{id}_{(q,p)}$ provides an identity element as can be verified by a quick calculation. That (10) does indeed provide an inverse is checked in this way: Clearly $s((z,v,q,p)^{-1}) = t((z,v,q,p))$ by reading off the last two entries. Also

$$t((z,v,q,p)^{-1}) = (qe^{\overline{q}z}e^{\overline{qe^{\overline{q}z}}(-ze^{-q\overline{z}})}, p) = (q,p) = s((z,v,q,p)).$$

Therefore composition with the inverse on both sides is well-defined and we can compute:

$$(z,v,q,p)(-ze^{-q\overline{z}}, p, qe^{\overline{q}z}, v) = \left(e^{qe^{\overline{q}z}\left(\overline{-ze^{-q\overline{z}}}\right)}z - ze^{-q\overline{z}}, v, qe^{\overline{q}z}, v\right) = \mathrm{id}_{(qe^{\overline{q}z},v)}$$
$$(-ze^{-q\overline{z}}, p, qe^{\overline{q}z}, v)(z,v,q,p) = \left(e^{q\overline{z}}(-ze^{-q\overline{z}} + z, p, q, p)\right) = \mathrm{id}_{(q,p)}$$

This finishes the verification that G is a groupoid. Equipping G with the standard smooth structure, we see that all structure maps are smooth. s is a projection and thus clearly a submersion. t is also a submersion since $(z,q) \mapsto (qe^{\overline{q}z})$ is a submersion. Therefore G admits a smooth Lie groupoid structure.
We are left to show that the Lie algebroid $\mathrm{Lie}(G) = T^tG|_{T^*\mathbb{R}^2}$ is isomorphic to L. Let $X \in \mathfrak{X}(T^tG)$ be a left-invariant vector field. X is determined by its values along the identity bisection $\mathrm{id}_{(q,p)}$. We consider the flow $\Theta^{(q,p)}$ of X along this bisection. Write $\Theta^{(q,p)} = (\Theta_1^{(q,p)}, \Theta_2^{(q,p)}, \Theta_3^{(q,p)}, \Theta_4^{(q,p)})$. By definition of the flow starting at $\mathrm{id}_{(q,p)}$ we have

$$\Theta_1^{(q,p)}(0) = 0, \quad \Theta_2^{(q,p)}(0) = \Theta_4^{(q,p)}(0) = p, \quad \Theta_3^{(q,p)}(0) = q.$$

Since X is tangent to the t-fibers the flow will stay inside the t-fibers and thus $(q,p) = t(\Theta^{(q,p)}) = (\Theta_3^{(q,p)} e^{\overline{\Theta_3^{(q,p)}}\Theta_1^{(q,p)}}, \Theta_2^{(q,p)})$. Therefore $\Theta_2^{(q,p)}(s) = p$ for all s in the flow domain.
Differentiating the equality $q = \Theta_3 e^{\overline{\Theta_3}\Theta_1}$ at $s = 0$ gives:

$$\begin{aligned} 0 &= \left(\Theta_3'(0) + \Theta_3(0)\overline{\Theta_3'(0)}\Theta_1(0) + \Theta_3(0)\overline{\Theta_3(0)}\Theta_1'(0)\right) e^{\overline{\Theta_3(0)}\Theta_1(0)} \\ &= \left(\Theta_3'(0) + 0 + q\overline{q}\Theta_1'(0)\right) e^0 = X_3 + |q|^2 X_1, \end{aligned}$$

where we denoted the components of X at $\mathrm{id}_{(q,p)}$ by X_i. It follows from this discussion that left-invariant vector fields along the identity bisection are exactly of the form

$$X_{(q,p)} = \left(X_1, 0, -|q|^2 X_1, X_2\right),$$

where X_1 is in $T\mathbb{C} \cong T\mathbb{R}^2 \subset TT^*\mathbb{R}^2$ and X_2 is in the vertical part of $TT^*\mathbb{R}^2$, i.e. $X_1 \in \mathrm{span}(\frac{\partial}{\partial q^i})$ and $X_2 \in \mathrm{span}(\frac{\partial}{\partial p_i})$. We get a global trivialization frame for the vector bundle $\mathrm{Lie}(G)$ by defining

$$\begin{aligned} a_1 &= (\tfrac{\partial}{\partial q^2}, 0, -|q|^2\tfrac{\partial}{\partial q^2}, 0) & b_1 &= (0,0,0,\tfrac{\partial}{\partial p_1}) \\ a_2 &= (-\tfrac{\partial}{\partial q^1}, 0, |q|^2\tfrac{\partial}{\partial q^1}, 0) & b_2 &= (0,0,0,\tfrac{\partial}{\partial p_2}) \end{aligned}$$

