Cohomological Extension of Groupoid Convolution

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1 Introduction

This thesis investigates the Hochschild cohomology of the convolution algebra of Lie groupoids as an invariant of the underlying differentiable stack as well as its relation to deformation cohomology of the Lie groupoid. We provide a bornological framework in which the noncommutative convolution algebra can be viewed as the function algebra of a differentiable stack.

In order to motivate the objects of interest, I will first sketch some points of view from which the theory is naturally interesting.

1.1 Motivation

In classical noncommutative geometry, one is interested in geometric models for noncommutative C*-algebras. The category of commutative C*-algebras is equivalent to the category of locally compact Hausdorff spaces by the Gelfand-Naimark duality. The idea is now that noncommutative algebras correspond to generalized spaces such as quotients of group actions M/G that have a strange or trivial topology. The class of geometric models that can incorporate this are topological groupoids. Their space of compactly supported continuous functions come with a convolution product that is noncommutative in general. One then completes this algebra to a C*-algebra. It is well-known that in this picture Morita equivalent topological groupoids are mapped to Morita equivalent C*-algebras. [1] The Morita equivalence of C*-algebras for example implies that they are KK-equivalent.

The notion of Morita equivalence of Lie groupoids will be central to this thesis. Any Lie groupoid represents a differentiable stack and Morita equivalent groupoids represent the same stack. Hence, we can think of differentiable stacks as equivalence classes of Lie groupoids. Differentiable stacks are a categorified and generalized class of smooth spaces. Manifolds embed into differentiable stacks and the quotient stack replaces the singular quotient space M/G.

Coming from the geometric side, Lie groupoids or differentiable stacks are inherently interesting in the areas of equivariant differential geometry, Poisson geometry and mathematical physics. The equivariant cohomology of a manifold with a Lie group action is a special case of the de Rham cohomology of the action groupoid. [2] In Poisson geometry, symplectic groupoids are the global counterparts to Poisson manifolds. [3]

Having established stacks as geometric objects, one might want to ask if there is some kind of algebra of smooth functions on a differentiable stack. Such an algebra should:

(i) not forget the smooth structure, so a C*-algebra (think of continuous functions) would be too big. [4]

(ii) only depend on the Morita equivalence class of a Lie groupoid.

(iii) just be the ordinary algebra of smooth functions with pointwise multiplication for a smooth manifold M.

The solution we will advocate is the bornological convolution algebra $\mathcal{A}_G$ of a Lie groupoid G. In this thesis, we show that the Morita equivalence class of this function algebra only depends on the Morita equivalence class of the Lie groupoid.

[1] A good reference for this is [Lan00].

[2] An introduction to the cohomology of stacks can be found in the lecture notes [Beh02].

[3] For a treatment of Poisson geometry using groupoids see the recent textbook [CFM21]. The connection to physics is explained in [Lan06].

[4] Another important point why using C*-algebras is not possible here is that the Hochschild cohomology often vanishes, c.f. [Kha13, Remark 3.4.7]. The continuous functions on a topological space do not admit any derivations.

The right notion of a vector field on a differentiable stack (a section of the tangent stack) actually recovers the notion of a multiplicative vector field on a Lie groupoid. Hence, vector fields on a differentiable stack are multiplicative vector fields on a Lie groupoid and they should morally act as derivations on the function algebra. This is indeed the case and was first observed in [KP21]. Due to the noncommutativity of the function algebra there will also be inner derivations. Since we only work with the function algebra up to Morita equivalence it might be Morita equivalent to a commutative algebra which has no inner derivations. The most one could expect is that the outer derivations of these algebras agree. This is encoded in the first Hochschild cohomology of the algebras. Whether the Hochschild cohomology $H^*(A, A)$ of an algebra A is a Morita invariant is still an open question, but we could achieve partial results. Let us point out, that we use Hochschild cohomology with continuous (or bounded) cocycles and that the algebras are generally nonunital.

More generally, the multiplicative vector fields on a Lie groupoid G are actually 1-cocycles in the deformation complex $C^*_{\text{def}}(G)$ of the groupoid and derivations are 1-cocycles in the Hochschild complex. Hence, one might conjecture the existence of a cochain map relating the deformation cohomology of a Lie groupoid and the Hochschild cohomology of its convolution algebra. This was recently accomplished by Kosmeijer and Posthuma in [KP21]. We will fill in some details and reinterpret the cochain map $\Phi : C^*_{\text{def}}(G) \to C^*(\mathcal{A}_G, \mathcal{A}_G)$ in our setting of Hochschild cohomology of bornological algebras.

Another heuristic for the existence of a cochain map is the following: a smooth deformation of a Lie groupoid gives rise to a smooth deformation of its convolution algebra. Deformations of Lie groupoids are infinitesimally cochains in the deformation cohomology. Deformations of algebras are infinitesimally cochains in the Hochschild complex. Hence, by taking derivatives there should be a cohomological extension of the convolution operation.

1.2 Overview

In Section 2 we start with a reminder on Lie groupoids and Lie algebroids. We review the different equivalent ways to make Lie groupoids into a bicategory and sketch briefly the equivalence to differentiable stacks whereby a groupoid is mapped to its stack of torsors. Most important to our purposes is the description of the bicategory of Lie groupoids where the morphisms are right-principal bibundles following [Blo08].

In Section 3 we discuss integration on Lie groupoids and on the simplicial nerve. We start by investigating an intrinsic way of defining convolution on Lie groupoids. This is accomplished by using densities along the fibers of the source submersion. We show how this recovers convolution by means of a chosen Haar system. We thus obtain an associative algebra $\mathcal{A}_G$ on a groupoid G. In addition, we provide in Section 3.4 a simplicial way to prove the existence of a cochain map Φ relating deformation cohomology and Hochschild cohomology of $\mathcal{A}_G$. This uses the total convolution on the groupoid nerve (Section 3.3) and the combinatorics of the Lie derivative introduced in Section 3.1. This was originally accomplished in [KP21], but they only define the Lie derivative $\mathcal{L}_c$ along deformation cocycles implicitly. This result is included early on since the entire section builds up the necessary techniques. The detailed standalone treatment of the deformation and Hochschild cochain complexes is deferred to later chapters.

In Section 4, we give a short introduction to deformation cohomology based on the original paper [CMS15]. We include short treatments of differentiable cohomology and representations up to homotopy. We compute the low degree terms of the deformation cohomology in terms of isotropy and normal parts in Section 4.4 and we include the vanishing result for proper groupoids in Section 4.5.

We continue to consider the convolution algebra $\mathcal{A}_G$ throughout the whole text, although we single out some of its properties and investigate those abstractly and independently.

The central result of this thesis is that the convolution algebra is a Morita invariant in the following sense:

Theorem 6.6. The assignment of the bornological convolution algebra to a proper Lie groupoid is a weak 2-functor:

$$\begin{aligned} \mathsf{GrpdBiBun}^{\text{proper}} &\longrightarrow \mathsf{SAlg}^{\text{bi}}(\mathsf{CBorn}) \\ G &\longmapsto \mathcal{A}_G \end{aligned} \tag{1.1}$$

Hence, Morita equivalent proper Lie groupoids are mapped to Morita equivalent quasi-unital bornological algebras.

In order to properly state this theorem, Section 5 will provide an introduction to bornological algebras. They are a convenient variant of locally convex topological vector spaces which are recapped briefly in Section 5.1 together with a discussion of different topologies on locally convex tensor products. Locally convex spaces and bornological vector spaces are related by an adjunction that is an equivalence on Fréchet spaces and preserves the tensor products in many interesting situations. Our treatment will be more categorically minded, with the guiding principle that $\mathcal{A}_G$ should be an algebra object in the symmetric monoidal cartesian closed category CBorn of complete bornological vector spaces. The goal is to keep the functional analysis at a minimum and to enable algebraic manipulations. We still need subsection Section 5.2 to show various continuity properties.

Following [Mey07, Appendix A] we develop a basic module theory over nonunital algebras in symmetric monoidal preabelian categories in 5.4. In particular, it is troublesome to not have a unit in the algebra. We show in 5.4.7 that convolution algebras $\mathcal{A}_G$ actually have a quasi-unit in the form of an $\mathcal{A}_G$-linear splitting of the multiplication $\mathcal{A}_G \otimes \mathcal{A}_G \to \mathcal{A}_G$. They are in particular self-induced ($\mathcal{A}_G \otimes_{\mathcal{A}_G} \mathcal{A}_G \cong \mathcal{A}_G$) in the sense of [Mey11]. This allows us to refine the notion of Morita equivalence in loc.cit. to a Morita bicategory of self-induced algebras $\mathsf{SAlg}^{\text{bi}}$ in which convolution algebras of proper Lie groupoids naturally live. The construction of this bicategory is done in 5.4.2. With these tools we are already able to formulate Theorem 6.6. [5]

In parallel, we develop aspects of homological algebra for the category of modules over an algebra in an arbitrary symmetric monoidal category with an eye towards a working interpretation of Hochschild cohomology as a derived functor.

We call a sequence of A-modules exact, if it is split as a sequence of bornological vector spaces. This amounts to doing relative homological algebra. In the language of Quillen we are defining an exact category structure. [6] In this world, there is a well-developed construction of derived functors. We will however develop the existence and uniqueness of projective resolutions in Section 5.4.3 in an ad-hoc manner that keeps our treatment self-contained. This is a longer reformulation of the original work of Connes [Con85] that uses projective resolutions to prove the Hochschild-Kostant-Rosenberg theorem for compact manifolds in the setting of Fréchet algebras. It is ultimately a generalisation of these methods that we seek to develop following [Mey07]. We also revisit the HKR theorem later. In Section 5.4.6, we show that quasi-unitality implies that the bar complex is a projective bimodule resolution. We are not aware of this being stated in the literature. Combining

[5] We use *bicategory* and *weak 2-category* synonymously in the sense of Bénabou's definition.

[6] c.f. the survey [Bü10]

this with the quasi-unitality of convolution algebras that we prove in Section 5.4.7, we obtain a generalisation of a result of Crainic and Moerdijk in [CM01], which says that groupoid convolution algebras are H-unital. We actually obtain strong H-unitality (continuous contractibility instead of mere exactness) for proper Lie groupoids. This is interesting for computing Hochschild cohomology in general: We may choose any projective resolution for computation. The next computationally interesting question is whether Morita equivalence of algebras gives an isomorphism on Hochschild cohomology. Or, in the groupoid case, whether Morita equivalent groupoids have isomorphic Hochschild cohomology. We define Hochschild cohomology by the "naive" complex in Section 5.5 internal to a cartesian closed symmetric monoidal category. We advocate however to not take cohomology and land in Ab but to rather think about the complex up to homotopy. We give sufficient conditions on when Hochschild cohomology is a derived Hom functor. We also present sufficient conditions for a Morita equivalence of algebras to introduce a homotopy equivalence of Hochschild complexes with values in a bimodule.

Theorem 5.79. Let A, B be projectively Morita equivalent quasi-unital algebras. Let $M \cong RM$ be a rough A-bimodule such that also $Q \otimes_A M \otimes_A P$ is a rough B-bimodule. Let $P \in \mathrm{SMod}(A, B)$ and $Q \in \mathrm{SMod}(B, A)$ be the projective bimodules inducing the Morita equivalence $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$. Then, their Hochschild cochain complexes with values in the rough bimodule are homotopy-equivalent.

$$C^*(A, M) \simeq C^*(B, Q \otimes_A M \otimes_A P) \tag{1.2}$$

In particular, their Hochschild cohomology with values in M is isomorphic.

In Lemma 6.8, we show that these hypotheses indeed apply in the case of groupoid convolution algebras. It is however by these methods not possible to prove the homotopy invariance of $C^*(A, A)$ for nonunital A, c.f. Lemma 5.47.

In Section 6 we finally prove the functoriality theorem. We start by writing down an explicit Morita equivalence between the convolution algebras of a trivial groupoid and the Čech groupoid to make the reader used to the notation and constructions. After discussing the theorem and its proof, we point out a few technicalities such as projectivity of the bimodules. This helps to connect it to homological algebra and Hochschild cohomology. We provide a schematic of how to use this reasoning to conclude that certain Morita equivalences between Lie groupoids induce isomorphisms in Hochschild cohomology.

In Section 7, we provide explicit computations of Hochschild cohomology and link the previous chapters with examples. In Section 7.2 we discuss the classical HKR theorem as first proven in [Con85] in the setting of Fréchet algebras, which equates the Hochschild cohomology of $C^\infty(M)$ with the multivector fields. We will provide a much more detailed proof and also a proof of (cohomological) formality. [7] The original computations carry over verbatim by the abstract setup since treating Fréchet spaces as bornological spaces or topological vector spaces is monoidally equivalent. Using Euler-like vector fields, we exhibit a projective resolution of the algebra of smooth functions $C^\infty(S)$ on a submanifold $S \subset M$ as a module over $C^\infty(M)$, which is a generalisation of the construction in loc.cit. for the diagonal embedding $M \subset M \times M$. This was also published without proof in [PPT20, B.8].

Another example we cover in Section 7.3 is the Hochschild cohomology of the action groupoid of a finite group on a compact manifold following [NPPT06]. If the quotient manifold exists we can apply our previous theorems that show that Hochschild cohomology is a Morita invariant, together with the HKR theorem, to identify the Hochschild

[7] To be distinguished from Kontsevich's L_∞-formality.

cohomology with multivector fields on the quotient. This is interesting since we can compare it with the findings of this subsection, showing that both methods agree. [8] In this section we do work bornologically, but in order to have the more powerful tools such as spectral sequences we need to work compute the cohomology internal to the abelian category Vect of complex vector spaces. We also fix an error in [NPPT06] in the definition of a cochain map. To finally link everything together, we can explicitly characterise the image of the deformation cohomology of a proper étale action groupoid inside the Hochschild cohomology of its convolution algebra under the cochain map constructed earlier. Because of the vanishing due to properness, it is only nontrivial in degree one, i.e. does not see fixed point sets and higher multivector fields.

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[8] The formality result is not recovered, which means that our own methods prove a stronger result.

2 Lie Groupoids, Lie Algebroids and the Bicategory of Differentiable Stacks

This section covers well-known material. A good reference for the basics on groupoids and algebroids is [CdSW99]. The equivalence of the bicategory of Lie groupoids with differentiable stacks was first proven in [Blo08].

Definition 2.1. A *groupoid* G is a small category in which every morphism is invertible.

Explicitly, this means that a groupoid consists of a set $G = G^{(1)}$ of arrows or morphisms and a set $M = G^{(0)}$ of objects. Any arrow $g \in G$ has a source $s(g)$ and a target $t(g)$ in M. Two arrows $g, h \in G$ are composable if $s(g) = t(h)$ and we denote their composition by $m(g, h) = gh$. By definition, any $g \in G$ is invertible, so there is an inverse $i(g) = g^{-1}$. For any object $x \in M$ there is a unit morphism $u(x) = \mathrm{id}_x \in G$. The structure maps will be important. Diagrammatically, these are:

$$G^{(2)} = G \times_M^{s,t} G \xrightarrow{m} G \xrightarrow{i} G \overset{s}{\underset{t}{\rightrightarrows}} M \xrightarrow{u} G \tag{2.1}$$

The axioms that these maps need to satisfy in order to make G a groupoid are:

- $s(\mathrm{id}_x) = t(\mathrm{id}_x) = x$
- $\mathrm{id}_{t(g)} g = g \mathrm{id}_{s(g)} = g$
- $s(g^{-1}) = t(g)$ and $t(g^{-1}) = s(g)$
- $gg^{-1} = \mathrm{id}_{t(g)}$ and $g^{-1}g = \mathrm{id}_{s(g)}$
- $s(gh) = s(h)$ and $t(gh) = t(g)$ for all $(g, h) \in G^{(2)}$
- $g(hk) = (gh)k$ for all $(g, h) \in G^{(2)}$ and $(h, k) \in G^{(2)}$

We write $g : s(g) \to t(g)$ to clarify source and target.

Definition 2.2. A *Lie groupoid* is a groupoid object in the category of manifolds such that additionally the source and target maps s, t are submersions. This means all structure maps are supposed to be smooth maps between smooth manifolds. A homomorphism of Lie groupoids $F : G \to H$ is a smooth functor, i.e. a map satisfying $F(g_1 g_2) = F(g_1)F(g_2)$ for composable g_1, g_2. We refer to this category as Grpd.

Note that the condition on s, t implies that the fiber product $G^{(2)}$ is a manifold.

Example 2.3. The following examples are good to have in mind.

1. Any Lie group G can be regarded as a Lie groupoid $G \rightrightarrows pt$ over a point.
2. Any manifold M can be regarded as a Lie groupoid $M \rightrightarrows M$ over itself with all arrows being identities.
3. Let G be a Lie group acting on a manifold M. Then $G \ltimes M \rightrightarrows M$ is a Lie groupoid called the *action groupoid*. Here, an element (g, m) is an arrow $m \to g.m$. The multiplication is given by $(g, h.m)(h, m) = (gh, m)$.
4. The *pair groupoid* $M \times M \rightrightarrows M$ has arrows $(m, n) : n \to m$. Multiplication is given by $(m, n)(n, k) = (m, k)$.

5. Let G be a Lie group and let $P \to M$ be a principal G-bundle. The *Atiyah groupoid* $At(P) \rightrightarrows M$ is the quotient of the pair groupoid $P \times P \rightrightarrows P$ by the G-action. Explicitly, G acts diagonally on $P \times P$ and the source and target maps to P are equivariant with respect to this action.

6. The *Čech groupoid* associated to an open cover $\{U_i\}$ of M is the groupoid

$$\bigsqcup_{i,j} U_i \cap U_j \rightrightarrows \bigsqcup_i U_i \,.$$

The arrow (i, x, j) corresponding to $x \in U_i \cap U_j$ has source (x, j) and target (x, i). Composition is defined via $(i, x, j)(j, x, k) = (i, x, k)$.

The first two examples above are special cases of the third.

Definition 2.4. A Lie groupoid $G \rightrightarrows M$ is called *proper* if the map $(s, t) : G \to M \times M$ is proper. It is called *étale* if the source and target maps are local diffeomorphisms.

An action groupoid is proper if and only if the action is proper. An action groupoid is étale if and only if the group is discrete. The Čech groupoid is étale.

Definition 2.5. Any groupoid $G \rightrightarrows M$ induces an equivalence relation on M identifying $s(g) \sim t(g)$. The equivalence classes are called *orbits*.

In general, M/G is not a manifold and the Lie groupoid resolves this singular quotient. The orbits of an action groupoid are precisely the orbits of the group action. The pair groupoid and Atiyah groupoid are transitive, i.e. only have one orbit.

Let $\pi : N \to M$ be a submersion and $G \rightrightarrows M$ a Lie groupoid. The *pullback groupoid* $\pi^*G \rightrightarrows N$ is the manifold $N \times_M G \times_M N$. Its multiplication is given by $(n_1, g, n_2)(n_2, h, n_3) = (n_1, gh, n_3)$ whenever defined. It fits into a cartesian diagram of smooth manifolds:

$$\begin{tikzcd}
\pi^*G \arrow[r] \arrow[d, "(s,t)"] & G \arrow[d, "(s,t)"] \\
N \times N \arrow[r, "\pi"] & M \times M
\end{tikzcd} \tag{2.2}$$

The Čech groupoid is the pullback of $M \rightrightarrows M$ along the map $\pi : \bigsqcup U_i \to M$.

Definition 2.6. A *Lie algebroid* is a vector bundle $A \to M$ with a Lie bracket on $\Gamma(A)$ and an *anchor map* $\sharp : A \to TM$ satisfying the Leibniz rule

$$[\sigma, f\tau] = f[\sigma, \tau] + \sharp(\sigma)f \cdot \tau \qquad \forall \sigma, \tau \in \Gamma(A), f \in C^\infty(M) \tag{2.3}$$

Any Lie groupoid $G \rightrightarrows M$ has an associated Lie algebroid $A = \mathrm{Lie}(G)$. Its underlying vector bundle is given by $A = \ker(ds)|_M$. Any section α of A gives rise to a right-invariant vector field $\vec{\alpha}$ on G via $\vec{\alpha}_g := dr_g \alpha_{t(g)}$ and, vice versa, any right-invariant vector field determines a section of A. Right translation by g only makes sense along the s-fiber $s^{-1}(t(g))$. Hence, the differential dr_g is defined on the tangent space to this fiber which equals $\ker(ds)$. The ordinary Lie bracket of right-invariant vector fields is right-invariant. Then $[\vec{\alpha}, \vec{\beta}] = \overrightarrow{[\alpha, \beta]}$ implicitly defines a bracket on $\Gamma(A)$. The anchor is finally given by $\sharp = dt : \ker(ds)|_M \to TM$. It is a consequence of the axioms that $\sharp : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.