As s is just the projection onto the last two components, the anchor $\rho = ds$ will also be the projection onto the last components. The bracket is uniquely determined by the fact that ρ is a Lie algebra homomorphism and continuity. We therefore see that

$$\begin{aligned} \rho(a_1) &= -|q|^2\tfrac{\partial}{\partial q^2} & \rho(b_1) &= \tfrac{\partial}{\partial p_1} & [a_1, a_2] &= -2q^1 a_1 - 2q^2 a_2 \\ \rho(a_2) &= |q|^2\tfrac{\partial}{\partial q^1} & \rho(b_2) &= \tfrac{\partial}{\partial p_2} & [a_i, b_j] &= [b_i, b_j] = 0. \end{aligned}$$

Comparing this to the magnetic algebroid $L \to T^*\mathbb{R}^2$ we get an obvious isomorphism $\mathrm{Lie}(G) \cong L$. □

The resulting groupoid has two orbits: $T_0^*\mathbb{R}^2$ and $T^*\mathbb{R}^2_\times$.

5.3 The noncommutative geometry of the magnetic monopole

Our next goal is to compute the C*-algebra of the magnetic monopole which can be interpreted as the noncommutative geometry arising from it. For this we have to find and choose a (left) Haar system. It turns out that finding a right Haar system is slightly easier, but as the two are related by inversion we are going to stick with the associated left Haar system.

Lemma 18. *A left Haar system on $G \rightrightarrows T^*\mathbb{R}^2$ is given by:*

$$\int f d\mu_{(q,p)} = \int f(-ze^{-q\overline{z}}, p, qe^{\overline{q}z}, v)|e^{-\overline{q}z}|dvdz,$$

where dv, dz are two dimensional Lebesgue measures on $\mathbb{C} \cong \mathbb{R}^2$.

Proof. The Riesz representation theorem guarantees the existence of such a system of measures. Note that we integrate over $(z, v, q, p)^{-1} = (-ze^{-q\overline{z}}, p, qe^{\overline{q}z}, v)$. Therefore $t((z, v, q, p)^{-1}) = (q, p)$ and this is also a surjective parametrization for the t-fiber $t^{-1}(q, p)$. This shows that $\operatorname{supp}(\mu_{(q,p)}) = t^{-1}(q, p)$.
For left-invariance we compute:

$$\begin{aligned}\int f \circ L_{(z',v',q,p)} d\mu_{(q,p)} &= \int f((z', v', q, p)(-ze^{-q\overline{z}}, p, qe^{\overline{q}z}, v))|e^{\overline{q}z}|dvdz \\ &= \int f((z' - z)e^{-q\overline{z}}, v', qe^{\overline{q}z}, v)|e^{\overline{q}z}|dvdz.\end{aligned}$$

While on the other hand $t(z', v', q, p) = (qe^{\overline{q}z'}, v')$ and

$$\begin{aligned}\int f d\mu_{(qe^{\overline{q}z'},v')} &= \int f(-ze^{-qe^{\overline{q}z'}\overline{z}}, v', qe^{\overline{q}z'}e^{\overline{qe^{\overline{q}z'}}z}, v)|e^{\overline{qe^{\overline{q}z'}}z}|dvdz \\ &= \int f(-(w - z')e^{-q\overline{z}'}e^{-q(\overline{w}-\overline{z}')}, v', qe^{\overline{q}w}, v)|e^{-\overline{q}(w-z')}||e^{-q\overline{z}'}|dvdw,\end{aligned}$$

where we have substituted $w = z' + ze^{q\overline{z}'}$ with $dw = |e^{q\overline{z}'}|dz$. These expressions obviously match. Lastly, we need to verify smoothness of $(q, p) \mapsto \int f d\mu_{(q,p)}$ for $f \in C_c^\infty(G)$. For this we just rewrite $|e^{\overline{q}z}| = e^{q\cdot z}$. Then the integrand is smooth in (z, v, q, p) and thus the integral will depend smoothly on (q, p). □

This Haar system induces the convolution

$$f * g(z, v, q, p) = \int f((z - w)e^{-q\overline{w}}, v, qe^{\overline{q}w}, u)g(w, u, q, p)e^{q\cdot w}dudw \tag{11}$$

and involution

$$f^*(z, v, q, p) = \overline{f(-ze^{-q\overline{z}}, p, qe^{\overline{q}z}, v)}. \tag{12}$$

Note that $f * g(z, v, 0, p) = \int f(z - w, v, 0, u)g(w, u, 0, p)dudw$ takes a much nicer form when restricted to the singular fiber $T_0^*\mathbb{R}^2$. Also $f^*(z, v, 0, p) = \overline{f(-z, p, 0, v)}$. This motivates the following lemma.