Example 2.7. 1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, the Lie algebroid of $G \rightrightarrows pt$ is $\mathfrak{g} \to pt$ with the ordinary Lie bracket.

2. The Lie algebroid of the groupoid $M \rightrightarrows M$ is the trivial zero dimensional vector bundle over M. This remains true for all étale groupoids.

3. The Lie algebroid of the action groupoid $G \ltimes M \rightrightarrows M$ is the *action algebroid* $\mathfrak{g} \ltimes M \to M$. Its anchor is induced from the infinitesimal Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M)$ that maps an element $X \in \mathfrak{g}$ to the fundamental vector field $\hat{X}$. Then, $\sharp(X, m) = \hat{X}(m)$. The bracket of the action algebroid is induced from the bracket on $\mathfrak{g}$ and the Leibniz rule.

For the following fix a groupoid $G \rightrightarrows G^{(0)}$.

Definition 2.8. A *right action* of G on a manifold P consists of a moment map $\mu : P \to G^{(0)}$ and a map $a : P \times^{\mu,t}_{G^{(0)}} G \to P$ denoted $a(p, g) = p.g$ that satisfies the usual $p.g.h = p.(gh)$ whenever defined.

Analogously we can define left actions.

Definition 2.9. A *representation* of G is a vector bundle E over $G^{(0)}$ with a G-action where the moment map is the bundle projection.

The following definition generalizes principal bundles over Lie groups.

Definition 2.10. A *G-torsor* or *(right) principal G-bundle* over B is a submersion $\pi : P \to B$ together with a free and transitive G-action on the fibers. That is, there is a moment map $\mu : P \to G^{(0)}$ and an action $P \times_{G^{(0)}} G \to P$ such that

$$\begin{aligned} P \times^{\mu,t}_{G^{(0)}} G &\longrightarrow P \times^{\pi,\pi}_{B} P \\ (p, g) &\longmapsto (p, p.g) \end{aligned} \tag{2.4}$$

is a diffeomorphism.

We remark that Equation (2.4) indeed implies that G acts on the π-fibers and that the action is free and transitive. It also helps to have a graphical representation of the data as follows:

$$\begin{tikzcd} P \arrow[d, "\pi"] \arrow[drr, "\mu"] & \circlearrowleft & G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ B & & G^{(0)} \end{tikzcd} \tag{2.5}$$

Remark 2.11. The map 2.4 is an isomorphism if and only if it is an isomorphism for each $P|_{U_i}$ on an open cover $\{U_i\}$ of B. This is true if and only if P trivializes over an open cover, i.e. there is an isomorphism of G-torsors $P|_{U_i} \cong U_i \times_{G^{(0)}} G$ for some maps $U_i \to G^{(0)}$. We construct the local trivialisations in the proof of Lemma 2.15. Conversely, note that trivial G-torsors indeed satisfy Equation (2.4).

Definition 2.12. A morphism of G-torsors is a G-equivariant smooth map between them. It necessarily intertwines the moment maps and the projections to the base manifolds.

The following is analogous to the case of principal G-bundles for Lie groups.

Remark 2.13. Let $\Phi : P \to P'$ be a morphism of G-torsors over the same base. Then, Φ is automatically invertible.

$$\begin{tikzcd} & P' \arrow[dl, "\pi'"'] \arrow[drr, "\mu'"] & & \\ P \arrow[ur, "\Phi"] \arrow[d, "\pi"] \arrow[drrr, "\mu"] & & & G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ B & & & G^{(0)} \end{tikzcd} \tag{2.6}$$

Φ is easily seen to be a bijection since it covers the identity, is G-equivariant and the G-action is free and transitive on fibers. In local trivialisations Φ is given by multiplication by some $g : U \to G$:

$$\begin{tikzcd}
P|_U \arrow[rr, "\Phi"] \arrow[d, "\cong"] & & P'|_U \arrow[d, "\cong"] \\
U \times_{G^{(0)}} G \arrow[rr, "(x,h)\mapsto(x,g(x)h)"'] & & U \times_{G^{(0)}} G
\end{tikzcd} \tag{2.7}$$

Φ^{-1} is then just multiplication by $g(x)^{-1}$ and hence smooth.

We now define the analogue of a bimodule for groupoids. We will need some additional assumptions to build the analogue of the tensor product of bimodules - the composition in the bicategory. Let $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ be Lie groupoids.

Definition 2.14. A *right principal X-Y-bibundle ${}_XQ_Y$* is a right Y-torsor with an additional left X-action that has a submersive left moment map. Both actions are required to commute, i.e. $(x.q).y = x.(q.y)$ whenever defined.

$$\begin{tikzcd}
X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & Q \arrow[dl, "l_Q"] \arrow[dr, "r_Q"'] & Y_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
X_0 & & Y_0
\end{tikzcd} \tag{2.8}$$

A morphism of X-Y-bibundles is a biequivariant map. Given two bibundles ${}_XQ_Y, {}_YR_Z$ we can define their composition:

$$Q \circ R = (Q \times_{Y_0}^{r_Q,l_R} R)/Y \tag{2.9}$$

The quotient is with respect to the diagonal action $(q, r).y = (q.y, y^{-1}.q)$.

$$\begin{tikzcd}
 & & Q \circ R \arrow[dll, bend right, "l_{Q\circ R}"'] \arrow[drr, bend left, "r_{Q\circ R}"] & & \\
X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & Q \arrow[dl, "l_Q"] \arrow[dr, "r_Q"'] & Y_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & R \arrow[dl, "l_R"] \arrow[dr, "r_R"'] & Z_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
X_0 & & Y_0 & & Z_0
\end{tikzcd} \tag{2.10}$$

Lemma 2.15. *If the two bibundles ${}_XQ_Y, {}_YR_Z$ are right principal then $Q \circ R$ is a right principal X-Y-bibundle. For a composition of three bibundles, there is a natural isomorphism of bibundles:*

$$\mathrm{ass}_{Q,R,S} : (Q \circ R) \circ S \to Q \circ (R \circ S)$$

Proof. The following structure maps are easily verified to be well-defined on the equivalence class $[q, r] \in Q \circ R$:

$$l_{Q\circ R}([q, r]) = l(q) \tag{2.11}$$

$$r_{Q\circ R}([q, r]) = r_R(r) \tag{2.12}$$

$$x.[q, r].z = [x.q, r.z] \tag{2.13}$$

Since all maps are submersions, $Q \times_{Y_0} R$ is a smooth manifold. Let $P \to Y_0$ be *any* manifold equipped with a left Y-action. We will now show that $Q \circ P$ is also a manifold and that $Q \times_{Y_0} P \to Q \circ P$ is a submersion. All maps above are automatically smooth by the universal property of a submersion. The proof is a generalisation of the construction of local trivialisations for associated bundles to principal bundles.

The map $l_Q : Q \to X_0$ is a submersion and hence admits a local section $\sigma : U \to Q$ on an open set $U \subset X_0$ around any point. This yields a local trivialisation of Q as a Y-torsor via

$$\begin{aligned} U \times_{X_0} Y &\longrightarrow Q|_U \\ (x_0, y) &\longmapsto \sigma(x_0).y \end{aligned} \quad . \tag{2.14}$$

Then $Q|_U \times_{Y_0} P \cong U \times_{Y_0} Y \times_{Y_0} P$. The diagonal Y-action on the left corresponds to the diagonal action on $Y \times_{Y_0} P$ on the right hand side. Applying the Y-action $Y \times_{Y_0} P \to P$ gives a Y-invariant map

$$Q|_U \times_{Y_0} P \longrightarrow U \times_{Y_0} P\,. \tag{2.15}$$

There is a section of this map in the other direction via $(x_0, p) \mapsto (\sigma(x_0), p)$ and, by applying the Y-action to σ, there is actually a section through every point in the fiber. Thus, we see that the quotient by Y is locally given by

$$(Q|_U \times_{Y_0} P)/Y \cong U \times_{Y_0} P\,. \tag{2.16}$$

Hence, the global quotient has a natural smooth structure for which the map $Q \times_{Y_0} P \to Q \circ P$ is a submersion. For $P = R$, we have a local trivialisation $R|_V \cong V \times_{Z_0} Z$. By shrinking U we may assume that the image of U in Y_0 is entirely contained in V. Then $U \times_{Y_0} R \cong U \times_V V \times_{Z_0} Z \cong U \times_{Z_0} Z$ and hence $Q \circ R$ trivializes over U. This shows that it is indeed a right Z-torsor. Finally, the moment map $l_{Q\circ R}$ is a submersion since in the trivialisation $U \times_{Z_0} Z$ it is given by the projection onto U.

Let Q, R, S be composable bibundles. Then, $(Q \times_{Y_0} R) \times_{Z_0} S \cong Q \times_{Y_0} (R \times_{Z_0} S)$ as submanifolds of $Q \times R \times S$. The diagonal actions of Y and Z commute. Hence the following maps are diffeomorphisms:

$$\begin{aligned} \mathrm{ass} : (Q \circ R) \circ S &\xrightarrow{\ \cong\ } (Q \times_{Y_0} R \times_{Z_0} S)/(Y \times Z) \xrightarrow{\ \cong\ } Q \circ (R \circ S) \\ [[q, r], s] &\longmapsto [q, r, s] \longmapsto [q, [r, s]] \end{aligned} \tag{2.17}$$

Let $F : Q \to Q'$ be a morphism of bibundles. Then, there is a map $F \circ \mathrm{id}_R : Q \circ R \to Q' \circ R$ given by $[q, r] \mapsto [F(q), r]$. Naturality of the associator translates to commutativity of the following diagram:

$$\begin{tikzcd} (Q \circ R) \circ S \arrow[r, "\mathrm{ass}_{Q,R,S}"] \arrow[d, "(F\circ\mathrm{id}_R)\circ\mathrm{id}_S"'] & Q \circ (R \circ S) \arrow[d, "F\circ(\mathrm{id}_{R\circ S})"] \\ (Q' \circ R) \circ S \arrow[r, "\mathrm{ass}_{Q',R,S}"] & Q' \circ (R \circ S) \end{tikzcd} \tag{2.18}$$

Both maps are just given by $[[q, r], s] \mapsto [F(q), [r, s]]$. The naturality in R, S is similarly straightforward. □

Here and in the following we use the terms bicategory and weak 2-category interchangeably.

Proposition 2.16. *There is a bicategory* $\mathsf{GrpdBiBun}$ *of Lie groupoids. The objects are Lie groupoids. The 1-morphisms are right principal groupoid bibundles and the composition is given by Equation* (2.9)*. 2-morphisms are biequivariant morphisms of bibundles. All 2-morphisms are invertible.*

We call two Lie groupoids that are isomorphic in $\mathsf{GrpdBiBun}$ *Morita equivalent.*

Proof. We need to check a few things. Firstly, it is clear that the Hom-set $\mathsf{GrpdBiBun}(X,Y)$ is a category. It has objects X-Y-bibundles and biequivariant morphisms as arrows. Any such biequivariant map is in particular a morphism of Y-torsors, hence automatically invertible. Secondly, the composition $\circ : \mathsf{GrpdBiBun}(X,Y) \times \mathsf{GrpdBiBun}(Y,Z) \to \mathsf{GrpdBiBun}(X,Z)$ is a bifunctor. For $\Phi : Q \to Q'$ and $\Psi : R \to R'$ we get $\Phi \circ \Psi : Q \circ R \to Q' \circ R'$ by $[q,r] \mapsto [\Phi(q), \Psi(r)]$. The associator ass is natural and satisfies the pentagon relation. The proof is just rebracketing. There is a unit bibundle Id_X for any Lie groupoid X:

$$
\begin{tikzcd}
X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & & X_1 \arrow[dll, "t"] \arrow[drr, "s"'] & & X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
X_0 & & & & X_0
\end{tikzcd}
\tag{2.19}
$$

Then, there is a natural isomorphism $\mathrm{Id}_X \circ Q \cong Q$ where $[x,q] \mapsto x.q$ and $q \mapsto [\mathrm{id}_{l(q)}, q]$. This is the left unitor $\lambda_Q : \mathrm{Id}_X \circ Q \to Q$. There is also a right unitor ρ_Q. Together these maps satisfy the triangle identity:

$$
\begin{tikzcd}
(Q \circ \mathrm{Id}_Y) \circ R \arrow[rr, "\mathrm{ass}_{Q,\mathrm{Id}_Y,R}"] \arrow[dr, "\rho_Q \circ \mathrm{id}_R"'] & & Q \circ (\mathrm{Id}_Y \circ R) \arrow[dl, "\mathrm{id}_Q \circ \lambda_R"] \\
& Q \circ R &
\end{tikzcd}
\tag{2.20}
$$

This concludes the proof that $\mathsf{GrpdBiBun}$ is a bicategory. All 2-morphisms are biequivariant maps between right principal bibundles, i.e. in particular maps of right torsors. Hence, they are automatically invertible. $\square$

The following shows that Morita equivalences of Lie groupoids have a special form. We call a $Q \in \mathsf{GrpdBiBun}(X,Y)$ *biprincipal* if it is left and right principal. In this case $Q^{\mathrm{op}} \in \mathsf{GrpdBiBun}(Y,X)$ is well-defined and biprincipal.

Lemma 2.17. *A bibundle ${}_XQ_Y$ is an isomorphism in* $\mathsf{GrpdBiBun}$ *if and only if it is biprincipal. In this case, its opposite bundle ${}_YQ^{\mathrm{op}}{}_X$ is an inverse.*

Proof. Let Q be biprincipal. We just need to find a biequivariant map $Q \circ Q^{\mathrm{op}} \to \mathrm{Id}_X$. This will automatically be an isomorphism. The other composition is dealt with completely analogously. The following bibundle pairing, built from the left action map, is invariant under the diagonal Y-action:

$$
\begin{array}{c}
Q \times_{Y_0}^{r_Q, l_{Q^{\mathrm{op}}}} Q^{\mathrm{op}} \xlongequal{\simeq} Q \times_{Y_0} Q \xrightarrow{\cong} X_1 \times_{X_0} Q \longrightarrow X_1 \\
(x.q, q) \longleftarrow\!\!\shortmid (x,q)
\end{array}
\tag{2.21}
$$

Explicitly, it maps a pair $(x.q, q)$ to x. If $(q', q) = (x.q, q)$, then $(q'.y, q.y) = (x.q.y, q.y)$ and both are mapped to x.
This map is also equivariant under the left and right X-actions: If $(q',q) = (x.q,q)$ maps to x then $(x_1.q', x_2^{-1}.q) = (x_1 x x_2 . x_2^{-1}.q, x_2^{-1}.q)$ is mapped to $x_1 x x_2$.
The map hence descends to a bibundle map $Q \circ Q^{\mathrm{op}} \to \mathrm{Id}_X$ which is the desired isomorphism.
Conversely, suppose there is ${}_YR_X$ such that $R \circ Q \cong \mathrm{Id}_Y$ and $Q \circ R \cong \mathrm{Id}_X$. We need to show that Q is also left principal. First, the right moment map of $R \circ Q$ is surjective. Hence, the right moment map of Q is surjective. Assume the left X-action on Q was not free with $x.q = q$. Then, there is an element $r \in R$ such that $r_Q(q) = l_R(r)$. We arrive at a contradiction $x.[q,r] = [x.q, r] = [q,r]$ to the freeness of the left X-action on $Q \circ R$. Let

$r_Q(q) = r_Q(q')$ lie in the same fiber of the right moment map. Then, as above, there is an $r \in R$ such that $[q, r]$ and $[q', r]$ are elements of $Q \circ R$. There is a unique $x \in X$ such that $x.[q, r] = [q', r]$. Then there must be $y \in Y$ such that $(x.q.y, y^{-1}.r) = (q', r)$. Since the action is free, $y = \mathrm{id}_{r(q)}$ and $x.q = q'$. Hence, we have a free and transitive action along fibers of the submersion r_Q. This shows that Q is also left principal, i.e. a biprincipal bibundle. □

Proposition 2.18. *There is a bundlization functor* $\mathsf{Grpd} \to \mathsf{GrpdBiBun}$ *which is the identity on objects and sends homomorphisms* $F : X \to Y$ *to their bundlization* $\mathrm{Bun}(F)$ *defined by the following diagram:*

$$\begin{tikzcd} X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathrm{Bun}(F) = X_0 \times^{F,t}_{Y_0} Y_1 \arrow[dl, "\mathrm{pr}_{X_0}"] \arrow[dr, "s\circ\mathrm{pr}_{Y_1}"'] & Y_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ X_0 & & Y_0 \end{tikzcd} \tag{2.22}$$

The right Y-action is obvious and the left X-action is diagonal via $x.(s(x), y) = (t(x), F(x)y)$.

It is straightforward to verify that $\mathrm{Bun}(F)$ is indeed a right principal bibundle and that $\mathrm{Bun}(F) \circ \mathrm{Bun}(G) \cong \mathrm{Bun}(F \circ G)$. However, the domain category is a 1-category and the target category is a weak 2-category.

Lemma 2.19. *A bibundle is the bundlization of a smooth homomorphism if and only if the left moment map has a smooth section.*

Proof. Clearly $x_0 \mapsto (x_0, F(\mathrm{id}_{x_0}))$ is a section of the left moment map in diagram 2.22. Conversely, suppose that ${}_XQ_Y$ has a section $\sigma : X_0 \to Q$. Then we consider the composition

$$\begin{tikzcd} X_1 \arrow[r] \arrow[rrr, bend left, "F"] & Q \times_{X_0} Q & Q \times_{Y_0} Y_1 \arrow[l, "\cong"] \arrow[r] & Y_1 \\ x \arrow[r, mapsto] & (\sigma_{t(x)}, x.\sigma_{s(x)}) & (q, y) \arrow[r, mapsto] & y \end{tikzcd} \tag{2.23}$$

Translating this into equations we define F implicitly by $x.\sigma_{s(x)} = \sigma_{t(x)}.F(x)$. Then we use associativity and commutativity of the actions to conclude

$$\sigma_{t(x)}.F(xx') = xx'.\sigma_{s(x')} = x.\sigma_{t(x')}.F(x') = \sigma_{t(x)}.F(x).F(x'). \tag{2.24}$$

Hence, $F(xx') = F(x)F(x')$ since the Y-action is free and we have indeed found a homomorphism $X \to Y$. Now the equivariant isomorphism $\mathrm{Bun}(F) \cong Q$ is given by $(x_0, y) \mapsto \sigma_{x_0}.y$. □

Definition 2.20. A *Morita morphism* $F : X \to Y$ is a homomorphism of Lie groupoids that is:

- A surjective submersion
- Fully faithful, i.e. the following diagram is cartesian/ a pullback diagram in Mfld:

$$\begin{tikzcd} X_1 \arrow[r, "F"] \arrow[d, "(s,t)"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "(s,t)"] \\ X_0 \times X_0 \arrow[r, "(F,F)"] & Y_0 \times Y_0 \end{tikzcd} \tag{2.25}$$

So if $y : F(x) \to F(x')$, we get a unique induced map $* \to X_1$ which maps to y under F. Note that this implies that F is fully faithful and essentially surjective as a map of set-theoretical groupoids, i.e. an equivalence. We are thus dealing with an appropriate notion of equivalence of categories in the setting of a smooth structure on the arrows. Note that X_1 is in this case a pullback groupoid along the surjective submersion $F : X_0 \to Y_0$.

Lemma 2.21. *The bundlization functor sends Morita morphisms in* Grpd *to isomorphisms in* $\mathsf{GrpdBiBun}$.

Proof. Let $F : X \to Y$ be a Morita morphism. Then, $\mathrm{Bun}(F) = X_0 \times_{Y_0} Y_1$ is also a left X-torsor: The right moment map $s \circ \mathrm{pr}_{Y_1}$ is a submersion since it is a composition of submersions. Indeed, to show that pr_{Y_1} is a submersion we can use that $F : X_0 \to Y_0$ admits local sections inducing local sections of $\mathrm{pr}_{Y_1} : X_0 \times_{Y_0} Y_1 \to Y_1$.
Transitivity on the fibers of the right moment map is shown as follows: Let (x_0, y) and (x_0', y') be elements of the same right moment map fiber. They satisfy $s(y) = s(y') = y_0$ and of course $F(x_0) = t(y)$ and $F(x_0') = t(y')$. Then, $y'y^{-1} : F(x_0) \to F(x_0')$ is an arrow in Y_1. There is a unique $x \in X_1$ with $F(x) = y$ and $s(x) = x_0$, $t(x) = x_0'$. Clearly $x.(x_0, y) = (x_0', y')$. The uniqueness of x also shows that the action is free.
We thus have a biprincipal bitorsor. Using Lemma 2.17 this indeed an isomorphism in $\mathsf{GrpdBiBun}$. ∎

Lemma 2.22. *Any bibundle ${}_XQ_Y$ is isomorphic to a zig-zag of the form* $\mathrm{Bun}(G) \circ \mathrm{Bun}(F)^{-1}$ *where* $F : Z \to X$ *is a Morita morphism and* $G : Z \to Y$ *a homomorphism of groupoids.*

$$\begin{tikzcd} & Z \arrow[dl, "F"', "\simeq"] \arrow[dr, "G"] & \\ X & & Y \end{tikzcd} \tag{2.26}$$

In fact, we can arrange for $Z = \pi^* X$, *where* $\pi : \bigsqcup_i U_i \to X_0$ *and* (U_i) *is an open covering of* X_0.

Proof. Let U_i be a cover of X, s.t. Q admits local sections $\sigma_i : U_i \to Q$. Then let π be the projection $\bigsqcup_i U_i \to X$ and also note $F : \pi^* X \to X$ is a Morita morphism. The left moment map of the composition $\mathrm{Bun}(F) \circ Q$ has a section and is hence induced by a homomorphism $G : \pi^* X \to Y$. ∎

Remark 2.23. The algebraic induction of representations via tensoring finds its analogue in the geometric setting via a similar process: Note that an X-torsor over M is the same as a right principal M-X-bibundle, where we view $M \rightrightarrows M$ as a trivial groupoid. Define $[X](M)$ to be the groupoid of all X-torsors over M. The 2-categorical composition then allows us to view X-Y-bibundles Q as maps:

$$\begin{aligned} [X](M) &\longrightarrow [Y](M) \\ P &\longmapsto P \circ Q \end{aligned} \tag{2.27}$$

If $Q = \mathrm{Bun}(F)$, then $P \circ Q = (P \times_{Y_0}^{F\circ\mu, t} Y_1)/X_1$. Note that when restricted to Lie groups, the construction we end up with is called reduction of structure group or associated bundle construction, depending on whether we are interested in the fiber of this map or the image. Depending on the context, the object $[X]$ [9] might be called the *differentiable stack* associated to X. As we saw, two Morita equivalent Lie groupoids give rise to equivalent differentiable stacks. The converse is also true. The bicategories of differentiable stacks

[9] $[X]$ is a 2-sheaf with values in groupoids on the site Mfld.

and Lie groupoids are equivalent. For translation purposes there are the “dictionary lemmata” in [BX11, 2.6].

However, the bicategory $\mathsf{GrpdBiBun}$ is more accessible for differential geometric constructions. If such a construction is Morita-invariant, it is a construction on the stack.

We can summarize the discussion about bundlizations as follows. Let W be the class of Morita morphisms.

Theorem 2.24. *The bundlization functor* $\mathsf{Grpd} \to \mathsf{GrpdBiBun}$ *induces an equivalence of bicategories* $\mathsf{Grpd}[W^{-1}] \to \mathsf{GrpdBiBun}$.

Both of the latter are actually equivalent to the bicategory of stacks, as defined e.g. in [Blo08] and hinted upon in Remark 2.23. From now on we will only use $\mathsf{GrpdBiBun}$, but it helps to have in mind the relation to Morita morphisms.

There is still a lot to say about why this bicategory is inherently interesting. For example, n-dimensional representations of a groupoid are classified by the stack BO_n represented by the Lie groupoid $O(n) \rightrightarrows pt$. This is essentially by definition. This also shows how representations are manifestly a Morita invariant or invariant of the underlying stack. For a more detailed treatment of these bicategories we refer to [HG07, BX11].

3 Integration on Lie Groupoids

This section discusses integration on Lie groupoids from a variety of perspectives. This includes introducing the convolution algebra, both via systems of measures and intrinsically via densities. The intrinisic viewpoint is important for general constructions such as deforming the groupoid structure. Otherwise, the choice of a Haar system remains as an artefact. We will define the convolution operation. Section 3.1 introduces Lie derivatives. Using this we will prove in Section 3.3 the existence of a cochain map relating the deformation complex of a Lie groupoid to the Hochschild cochain complex of its convolution algebra. With a few differences in exposition, we follow [KP21]. The main difference is that we explicitly define the Lie derivative $\mathcal{L}_c$ along deformation cocycles $c \in C^k_{\mathrm{def}}(G)$ and study their combinatorics. We also provide more details.

Definition 3.1. A *left Haar system* on a Lie groupoid $G \rightrightarrows M$ is a set $\{\lambda_x\}_{x\in M}$ of Radon measures with $\mathrm{supp}\lambda_x = t^{-1}(x)$ such that

1. The integral
$$\varphi \mapsto \left[x \mapsto \int_{t^{-1}(x)} \varphi \mathrm{d}\lambda_x\right] \tag{3.1}$$
maps smooth functions in $C_c^\infty(G)$ to smooth functions in $C_c^\infty(M)$.

2. The integration is left-invariant:
$$\int_{t^{-1}(t(g))} \varphi(h)\mathrm{d}\lambda(h) = \int_{t^{-1}(s(g))} \varphi(gh)\mathrm{d}\lambda(h) \tag{3.2}$$

Analogously we define right Haar systems. In the integral we will usually just write $\mathrm{d}h$ for $\mathrm{d}\lambda_x(h)$. Starting with a left Haar system on G, we can define a convolution product on $C_c^\infty(G)$ by
$$\varphi_1 * \varphi_2(g) = \int \varphi_1(gh)\varphi_2(h^{-1})\mathrm{d}h\,. \tag{3.3}$$

Here it is clear that we can only integrate over $t^{-1}(s(g))$, so we spare this from the notation. Similarly, a right Haar system yields a convolution product
$$\varphi_1 * \varphi_2(g) = \int \varphi_1(gh^{-1})\varphi_2(h)\mathrm{d}h\,. \tag{3.4}$$

The integration is clearly over $s^{-1}(s(g))$.

Example 3.2. 1. For a Lie group G this reduces to the smooth Lie group convolution algebra. Integration is done with respect to the Haar measure on the group.
$$\varphi_1 * \varphi_2(g) = \int_G \varphi_1(gh^{-1})\varphi_2(h)\mathrm{d}h\,. \tag{3.5}$$
The case of $(\mathbb{R}^n, +)$ should be familiar where we get
$$\varphi_1 * \varphi_2(x) = \int_{\mathbb{R}^n} \varphi_1(x-y)\varphi_2(y)\mathrm{d}y\,. \tag{3.6}$$

2. If $G \rightrightarrows M$ is an étale groupoid, then the counting measure is a Haar system. The convolution becomes a summation:
$$f_1 * f_2(g) = \sum_{h:s(h)=s(g)} f_1(gh^{-1})f_2(h) = \sum_{g_1g_2=g} f_1(g_1)f_2(g_2)\,. \tag{3.7}$$

The problem here is that this choice of a Haar system is not intrinsic. We will now see how an intrinsic version of the convolution product is defined.

Recall that a *density* on an n-dimensional vector space V is an alternating multilinear function $a : \bigwedge^n V \to \mathbb{R}$ such that $a(Av_1, \ldots, Av_n) = |\det(A)| a(v_1, \ldots, v_n)$. To any vector bundle E there is a natural bundle $\mathcal{D}_E$ of densities on E which is always a trivial vector bundle. One can always integrate densities on TM to smooth functions on M. More generally, given any surjective submersion $f : M \to N$, we let $\mathcal{D}_f = \mathcal{D}_{\ker(df)}$. There is an associated *fiber integral*

$$\int_f : \Gamma_c(\mathcal{D}_f) \to C_c^\infty(N)\,, \tag{3.8}$$

which just integrates the densities fiberwise. Even more generally, we can integrate sections of pullback bundles

$$\int_f : \Gamma_c(\mathcal{D}_f \otimes f^* E) \to \Gamma_c(E)\,. \tag{3.9}$$

Lemma 3.3. *There is a natural isomorphism of vector bundles:*

$$\begin{aligned} f^*\mathcal{D}_E &\xrightarrow{\cong} \mathcal{D}_{f^*E} \\ a &\mapsto \hat{a}\,. \end{aligned} \tag{3.10}$$

Proof. Denote the canonical map by $F : f^*E \to E$. Let $a \in (f^*\mathcal{D}_E)_m$, i.e. $a \in (\mathcal{D}_E)_{f(m)}$. Then define $\hat{a}(v_1, \ldots, v_n) := a(Fv_1, \ldots, Fv_n)$ for $v_1, \ldots, v_n \in (f^*E)_m$. This is clearly a vector bundle isomorphism since F is invertible on each fiber. ☐

Lemma 3.4. *Let G be a Lie groupoid, $A = Lie(G)$. There are natural vector bundle isomorphisms of vector bundles over $G^{(2)} = G \times_M G$:*

$$\mathrm{pr}_1^* \mathcal{D}_s \cong m^* \mathcal{D}_s \tag{3.11}$$

$$\mathrm{pr}_2^* \mathcal{D}_s \cong \mathcal{D}_m \tag{3.12}$$

Over G, we also have $\ker(ds) \cong t^* A$.

Proof. The first one is induced by the vector bundle isomorphism $\mathrm{pr}_1^* \ker(ds) \cong m^* \ker(ds)$ given by right multiplication:

$$dR_{g_2} : \ker(ds)_{g_1} \to \ker(ds)_{g_1 g_2}\,. \tag{3.13}$$

While the second one is induced from the vector bundle isomorphism $\mathrm{pr}_2^* \ker(ds) \cong \ker(dm)$:

$$\begin{aligned} d\,\mathrm{pr}_2 : \ker(dm) &\to \mathrm{pr}_2^* \ker(ds) \\ (X_{g_1}, X_{g_2}) &\mapsto X_{g_2} \end{aligned} \tag{3.14}$$

The inverse of this map is $X_{g_2} \mapsto (dL_{g_1} di(X_{g_2}), X_{g_2})$.
The final isomorphism is also given by right multiplication. One can also pullback the first isomorphism along $\mathrm{id} \times i : G \to G^{(2)}$, which induces an isomorphism $\ker(ds) \cong t^* \ker(ds)$ since $\mathrm{pr}_1 \circ \mathrm{id} \times i = \mathrm{id}$ and $m \circ \mathrm{id} \times i = t$. ☐

Definition 3.5. The *convolution algebra* $\mathcal{A}_G$ is the space $\Gamma_c(\mathcal{D}_s)$ with the convolution product written symbolically as

$$a_1 * a_2 = \int_m a_1 \otimes a_2\,. \tag{3.15}$$

It is the following composition of operations:

$$* : \Gamma_c(\mathcal{D}_s) \otimes \Gamma_c(\mathcal{D}_s) \xrightarrow{\otimes} \Gamma_c(G^{(2)}, \mathrm{pr}_1^* \mathcal{D}_s \otimes \mathrm{pr}_2^* \mathcal{D}_s) \xrightarrow{\cong} \Gamma_c(G^{(2)}, m^*\mathcal{D}_s \otimes \mathcal{D}_m) \xrightarrow{\int_m} \Gamma_c(\mathcal{D}_s) \tag{3.16}$$

Choose a density ω_0 on A. Via pullback this gives a density $\omega(g) = dR_g^*\omega_0(t(g))$ on $t^*A \cong \ker(ds)$ which is right-invariant as a section of $\mathcal{D}_s$. The density ω also serves as a trivialisation of $\mathcal{D}_s$ and hence induces an isomorphism $C_c^\infty(G) \to \Gamma_c(\mathcal{D}_s)$, $f \mapsto f\omega$. Lastly, ω also provides a right Haar system on G via the functional

$$\varphi \mapsto \int_{s^{-1}(x)} \varphi\omega =: \int_{s^{-1}(x)} \varphi(h)\mathrm{d}h\,, \tag{3.17}$$

which provides a set of Radon measures on the source fibers by the Riesz representation theorem.
We now want to calculate $*$ on $\varphi_1, \varphi_2 \in C_c^\infty(G)$: Under the isomorphism $\mathrm{pr}_1^* \mathcal{D}_s \otimes \mathrm{pr}_2^* \mathcal{D}_s \cong m^*\mathcal{D}_s \otimes \mathcal{D}_m$ we have:

$$\begin{aligned} \varphi_1(g_1)\omega_{g_1} \otimes \varphi_2(g_2)\omega_{g_2} \longmapsto &\varphi_1(g_1) dR^*_{g_2^{-1}}\omega_{g_1} \otimes \varphi_2(g_2) d\,\mathrm{pr}_2^*\,\omega_{g_2} \\ =&\varphi_1(g_1)\omega_{g_1g_2} \otimes \varphi_2(g_2)(\mathrm{pr}_2^*\,\omega)_{(g_1,g_2)}\,. \end{aligned} \tag{3.18}$$

We used right invariance of ω to conclude $\omega_{g_1g_2} = (R^*_{g_2^{-1}}\omega)_{g_1g_2} = dR^*_{g_2^{-1}}\omega_{g_1}$.

$$\begin{aligned} \varphi_1\omega * \varphi_2\omega(g) =& \int_{(g_1,g_2)\in m^{-1}(g)} \varphi_1(g_1)\omega_{g_1g_2} \otimes \varphi_2(g_2)\,\mathrm{pr}_2^*\,\omega_{(g_1,g_2)} \\ =&\omega_g \otimes \int_{g_1g_2=g} \varphi_1(g_1)\varphi_2(g_2)\,\mathrm{pr}_2^*\,\omega_{(g_1,g_2)} \\ =&\left(\int_{s^{-1}(s(g))} \varphi_1(gh^{-1})\varphi_2(h)\omega_h\right)\omega_g \\ =&\left(\int_{s^{-1}(s(g))} \varphi_1(gh^{-1})\varphi_2(h)\mathrm{d}h\right)\omega_g \end{aligned} \tag{3.19}$$

Here we used that the integral does not change under pullback along $m^{-1}(g) \cong s^{-1}(s(g))$ via pr_2 and $h \mapsto (gh^{-1}, h)$ in the reverse direction. To conclude, this hence indeed provides an intrinsic convolution operation on s-densities. We record:

Proposition 3.6. *For different choices of right Haar system, the algebras $\mathcal{A}_G$ and $(C_c^\infty(G), *)$ are isomorphic. They are also isomorphic to the one obtained by any left Haar system.*

Proof. The first part is dealt with in the previous computation. The latter isomorphism is given by sending a function f to $f \circ i$, the precomposition with inversion. This provides an isomorphism between the convolution algebras of a left Haar system and its associated right Haar system, related by inversion. □

3.1 Operations on Densities

This section is based on [KP21] and fills in a gap in their argument for the existence of Lie derivatives along deformation cochains.

Let $E \to M$ be a vector bundle. If $\Psi : E \to E$ is a vector bundle isomorphism covering a diffeomorphism $\Phi : M \to M$, we can pull back a section a of $\mathcal{D}_E$ via

$$(\Psi^*a)_x(v_1, \ldots, v_n) = a_{\Phi(x)}(d\Psi(v_1), \ldots, d\Psi(v_n))\,. \tag{3.20}$$

Let $f : M \to N$ be a submersion. Recall that two vector fields X on M and Y on N are said to be *f-related* if $df(X_m) = Y_{f(m)}$. If for X there is such a Y, we call X *f-relatable* and denote the Lie algebra of f-relatable vector fields by $\mathfrak{X}_f(M)$. That X and Y are f-related is equivalent to $f \circ \Phi^t_X = \Phi^t_Y \circ f$ for small t, where Φ^t_X is the flow of X. Hence, $d\Phi^t_X$ maps $\ker(df)$ to itself while covering the diffeomorphism Φ^t_X:

$$df \circ d\Phi^t_X = d\Phi^t_Y \circ df \,. \tag{3.21}$$

Then, any f-relatable vector field $X \in \mathfrak{X}_f(M)$ acts on $a \in \Gamma(\mathcal{D}_f)$ via

$$\mathcal{L}_X a = \frac{d}{dt}\bigg|_{t=0} (\Phi^t_X)^* a \,. \tag{3.22}$$

Lemma 3.7. *Let $a \in \Gamma(\mathcal{D}_f)$, $y \in N$ and $X \in \mathfrak{X}_f(M)$ be an f-relatable vector field. If X vanishes along $f^{-1}(y)$, then $\mathcal{L}_X a$ also vanishes along $f^{-1}(y)$.*

Proof. If $X = 0$ on the fiber we have $\Phi^t_X(x) = x$ for $x \in f^{-1}(y)$. In particular, $d\Phi^t_X(v) = v$ for $v \in \ker(df)$. Hence, $(\Phi^t_X)^* a = a$ along $f^{-1}(y)$ and thus $\mathcal{L}_X a = 0$. □

This means that we can define $\mathcal{L}_X a$ for $X \in \Gamma_f(f^{-1}(n), TM)$ by just computing $\mathcal{L}_{\tilde{X}}$ for any smooth extension $\tilde{X}$ to an f-relatable vector field on all of M. The following question is not addressed in [KP21].

Question 3.8. Can we always find an f-relatable extension $\tilde{X}$ of X?

That this indeed exists is the content of the following lemmata.

Let $f : M \to N$ be a submersion. Then $f^{-1}(n)$ is a properly embedded submanifold of M. This means it is closed as a subspace or equivalently that the inclusion is proper. The following lemma is easy to prove using a partition of unity and the slice charts of an embedding. It appears as Lemma 5.34 in [Lee13].

Lemma 3.9. *Let $\iota : S \subset M$ be an embedded submanifold. The following are equivalent:*

1. *S is properly embedded.*
2. *The restriction $\iota^* : C^\infty(M) \to C^\infty(S)$ is surjective, i.e. any smooth function on S extends to M.*
3. *For any vector bundle $E \to M$, $\iota^* : \Gamma(E) \to \Gamma(\iota^* E)$ is surjective.*

Since f is a submersion, there is a short exact sequence of vector bundles

$$0 \longrightarrow VM \longrightarrow TM \longrightarrow f^*TN \longrightarrow 0 \ , \tag{3.23}$$

where $VM := \ker(df)$ is the vertical bundle. A splitting of this sequence is an Ehresmann connection which gives us a direct sum decomposition $TM \cong VM \oplus HM$ into the vertical and a horizontal bundle where $df : HM \xrightarrow{\cong} f^*TN$.

In particular, we have horizontal lifts

$$\begin{aligned} \mathfrak{X}(N) &\longrightarrow \mathfrak{X}_{\mathrm{hor}}(M) = \Gamma(HM) \\ Y &\longmapsto \tilde{Y} \,, \end{aligned} \tag{3.24}$$

where $\tilde{Y} = f^*Y$ under the isomorphism of the horizontal bundle and the pullback of TN. Horizontal lifts are unique. Note that $\mathfrak{X}_{\mathrm{hor}}(M) \subset \mathfrak{X}_f(M)$.

Consider the following diagram of embedded submanifolds and surjective submersions:

$$\begin{array}{ccc} S & \hookrightarrow & M \\ \big\downarrow f & & \big\downarrow f \\ T & \hookrightarrow & N \end{array} \tag{3.25}$$

Lemma 3.10. *Let S, T be properly embedded. The restriction $\iota^* : \mathfrak{X}_f(M) \to \Gamma_f(S, \iota^* TM)$ is surjective. That is, any vector field X defined only on S that is f-relatable to a vector field on T can be extended to an f-relatable vector field.*

Proof. Let X be as above and decompose $X = X_{\mathrm{hor}} + X_{\mathrm{ver}}$. Let $df(X) = Y \in \mathfrak{X}(T)$ and extend it arbitrarily to a vector field $Y \in \mathfrak{X}(N)$ using that T is properly embedded. Then $\tilde{Y}$ is the horizontal lift of Y and hence restricts to X_{hor} along S. Using that S is properly embedded, there is an extension of X_{ver} to a vector field Z on M. Then $Z = Z_{\mathrm{ver}} + Z_{\mathrm{hor}}$ and Z_{ver} is also an extension of X_{ver}. All in all, $\tilde{Y} + Z_{\mathrm{ver}}$ is an extension of X that is f-related to Y. $\square$

The exact same proof also proves the following slightly more general statement:

Lemma 3.11. *Let $E \to M$ and $E' \to N$ be vector bundles and $F : E \to E'$ a vector bundle homomorphism covering $f : M \to N$. Assume that F has constant rank and is surjective.*

$$\begin{tikzcd}
 & \iota^* E \arrow[rr] \arrow[dl] \arrow[dd, "F"] & & E \arrow[dl] \arrow[dd, "F"] \\
S \arrow[rr, hook, "\iota"] \arrow[dd, "f"] & & M \arrow[dd, "f"] & \\
 & \iota'^* E' \arrow[rr] \arrow[dl] & & E' \arrow[dl] \\
T \arrow[rr, hook, "\iota'"] & & N &
\end{tikzcd} \tag{3.26}$$

Then $\iota^* : \Gamma_F(E) \to \Gamma_F(\iota^* E)$ is surjective. Here $\Gamma_F(E)$ consists of sections X of E such that there is a section Y of E' with $F \circ X = Y \circ f$.