Lemma 19. *Let $G \rightrightarrows M$ be a Lie groupoid.*

*(i) Let $U \subset M$ be an open orbit of G and $H = s^{-1}(U) \cap t^{-1}(U)$. Then we get an injective *-homomorphism $i : C^*(H) \hookrightarrow C^*(G)$. Its image is an ideal.*

*(ii) Let $C \subset M$ be a closed orbit of G and $S = s^{-1}(C) \cap t^{-1}(C)$. Then we get a surjective *-homomorphism $r : C^*(G) \to C^*(S)$ by restriction.*

Proof. The map $i : C_c^\infty(H) \to C_c^\infty(G)$ is given by continuation by 0. $r : C_c^\infty(G) \to C_c^\infty(S)$ is given by restriction. For this we need the sets to be open and closed respectively.

First, we need to implicitly restrict the Haar systems to the subgroupoids. It is then easy to see that the convolution restricts when we are dealing with orbits (or unions thereof). Involution obviously restricts to subgroupoids, so that i, r are *-homomorphisms.
i is an isometry since the left regular representations of H trivially agree with their counterparts in G. The other representations will precisely be those of U^c where any $i(f)$ will vanish identically. The supremum norm will thus agree and i extends to an isometric embedding $C^*(H) \to C^*(G)$.
For r we are reducing the number of representations which makes it trivially continuous. It then extends to a *-homomorphism of the completions with dense image. Since the image of *-homomorphisms of C*-algebras is always closed, r extends to a surjective map. □

In our example we are interested in the singular closed subgroupoid S lying over $T_0^*\mathbb{R}^2$ and its regular open counterpart Γ lying over $T^*\mathbb{R}^2_\times$. Note that $0 \longrightarrow C_c^\infty(\Gamma) \longrightarrow C_c^\infty(G) \longrightarrow C_c^\infty(S) \longrightarrow 0$ is certainly not exact since functions vanishing at 0 need not vanish on a neighbourhood of 0. We could hope to get rid of this by passing to completions. Using the lemma we get the following short exact sequence: $0 \longrightarrow \ker(r) \longrightarrow C^*(G) \longrightarrow C^*(S) \longrightarrow 0$. Since $r(C_c^\infty(\Gamma)) = 0$ we have $C^*(\Gamma) \subset \ker(r)$ by continuity.
Proposition 5.1 in [LR01] says that we would have $C^*(\Gamma) = \ker(r)$ if S were amenable, i.e. $C^*_{\text{full}}(S) = C^*(S)$. Furthermore this sequence would *always* be exact when dealing with the full C*-algebras.
However, in our case the inclusion is strict, since bounding the L^2 norms in the singular representation is impossible. The groupoids S and Γ are transitive. This means that all left regular representations are unitarily equivalent and we can view them as C*-algebras of operators acting on $L^2(s^{-1}(0,0)) \cong L^2(\mathbb{C}^2)$ and $L^2(s^{-1}(1,0))$ respectively.
We can compute $C^*(S)$ directly: The convolution takes an easy form as seen above. It seems to be a simultaneous convolution of integral kernels and of functions on $\mathbb{C}$. We make this more precise using the following map:

$$\begin{aligned} C_c^\infty(\mathbb{C}) \otimes C_c^\infty(\mathbb{C}^2) &\to C_c^\infty(S) \\ f \otimes \varphi &\mapsto [f \otimes \phi(z, v, 0, p) = f(z)\varphi(v, p)] \end{aligned} \tag{13}$$

This has dense image. We can regard $\mathbb{C}$ as a group and $\mathbb{C}^2$ as a pair groupoid. The convolution is compatible with this:

$$\begin{aligned} (f \otimes \varphi) * (g \otimes \psi)(z, v, 0, p) &= \iint f(z - w)\varphi(v, u)g(w)\psi(u, p)du dw \\ &= \int f(z - w)g(w)dw \int \varphi(v, u)\psi(u, p)du \\ &= f * g(z) \cdot \varphi * \psi(v, p) = (f * g) \otimes (\varphi * \psi)(z, v, 0, p) \end{aligned}$$

A similar computation shows compatibility with the involution on the tensor product. We implicitly equipped the tensor product of two *-algebras with a

-algebra structure. Finding a C-norm on tensor products is more difficult. We will only remark that compact operators are a nuclear, i.e. that one finds a unique C*-norm on all tensor products with compact operators. (c.f. [Bla06]) Our map is an isometry with dense image and we thus get an induced isomorphism on the completed C*-algebras: $C^*(\mathbb{C}) \otimes K(H) \cong C^*(S)$. Here we used that $C^*(\mathbb{C} \times \mathbb{C}) \cong K(H)$. This fits neatly into the following theorem.

Theorem 20 ([MRW87]). *Let $\mathcal{G} \rightrightarrows M$ be a transitive Lie groupoid and $m \in M$ with isotropy group $\mathcal{G}_m$. Then $C^*(\mathcal{G}) \cong C^*(\mathcal{G}_m) \otimes K(L^2(s^{-1}(m)))$.*

Corollary 21. *$C^*(\Gamma) \cong C^*(\mathbb{Z}) \otimes K(H)$ and $C^*(S) \cong C^*(\mathbb{C}) \otimes K(H)$. The last isomorphism is given explicitly by (13).*

We now collect a few facts about $C^*(\mathbb{Z})$. It is a commutative and unital C*-algebra, since $\mathbb{Z}$ is abelian and discrete. Hence, the Gelfand representation gives us $C^*(\mathbb{Z}) \cong C(Spec(C^*(\mathbb{Z}))$. The compact Hausdorff space $Spec(C^*(\mathbb{Z}))$ is just S^1: Any character is uniquely determined by its value at the identity $\phi(1) \in U(1) \cong S^1$. This is a continuous evaluation in the weak *-topology on the spectrum and bijective, thus a homeomorphism of compact Hausdorff spaces. We interpret this as a remnant of the singularity.

Proposition 22. $C^*(\Gamma) \cong C(S^1) \otimes K(H)$.

6 Morita equivalence

We remarked earlier that the C*-algebra of a Lie groupoid depends on the choice of a Haar system and that there may not be a isomorphism between two such choices. But what we do obtain are Morita equivalent C*-algebras in a sense that we will introduce now. For a more detailed discussion we refer to [RW98] and [Öc16].
The classical Morita theory deals with rings and their category of left modules. Two rings are said to be Morita equivalent if their categories of left modules are equivalent. Every such equivalence between rings S and R will be naturally equivalent to taking a tensor product with a S-R-bimodule ${}_SQ_R$ and an inverse ${}_RP_S$. [Mey97]
For a C*-algebra A the relevant category of representations is the category of Hermitian A-modules.

Definition 11. A **Hermitian A-Module** is a Hilbert space H together with a nondegenerate *-representation of A by which A acts from the left on H. A morphism of Hermitian A-modules is an A-equivariant continuous map between Hilbert spaces. We will denote this category by Rep(A).

Any representation (π, H) of A can be made nondegenerate by considering $\overline{\pi(A)H}$.

Definition 12. A **right Hilbert** B**-module** is a right B-module and $\mathbb{C}$-vector space X together with a sesquilinear B-valued inner product $\langle \cdot, \cdot \rangle : X \times X \to B$ that is linear in the second variable and satisfies:

(i) $\langle x, x \rangle \geqslant 0$ with equality only for $x = 0$ [2]

(ii) $\langle x, y \rangle^* = \langle y, x \rangle$

(iii) $\langle x, yb \rangle = \langle x, y \rangle b \qquad \forall x, y \in X, b \in B$

Furthermore X is assumed to be complete in the norm $\|\langle x, x \rangle\|^{\frac{1}{2}}$.
An operator $T \in B(X)$ is called **adjointable** if there is an operator $T^* \in B(X)$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$.
A **Hilbert** A**-**B**-bimodule** is a right Hilbert B-module X together with a nondegenerate *-representation $\pi : A \to B(X)$ which maps into the C*-algebra of adjointable operators.

Note that a Hilbert A-$\mathbb{C}$ bimodule is just a Hermitian A-module and that a $\mathbb{C}$-B bimodule is a right Hilbert B-module. Any C*-algebra A provides a canonical A-A-bimodule with the inner product $\langle a, b \rangle = a^* b$.
Given two bimodules ${}_AH_B$ and ${}_BK_C$ we can form a product A-C bimodule ${}_AH\widehat{\otimes}K_C$ as follows: The sesquilinear pairing $\langle h \otimes k, h' \otimes k' \rangle := \langle k, \langle h, h' \rangle_B k' \rangle_C$ on $H \otimes_{\mathbb{C}} K$ is compatible with the canonical right C-action. Factoring out by the space of isotropic vectors and completing with respect to the norm then yields a right Hilbert C-module. It is an easy verification that the canonical A-action on $H \otimes_{\mathbb{C}} K$ factors through the quotient, acts by adjointable operators and is nondegenerate.
This tensor product with an A-B-bimodule ${}_AH_B$ takes Hermitian B-modules to Hermitian A-modules: ${}_AH_B\widehat{\otimes}- : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$. Moreover the tensor product is associative up to isomorphism by the usual $h \otimes (k \otimes l) \mapsto (h \otimes k) \otimes l$ which preserves the sesquilinear pairing and thus factors as an isometry to the completions. The bimodule ${}_AA_A$ acts as an identity up to isomorphism. This gives us a weak 2-category **C*Bimod** with C*-algebras as objects, Hilbert bimodules as 1-morphisms and biequivariant maps as 2-morphisms. We define two C*-algebras A, B to be **Morita equivalent** if they are equivalent in this category and this equivalence is implemented by an *equivalence bimodule* ${}_AH_B$. [3] In this case, their categories of representations are equivalent and this equivalence is additive. Therefore Morita equivalence also preserves irreducibility of representations.