Corollary 3.12. *For $S = f^{-1}(n)$, $T = \{n\}$, we get that any vector field on a single fiber with constant projection can be extended to an f-relatable vector field.*

For the definition of $C^k_{\mathrm{def}}(G)$ we refer to Section 4. The nerve $G^{(k)}$ is discussed thoroughly in Section 3.2. As a manifold $G^{(k)}$ is the subset of $(g_1, \ldots, g_k) \in G^k$ with $s(g_i) = t(g_{i+1})$. For now, we just note $C^k_{\mathrm{def}}(G) = \Gamma_{ds}(G^{(k)}, \mathrm{pr}_1^* TG)$. For fixed $g_2, \ldots, g_k$ and any $c \in C^k_{\mathrm{def}}(G)$, the restriction $c(-, g_2, \ldots, g_k)$ has constant projection along ds.

Corollary 3.13. *Let G be a Lie groupoid. Then, taking any element $c \in C^k_{\mathrm{def}}(G)$ we can extend $c(\cdot, g_2, \ldots, g_k) : s^{-1}(t(g_2)) \to TG$ to an s-relatable vector field on G.*

We can make this even stronger. Consider the inclusion $G^{(k)} \subset G^k$.

Corollary 3.14. *Any $c \in C^k_{\mathrm{def}}(G)$ can be extended to*

$$\tilde{c} : G^k \to TG \,, \tag{3.27}$$

with $\tilde{c}(g_1, \ldots, g_k) \in T_{g_1} G$ and $ds \circ \tilde{c}(g_1, \ldots, g_k) = \tilde{s_c}(s(g_1), g_2, \ldots, g_k)$.
For fixed $(g_2, \ldots, g_k) \in G^{(k-1)}$ we hence have an s-related pair of vector fields with $\tilde{c}(-, g_2, \ldots, g_k) \in \mathfrak{X}_s(G)$ and $\tilde{s_c}(-, g_2, \ldots, g_k) \in \mathfrak{X}(M)$.

Proof. Take $M = G^k$, $N = M \times G^{k-1}$, $S = G^{(k)}$ and $T = M \times_M G^{(k-1)} \cong G^{(k-1)}$ with

$$\begin{aligned} G^k &\to M \times G^{k-1} \\ (g_1, \ldots, g_k) &\mapsto (s(g_1), g_2, \ldots, g_k) \,. \end{aligned} \tag{3.28}$$

As vector bundles, we let $E = \mathrm{pr}_1^* TG$ and $E' = (\mathrm{pr}_M)^* TM$.

$$
\begin{tikzcd}
& \mathrm{pr}_1^* TG \arrow[rr] \arrow[dl] \arrow[dd, "ds"] & & \mathrm{pr}_1^* TG \arrow[dl] \arrow[dd, "ds"] \\
G^{(k)} \arrow[rr, hook, "\iota"] \arrow[dd] & & G^k \arrow[dd, "s\times \mathrm{id}_{G^{k-1}}"] & \\
& (t\circ \mathrm{pr}_1)^* TM \arrow[rr] \arrow[dl] & & \mathrm{pr}_M^* TM \arrow[dl] \\
M \times_M G^{(k-1)} \cong G^{(k-1)} \arrow[rr, hook, "\iota'"] & & M \times G^{k-1} &
\end{tikzcd}
\tag{3.29}
$$

The datum of $c \in C^k_{\mathrm{def}}(G)$ is the same as $c \in \Gamma_{ds}(\iota^* \mathrm{pr}_1^* TG)$. It is extendable to G^k since the embeddings of fiber products are proper. □

All in all, given $c \in \Gamma_{ds}(G^{(k)}, \mathrm{pr}_1^* TG)$ and $a \in \Gamma(\mathcal{D}_s)$ we can define

$$\mathcal{L}_c a(g_1, \ldots, g_k) := \mathcal{L}_{\tilde{c}(-,g_2,\ldots,g_k)} a(g_1)\,, \tag{3.30}$$

where $\tilde{c}$ is any ds-relatable extension of c to G^k.

3.2 Simplicial Nerve of a Groupoid

Any Lie groupoid $G \rightrightarrows G^{(0)}$ naturally gives rise to a simplicial manifold, a simplicial model for its classifying space. For future reference we set notation and describe the structure here: The k-nerve $G^{(k)}$ is the space of k composable arrows. It is a smooth manifold as a fiber product along submersions.

$$G^{(k)} = \{(g_1, \ldots, g_k) : s(g_i) = t(g_{i+1})\}$$

Face and degeneracy maps are given by:

$$\delta_i^{(k)}(g_1, \ldots g_k) = \begin{cases} (g_2, \ldots, g_k) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_k) & i = 1, \ldots, k-1 \\ (g_1, \ldots, g_{k-1}) & i = k \end{cases} \tag{3.31}$$

$$\sigma_i^{(k)}(g_1, \ldots, g_k) = \begin{cases} (\mathrm{id}_{t(g_1)}, g_1, \ldots, g_k) & i = 0 \\ (g_1, \ldots, g_i, \mathrm{id}_{s(g_i)}, g_{i+1}, \ldots, g_k) & i = 1, \ldots, k-1 \\ (g_1, \ldots, g_k, \mathrm{id}_{s(g_k)}) & i = k \end{cases} \tag{3.32}$$

To be precise we should add $G^{(0)}$ and $\delta_0^{(1)}(g) = s(g)$, $\delta_1^{(1)} = t(g)$. The simplicial identities are:

$$\delta_i \delta_j = \delta_{j-1}\delta_i \qquad i < j \tag{3.33}$$

$$\delta_i \sigma_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases} \tag{3.34}$$

$$\sigma_i \sigma_j = \sigma_{j+1}\sigma_i \qquad i \leq j \tag{3.35}$$

The resulting structure is the nerve $G^{(\bullet)}$ which is a simplicial manifold

$$G^{(\bullet)} : \Delta^{\mathrm{op}} \to \mathsf{Mfld}.$$

It turns out that the geometric realization of this simplicial set is a model for the classifying space of G. [HG07, 2.22] Similarly, the geometric realization of an action groupoid $G \ltimes M \rightrightarrows M$ becomes the Borel construction $EG \times_G M$.
Looking at the graded function algebra $C^\bullet(G) = C^\infty(G^{(\bullet)})$ we easily see that the simplicial identities provide us with a cohomological differential

$$\delta f = \sum_{i=0}^{k} (-1)^i \delta_i^* f \tag{3.36}$$

that is a graded derivation with respect to the cup product $C^p(G) \otimes C^q(G) \to C^{p+q}(G)$.

$$f_1 \cup f_2(g_1, \ldots, g_p, \ldots, g_{p+q}) = f_1(g_1, \ldots, g_p) f_2(g_{p+1}, \ldots, g_{p+q}) \tag{3.37}$$

We will sometimes also write $f_1 \otimes f_2 = f_1 \cup f_2$.

3.3 A simplicial Look at Convolution

Upon choosing a right Haar system, note that we have canonical identifications

$$\Gamma_c^\infty(G^{(k)}, \mathrm{pr}_1^* \mathcal{D}_s \otimes \cdots \otimes \mathrm{pr}_k^* \mathcal{D}_s) \cong C_c^\infty(G^{(k)}) =: C_c^k(G).$$

Since the formulae look cleaner we will phrase the following simply in terms of the latter functions. We let $m_i : C_c^k(G) \to C_c^{k-1}(G)$ be the fiber integration along $\delta_i : G^{(k)} \to G^{(k-1)}$. Explicitly:

$$m_i f(g_1, \ldots, g_{k-1}) = \begin{cases} \int_{s^{-1}(t(g_1))} f(g, g_1, \ldots, g_{k-1}) \mathrm{d}g & i = 0 \\ \int_{s^{-1}(s(g_i))} f(g_1, \ldots, g_{i-1}, g_i g^{-1}, g, g_{i+1}, \ldots, g_{k-1}) \mathrm{d}g & i = 1, \ldots, k-1 \\ \int_{s^{-1}(s(g_{k-1}))} f(g_1, \ldots, g_{k-1}, g^{-1}) \mathrm{d}g & i = k \end{cases} \tag{3.38}$$

There are some easy-to-write-down compatibilities with the cup product such as $m_i(f \cup g) = (m_i f) \cup g$ for $f \in C_c^k(G), g \in C_c^l(G)$ and $i < k$. We can also recover a simplicial identity:

$$m_i m_j = m_{j-1} m_i, \quad i < j \tag{3.39}$$

$$\begin{array}{c} \vdots \\ \downarrow\downarrow\downarrow\downarrow\downarrow \\ C_c^3(G) \\ \downarrow\downarrow\downarrow\downarrow \\ C_c^2(G) \\ \downarrow\downarrow\downarrow \\ C_c^1(G) \\ \downarrow\downarrow \\ C_c^0(G) \end{array} \tag{3.40}$$

In the above diagram the most important are the inner m_i for $i = 1, \dots, k-1$ that indeed perform a convolution. m_0 and m_k do a sort of averaging. From the simplicial identities we know that all compositions of the inner m_i's (inductive convolutions) coincide as maps $C_c^k(G) \to C_c^1(G) = C_c^\infty(G)$, e.g. $m_1 \circ m_1 = m_1 \circ m_2$.
The convolution product on $C_c^\infty(G) = C_c^1(G)$ is actually given by $f * g = m_1(f \cup g)$. Hence, convolution is associative:

$$\begin{aligned} (f * g) * h &= m_1(m_1(f \cup g) \cup h) = m_1 m_1(f \cup g \cup h) \\ &= m_1 m_2(f \cup g \cup h) = m_1(f \cup m_1(g \cup h)) = f * (g * h) \end{aligned} \tag{3.41}$$

Proof of the identities. There are two cases to consider: $j = i+1$ and $j > i+1$. Let first $j > i+1$:

$$\begin{aligned} m_i m_j f(g_1, \dots, g_{k-2}) &= \int_{s^{-1}(s(g_i))} m_j f(g_1, \dots, g_i g^{-1}, g, g_{i+1}, \dots, g_{k-2}) \mathrm{d}g \\ &= \int_{s^{-1}(s(g_i))} \int_{s^{-1}(s(g_{j-1}))} f(g_1, \dots, g_i g^{-1}, g, g_{i+1}, \dots, g_{j-1} h^{-1}, h, g_j, \dots, g_{k-2}) \mathrm{d}h \mathrm{d}g \\ &= \int_{s^{-1}(s(g_{j-1}))} \int_{s^{-1}(s(g_i))} f(g_1, \dots, g_i g^{-1}, g, g_{i+1}, \dots, g_{j-1} h^{-1}, h, g_j, \dots, g_{k-2}) \mathrm{d}g \mathrm{d}h \\ &= m_{j-1} m_i f(g_1, \dots, g_{k-2}) \,. \end{aligned} \tag{3.42}$$

If $j = i+1$ we have

$$\begin{aligned} m_i m_{i+1} f(g_1, \dots, g_{k-2}) &= \int_{s^{-1}(s(g_i))} m_{i+1} f(g_1, \dots, g_i g^{-1}, g, g_{i+1}, \dots, g_{k-2}) \mathrm{d}g \\ &= \int_{s^{-1}(s(g_i))} \int_{s^{-1}(s(g_i))} f(g_1, \dots, g_i g^{-1}, g h^{-1}, h, g_{i+1}, \dots g_{k+2}) \mathrm{d}h \mathrm{d}g \\ &= \int_{s^{-1}(s(g_i))} \int_{s^{-1}(t(h))} f(g_1, \dots, g_i h^{-1} k^{-1}, k, h, g_{i+1}, \dots g_{k+2}) \mathrm{d}k \mathrm{d}h \\ &= m_i m_i f(g_1, \dots, g_{k-2}) \,. \end{aligned} \tag{3.43}$$

We used right invariance to change variables here to $k = gh^{-1}$. For $i = 0, k$ we do not have to make new computations since they are formally identical when we just think of $(g_1, \dots, g_k)$ as an enlarged tuple $(\mathrm{id}_{t(g_1)}, g_1, \dots, g_k, \mathrm{id}_{s(g_k)})$.

For the claim about independence of ordering we can use an inductive argument. Starting with $m_1 m_2 = m_1 m_1$, this is a simplicial identity. Let $m_{i_1} \dots m_{i_k} : C_c^{k+1}(G) \to C_c^1(G)$ be any concatenation of convolutions. Then $m_{i_1} \dots m_{i_{k-1}} = m_1 \dots m_1$ by induction. Therefore, using the induction hypothesis twice, we can conclude that all such compositions agree:

$$m_{i_1} \dots m_{i_k} = m_1 \dots m_1 m_{i_k} = m_1 \dots m_{i_k - 1} m_1 = m_1 \dots m_1 m_1 \,. \tag{3.44}$$

□

We denote the total convolution by $m : C_c^k(G) \to C_c^1(G)$.

3.4 Construction of the Cochain Map relating Deformation Cohomology and Hochschild Cohomology

Here we show how to construct the cochain map $\Phi : C^*_{\mathrm{def}}(G) \to C^*(\mathcal{A}_G, \mathcal{A}_G)$, originally achieved in [KP21]. We take a more combinatorial approach using the (co)simplicial structure maps.

Let A be an algebra over $\mathbb{C}$ or $\mathbb{R}$. Below we work with the convolution algebra $A = \mathcal{A}_G$. Let us start with a working definition of the Hochschild cochain complex $C^*(A, A)$. In degree k its cochains are multilinear maps $f : A^k \to A$ or, equivalently, linear maps $A^{\otimes k} \to A$. It is easier to define multilinear maps than to define a tensor product. But considerations like this are reserved for later chapters. the differential $d : C^k(A, A) \to C^{k+1}(A, A)$ is given by

$$\begin{aligned} df(a_1, \ldots, a_{k+1}) =& a_1 f(a_2, \ldots, a_{k+1}) + \sum_{i=1}^{k} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{k+1}) \\ &+ (-1)^{k+1} f(a_1, \ldots, a_k) a_{k+1} \,. \end{aligned} \tag{3.45}$$

The differential d is the alternating sum of the cosimplicial face maps that we denote by $d_i : C^k(A, A) \to C^{k+1}(A, A)$. Similarly, we start with a working definition of the deformation complex $C^*_{\text{def}}(G)$ of a Lie groupoid $G \rightrightarrows M$. A cochain in degree k, $k \geq 1$, is an element $c \in \Gamma_{ds}(G^{(k)}, \mathrm{pr}_1^* TG)$, i.e. $ds(c(g_1, \ldots, g_k)) \in T_{t(g_2)}M$ is independent of g_1. Denote the division map by $\bar{m}(g, h) = gh^{-1}$ whenever $s(g) = s(h)$. The differential of $C^*_{\text{def}}(G)$ is given by:

$$\begin{aligned} \delta c(g_1, \ldots, g_{k+1}) =& - d\bar{m}\,(c(g_1 g_2, g_3, \ldots, g_{k+1}), c(g_2, \ldots, g_{k+1})) \\ &+ \sum_{i=2}^{k} (-1)^k c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) + (-1)^{k+1} c(g_1, \ldots, g_k) \,. \end{aligned} \tag{3.46}$$

This differential has the form $\delta c = -\bar{m}c + \sum_{i=1}^{k+1} (-1)^i \delta_i^* c$.

Having defined total convolution and the Lie derivative along deformation cocycles as well as Hochschild cohomology and deformation cohomology we can now define the cochain map relating the two.

$$\begin{aligned} \Phi : C^k_{\text{def}}(G) &\to C^k(\mathcal{A}_G, \mathcal{A}_G) = \mathrm{Hom}(\mathcal{A}_G^{\otimes k}, \mathcal{A}_G) \\ c &\mapsto [a_1 \otimes \cdots \otimes a_k \mapsto m(\mathcal{L}_c a_1 \otimes a_2 \otimes \cdots \otimes a_n)] \end{aligned} \tag{3.47}$$

Explicitly,

$$[\Phi(c)(a_1, \ldots, a_n)]\,(g) = \int_{g_1 \cdots g_k = g} \mathcal{L}_c a_1(g_1, \ldots, g_k) \otimes a_2(g_2) \otimes \cdots \otimes a_k(g_k) \,. \tag{3.48}$$

Remark 3.15. Φ is *not* defined in degree 0. In [KP21, 2.4], the authors even show that Φ cannot be extended to a cochain map. If it did, $\Phi(\vec{\alpha} + \overleftarrow{\alpha})$ would need to be an inner derivation for any section $\alpha \in \Gamma(A)$ of the Lie algebroid. They construct explicit counterexamples that do not have compact support.

To show that Φ is indeed a cochain map we first prove the following lemma.

Lemma 3.16. *Let $c \in C^k_{\text{def}}(G)$.*

1. $\Phi(\delta_i^* c) = d_i \Phi(c)$ *for* $i = 2, \ldots, k+1$.
2. $\mathcal{L}_c(a_1 * a_2) = m_1(\mathcal{L}_{\bar{m}c} a_1 \otimes a_2) + m_1(a_1 \otimes \mathcal{L}_c a_2)$

Proof. 1. We first show $\mathcal{L}_{\delta_i^* c} a = \delta_i^* \mathcal{L}_c a$ for $i = 2, \ldots, k$.

$$\begin{aligned} \mathcal{L}_{\delta_i^* c} a(g_1, \ldots, g_{k+1}) &= (\mathcal{L}_{c(\cdot, g_2, \ldots, g_i g_{i+1}, \ldots, g_{k+1})} a)(g_1) \\ &= \mathcal{L}_c a(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) \\ &= (\delta_i^* \mathcal{L}_c a)(g_1, \ldots, g_{k+1}) \end{aligned} \tag{3.49}$$

For $i = k+1$ the computation is analogous and both sides, $\mathcal{L}_{\delta^*_{k+1}c}a$ and $\delta^*_{k+1}\mathcal{L}_c a$, are equal to $(\mathcal{L}_{c(\cdot,g_2,\dots,g_k)}a)(g_1)$.

For $i = 2,\dots k$ we compute:

$$\begin{aligned}
\Phi(\delta_i^* c)(a_1,\dots,a_{k+1}) &= m(\delta_i^*\mathcal{L}_c a_1 \otimes \cdots \otimes a_{k+1}) \\
&= m(m_i(\delta_i^*\mathcal{L}_c a_1 \otimes \cdots \otimes a_{k+1})) \\
&= m(\mathcal{L}_c a_1 \otimes m_{i-1}(a_2 \otimes \cdots \otimes a_{k+1})) \\
&= m(\mathcal{L}_c a_1 \otimes \cdots \otimes a_i * a_{i+1} \otimes \cdots \otimes a_{k+1}) \\
&= \Phi(c)(b_i'(a_1 \otimes \cdots \otimes a_{k+1})) \\
&= (d_i\Phi(c))(a_1,\dots,a_{k+1})
\end{aligned} \tag{3.50}$$

Where we use the projection formula $m_i(\delta_i^* a \otimes b) = a \otimes m_{i-1}b$. Here, a can be pulled out of the integral since we integrate over the set where $g_i g_{i+1} = g$ and we plug in this product in the i-th slot of a. For $i = k+1$ we compute:

$$\begin{aligned}
\Phi(\delta_{k+1}^* c)(a_1,\dots,a_{k+1}) &= m(\delta_{k+1}^*\mathcal{L}_c a_1 \otimes \cdots \otimes a_{k+1}) \\
&= m_k(m(\delta_{k+1}^*\mathcal{L}_c a_1 \otimes \cdots \otimes a_k) \otimes a_{k+1}) \\
&= m_k(\Phi(c)(a_1,\dots,a_k) \otimes a_{k+1}) \\
&= \Phi(c)(a_1,\dots,a_k) * a_{k+1} \\
&= d_{k+1}\Phi(c)(a_1,\dots,a_{k+1})
\end{aligned} \tag{3.51}$$

2. We have to compute $\mathcal{L}_c(a_1 * a_2)$. Recall, that the Lie derivative along c uses an extension $\tilde{c}$ to G^k that is still s-projectable. For notational convenience, define $X := \tilde{c}(\cdot, g_2,\dots,g_k)$ for fixed $g_2,\dots,g_k$. Then, $X \in \mathfrak{X}_s(G)$ is s-related to s_X. Denote its flow by ϕ_t^X. Without loss of generality this flow is defined for all $t \in (-\epsilon,\epsilon)$ since we are just computing on a compact subset of G^2. Also, recall that the convolution of two densities is given by

$$a_1 * a_2(g) = \int_{s^{-1}(s(g))} (r_{h^{-1}}^* a_1)(g) a_2(h)\,. \tag{3.52}$$

We are now ready to compute:

$$\begin{aligned}
\mathcal{L}_c(a_1 * a_2)(g_1,\dots,g_k) &= \left[\mathcal{L}_{c(\cdot,g_2,\dots,g_{k+1})}(a_1 * a_2)\right](g_1) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(d\phi_t^{\tilde{c}(\cdot,g_2,\dots,g_k)}\right)^* (a_1 * a_2)(\phi_t^{\tilde{c}(\cdot,g_2,\dots,g_k)}(g_1)) \\
&= \left.\frac{d}{dt}\right|_{t=0} (d\phi_t^X)^*(a_1 * a_2)_{\phi_t^X(g_1)} \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{h\in s^{-1}(s(\phi_t^X(g)))} (d\phi_t^X)^*(r_{h^{-1}}^* a_1)_{\phi_t^X(g)}(a_2)_h \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{h\in s^{-1}(\phi_t^{sX}(s(g)))} \left[(\phi_t^X)^* r_{h^{-1}}^* a_1\right](g) a_2(h)
\end{aligned} \tag{3.53}$$

Note that $s \circ \phi_t^X = \phi_t^{sX} \circ s$. Now, using that $\phi_t^X : s^{-1}(s(g)) \to s^{-1}(\phi_t^{sX}(s(g)))$ is a diffeomorphism, we can pullback the integral to get:

$$\begin{aligned}
&= \left.\frac{d}{dt}\right|_{t=0} \int_{s^{-1}(s(g))} \left[(\phi_t^X)^* r^*_{\phi_t^X(h)^{-1}} a_1\right](g) \left((\phi_t^X)^* a_2\right)_h \\
&= \int_{s^{-1}(s(g))} \left.\frac{d}{dt}\right|_{t=0} \left[(\phi_t^X)^* r^*_{\phi_t^X(h)^{-1}} a_1\right](g) (a_2)_h \\
&\quad + \int_{s^{-1}(s(g))} r^*_{h^{-1}} a_1(g) \left.\frac{d}{dt}\right|_{t=0} \left((\phi_t^X)^* a_2\right)_h \\
&= \int_{s^{-1}(s(g))} \mathcal{L}_{\bar{m}X} a_1(gh^{-1}, h) a_2(h) + a_1 * \mathcal{L}_X a_2(g)\,.
\end{aligned} \tag{3.54}$$

This last line is indeed $m_1(\mathcal{L}_{\bar{m}c} a_1 \otimes a_2) + m_1(a_1 \otimes \mathcal{L}_c a_2)$ and we are done, if we can justify the last equality. Only the first summand deserves justification; rewriting the second summand is just the definition of $\mathcal{L}_X$. Now for the first summand:

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t^X(g)\phi_t^X(h)^{-1} = d\bar{m}(X(gh^{-1}), X(h)) = \bar{m}X(gh^{-1}, h)\,. \tag{3.55}$$

Hence, a_1 is pulled back by the family of diffeomorphisms

$$\Psi_{t,h}(g) = \phi_t^X(g)\phi_t^X(h)^{-1} = r_{\phi_t^X(h)^{-1}} \phi_t^X(g)\,, \tag{3.56}$$

and this can be used to compute the Lie derivative along $\bar{m}c$. Note, that this family is in general *not* the flow of $\bar{m}c$, but the Lie derivative only depends on the derivative of such a family at $t = 0$. This fact is essentially a computation in local coordinates. Then:

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \left[(\phi_t^X)^* r^*_{\phi_t^X(h)^{-1}} a_1\right](g) &= \left.\frac{d}{dt}\right|_{t=0} \Psi^*_{t,h} a_1(g) \\
&= \mathcal{L}_{\bar{m}X} a_1(gh^{-1}, h)
\end{aligned} \tag{3.57}$$

This proves all of the formulas in the lemma. □

Theorem 3.17. $\Phi : C^*_{\mathrm{def}}(G) \to C^*(\mathcal{A}_G, \mathcal{A}_G)$ *is indeed a cochain map.*

Proof. We investigate $d \circ \Phi$.

$$\begin{aligned}
d\Phi(c)(a_1, \ldots, a_{k+1}) =& a_1 * \Phi(c)(a_2, \ldots, a_{k+1}) - \Phi(c)(a_1 * a_2, \ldots, a_{k+1}) \\
&+ \sum_{i=2}^{k} (-1)^i \Phi(c)(a_1, \ldots, a_i * a_{i+1}, \ldots, a_{k+1}) \\
&+ (-1)^{k+1} \Phi(c)(a_1, \ldots, a_k) * a_{k+1}
\end{aligned} \tag{3.58}$$

The summands in line 2 and 3 are easy to deal with: They combine to

$$\sum_{i=2}^{k+1} (-1)^i d_i \Phi(c) = \sum_{i=2}^{k+1} (-1)^i \Phi(\delta_i^* c). \tag{3.59}$$

The first line becomes:

$$a_1 * \Phi(c)(a_2, \ldots, a_{k+1}) = m_1(a_1 \otimes m(\mathcal{L}_c a_2 \otimes \cdots \otimes a_{k+1})) \tag{3.60}$$

$$\Phi(c)(a_1 * a_2, \ldots, a_{k+1}) = m(\mathcal{L}_c(a_1 * a_2) \otimes \cdots \otimes a_{k+1}) \tag{3.61}$$

Where we use the above lemma to calculate:

$$\begin{aligned} m(\mathcal{L}_c(a_1 * a_2) \otimes \cdots \otimes a_{k+1}) &= m(m_1(\mathcal{L}_{\bar{m}c} a_1 \otimes a_2) + m_1(a_1 \otimes \mathcal{L}_c a_2) \otimes \cdots \otimes a_{k+1}) \\ &= m(\mathcal{L}_{\bar{m}c} a_1 \otimes a_2 \otimes \cdots \otimes a_{k+1})) + m(a_1 \otimes \mathcal{L}_c a_2 \otimes \ldots a_{k+1}) \\ &= \Phi(\bar{m}c)(a_1, \ldots, a_{k+1}) + m(a_1 \otimes \mathcal{L}_c a_2 \otimes \ldots a_{k+1}) \end{aligned} \tag{3.62}$$

Hence in the first line only the $-\Phi(\bar{m}c)$ term survives and we can conclude:

$$d\Phi(c) = \Phi(-\bar{m}c) + \sum_{i=2}^{k+1} (-1)^i \Phi(\delta_i^* c) = \Phi(\delta c)\,. \tag{3.63}$$

□

Remark 3.18. Suppose X is a multiplicative vector field on G. We will see in Equation (4.21) that this is equivalent to $\delta X = 0$ in $C^2_{\mathrm{def}}(G)$. Then $\Phi(X) : \mathcal{A}_G \to \mathcal{A}_G$ is a derivation since $0 = \Phi(\delta X) = d\Phi(X)$ and the cocycles in $C^1(\mathcal{A}_G, \mathcal{A}_G)$ are precisely the derivations. In conclusion, multiplicative vector fields are derivations that act on the convolution algebra via the Lie derivative $\mathcal{L}_X$.

Example 3.19. Let $G \rightrightarrows M$ be an étale groupoid. We have seen that convolution on $C^\infty_c(G)$ takes the form

$$\varphi_1 * \varphi_2(g) = \sum_{g_1 g_2 = g} \varphi_1(g_1)\varphi_2(g_2)\,. \tag{3.64}$$

Under Φ, a cochain $c \in C^k_{\mathrm{def}}(G)$ maps to the Hochschild cochain

$$\begin{aligned} \Phi(c) : C^\infty_c(G)^k &\longrightarrow C^\infty_c(G) \\ \Phi(c)(\varphi_1, \ldots, \varphi_k)(g) &= \sum_{g_1 \cdots g_k = g} c(g_1, \ldots, g_k)\varphi_1 \cdot \varphi_2(g_2) \ldots \varphi_k(g_k)\,. \end{aligned} \tag{3.65}$$

The tangent vector $c(g_1, \ldots, g_k) \in T_{g_1}G$ acts on φ_1 as a derivation.

We will see later in the case of a proper étale action groupoid that in cohomology Φ is only the inclusion of invariant vector fields on G into the Hochschild cohomology of $\mathcal{A}_G$. The Hochschild cohomology however also sees multivector fields and even multivector fields on fixed point sets.

4 Deformation Cohomology

This section introduces deformation cohomology. To accomplish this, we follow the original source [CMS15] closely. We include definitions of differentiable cohomology and representations up to homotopy. Because proper groupoids are interesting to us later, we include the computation of low degree deformation cohomology in terms of isotropy and normal representation in Section 4.4 and the vanishing result in Section 4.5. Proper étale groupoids are very rigid and their deformation cohomology is only nontrivial in degree 1, where $H^1_{\mathrm{def}}(G) \cong \mathfrak{X}(G)^{\mathrm{inv}}$ consists of invariant vector fields. In Section 4.6 we briefly discuss Morita invariance of deformation cohomology, i.e., deformation cohomology is a stack invariant.

4.1 Differentiable Cohomology

Let $G \rightrightarrows M$ be a groupoid and $E \to M$ be a representation. Then any $g : x \to y$ acts by a linear map $E_x \to E_y, v \mapsto g.v$. The *differentiable cohomology* of G with coefficients in E is the cohomology of the cochain complex $C^k(G, E) = \Gamma(G^{(k)}, (t \circ \mathrm{pr}_1)^* E)$ with differential

$$\begin{aligned}\delta c(g_1, \ldots, g_{k+1}) =& g_1.c(g_2, \ldots, g_{k+1}) + \sum_{i=1}^{k} c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) \\ &+ (-1)^{k+1} c(g_1, \ldots, g_k)\,.\end{aligned} \tag{4.1}$$

We can rewrite this as

$$\delta c = \rho \delta_0^* c + \sum_{i=1}^{k+1} (-1)^i \delta_i^* c\,, \tag{4.2}$$

where ρ is the action of G on E. This will be made more explicit in Proposition 4.3. For the trivial representation on a trivial line bundle we recover the complex $C^\bullet(G)$.

4.2 Representations up to Homotopy

Here we introduce the notion of representation up to homotopy of Lie groupoids. This section is not strictly necessary for any of the following discussion, but it fits here as a generalisation of representations that also encompasses deformation cohomology. The original treatment is [AAC13]. It is a straightforward algebraic generalization of usual representation to a derived or differential graded setting. Fix a Lie groupoid $G \rightrightarrows G^{(0)}$.

Let $E_\bullet = \bigoplus E_i$ be a graded vector bundle over $G^{(0)}$. We can consider

$$\begin{aligned}C^k(G, E) &= \bigoplus_{p-q=k} C^p(G, E_q) \\ &= \bigoplus_{p-q=k} \Gamma(G^{(p)}, (t \circ \mathrm{pr}_1)^* E_q)\end{aligned} \tag{4.3}$$

as particular graded sections over the nerve.

Notice that this becomes a graded $C^*(G)$-module by the cup product: $\cup : C^k(G, E) \otimes C^l(G) \to C^{k+l}(G, E)$:

$$(c \cup \varphi)(g_1, \ldots, g_k, g_{k+1}, \ldots, g_{k+l}) = c(g_1, \ldots, g_l)\varphi(g_{k+1}, \ldots, g_{k+l}) \tag{4.4}$$

Remark 4.1. We can even make sense of a (cup) product $C^k(G, E \to C) \otimes C^l(E) \to C^{k+l}(C)$. Here $\gamma \in C^k(G, E \to C)$ is a section of the bundle $\mathrm{Hom}((s \circ \mathrm{pr}_k)^* E, (t \circ \mathrm{pr}_1)^* C)$. This means explicitly $\gamma(g_1, \ldots, g_k) : E_{s(g_k)} \to E_{t(g_1)}$ is a linear map.

A cochain is *normalized* if it vanishes on degenerate tuples, i.e. whenever one of the inputs is an identity. There is *no* condition on cochains defined on $G^{(0)}$.

Definition 4.2. A *representation up to homotopy* (abbreviated *ruth*) is a differential D of degree $+1$ on the complex $C^\bullet(G, E)$ preserving the normalized complex that satisfies a Leibniz rule w.r.t. the cup product module structure:

$$D(c \cup \varphi) = Dc \cup \varphi + (-1)^{\deg(c)} c \cup \delta\varphi \tag{4.5}$$

A morphism of ruths is a total degree 0 $C^\bullet(G)$-linear chain map.

The reason for defining ruths is the following proposition. Note that any representation of $\rho : G \curvearrowright E$ on a vector bundle defines an element $\rho \in C^1(G, E \to E)$ since it acts as $G \to \mathrm{Hom}(s^*E, t^*E)$.

Proposition 4.3. *There is an equivalence of categories between linear representations of G and 1-term representations up to homotopy*

$$Rep(G) \to Rep^\infty_{1-term}(G),$$

where to a representation ρ on E we associate the differential:

$$D\omega = \rho\delta_0^*\omega + \sum_{i=1}^{k} (-1)^i \delta_i^* \omega \tag{4.6}$$

Proof. Let $\rho : G \curvearrowright E$ be a linear representation. Let us first verify that $D^2 = 0$. Notice that the cosimplicial identities still hold for this differential operator. A key point to observe is that $(\rho\delta_0^*) \circ (\rho\delta_0^*) = \delta_1^* \circ (\rho\delta_0^*)$ is equivalent to $\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$. This immediately shows that $D^2 = 0$. The Leibniz rule and preservation of normalized chains is a very quick computation. Let now $\Phi : E \to E'$ be a morphism of representations. This means Φ commutes with ρ and thus Φ_* commutes with all operations in the definition of D. $\Phi_* : C(G, E) \to C(G, E')$ is then a morphism of ruths.
We now show that this functor is essentially surjective and fully faithful. Let E be a bundle over G_0 and D be a differential so that we have a 1-term ruth $(C^\bullet(G, E), D)$. Define an action of G on E by considering any section ϵ of E over $G^{(0)}$ as an element in $C^0(G, E)$.

$$\rho_g.\epsilon_{s(g)} := D\epsilon(g) + \epsilon_{t(g)} \tag{4.7}$$

For this to be well-defined on individual vectors we use the usual trick of $C^\infty(G^{(0)})$-linearity employing the Leibniz rule for $C^0(G)$:

$$\begin{aligned} D(\epsilon f)(g) + (\epsilon f)_{t(g)} &= D\epsilon(g) f(s(g)) + \epsilon_{t(g)} \delta f(g) + \epsilon_{t(g)} f(t(g)) \\ &= (D\epsilon(g) + \epsilon_{t(g)}) f(s(g)). \end{aligned} \tag{4.8}$$

Equation (4.7) clearly defines a linear map. It is unital precisely when $D\epsilon(\mathrm{id}_x) = 0$, i.e. if D preserves the normalized complex. The condition for the action to be associative is:

$$\begin{aligned} \rho_g \rho_h \epsilon_{s(h)} = \rho_g(D\epsilon(h) + \epsilon_{t(h)}) = \rho_g D\epsilon(h) + D\epsilon(g) + \epsilon_{t(g)} \overset{!}{=} D\epsilon(gh) + \epsilon_{t(g)} = \rho_{gh}\epsilon_{s(h)} \\ \iff \rho_g D\epsilon(h) - D\epsilon(gh) + D\epsilon(g) = 0 \end{aligned} \tag{4.9}$$

If we show that D has indeed the form of Equation (4.6) then the last line just says $D^2 = 0$. In fact, D is uniquely determined by its degree 0 component and the Leibniz identity: Locally any $\omega \in C^k(G, E)$ has the form $\sum_i (t \circ pr_1)^* \epsilon_i \cup f_i$ with $\epsilon_i \in C^0(G, E)$

and $f_i \in C^k(G)$. Using the Leibniz rule and locality of D we deduce that indeed D has the form 4.6. This shows essential surjectivity.
If Ψ is a chain map between $C^\bullet(G,E)$ and $C^\bullet(G,E')$ its degree 0 part is a $C^\infty(G^{(0)})$-linear map between sections. Hence it is induced by a smooth bundle map $\Phi : E \to E'$ by the classical Serre-Swan theorem. Since $\Psi = \Phi_*$ commutes with D, Φ intertwines the action defined in Equation (4.7). This proves full faithfulness. $\square$

4.3 Deformation Cohomology

This section follows [CMS15]. There is no adjoint representation of a Lie groupoid. There is only an adjoint representation up to homotopy. Deformation cohomology is the substitute for differentiable cohomology with values in the adjoint representation. It is intrinsically defined, that is, the deformation complex can be written down in terms of structure maps of the Lie groupoid without additional choices, e.g., of a connection.

Let $G \rightrightarrows M$ be a Lie groupoid. Most of the groupoid structure is encoded in the multiplication and inversion map. The division map

$$\begin{aligned} \bar{m} : G \times_M^{s,s} G &\longrightarrow G \\ (g,h) &\longmapsto gh^{-1} \end{aligned} \tag{4.10}$$

encodes both. The differential $d\bar{m}$ is then defined on $T_{(g,h)}(G \times_{s,s} G) \cong T_g G \times_{ds,ds} T_h H$. Hence, we can compute $d\bar{m}(v,w)$ whenever $ds(v) = ds(w)$.

Definition 4.4. In degree $k \geq 1$, the *deformation complex* $C^k_{\mathrm{def}}(G)$ consists of sections $c \in \Gamma(G^{(k)}, \mathrm{pr}_1^* TG)$ that are s-relatable, i.e., there is a section s_c in $\Gamma(G^{(k-1)}, t^*TM)$ satisfying:

$$ds \circ c(g_1, \ldots, g_k) = s_c(g_2, \ldots, g_k) \tag{4.11}$$

This means that $ds \circ c$ must be independent of g_1. The differential $\delta : C^k_{\mathrm{def}}(G) \to C^{k+1}_{\mathrm{def}}(G)$ is given by

$$\delta c(g_1, \ldots, g_{k+1}) = -d\bar{m}(c(g_1 g_2, \ldots, g_{k+1}), c(g_2, \ldots, g_{k+1})) + \sum_{i=2}^{k+1} (-1)^i \delta_i^* c(g_1, \ldots, g_{k+1})\,. \tag{4.12}$$

We can also write this as:

$$\delta c = -\bar{m}c + \sum_{i=2}^{k+1} (-1)^i \delta_i^* c\,. \tag{4.13}$$

We can define a degree zero part $C^0_{\mathrm{def}}(G) = \Gamma(G^{(0)}, A)$ with differential $\delta\alpha = \vec{\alpha} + \overleftarrow{\alpha}$.

Here, $\vec{\alpha}_g = dr_g \alpha_{t(g)}$ and $\overleftarrow{\alpha}_g = dl_g di \alpha_{s(g)}$. We denote the cohomology of the deformation complex by $H^*_{\mathrm{def}}(G)$.

Remark 4.5. We can encode s-relatability in the following diagram:

$$\begin{tikzcd} \Gamma(G^{(k)}, \mathrm{pr}_1^* TG) \arrow[r, "ds"] & \Gamma(G^{(k)}, (s \circ \mathrm{pr}_1)^* TM) \\ & \Gamma(G^{(k-1)}, t^*TM) \arrow[u, "\delta_0^*"] \end{tikzcd} \tag{4.14}$$

An element of $C^k_{\mathrm{def}}(G)$ can be thought of as a tuple (c, s_c) where $ds \circ c = \delta_0^* s_c$. Obviously, s_c is uniquely determined by c. However, *choosing* a section of ds allows us to subtract the redundant information from c. Thus, the datum of $c \in C^k_{\mathrm{def}}(G)$ is the same as a tuple $(\tilde{c}, s_c) \in \Gamma(G^{(k)}, \mathrm{pr}_1^* \ker ds) \oplus \Gamma(G^{(k-1)}, (t \circ \mathrm{pr}_1)^* TM)$. This provides us with a link to representations up to homotopy since $\ker(ds) \cong t^*A$. The following Lemma then shows that the graded 2-term vector bundle $E_\bullet = A \oplus TM[-1]$ becomes a 2-term ruth under the above identification of $C^k(G, E)$ with $C^k_{\mathrm{def}}(G)$. Note that this depends on the choice of a splitting.

The following are Lemma 2.2 and 2.5 in [CMS15].