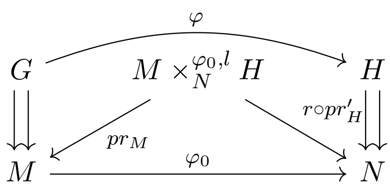
[2] An element $b \in B$ is *positive*, denoted $b \geqslant 0$, if $b = c^* c$ or equivalently, if it is self-adjoint and $spec(b) \subset [0, \infty)$.

[3] An A-B *-equivalence bimodule* ${}_AH_B$ is both a left Hilbert A- and right Hilbert B-module with ${}_A\langle x, y \rangle z = x \langle y, z \rangle_B$. The actions need to be adjointable in both inner products and the images ${}_A\langle H, H \rangle \subset A, \langle H, H \rangle_B \subset B$ are required to be dense.
Every equivalence bimodule is canonically invertible by ${}_BH^{\mathrm{op}}{}_A$ consisting of the same underlying set. The left B action on H^{op} is given by the right action by the adjoint, etc.

A *-homomorphism of C*-algebras $\psi : A \to B$ induces a A-B bimodule X_ψ as follows: $a.b := \psi(a)b$ defines a *-representation of A on B which is canonically a right B-module. We can make the A-representation nondegenerate by restricting to $\overline{\psi(A)B}$. A straightfroward computation shows that this provides a functor **C*Alg** to **C*Bimod**. Thus isomorphic C*-algebras are Morita equivalent.
For Lie groupoids we can also define a notion of Morita equivalence by means of bibundles.

Definition 13. A ***G*-*H*-bibundle** is a manifold M together with maps $l_m : M \to G^{(0)}, r_m : M \to H^{(0)}$ that is equipped with commuting left and right G and H actions respectively. It is **left-principal** if r_M is a surjective submersion whose fibres are the free G-orbits of M.

For left-principal bibundles ${}_GM_H$ and ${}_HN_K$ we can define their tensor product $M \otimes_H N := (M \times_{H^{(0)}} N)/H$ where the quotient is with respect to the H-action given by $(m, n).h = (m.h, h^{-1}.n)$. Left-principality is sufficient to make $M \otimes_H N$ a smooth manifold (c.f. [Öc16]). Furthermore, the product is associative up to isomorphism, i.e. up to a biequivariant diffeomorphism. To any group $G \to G^{(0)}$ we can associate the G-G-bibundle G with the G-actions given by left and right multiplication. It acts as an identity for the product. We thus get another weak 2-category **LGBimod**. Two Lie groupoids are called **Morita equivalent** if they are isomorphic in this bicategory.
There is a canonical way to associate to a morphism of Lie groupoids $\varphi : G \to H$ a G-H-bibundle given diagramatically as follows:

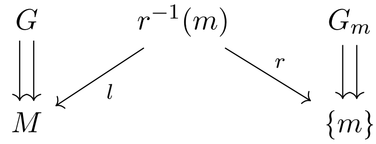


This actually provides a functor **LieGrpd** to **LGBimod** sending a composition of Lie groupoid morphisms to the product of bimodules. This shows that isomorphic groupoids are actually Morita equivalent.
A biprincipal bibundle ${}_GM_H$ is a G-H-bibundle that is both left and right principal, where the latter condition is essentially left principality mutatis mutandis. Such a bibundle is invertible in the following sense: We obtain an H-G-bibundle M^{op} by using the same underlying set but exchanging all relevant maps: The left H-action is given by the right H-action with the inverse, etc. Then $M \otimes_H M^{\mathrm{op}} \cong G$ and $M^{\mathrm{op}} \otimes_G M \cong H$ constitues a Morita equivalence. We see this by $M \times_{H^{(0)}} M^{\mathrm{op}} = \{(m, n) : r(m) = r(n), m, n \in M\} = \{(m, g.m) : m \in M, g \in G\}$. The H-action then identifies $(m, g.m) \sim (m.h, g.m.h)$. Since H acts freely and transitively on the l_M orbits this precisely identifies $(m, g.m) \sim (n, g.n)$. The desired isomorphism of bibundles is then finally given by $g \mapsto [g^{-1}.m, m]$ for an arbitrary

$m \in l_M^{-1}(l(g))$.
It is shown in [Blo08] that *any* Morita equivalence of Lie groupoids is actually given by a biprincipal bibundle and that $N \cong M^{\mathrm{op}}$.