Lemma 4.6. *$C^*_{\mathrm{def}}(G)$ is indeed a cochain complex, i.e. $\delta^2 = 0$. Furthermore, δ preserves normalized cochains and δ is $C^\bullet(G)$-linear.*

Example 4.7. Consider the groupoid $G = M \rightrightarrows M$. Then $G^{(k)} \cong M$ under the identification of $(x, \dots, x)$ with x. Also, $TG = TM$ and $ds = \mathrm{id}$. Since the Lie algebroid vanishes, we have $C^0_{\mathrm{def}}(G) = 0$. For $k \geq 1$, the space $C^k_{\mathrm{def}}(G)$ consists of all sections $c : M \to TM$ since s-projectability is a void condition. Finally, we are left to compute the differential. Note, that under the above identification $\delta_i^* c = c$. Also $\bar{m}c(x) = d\bar{m}(c(x), c(x)) = 0$. Hence:

$$\delta c = -\bar{m}c + \sum_{i=2}^{k+1} (-1)^i \delta_i^* c = \begin{cases} c & k \text{ odd} \\ 0 & \text{even} \end{cases} \tag{4.15}$$

The deformation complex $C^*_{\mathrm{def}}(G)$ starting in degree 0 is hence:

$$0 \longrightarrow \mathfrak{X}(M) \xrightarrow{0} \mathfrak{X}(M) \xrightarrow{\mathrm{id}} \mathfrak{X}(M) \xrightarrow{0} \mathfrak{X}(M) \xrightarrow{\mathrm{id}} \dots \tag{4.16}$$

The deformation cohomology vanishes except in degree 1, where it is $H^1_{\mathrm{def}}(G) = \mathfrak{X}(M)$.

Example 4.8. We want to compute $H^1_{\mathrm{def}}(G)$ for arbitrary groupoids. Note, that $C^1_{\mathrm{def}}(G) = \mathfrak{X}_s(G)$ consists of s-relatable vector fields. The differential in degree 1 becomes

$$\delta X(g, h) = -d\bar{m}(X(gh), X(h)) + X(g)\,, \tag{4.17}$$

so that cocycles in degree 1 satisfy

$$X(g) = d\bar{m}(X(gh), X(h)\,. \tag{4.18}$$

Thus, we get

$$\begin{aligned} X(\mathrm{id}_x) &= d\bar{m}(X(\mathrm{id}_x), X(\mathrm{id}_x)) = 0 && (4.19)\\ X(g^{-1}) &= d\bar{m}(X(\mathrm{id}_{t(g)}), X(g)) = d\bar{m}(0, X(g)) = diX(g) && (4.20)\\ X(gh) &= d\bar{m}(X(g), X(h^{-1}) = d\bar{m}(X(g), diX(h)) = dm(X(g), X(h)) && (4.21) \end{aligned}$$

If X satisfies Equation (4.21), it is called *multiplicative* vector field. This is equivalent to $X : G \to TG$ being a Lie groupoid homomorphism. Conversely, one checks that multiplicative vector fields are also cocycles. We call vector fields of the form $\vec{\alpha} + \overleftarrow{\alpha}$ *inner multiplicative*. We arrive at

$$H^1_{\mathrm{def}}(G) = \frac{\text{multiplicative vector fields}}{\text{inner multiplicative vector fields}}\,. \tag{4.22}$$

4.4 Isotropy and normal Representations

This section again closely follows [CMS15]. Morally, we show how replacing the ruth $A \to TM$ by its cohomology $\ker(\sharp) \oplus \mathrm{coker}(\sharp)$ helps to compute deformation cohomology. Let $G \rightrightarrows M$ be a groupoid with Lie algebroid $A = \mathrm{Lie}(G)$ and anchor $\sharp : A \to TM$. Define the *isotropy bundle* $\mathfrak{i}$ as

$$\mathfrak{i} = \ker(\sharp : A \to M) = \ker(ds)|_M \cap \ker(dt)|_M \,. \tag{4.23}$$

Note that this will only be a smooth vector bundle if $\sharp$ has constant rank, that is if and only if G is *regular*. The vector space $\mathfrak{i}_x$ is the Lie algebra of the isotropy Lie group $G_x^x = \{g : s(g) = t(g) = x\}$. There is a conjugation action of G on $\mathfrak{i}$ that is the infinitesimal version of the conjugation on the isotropy Lie group bundle: We can represent $\alpha \in \mathfrak{i}_x$ as $\frac{d}{dt}|_{t=0}h(t)$ with $h(t) \in G_x^x$. Then $gh(t)g^{-1} \in G_y^y$ for all $g : x \to y$. We write $\mathrm{Ad}_g : \mathfrak{i}_x \to \mathfrak{i}_y$ for the map $dl_g \circ dr_{g^{-1}}$ given explicitely by

$$\mathrm{Ad}_g\alpha = \left.\frac{d}{dt}\right|_{t=0} gh(t)g^{-1} \,. \tag{4.24}$$

Note that this becomes a genuine G-representation if G is regular. Hence differentiable cohomology $H^*(G, \mathfrak{i})$ is defined. In general, it still makes sense to talk of smooth sections of $\mathfrak{i}$ via $\Gamma(\mathfrak{i}) := \ker(\sharp : \Gamma(A) \to \Gamma(TM))$. This is the correct kernel when regarded as sheaves. Now define

$$H^0(G, \mathfrak{i}) := \Gamma(\mathfrak{i})^{\mathrm{inv}} := \{\alpha \in \Gamma(\mathfrak{i}) : \mathrm{Ad}_g\alpha(x) = \alpha(y) \quad \forall g : x \to y\} \,. \tag{4.25}$$

This agrees with the computation of $H^0(G, \mathfrak{i})$ in the regular case.

Proposition 4.9. $H^0_{\mathrm{def}}(G) = H^0(G, \mathfrak{i}) = \Gamma(\mathfrak{i})^{\mathrm{inv}}$.

Proof. Let $\alpha \in C^0_{\mathrm{def}}(G)$ be a cocycle, that is $\alpha \in \Gamma(A)$ and $\delta\alpha(g) = \vec{\alpha}_g + \overleftarrow{\alpha}_g = 0$. Note that the left-invariant extension $\overleftarrow{\alpha}$ lies in the kernel of dt and that $\sharp = dt|_M$ as a map on $A = \ker(ds)|_M$. Then $\sharp(\alpha) = 0$ since $0 = dt(\delta\alpha) = dt(\vec{\alpha}) + 0$ and the right hand side is precisely $\sharp(\alpha)$ when restricted to M. Hence $\alpha \in \Gamma(\mathfrak{i})$. Spelling out the cocycle condition we have $0 = \delta\alpha(g) = dr_g\alpha_{t(g)} - dl_g\alpha_{s(g)}$ which is equivalent to the invariance condition $\mathrm{Ad}_g\alpha_{s(g)} = \alpha_{t(g)}$. ∎

Definition 4.10. The complex $(C^*(G, \mathfrak{i}), \delta)$ that consists in degree k of maps $G^{(k)} \to \mathfrak{i} \subset A$ can be defined as in differentiable cohomology. A cochain needs to be smooth as a map $c : G^{(k)} \to A$. Explicitly, the differential $\delta : C^k(G, \mathfrak{i}) \to C^{k+1}(G, \mathfrak{i})$ is given by

$$\begin{aligned}\delta c(g_1, \ldots, g_{k+1}) =& \mathrm{Ad}_{g_1}c(g_2, \ldots, g_{k+1}) + \sum_{i=1}^{k}(-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) \\ &+ (-1)^{k+1}c(g_1, \ldots, g_k) \,.\end{aligned} \tag{4.26}$$

Note that for such a cochain we have $c(g_1, \ldots, g_k) \in A_{t(g_1)}$. There is an injective cochain map $r : C^*(G, \mathfrak{i}) \to C^*_{\mathrm{def}}(G)$ given by

$$r(c)(g_1, \ldots, g_k) = dr_{g_1}c(g_1, \ldots, g_k) \,. \tag{4.27}$$

This identifies $C^*(G, \mathfrak{i})$ with the subcomplex of $C^*_{\mathrm{def}}(G)$ taking values in $\ker(ds) \cap \ker(dt)$. The inverse is easily written down by composing with $dr_{g_1^{-1}}$. There is a computation to be made here which shows compatibility of the differentials. We skip this due to space constraints.

We now also define the *normal representation* ν of G where

$$\nu := \operatorname{coker}(\sharp) = TM/\sharp(A)\,. \tag{4.28}$$

Again, this is only a genuine representation if G is regular. Denote by $[v]$ the equivalence class of $v \in T_xM$ modulo $\sharp(A_x)$. There is an action of G on ν by which $g : x \to y$ acts on $[v] \in \nu_x$ by choosing a lift $w \in T_gG$ with $ds(w) = v$ and defining

$$\mathrm{Ad}_g[v] := [dt(w)]\,. \tag{4.29}$$

This is well-defined since $ds(w) = 0$ implies $[dt(w)] = [\sharp(r_{g^{-1}}w)] = 0$ and since $v = \sharp(\alpha)$ implies that $v = dt(\alpha) = ds(di\alpha) = ds(dl_g di\alpha)$ is a lift. This lift satisfies $dt(dl_g di\alpha) = d(t \circ l_g)(di\alpha) = 0$ since $t(g)$ is constant. In conclusion, we indeed get a map $\mathrm{Ad}_g : T_xM/\sharp(A_x) \to T_yM/\sharp(A_y)$.
We can now define $\Gamma(\nu) := \mathfrak{X}(M)/\operatorname{Im}(\sharp)$ as equivalence classes $[V]$ of vector fields V on M. An s-t-lift of a vector field V on M is $X \in \mathfrak{X}(G)$ which is s- and t-related to V.

$$\Gamma(\nu)^{\mathrm{inv}} := \{[V] : V \in \mathfrak{X}(M) \text{ has an } s\text{-}t\text{-lift } X \in \mathfrak{X}(G)\} \tag{4.30}$$

Indeed, $\mathrm{Ad}_g[V_{s(g)}] = \mathrm{Ad}_g[ds(X_g)] = [dt(X_g)] = [V_{t(g)}]$ is invariant in the usual sense. We needed to make this definition to make sense of smoothness. Since the cocycles in $C^1_{\mathrm{def}}(G)$ are multiplicative vector fields which by definition are s,t-relatable, there is a canonical "projectio" map:

$$\pi : H^1_{\mathrm{def}}(G) \to \Gamma(\nu)^{\mathrm{inv}} \tag{4.31}$$

Without restriction on the class of groupoids we can now establish a result on the low degree deformation cohomology.

Proposition 4.11. *There is an exact sequence:*

$$0 \longrightarrow H^1(G,\mathfrak{i}) \xrightarrow{\;r\;} H^1_{\mathrm{def}}(G) \xrightarrow{\;\pi\;} \Gamma(\nu)^{\mathrm{inv}} \xrightarrow{\;\kappa\;} H^2(G,\mathfrak{i}) \xrightarrow{\;r\;} H^2_{\mathrm{def}}(G) \tag{4.32}$$

Proof. We first construct κ. Let $[V] \in \Gamma(\nu)^{\mathrm{inv}}$ have s-t-lift $X \in \mathfrak{X}(G)$. We claim that δX lies in the subcomplex $C^2(G,\mathfrak{i})$ of $C^*_{\mathrm{def}}(G)$ and that $\kappa([V]) := \delta X$ is well-defined.
By the previous discussion we need to show that $ds(\delta X) = dt(\delta X) = 0$.

$$\delta X(g,h) = -d\bar{m}(X_{gh}, X_h) + X_g \tag{4.33}$$

Now using $s(\bar{m}(g,h)) = t(h)$ and $t(\bar{m}(g,h)) = t(g)$ and the projectability we arrive at:

$$ds(\delta X(g,h)) = -dt(X_h) + ds(X_g) = -V_{t(h)} + V_{s(g)} = 0 \tag{4.34}$$

$$dt(\delta X(g,h)) = -dt(X_{gh}) + dt(X_g) = -V_{t(gh)} + V_{t(g)} = 0 \tag{4.35}$$

If X' is another s-t-lift then $X - X'$ is an s-t-lift of 0 and hence lies in the subcomplex $C^1(G,\mathfrak{i})$. Hence the class of $\delta X \in H^2(G,\mathfrak{i})$ is independent of the lift X and κ is well-defined.

1. r is injective: Let $c \in H^1(G,\mathfrak{i})$ with $r(c) = 0$, i.e., upon regarding r as an inclusion of a subcomplex we have $c = \delta\alpha$ for some $\alpha \in \Gamma(A) = C^0_{\mathrm{def}}(G)$. Then automatically $dt(\delta\alpha) = \sharp(\alpha) = 0$ along M. Hence α already lies in $C^0(G,\mathfrak{i})$, as $ds(\alpha) = dt(\alpha) = 0$. Hence $c = \delta\alpha = 0$ in $H^1(G,\mathfrak{i})$.

2. $\ker(\pi) = \operatorname{Im}(r)$: Let $X \in H^1_{\mathrm{def}}(G)$ with $\pi(X) = 0$. Then X is a multiplicative vector field that is the s-t lift of $\sharp(\alpha)$ for some $\alpha \in \Gamma(A)$. $X - \delta\alpha$ is an s-t-lift of 0 which is equivalent to $ds(X - \delta\alpha) = dt(X - \delta\alpha) = 0$, i.e. $X - \delta\alpha \in \operatorname{Im}(r)$.

3. $\ker(\kappa) = \mathrm{Im}(\kappa)$: Let $[V] \in \Gamma(\nu)^{\mathrm{inv}}$ have s-t-lift X. $\kappa([V]) = \delta X = 0$ in $H^2(G, \mathfrak{i})$ means $\delta X = \delta c$ for some $c \in C^1(G, \mathfrak{i})$ which we can identify with its image in $C^1_{\mathrm{def}}(G)$ satisfying $ds(c) = dt(c) = 0$. Therefore $X - c$ is also an s-t-lift of $[V]$. Hence we have $[V] \in \ker(\pi)$ if and only if it has an s-t-lift X with $\delta X = 0$ which happens if and only if it lies in the image π.

4. $\ker(r) = \mathrm{Im}(\pi)$: Suppose $c \in H^2(G, \mathfrak{i})$ lies in the kernel of r, i.e. $c = \delta X$ for $X \in C^1_{\mathrm{def}}(G)$ where we again identify c with a cochain in $C^2_{\mathrm{def}}(G)$ satisfying $ds(c) = dt(c) = 0$. For all composable (g, h) we then have:

$$0 = dt(c(g,h)) = -dt(X_h) + ds(X_g) \tag{4.36}$$

$$0 = ds(c(g,h)) = -dt(X_{gh}) + dt(X_g) \tag{4.37}$$

From this one reads off that X is s-t-lift of $[V_{t(g)}] = [dt(X_g)]$. This means $c = \kappa([V])$. Conversely, clearly $r \circ \kappa(V) = \delta X = 0$. $\square$

Example 4.12. Let $G \rightrightarrows M$ be an étale groupoid. Then, the Lie algebroid is trivial. Hence, $\mathfrak{i} = 0$ and $H^0_{\mathrm{def}}(G) = 0$. By the short exact sequence, also $H^1_{\mathrm{def}}(G) \cong \Gamma(\nu)^{\mathrm{inv}}$. But $\Gamma(\nu)^{\mathrm{inv}}$ consists of vector fields V on M with an s-t-lift X. Since G is étale, $ds = dt = \mathrm{id}$ along the units $M \subset G$ and hence $V = X|_M$. The vector field X is now itself invariant by its $s - t$-projectability. Hence, $H^1_{\mathrm{def}}(G) \cong \mathfrak{X}(G)^{\mathrm{inv}}$.

4.5 Vanishing of H^k_{def} for proper Groupoids and $k > 1$

The following theorem is Theorem 6.1 in [CMS15].

Theorem 4.13. *Let $G \rightrightarrows M$ be a proper Lie groupoid. Then $H^k_{\mathrm{def}}(G) \cong 0$ for $k > 1$ and*

$$H^0_{\mathrm{def}}(G) \cong \Gamma(\mathfrak{i})^{\mathrm{inv}} \quad H^1_{\mathrm{def}}(G) \cong \Gamma(\nu)^{\mathrm{inv}} \tag{4.38}$$

Proof. Since G is proper there exists a left Haar system and a function $\lambda \in C^\infty(G)$ with the property that $s : \mathrm{supp}(\lambda) \to M$ is proper and $\int_{t^{-1}(x)} \lambda(g)\mathrm{d}g = 1$ for all $x \in M$. (c.f. 8.1)
For $k > 1$ and $c \in C^k_{\mathrm{def}}(G)$ define $H(c) \in C^{k-1}_{\mathrm{def}}(G)$ by

$$H(c)(g_1, \ldots, g_{k-1}) = (-1)^k \int_{t^{-1}(s(g_{k-1}))} c(g_1, \ldots, g_{k-1}, h)\lambda(h)\mathrm{d}h\,. \tag{4.39}$$

Calculate:

$$
\begin{aligned}
\delta H(c)(g_1,\dots,g_k) = &-d\bar{m}(H(c)(g_1g_2,\dots,g_k), H(c)(g_2,\dots,g_k)) \\
&+\sum_{i=2}^{k-1}(-1)^i H(c)(g_1,\dots,g_ig_{i+1},\dots,g_k) \\
&+(-1)^k H(c)(g_1,\dots,g_{k-1}) \\
=&-d\bar{m}\left((-1)^k\int_{s(g_k)} c(g_1g_2,\dots,g_k,h)\lambda(h)\mathrm{d}h, (-1)^k\int_{s(g_k)} c(g_2,\dots,g_k,h)\lambda(h)\mathrm{d}h\right) \\
&+\sum_{i=2}^{k-1}(-1)^{i+k}\int_{s(g_k)} c(g_1,\dots,g_ig_{i+1},\dots,g_k,h)\lambda(h)\mathrm{d}h \\
&+\int_{s(g_{k-1})} c(g_1,\dots,g_{k-1},h)\lambda(h)\mathrm{d}h \\
=&(-1)^k\int_{s(g_k)}\Big(\delta c(g_1,\dots,g_k,h)\lambda(h)-(-1)^{k+1}c(g_1,\dots,g_k)-(-1)^kc(g_1,\dots,g_kh)\Big)\lambda(h)\mathrm{d}h \\
&+(-1)^k\int_{s(g_{k-1})} c(g_1,\dots,g_{k-1},h)\lambda(h)\mathrm{d}h \\
=&c(g_1,\dots,g_k)-H(\delta c)(g_1,\dots,g_k)
\end{aligned}
\tag{4.40}
$$

Here we used the properties of λ and added and subtracted terms to make the term δc appear. Thus, we get an explicit contraction. We indeed have $H(c) \in C^{k-1}_{\mathrm{def}}(G)$ since

$$
ds(H(c))(g_1,\dots,g_{k-1}) = (-1)^k\int_{s(g_{k-1})} s_c(g_2,\dots,g_{k-1},h)\lambda(h)\mathrm{d}h \tag{4.41}
$$

only depends on $g_2,\dots,g_k$. Note that this does not work for $k=1$ since $C^0_{\mathrm{def}}(G)=\Gamma(A)$ and $ds(H(c))$ need not vanish. However the exact same formulas do work for $C^k(G,\mathfrak{i})$ for $k>0$ and show $H^k(G,\mathfrak{i})=0$ for $k>0$. Similarly, all differentiable cohomologies vanish. (c.f. [Cra03, 2.1]) Now the exact sequence in Proposition 4.11 shows the claimed isomorphism in degree 1. Furthermore, we always have $H^0_{\mathrm{def}}(G)\cong\Gamma(\mathfrak{i})^{\mathrm{inv}}$. □

Example 4.14. For an étale groupoid $G \rightrightarrows M$ we now know the entire deformation cohomology. It is given by $H^1_{\mathrm{def}}(G)\cong\mathfrak{X}(G)^{\mathrm{inv}}$ and $H^k_{\mathrm{def}}(G)=0$ otherwise.

4.6 Morita Invariance of Deformation Cohomology

To relate the deformation complexes of Morita equivalent Lie groupoids we need to have an intermediate cochain complex. To prove Morita invariance of deformation cohomology it is easiest to consider Morita equivalence of groupoids as a zig-zag of weak equivalences, i.e. to consider it in $\mathsf{Grpd}[W^{-1}]$. Recall that the weak equivalences were precisely the Morita morphisms, i.e. pullbacks of groupoids along surjective submersions. If we want to prove that some construction on Grpd, such as H^*_{def}, is Morita invariant, then we only have to show that weak equivalences give rise to isomorphisms.

Definition 4.15. Let $F: G\to H$ be a homomorphism of Lie groupoids. Then we can define $C^*_{\mathrm{def}}(F)$ to be the cochain complex $C^k_{\mathrm{def}}(F)\subset\Gamma(G^{(k)},F^*(TH))$ consisting of s-relatable cochains. Explicitly, this means that $ds\circ c(g_1,\dots,g_k)$ does not depend on g_1. The differential $C^k_{\mathrm{def}}(F)\to C^{k+1}_{\mathrm{def}}(F)$ is given by

$$
\delta c = -\bar{m}_H c + \sum_{i=2}^{k+1}(-1)^i\delta_i^* c\,. \tag{4.42}
$$

Here, $\bar{m}_H$ is the division map on H. The proof that this is indeed a cochain complex is formally identical to the ordinary deformation complex. Note that $C^*_{\mathrm{def}}(\mathrm{id}_G) = C^*_{\mathrm{def}}(G)$.

If $F : G \to H$ is a homomorphism of Lie groupoids there are cochain maps

$$C^*_{\mathrm{def}}(G) \xrightarrow{F_*} C^*_{\mathrm{def}}(F) \xleftarrow{F^*} C^*_{\mathrm{def}}(H)\,, \tag{4.43}$$

given by

$$F_*(c)(g_1, \dots, g_k) = dF(c(g_1, \dots, g_k)) \tag{4.44}$$

$$F^*(c')(g_1, \dots, g_k) = c'(F(g_1), \dots, F(g_k))\,. \tag{4.45}$$

The following is Theorem 11.6 in [CMS15].

Proposition 4.16. *If two Lie groupoids are Morita equivalent, then their deformation cohomologies are isomorphic.*

Idea of proof. As remarked above, we only need to show that Morita morphisms induce isomorphisms on deformation cohomology. A Morita morphism $H \to G$ is a surjective submersion f on the base manifolds such that $H \cong f^*G$. So let $f : P \to M$ be a surjective submersion and $F : f^*G \to G$ the canonical map. Now one shows that $F_* : C^*_{\mathrm{def}}((f^*G)) \to C^*_{\mathrm{def}}(F)$ is a quasi-isomorphism. Secondly, one shows that $F^* : C^*_{\mathrm{def}}(G) \to C^*_{\mathrm{def}}(F)$ is a quasi-isomorphism. This latter step requires a large computation and several reductions. $\square$

5 Bornological Algebras and Hochschild Cohomology

In this section we introduce the theory of bornological algebras as a means to study convolution algebras. This includes a self-contained introduction to bornological vector spaces in Section 5.3, their relation to locally convex topological vector spaces - introduced in Section 5.1 - and their categorical properties. After that, we discuss bornological algebras in Section 5.4 and how to do homological algebra with bornological algebras in Section 5.4.3. None of this is new, but the presentation is streamlined and parts of it can be difficult to find anywhere. Special emphasis is needed to treat the generally nonunital convolution algebras. The notion of self-induced or quasi-unital algebras helps to remedy this. The parts in Sections 5.4.2 and 5.4.5 to 5.4.8 on the Morita bicategory, smoothly projective modules, the contractibility of the bar complex, quasi-unitality and strong H-unitality of groupoid convolution algebras are all original work of the author. Using that, Hochschild cohomology is introduced in Section 5.5 and sufficient conditions for Hochschild cohomology to be a derived Hom-functor are given in Section 5.5.3. It is not important for these sections to work bornologically and we will often work in any symmetric monoidal preabelian category.

5.1 Locally Convex Algebras

In this section we recall some of the basics on locally convex vector spaces and establish notation. A locally convex vector space (lcs) is vector space over $\mathbb{R}$ or $\mathbb{C}$ that is equipped with the initial topology induced by a family of seminorms $\{p_i\}_{i\in I}$. Morphisms of lcs are continuous linear maps.
Equivalently, an lcs is a topological vector space with a convex neighbourhood basis of zero. A Fréchet space is a complete Hausdorff lcs that admits a countable family of seminorms generating the topology. Equivalently, a Fréchet space is a lcs whose topology is induced by a complete translation invariant metric. An LF-space is a countable direct limit of Fréchet spaces. Fréchet spaces are arguably the nicest locally convex spaces one encounters.

There are two important locally convex algebras of smooth functions.

0. The Fréchet algebra $C^\infty(U)$ for $U \subset \mathbb{R}^n$ open. It is equipped with the locally convex topology induced by the seminorms

$$p_{K,k}(f) = \sum_{|\alpha|\le k} \frac{1}{\alpha!}\|\partial^\alpha f\|\,, \tag{5.1}$$

where $K \subset U$ is a compact subset. Multiplication is continuous:

$$\begin{aligned} p_{K,k}(fg) &= \sum_{|\alpha|\le k} \frac{1}{\alpha!}\|\partial^\alpha (fg)\| \\ &= \sum_{|\alpha|\le k} \frac{1}{\alpha!}\left\|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta f \partial^{\alpha-\beta} g\right\| \\ &\le \sum_{|\alpha|\le k}\sum_{\beta\le\alpha} \frac{1}{\beta!(\alpha-\beta)!}\|\partial^\beta f\|\|\partial^{\alpha-\beta} g\| \\ &\le \sum_{|\beta|\le k} \frac{1}{\beta!}\|\partial^\beta f\| \sum_{|\gamma|\le k}\frac{1}{\gamma!}\|\partial^\gamma g\| = p_{K,k}(f)p_{K,k}(g) \end{aligned} \tag{5.2}$$

It is a special property, that every seminorm can bound itself submultiplicatively. These algebras are referred to as *m-algebras* in [CST04]. Note that $p_{K,k}$ actually makes $C^k(K)$ a Banach algebra and $C^\infty(K) = \varprojlim C^k(K)$.

1. The Fréchet algebra $C^\infty(M)$ for a manifold M. Its defining seminorms are given by the ones above for all coordinate patches U.

2. The LF algebra $C_c^\infty(M)$ of compactly supported functions, sometimes also referred to as test functions. As a locally convex space is the direct limit of C_K^∞ for K ranging through all compact subsets of M. Here, $C_K^\infty = \{f \in C_c^\infty(M) : \mathrm{supp} f \subset K\}$ with the same seminorms as above. A sequence in $C_c^\infty(M)$ converges, if and only if they are all supported inside a compact set K and they converge in C_K^∞. The dual of $C_c^\infty(M)$ are the distributions.

The groupoid convolution algebras also have the underlying LF space $C_c^\infty(G)$ but in general no longer have a jointly continuous multiplication. (c.f. Example 5.10).

5.1.1 Locally Convex Tensor Products

There are a large number of ways to define locally convex topologies on algebraic tensor products of lcs. We discuss here the projective, injective and inductive tensor products. Let U, V be locally convex. Let p be any seminorm on U, q a seminorm on V.

Definition 5.1. The *injective* tensor product $U \otimes_\epsilon V$ is equipped with the tensor product seminorms

$$p \otimes_\epsilon q(\phi) = \sup \left\{ q\left(\sum \xi(f_i) g_i\right) : \phi = \sum_i f_i \otimes g_i, \xi \in U^*, p^*(\xi) \leq 1 \right\}, \tag{5.3}$$

where $p^*(\xi) \leq 1$ means that $|\xi(f)| \leq p(f)$ and we take the supremum over all decompositions as elementary tensors.

The *projective* tensor product $\otimes_\pi$ is best defined to be universal with respect to jointly continuous maps. That is, any continuous bilinear map on the product $U \times V \to W$ should factor through a linear map $U \otimes_\pi V \to W$.
The *inductive* tensor product $\otimes_\iota$ is universal with respect to separately continuous bilinear maps in the same way. A space is *nuclear* when the projective and injective tensor product topologies agree. We will use $\hat{\otimes}_\pi$ etc. to denote the completion of the tensor product in the respective topology. The universal properties carry over to the reflexive subcategory of complete lcs.

Remark 5.2. The inductive tensor product is *not* assocative on arbitrary lcs, c.f. [Mey07, 1.94], due to its universal property.

Theorem 5.3 (Nachbin's Theorem [Nac49])**.** *A subalgebra $\mathcal{A}$ of $C^\infty(M)$ is dense if and only if*

- *it separates points, i.e. for all $x \neq y \in M$ there is $f \in \mathcal{A}$ with $f(x) \neq f(y)$*
- *it separates derivations, i.e. for all $v \in TM$ there is $f \in \mathcal{A}$ with $df(v) \neq 0$*

Corollary 5.4 (Stone-Weierstraß, smooth version)**.** *The following maps have dense image:*

$$\begin{aligned} C^\infty(M) \otimes C^\infty(N) &\longrightarrow C^\infty(M \times N) \\ C_c^\infty(M) \otimes C_c^\infty(N) &\longrightarrow C_c^\infty(M \times N) \\ f \otimes g &\longmapsto [(x, y) \mapsto f(x) g(y)] \end{aligned} \tag{5.4}$$

Theorem 5.5.

$$C^\infty(M)\hat{\otimes}_\pi C^\infty(N) = C^\infty(M)\hat{\otimes}_\epsilon C^\infty(N) \cong C^\infty(M\times N) \tag{5.5}$$

via the map $f\otimes g \mapsto [(x,y)\mapsto f(x)g(y)]$

Proof. We first check that the bilinear map from the algebraic tensor product is actually continuous. For that we just note that every compact set in $M\times N$ is contained inside some compact product $K\times L$ and that

$$p_{K\times L,k}(f\otimes g) = \sup_{x\in K, y\in L, |\alpha|+|\beta|\le k} |\partial^\alpha f(x)||\partial^\beta g(y)| \le p_{K,k}(f)p_{L,k}(g) \tag{5.6}$$

By the universal property of the projective tensor product we know that the maps from left to right are continuous.

Now we check that the algebraic injective ϵ-tensor product is actually equipped with the subspace topology. For this note that for $\phi\in C^\infty(M)\otimes C^\infty(N)$ and $p=p_{K,k}$, $q=p_{L,l}$ seminorms on M,N respectively:

$$\begin{aligned}
p\otimes_\epsilon q(\phi) &= \sup\left\{q\left(\sum_i \xi(f_i)g_i\right) : \phi=\sum f_i\otimes g_i, p^*(\xi)\le 1\right\}\\
&= \sup_\xi \sup_{\phi=\sum f_i\otimes g_i} \sup_{y\in L,|\beta|\le l}\left|\sum_i \xi(f_i)\partial^\beta g_i(y)\right|\\
&= \sup_\xi \sup_{\phi=\sum f_i\otimes g_i} \sup_{y\in L,|\beta|\le l}\left|\xi\left(\sum_i f_i\partial^\beta g_i(y)\right)\right|\\
&\le \sup_{\phi=\sum f_i\otimes g_i} \sup_{y\in L,|\beta|\le l} p\left(\sum_i f_i\partial^\beta g_i(y)\right)\\
&= \sup_{\phi=\sum f_i\otimes g_i} \sup_{y\in L,|\beta|\le l} \sup_{x\in K,|\alpha|\le k}\left|\sum_i \partial^\alpha f_i(x)\partial^\beta g_i(y)\right|\\
&= \sup_{K\times L,|\alpha|\le k,|\beta|\le l}\left|\partial^{(\alpha,\beta)}\phi\right| \le p_{K\times L,k+l}(\phi)\,.
\end{aligned} \tag{5.7}$$

From this inequality we see that on algebraic tensor product the ϵ-topology agrees with the subspace topology. Hence the completion of the injective tensor product agrees with the closure of the algebraic tensor product inside $C^\infty(M\times N)$. In particular, the map $C^\infty(M)\hat{\otimes}_\epsilon C^\infty(N)\to C^\infty(M\times N)$ is injective.
To conclude that this is an isomorphism it suffices to show that the image is dense, which is precisely the content of the smooth version of the Stone-Weierstraß Theorem 5.4. Now it remains to show that the injective and projective topology on the algebraic tensor product agree. This belongs to the deep and difficult theory of nuclearity. A comprehensive treatment can be found in chapters 50 and 51 of [Trè06]. ☐

The following is a consequence of the above and passing to direct limits. It is also Example 1.95 in [Mey07].

Theorem 5.6. *Let M,N be smooth manifold. There is a natural isomorphism:*

$$C_c^\infty(M)\hat{\otimes}_\iota C_c^\infty(N)\cong C_c^\infty(M\times N) \tag{5.8}$$

via the map $f\otimes g\mapsto [(x,y)\mapsto f(x)g(y)]$.

5.2 Separate Continuity of Convolution

This subsection is technical and shows various continuity properties. We sometimes use bornological terminology that is only introduced in the next subsection.

Lemma 5.7. *Let $p : M \to N$ be proper. Then $p^* : C_c^\infty(N) \to C_c^\infty(M)$ is a continuous map of LF spaces.*

Proof. Note that $p_K^* : C_K^\infty(N) \to C_{f^{-1}(K)}^\infty(M)$ is continuous for $K \subset N$ compact.

$$\begin{tikzcd} C_K^\infty(N) \arrow[r, "p_K^*"] \arrow[d, hook] & C_{f^{-1}(K)}^\infty(M) \arrow[d, hook] \\ C_c^\infty(N) = \varinjlim C_K^\infty(N) \arrow[r, "p^*"] & C_c^\infty(M) \end{tikzcd} \tag{5.9}$$

The continuity is immediate as a colimit of continuous maps. □

Lemma 5.8. *Let $f : M \to N$ be a submersion and let $\lambda \in \Gamma(\mathcal{D}_f)$ be a fixed nowhere vanishing density. Then the fiber integration*

$$\begin{aligned} \int_f : C_c^\infty(M) &\to C_c^\infty(N) \\ \varphi &\mapsto \left[x \mapsto \int_{f^{-1}(x)} \varphi\lambda \right] \end{aligned} \tag{5.10}$$

is a continuous map of LF spaces.

Proof. It suffices to show that for any compact set $K \subset M$ fiber integration is continuous as a map $C_K^\infty(M) \to C_{f(K)}^\infty(N)$. Let $\{U_i\}$ be a finite cover of K by coordinate charts in which the submersion is in standard form, i.e. there is a diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d, "f"] & \mathbb{R}^n \times \mathbb{R}^k \arrow[d, "\mathrm{pr}_{\mathbb{R}^n}"] \\ f(U_i) \arrow[r] & \mathbb{R}^k \end{tikzcd}, \tag{5.11}$$

in which the horizontal arrows are diffeomorphisms onto open subsets on the right. Let $\{\chi_i\}$ be a subordinate partition of unity. We have a continuous map

$$\begin{aligned} C_K^\infty(M) &\to \bigoplus_{i=1}^n C_c^\infty(U_i) \\ \varphi &\mapsto (\chi_i\varphi), \end{aligned} \tag{5.12}$$

for which $\int_f \varphi = \sum_i \int_f \chi_i\varphi$. So we only need to show that fiber integration is continuous as a map $C_c^\infty(U_i) \to C_c^\infty(f(U_i))$.
By the previous considerations this reduces to showing that this is true for the standard submersion $\mathbb{R}^{n+k} \to \mathbb{R}^n$. We can also assume that $\lambda = dy^1 \wedge \cdots \wedge dy^k =: \mathrm{d}y$ by using that otherwise $\lambda = \lambda_0(x, y)\mathrm{d}y$ for a smooth nonzero function λ_0. Multiplication by λ_0 then induces an isomorphism of LF spaces $C_c^\infty(\mathbb{R}^{n+k}) \to C_c^\infty(\mathbb{R}^{n+k})$ which translates between the fiber integral with respect to λ and with respect to $\mathrm{d}y$.

$$\begin{tikzcd} C_c^\infty(\mathbb{R}^{n+k}) \arrow[rr, "\lambda_0\cdot(-)"] \arrow[dr, "\int_{\mathrm{pr}_{\mathbb{R}^n}}(-)\lambda"'] & & C_c^\infty(\mathbb{R}^{n+k}) \arrow[dl, "\int(-)\mathrm{d}y"] \\ & C_c^\infty(\mathbb{R}^n) & \end{tikzcd} \tag{5.13}$$

Now if f has compact support in the compact set $K \times L \subset \mathbb{R}^n \times \mathbb{R}^k$ we can estimate the semi-norms of the fiber integral by

$$\begin{aligned} \sup_{x\in K}\left|\partial^\alpha \int_{\mathbb{R}^k} f(x,y)\mathrm{d}y\right| &= \sup_{x\in K}\left|\int_{\mathbb{R}^k} \partial_x^\alpha f(x,y)\mathrm{d}y\right| \\ &\leq \int_{\mathbb{R}^k} \sup_{x\in K\times L} |\partial_x^\alpha f|\,\mathrm{d}y \\ &\leq C p_{K\times L,|\alpha|} f\,. \end{aligned} \tag{5.14}$$

This shows continuity of $C^\infty_{K\times L}(M) \to C^\infty_K(N) \to C^\infty_c(N)$ and by a colimit argument continuity of the entire fiber integration. □

Corollary 5.9. *The convolution $*$ is separately continuous on $C^\infty_c(G) \times C^\infty_c(G)$.*

Proof. Note that convolution $*$ factors as a map as

$$C^\infty_c(G) \otimes C^\infty_c(G) \longrightarrow C^\infty_c(G\times G) \xrightarrow{\iota^*} C^\infty_c(G^{(2)}) \xrightarrow{\int_m} C^\infty_c(G)\ . \tag{5.15}$$

By Theorem 5.6 we know that the inductive tensor product of compactly supported functions are the compactly supported functions on the product. Hence the first map is continuous in the topology of the inductive tensor product and even an isomorphism if we pass to the completion. Since the inductive tensor product is universal with respect to separately continuous maps, this map is in general only separately continuous. All other maps are continuous by the previous lemmata. □

The following is a counterexample where continuity is not jointly continuous. It is taken from [Mey07, Example 1.34].

Example 5.10. Consider the additive group $\mathbb{R}$. Its group(oid) convolution algebra is $C^\infty_c(\mathbb{R})$ with the convolution product

$$f * g(x) = \int f(x-y)g(y)\mathrm{d}y \tag{5.16}$$

Since we are dealing with compactly supported functions, the following provides a continuous seminorm:

$$\nu(f) = \sum_{n\in\mathbb{N}} |f^{(n)}(n)|\,. \tag{5.17}$$

If convolution was jointly continuous, there would be seminorms ν_1, ν_2 with

$$\nu(f*g) \leq \nu_1(f)\nu_2(g)\,. \tag{5.18}$$

For all f supported in a fixed compact interval, say $[-1,1]$, $\nu_1(f)$ only depends on finitely many derivatives of f, say up to order $N-1$. It is hence possible to choose $f \in C^\infty_c([-1,1])$ with $\int_{-1}^1 |f^{(N)}|\mathrm{d}y > K$ such that also $\nu_1(f) \leq 1$. Then, $(f*g)^{(N)} = f^{(N)} * g$. Choose g with $g(x) = 1$ for $x \in [N-1, N+1]$

$$\nu(f*g) \geq |(f*g)^{(N)}(N)| \geq \int |f^{(N)}(N-y)||g(y)|\mathrm{d}y \geq K \tag{5.19}$$

Since f and K were arbitrary, this contradicts $\nu(f*g) \leq 1\cdot\nu_2(g)$.

Corollary 5.11. *The LF space $\mathcal{A}_G$ equipped with the von Neumann bornology and the convolution product $*$ is a bornological algebra.*

Proof. Separate continuity of $* : \mathcal{A}_G \times \mathcal{A}_G \to \mathcal{A}_G$ is equivalent to boundedness of $\mathcal{A}_G \otimes \mathcal{A}_G \to \mathcal{A}_G$ since the inductive tensor product on LF spaces agrees with the bornological tensor product. □

Lemma 5.12. *Let X be a vector field on M.*

(i) $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ *is continuous.*

(ii) $\mathcal{L}_X : C^\infty_c(M) \to C^\infty_c(M)$ *is continuous.*

(iii) $\mathcal{L} : \mathfrak{X}(M) \times C^\infty_c(M) \to C^\infty_c(M)$ *is continuous.*

(iv) $\mathcal{L} : \mathfrak{X}(M) \to \underline{\mathrm{Hom}}(C^\infty_c(M), C^\infty_c(M))$ *is bounded.*

Proof. This is essentially a statement about local coordinates. Let K be a compact subset of M that is contained in a coordinate chart U. Write $X = X^i \frac{\partial}{\partial x^i}$. Then,

$$\begin{aligned} p_{K,k}(\mathcal{L}_X f) &= \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \|\partial^\alpha (Xf)\| \\ &\leq \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \left(\|\partial^\alpha X^i\| \|\partial_i f\| + \|X^i\| \|\partial^\alpha \partial_i f\| \right) \\ &\leq C_1 p_{K,k}(X) p_{K,1}(f) + C_2 p_{K,0}(X) p_{K,k+1}(f) \\ &\leq C p_{K,k}(X) p_{K,k+1}(f) \,. \end{aligned} \tag{5.20}$$

Hence, $\mathcal{L}_X$ is continuous in the locally convex topology on $C^\infty(M)$. The same computation also proves (*ii*) and (*iii*). Statement (*iv*) follows from the tensor-hom adjunction on bornological spaces. □

Let us emphasize that the *bornological* Hochschild complex $C^*(A, A)$ consists of only *bounded* multilinear cocycles $A^{\times n} \to A$. We will later drop the adjective "bornological".