Proposition 23. *Let $G \rightrightarrows M$ be a transitive Lie groupoid and $m \in M$. Then $G \rightrightarrows M$ and the isotropy group $G_m \rightrightarrows \{m\}$ are Morita equivalent Lie groupoids.*

Proof.



We claim that $r^{-1}(m)$ provides an biprincipal bibundle. The left and right actions are given by ordinary multiplication whenever defined. For left principality we note that r is trivially a surjective submersion. The fiber of this map is the full $r^{-1}(m)$. Multiplication from the left stays inside this fiber. It is transitive since for $h, h' \in r^{-1}(m)$ we have $h' = (h'h^{-1}).h$.
Right principality is proven as follows: l is surjective since G is transitive. It has constant rank by Proposition 1 and is thus a submersion. Its fibers are $l^{-1}(n) \cap r^{-1}(m)$ on which G_m acts transitively from the right since for any two elements h, h' we have $h' = h.(h^{-1}h')$. □

A proof of the following can be found in [Öc16]. It also proves (using the trivial G-G-bibundle) that different Haar systems lead to the same C*-algebra.

Theorem 24. *Morita equivalent Lie groupoids have Morita equivalent C*-algebras. Furthermore, we get a functor* **LGBimod → C*Bimod** *mapping a Lie groupoid to its C*-algebra (with a chosen Haar system).*

The proof constructs a $C^*(G)$-$C^*(H)$-bimodule from a given G-H-bibundle M by equipping $C_c^\infty(M)$ with a $C_c^\infty(H)$-valued inner product and bilateral actions by convolution. It is then shown to be compatible with composition of bimodules by a series of computations.
The theorem gives a fancy way of showing the Morita equivalence between compact operators $K(H)$ and $\mathbb{C}$ by considering any transitive pair groupoid $M \times M$ with $C^*(M \times M) \cong K(L^2(s^{-1}(m)))$. Its isotropy group is trivial and thus has $\mathbb{C}$ as its convolution algebra. This also shows that Morita equivalence is strictly weaker than isomorphism.
Generally, a transitive groupoid C*-algebra will be Morita equivalent to the C*-algebra of any of its isotropy groups. This is a (slightly) weaker version of Theorem 20 which we could recover by stable isomorphism [BGR77].

6.1 The magnetic C*-algebra up to Morita equivalence

We will now discuss the Morita equivalence classes in our example.
$C^*(\Gamma) \cong C^*(\mathbb{Z}) \otimes K(H) \cong C(S^1) \otimes K(H)$ is Morita equivalent to $C^*(\mathbb{Z})$ and $C(S^1)$. The former is an abelian and unital group C*-algebra and thus has easy representation theory. Every nondegenerate representation is induced by a unitary representation of $\mathbb{Z}$ (Theorem 11) and by Schur's lemma all irreducible representations are one dimensional characters. A unitary representation of $\mathbb{Z}$ is determined by any unitary operator $\pi(1) = U \in U(H)$. This describes the Hermitian $C^*(\mathbb{Z})$-modules completely.
Actually, the above considerations work for arbitrary abelian groups.

Proposition 25. *Let H be an abelian topological locally compact group. $C^*(H) \cong C_0(\widehat{H})$ where $\widehat{H}$ is the Pontryagin dual of H.*
All nondegenerate representations of $C^*(H)$ are induced by unitary representations of H. This correspondence preserves irreducibility. All irreducible representations are characters.

Proof. The Gelfand representation gives us $C^*(H) \cong C_0(Spec(C^*(H)))$. Now every irreducible representation of $C^*(H)$ is induced by one of H (Theorem 11). This gives $\widehat{H} \cong Spec(C^*(H))$. It is worthwile to remark here that *all* abelian groups are amenable, i.e. $C^*(H) = C^*_{\text{full}}(H)$. The last assertions follow immediately. □

Morita equivalence preserves representation categories and irreducibility. Furthermore, Morita equivalence also preserves **liminal** C*-algebras [AHRW07], that is C*-algebras for which every irreducible representation acts by compact operators. In finite dimensions all operators are compact.

Corollary 26. *$C^*(\Gamma)$ is Morita equivalent to $C(S^1)$ and $C^*(S)$ is Morita equivalent to $C_0(\mathbb{C})$. Proposition 25 gives us an abstract classification of $Rep(C^*(\Gamma))$ and $Rep(C^*(S))$ and especially of irreducible representations. They are liminal.*

Morita equivalence of C*-algebras induces an isomorphism on the *lattice* of closed ideals [RW98]. The closed ideals $\mathcal{I}$ are partially ordered by inclusion and every pair $I, J \in \mathcal{I}$ has a greatest lower and least upper bound given by $I \cap J$ and $I \vee J = \overline{(I \cup J)}$. We can describe some of the lattices explicitly in the present situation.
Let X be Hausdorff and locally compact, $A \subset X$ closed and $I \subset C_0(X)$ a closed ideal. Define $V(I) = \{x \in X : f(x) = 0 \quad \forall f \in I\}$ and $I(A) = \{f \in C_0(X) : f(x) = 0 \quad \forall x \in A\}$.

Theorem 27. *$V(I(A)) = A$ and $I(V(I)) = I$ for all closed subsets $A \subset X$ and closed ideals $I \subset C_0(X)$.*
If $A \subset B$ then $I(B) \subset I(A)$. If $I \subset J$ then $V(J) \subset V(I)$.

(i) $I(A \cup B) = I(A) \cap I(B)$ *and* $I(A \cap B) = I(A) \vee I(B)$

(ii) $V(I \cap J) = V(I) \cup V(J)$ *and* $V(I \vee J) = V(I) \cap V(J)$

That is, $V : (\mathcal{I}, \vee, \cap) \to (\mathit{Closed}(X), \cap, \cup)$ *and* I *are inverse lattice isomorphisms where we equip Closed*(X) *with the reverse ordering* $A \leqslant B \iff A \supset B$ *.*

Proof. Let $A \subset X$ be closed and $x \in A^c$. Then by Urysohn's lemma $\exists f \in I(A)$ with $f(x) = 1$. Therefore $V(I(A)) = A$.
$I \subset I(V(I))$ is immediate. Applying Urysohn's lemma again we can see that I seperates points of $X \backslash V(I)$ and thus by Stone-Weierstraß, we have $I = C_0(X \backslash V(I)) = \{f|_{X \backslash V(I)} : f(x) = 0 \quad \forall x \in V(I)\} = I(V(I))$. [Fol16]
The remaining claims are easy computations some of which need to be shown by using that I, V are mutual inverses. □

Corollary 28. *The lattice of closed ideals in* $C^*(\Gamma)$ *is isomorphic to the lattice of closed subsets of* S^1*.*
The lattice of closed ideals in $C^*(S)$ *is isomorphic to the lattice of closed subsets of* $\mathbb{C}$*.*

Morita equivalence preserves nuclearity. [AHRW07] For C*-algebras with countable approximate identities this follows from the nuclearity of the compact operators and stable isomorphism [BGR77] ($C^*(S), C^*(\Gamma)$ are stable). Also, all commutative C*-algebras are nuclear. [RW98] Using either characterization we get:

Proposition 29. $C^*(S)$ *and* $C^*(\Gamma)$ *are nuclear.*

Proposition 30. *The canonical inclusion* $C^*(\Gamma) \hookrightarrow C^*(G)$ *does not induce a Morita equivalence.*

Proof. Consider the sequence $C^*(\Gamma) \xrightarrow{i} C^*(G) \xrightarrow{r} C^*(S)$ in **C*Alg** inducing $C^*(\Gamma) \xrightarrow{X_i} C^*(G) \xrightarrow{X_r} C^*(S)$ in **C*Bimod**. Since $r \circ i = 0$ we have $X_i \widehat{\otimes} X_r = 0$ by functoriality. Since $X_r \neq 0$, X_i cannot be invertible in **C*Bimod**. □

It is not yet known whether $C^*(\Gamma)$ and $C^*(G)$ are Morita equivalent or not. In this section we have listed some invariants of $C^*(\Gamma)$ that could help answering this question.

Remark 3. Somewhat counterintuitively, $C^*(\Gamma)$ and $C^*(S)$ are Morita equivalent. For commutative C*-algebras the equivalence class only depends on the cardinality of their spectra. [Rie74, Bee82]

7 Conclusion and Outlook

In the preceding chapters, we have shown an example of how to take a singular symplectic form on a manifold M, remove its singularity by a Dirac structure, integrate the Lie algebroid to a Lie groupoid and compute its convolution algebra.

Other removable singularities Some of our propositions are of a easily generalizable nature, but the integration is certainly not. We suspect that for M simply connected, $\dim M \geqslant 3$ some simplifications occur, as the pair groupoid $T^*M_\times \times T^*M_\times$ will be the unique t-simply connected integration of the regular part of the resulting Dirac structure.
If integrability is of concern, it might be possible to describe the Morita equivalence class of the C*-algebra more directly from the Lie algebroid or even the symplectic form, alleviating the need for integrability, perhaps even foregoing the removability hypothesis.

Prequantization In 5.2 we only used the Lie algebroid structure of the Dirac structure. However, the antisymmetric pairing $(X + \alpha, Y + \beta) = \beta(X) - \alpha(Y)$ on Dirac structures generally restricts to a closed 2-form in the Lie algebroid cohomology. Such cohomology classes are in one-to-one correspondence with central extensions of the Lie algebroid similar to example 3.d. We were not able to find an integration of this extended Lie algebroid in the form of a central extension of Lie groupoids, which are also described by 2-cocycles in the groupoid cohomology. The abstract integrability question is covered in [Cra04]. This again would yield an extension of C*-algebras and it would be interesting to characterize this in terms of cocycles.

References

[AHRW07] Astrid An Huef, Iain Raeburn, and Dana P. Williams. Properties preserved under Morita equivalence of C^*-algebras. *Proc. Amer. Math. Soc.*, 135(5):1495–1503, 2007.

[Bee82] Walter Beer. On Morita equivalence of nuclear C^*-algebras. *J. Pure Appl. Algebra*, 26(3):249–267, 1982.

[BGR77] Lawrence G. Brown, Philip Green, and Marc A. Rieffel. Stable isomorphism and strong Morita equivalence of C^*-algebras. *Pacific J. Math.*, 71(2):349–363, 1977.

[Bla06] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[Blo08] Christian Blohmann. Stacky Lie groups. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 082, 51, 2008.

[Blo17] Christian Blohmann. Removable presymplectic singularities and the local splitting of Dirac structures. *Int. Math. Res. Not. IMRN*, (23):7344–7374, 2017.

[Bur13] Henrique Bursztyn. A brief introduction to Dirac manifolds. In *Geometric and topological methods for quantum field theory*, pages 4–38. Cambridge Univ. Press, Cambridge, 2013.

[CdSW99] Ana Cannas da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[CF03] Marius Crainic and Rui Loja Fernandes. Integrability of Lie brackets. *Ann. of Math. (2)*, 157(2):575–620, 2003.

[CF11] Marius Crainic and Rui Loja Fernandes. Lectures on integrability of Lie brackets. In *Lectures on Poisson geometry*, volume 17 of *Geom. Topol. Monogr.*, pages 1–107. Geom. Topol. Publ., Coventry, 2011.

[Cra04] Marius Crainic. Prequantization and Lie brackets. *J. Symplectic Geom.*, 2(4):579–602, 2004.

[Fol16] Gerald B. Folland. *A course in abstract harmonic analysis*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.

[HM90] Philip J. Higgins and Kirill Mackenzie. Algebraic constructions in the category of Lie algebroids. *J. Algebra*, 129(1):194–230, 1990.

[LR01] N. P. Landsman and B. Ramazan. Quantization of Poisson algebras associated to Lie algebroids. 282:159–192, 2001.

[Mey97] Ralf Meyer. Morita equivalence in algebra and geometry. `https://math.berkeley.edu/~alanw/277papers/meyer.tex`, 1997. Accessed: 2020-07-13.

[MRW87] Paul S. Muhly, Jean N. Renault, and Dana P. Williams. Equivalence and isomorphism for groupoid C^*-algebras. *J. Operator Theory*, 17(1):3–22, 1987.

[Muh97] Paul S. Muhly. Coordinates in operator algebra. `https://operatoralgebras.org/resources-resources/Groupoids-Book-Muhly.pdf`, 1997. Accessed: 2020-07-13.

[Put19] Ian Putnam. Lecture notes on c*-algebras. `http://www.math.uvic.ca/faculty/putnam/ln/C*-algebras.pdf`, 2019. Accessed: 2020-07-13.

[Ren80] Jean Renault. *A groupoid approach to C^*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Rie74] Marc A. Rieffel. Morita equivalence for C^*-algebras and W^*-algebras. *J. Pure Appl. Algebra*, 5:51–96, 1974.

[RW98] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C^*-algebras*, volume 60 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[Öc16] Kaan Öcal. Symplectic singularities and convolution algebras. *Bachelor's thesis, Universität Bonn*, 2016.