Corollary 5.13. *The cochain map Φ defined in Equation* (3.47) *is a map to the bornological Hochschild complex.*

$$\Phi : C^*_{\mathrm{def}}(G) \longrightarrow C^*(\mathcal{A}_G, \mathcal{A}_G) \,. \tag{5.21}$$

5.3 Bornological Vector Spaces

We aim to explain the convenience of the category of (convex, complete) bornological vector spaces here. This includes a scaffolding of definitions. We do not aim for complete coverage. For more details we point to parts of the literature as needed, aiming to keep our own presentation self-contained. A classical account is [HN77], a thorough more modern introduction is given in [Mey07]. We can subsume a lot of the upcoming discussion by simply stating that $(\mathsf{CBorn}, \hat{\otimes}, \underline{\mathrm{Hom}})$ is a preabelian cartesian closed symmetric monoidal category having all limits and colimits.
Bornological spaces are closely related to locally convex spaces by an adjunction:

$$\gamma : \mathsf{Born} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{lcs} : \mathrm{vN} \tag{5.22}$$

The categorical pathologies in lcs seem to disappear upon passage to Born or CBorn and the plethora of tensor products on locally convex spaces is interchanged for a better behaved one. In special cases, e.g. Fréchet spaces with $\hat{\otimes}_\pi$, the functors will even be a monoidal equivalence. This lets us transport resolutions etc. and rephrase computations for Fréchet spaces in the bornological setting.
Unfortunately, maybe as a historical accident, we cannot merely view bornology as a substitute for topological vector spaces, but instead will contrast the two approaches. This only introduces more vocabulary.

Definition 5.14 (Elementary Notions)**.** Let E be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A subset $A \subset E$ is *circled* if $\lambda A \subset A$ for all $|\lambda| \leq 1$. A *disk* is a circled convex subset. For a disk D the space E_D spanned by D is equipped with a canonical seminorm $p_D = \|\cdot\|_D$ called *gauge of* D or *Minkowski functional* defined by:

$$p_D(x) = \inf\{\lambda : x \in \lambda D\} \tag{5.23}$$

The disk D is called *norming* if $\|\cdot\|_D$ is a norm. It is called *completant* if E_D is a Banach space, i.e. complete in the norm.

Definition 5.15. A *bornology* on a set X is a system of *small* or *bounded* subsets $\mathfrak{B} \subset \mathcal{P}(X)$ that satisfies:

(i) hereditary under inclusions, i.e. if $A \subset B$ and B is small, then A is small

(ii) stable under finite union, i.e. $A \cup B$ is small for A, B small

(iii) it covers X, i.e. each $x \in X$ is contained in a small set.

A *base* of the bornology is a subset of $\mathfrak{B}$ that is cofinal, i.e. every small set is contained inside one of the basis sets. We call X a *bornological space.* A map $f : X \to Y$ is called *bounded* if it maps bounded sets to bounded sets. This makes bornological spaces a category.
A *convex vector bornology* on a vector space V over $\mathbb{K}$ satisfies additionally:

1. stable under convex hulls
2. stable under circled hulls, i.e. $\bigcup_{|\lambda|<1} \lambda A$ is small for A small
3. stable under addition and scalar multiplication

We call V a *convex bornological vector space.*

Remark 5.16. A bornological vector space is just a vector space with a bornology and bounded structure maps. The category of (convex) bornological vector spaces is the subcategory of bornological spaces with bounded linear maps in between them.

We will now be working exclusively with convex bornological vector spaces. Similarly to point-set topology there are initial and final bornologies. Given a collection of maps $f_i : F \to F_i$, we call $B \subset F$ small, if $f_i(B)$ is small in F_i. Now one checks that $g : E \to F$ is bounded if and only if $f_i \circ g$ is bounded for all i.
Similarly for a collection of maps $f_i : F_i \to F$, we can define a basis for the bornology on F by the sets $f_i(B_i)$, for $B_i \subset F_i$ small. One again checks that $g : F \to E$ is bounded if and only if $g \circ f_i$ is bounded for all i.
This allows us to define bornologies on algebraic limits and colimits. This gives us bornological limits and colimits. Similarly, the algebraic tensor product $F \otimes F'$ can be equipped with the final bornology with respect to $F \times F' \to F \otimes F'$.

Remark 5.17. The forgetful functor $U : \mathsf{Born} \to \mathsf{Vect}$ has a left adjoint that is equipping a vector space V with the initial bornology with respect to all linear maps $V \to U(F)$. It is called the fine bornology.
It also has a right adjoint given by equipping V with the final bornology of all the maps $U(F) \to V$. Hence U preserves limits and colimits.

Example 5.18 (Von Neumann Bornology)**.** For a locally convex space E with seminorms $\{p_i\}_{i\in I}$ we call $B \subset E$ small if $p_i(B)$ is bounded in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Note that this actually constructs a convex bornology that is initial with respect to the maps $p_i : E \to \mathbb{K}$, where $\mathbb{K}$ gets its canonical bornology of bounded sets. The pair (E, vN) is called the *bounded* or *von Neumann* bornology of E.
Note that a continuous map $f : E \to E'$ is automatically bounded for the bounded bornologies and that we hence get a functor

$$\mathrm{vN} : \mathsf{lcs} \to \mathsf{Born}. \tag{5.24}$$

For the following, note that any convex bornological vector space has a basis of disks (by definition).

Example 5.19 (Bornivorous Topology)**.** Let F be a bornological vector space. A subset $A \subset F$ is called *bornivorous* if it absorbs every bounded set, i.e. there is a scalar λ such that each small B is contained in $\lambda' A$ for $\lambda' \geq \lambda$.
There is a natural locally convex topology γF on the bornological vector space F such that the convex bornivorous disks in F are a neighbourhood basis of zero for the topology. Since the preimage of (convex, disked) bornivorous sets is bornivorous (convex, disked), any bounded map is continuous in the bornivorous topology. Hence we get another functor

$$\gamma : \mathsf{Born} \to \mathsf{lcs}. \tag{5.25}$$

Proposition 5.20. *The functors in Equation* (5.24) *and Equation* (5.25) *are an adjoint pair.*

$$\gamma : \mathsf{Born} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{lcs} : \mathrm{vN} \tag{5.26}$$

Or, phrased differently, bounded maps out of a bornological F into a locally convex space E are automatically continuous for the topology on γF.
Furthermore, we have the following properties:

1. *The unit of the adjunction $F \to \mathrm{vN}\,\gamma F$ is an isomorphism for all convex bornological vector spaces F. This says that γ is fully faithful. We could hence say* Born *embeds into* lcs.

2. *The counit of the adjunction* $\gamma\,\mathrm{vN}(E) \to E$ *is an isomorphism for all metrizable locally convex spaces.*

3. *The counit is an isomorphism for all Fréchet spaces and for all LF spaces. Hence, a map between LF spaces is continuous if and only if it is bounded.*

4. *The adjunction restricts to the full subcategories of separated respectively Hausdorff spaces.*

5. vN *maps complete locally convex spaces to complete bornological spaces.*

We will properly introduce separated and complete spaces below.

Remark 5.21. 1. Note that the functors constitute an equivalence on a large subcategory. That they fail to be an equivalence justifies using bornologies. Otherwise, we could just stick to lcs.

2. γ can fail to preserve complete spaces. The completion of a bornological space can be different. Again, this is favorable when dealing with tensor products.

Proof. Let $f : \gamma F \to E$ be continuous. Let $B \subset F$ be bounded. $f(B)$ is bounded if and only if it is absorbed by any neighbourhood $U \subset E$ of zero. Since $f^{-1}(U)$ is open it contains a bornivorous disk A and $B \subset \lambda A$ for $\lambda \geq \lambda_0$. Hence, $f(B) \subset \lambda U$ for $\lambda \geq \lambda_0$ and $f : F \to \mathrm{vN}\,E$ is bounded.

Conversely, let $f : F \to \mathrm{vN}\,E$ be bounded. A neighbourhood $U \subset E$ of zero is bornivorous and hence $f^{-1}(U)$ is a bornivorous disk, i.e. open in γF. Hence $f : \gamma F \to E$ is continuous. This proves the adjunction.

1. $B \subset \mathrm{vN}\,\gamma F$ is bounded if and only if it is absorbed by all bornivorous sets if and only if $B \subset F$ is bounded.

2. This is [HN77, 4:1 Prop. (3)].

3. The subcategory where the counit $\gamma\,\mathrm{vN}(E) \to E$ is an isomorphism is closed under colimits: Both sides are the same vector space and the counit really is the identity as a vector space homomorphism. For it to be an isomorphism, we need $\mathrm{id} : E \to \gamma\,\mathrm{vN}(E)$ to be continuous. If $E = \varinjlim E_i$ and the counit is an isomorphism for E_i, then this is immediate by commutativity of

$$\begin{tikzcd} E_i & \gamma\,\mathrm{vN}(E_i) \arrow[l, "\cong"'] \arrow[d] \\ E \arrow[from=u] & \gamma\,\mathrm{vN}(E) \arrow[l] \end{tikzcd} . \tag{5.27}$$

The map $\mathrm{id} : E \to \gamma\,\mathrm{vN}(E)$ is continuous as the colimit of the continuous maps $\mathrm{id} : E_i \to \gamma\,\mathrm{vN}(E_i) \to \gamma\,\mathrm{vN}(E)$.

4. Let $F \in \mathsf{Born}$ be separated, i.e. $\{0\}$ is the only bounded subspace. Let $x \in F$. There is a bornivorous disk A that does not contain $\mathbb{K}x$; otherwise it would be bounded. We can hence assume $x \notin A$. Since A is a neighbourhood of zero, this shows Hausdorffness of γF. The converse is analogous.

5. This follows from the fact that Mackey-Cauchy sequences in $\mathrm{vN}\,E$ are in particular Cauchy sequences in E. $\square$

5.3.1 Inner Homs

There is a canonical bornology on the set of all bounded homomorphisms between bornological spaces F and F' that we denote $\underline{\mathrm{Hom}}(F, F')$. A set B of such maps is called *uniformly bounded* if for all small sets $A \subset F$ there is a small $A' \subset F'$ with $f(A) \subset A'$ for all $f \in B$. The uniformly bounded sets are the small sets in this bornology. If F' is convex, then $\underline{\mathrm{Hom}}(F, F')$ is also convex.
This is of course reminiscent of the compact-open topology.

Proposition 5.22 (Algebraic Tensor-Hom adjunction)**.** *The usual tensor-hom adjunction upgrades to an adjunction in* Born*:*

$$- \otimes F : \mathsf{Born} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Born} : \underline{\mathrm{Hom}}(F, -) \tag{5.28}$$

Proof. We just have to see that boundedness of maps is compatible on both sides. Given $u : F_1 \otimes F \to F_2$ corresponding to a bounded bilinear map $F_1 \times F \to F_2$. Then $\hat{u}(f_1) : F \to F_2$ is indeed bounded since $\hat{u}(f_1)(A) = u(\{f_1\} \times A)$ is small for any small A in F. Furthermore $\hat{u} : F_1 \to \underline{\mathrm{Hom}}(F, F_2)$ is bounded itself: For any small $A_1 \subset F_1$ we have that $\hat{u}(A_1)$ is a set of maps satisfying $\hat{u}(a_1)(A) \subset u(A_1 \times A)$ for all $a_1 \in A_1$ and the latter is a small set. Hence $\hat{u}(A_1)$ is uniformly bounded in $\underline{\mathrm{Hom}}(F, F_2)$.
Conversely let $v : F_1 \to \underline{\mathrm{Hom}}(F, F_2)$ be bounded. Then $f_1 \otimes f \mapsto v(f_1)(f)$ is bounded: Tensor products $A_1 \otimes A$ of small sets generate the bornology on $F_1 \otimes F$. Now note that $v(A_1)$ is small in $\underline{\mathrm{Hom}}(F, F_2)$ since v is bounded. Hence $v(A_1)(A)$ is small in F_2 by definition of the bornology on $\underline{\mathrm{Hom}}$. □

5.3.2 Bornological Completions

Definition 5.23. A bornological vector space is *separated* if 0 is the only small vector subspace or equivalently if it has a basis of norming disks. It is called *complete* if it has a basis of completant disks. [10] Since completant includes norming, complete spaces are separated.

Proposition 5.24 (Separation)**.** *There is a separation functor* Sep *that is left adjoint to the inclusion of the full subcategory of separated bornological vector spaces* SBorn*.*

$$\mathrm{Sep} : \mathsf{Born} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{SBorn} : \iota \tag{5.29}$$

The counit $\mathrm{Sep}\,\iota(F) \to F$ *is always an isomorphism for* F *separated.*
The inclusion is a right adjoint. Hence limits of separated spaces are separated.
It also holds that colimits of separated spaces with injective structure maps are separated.
Subspaces are always separated. $\underline{\mathrm{Hom}}(F, F')$ *is separated if* F' *is.*
A quotient F/F' *is separated if and only if the subspace* F' *is bornologically closed, that is* $\gamma F' \subset \gamma F$ *is closed. With this in mind,* $\mathrm{Sep}(F) = F/\overline{\{0\}}$ *is given by modding out the closure of zero in the bornivorous topology.*

The proofs can be found in [HN77, Chapter 2:10,2:11].

Remark 5.25. The category of separated convex bornological spaces is bicomplete, i.e. has all (co)limits. We already showed that all products exist and are separated. Clearly,

[10] There is an equivalent definition of completeness using *Mackey-Cauchy sequences*. A sequence is Mackey-Cauchy if it is bounded (i.e. entirely contained in a disk D) and Cauchy with respect to the Minkowski functional $\|\cdot\|_D$. Finally, a bornological space is complete if all Mackey-Cauchy sequences converge.

kernels also exist and are separated as subspaces. Cokernels are given by the quotient by the closure of the image, which is isomorphic to $\operatorname{coker}(f : F \to F') \cong \operatorname{Sep}(F'/\operatorname{Im}(f))$. This shows that all limits and colimits exist.

Proposition 5.26 (Completion)**.** *There is a completion functor C that is left adjoint to the inclusion of the full subcategory of complete bornological vector spaces.*

$$C : \mathsf{Born}_{separated} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CBorn} : \iota \tag{5.30}$$

The counit $C(\iota(F)) \to F$ is an isomorphism for F complete. The inclusion is a right adjoint. Hence limits of complete spaces are complete.
The inclusion is also a left adjoint for a different functor [HN77, Chapter 3:4]. Hence, if a colimit of complete spaces is separated, it is automatically complete.

Remark 5.27. The previous proposition hides the intuitive meaning of a completion. Of any type of completion we demand that bounded (continuous) maps into a complete space should extend boundedly (continuously) to the completion. This translates precisely into being left adjoint for the inclusion. "Extension" should be taken with a grain of salt as the next remark shows.

Remark 5.28. The adjunction unit $F \to C(F)$ need not be injective! There are cases where $C(F) = 0$ and F is nonzero. See for example exercise 3.E.5 in [HN77].

If a completion exists, it is of course unique up to natural isomorphism since it is a left adjoint, or, since it has the universal property that any map into a complete bornological space factors through it. That being said, the particular construction used is not too important. The quickest way is to note that any separated bornological space F is a colimit (inductive limit) of normed spaces F_D for D ranging through a basis of disks. This is called the *dissection* of F. Then the completion of F is the the colimit of the Banach space completions. The upshot is that bornological spaces are the full subcategory of inductive systems of Banach spaces with injective structure maps. The following diagram appears similarly in [Wal18] and packages a lot of the information. Parallel pairs are adjoint functors.

$$\begin{tikzcd}
Ind(SNorm_k) \arrow[r, shift left, "\lim"] \arrow[d, shift right] & \mathsf{Born} \arrow[l, shift left, "diss"] \arrow[r, shift left, "\gamma"] \arrow[d, shift right] & \mathsf{lcs} \arrow[l, shift left, "\mathrm{vN}"] \arrow[d, shift right] \\
Ind(Norm_k) \arrow[u, shift right] \arrow[r, shift left, "\lim"] \arrow[d, shift right] & \mathsf{SBorn} \arrow[u, shift right] \arrow[l, shift left, "diss"] \arrow[r, shift left, "\gamma"] \arrow[d, shift right] & \mathsf{lcs}_{\text{separated}} \arrow[u, shift right] \arrow[l, shift left, "\mathrm{vN}"] \arrow[d, shift right] \\
Ind(Ban_k) \arrow[u, shift right] \arrow[r, shift left, "\lim"] & \mathsf{CBorn} \arrow[u, shift right] \arrow[l, shift left, "diss"] \arrow[r, shift left, "C\gamma"] & \mathsf{lcs}_{\text{complete}} \arrow[u, shift right] \arrow[l, shift left, "\mathrm{vN}"]
\end{tikzcd} \tag{5.31}$$

5.3.3 The complete Tensor Product

We now discuss shortly how to get a tensor product on complete bornological spaces. The algebraic tensor product with its standard final bornology for $F \times F' \to F \otimes F'$ is a tensor product on the category of all convex bornological vector spaces.

Lemma 5.29. *If F, F' are separated, then $F \otimes F'$ is also separated.*

Proof. Let S be a bounded subspace and $x \in S$. This must be contained in some convex hull $Conv(D \otimes D') \subset E_D \otimes E_{D'}$ of disks D, D'. By Hahn-Banach there are functionals ϕ, ϕ' with $\|\phi\| \leq 1$ and $\phi \otimes \phi'(x) \neq 0$. Then $|\phi \otimes \phi'(\sum \lambda_i f_i \otimes f_i')| \leq \sum \lambda_i |\phi(f_i)||\phi(f_i')| \leq 1$ for $f_i \in D$ and $f_i' \in D'$ and $\sum \lambda_i = 1$. Hence $|\phi \otimes \phi'(\lambda x)| \leq 1$ for all $\lambda \in \mathbb{R}$ and hence $x = 0$. Thus $S = 0$ and consequently $F \otimes F'$ is separated. $\square$

As a cumulation of what we proved before we are now ready to do some abstract nonsense. Define $F\hat{\otimes}F' := CF \otimes F'$ to be the complete (bornological projective) tensor product.

Proposition 5.30 (Tensor-Hom adjunction for complete bornological spaces)**.** *There is an adjunction $-\hat{\otimes}F \dashv \underline{\mathrm{Hom}}(F,-)$ on the category of complete convex bornological spaces. $\underline{\mathrm{Hom}}(F,F')$ is complete if F' is complete.*

Proof. Let F, F_1, F_2 be complete convex bornological spaces. Then there are natural isomorphisms:

$$\begin{aligned}\mathsf{CBorn}(F_1\hat{\otimes}F, F_2) &\cong \mathsf{Born}(F_1 \otimes F, F_2)\\ &\cong \mathsf{Born}(F_1, \underline{\mathrm{Hom}}(F, F_2))\\ &\cong \mathsf{CBorn}(F_1, \underline{\mathrm{Hom}}(F, F_2))\end{aligned} \tag{5.32}$$

We used the adjunction $-\otimes F \dashv \underline{\mathrm{Hom}}(F,-)$, that completion is left adjoint to the inclusion of complete spaces and $F_1 \otimes F$ is separated. We conclude with the proof that $\underline{\mathrm{Hom}}$ is complete and so we have the desired adjunction in the full reflexive subcategory of complete bornological spaces.
Choose a basis $\{B'_j\}_{j\in J}$ of completant disks for F' and a basis of disks $\{B_i\}_{i\in I}$ for F. Then for any map $j : I \to J$ the set $D_j = \{f : F \to F' | f(B_i) \subset B'_{j(i)}\}$ is a uniformly bounded disk in $\underline{\mathrm{Hom}}(F,F')$ and their collection is a basis for the bornology. We want to show that $E_{D_j} = \mathrm{span}(D_j) = \mathbb{K}D_j$ is complete. For this let f_n be a Cauchy sequence in $\mathbb{K}D_j$. $\|f_n - f_m\|_{D_j} < \epsilon$ translates into $(f_n - f_m)(B_i) \subset \epsilon B'_{j(i)}$, i.e. $\sup_{B_i}\|f_n - f_m\|_{B'_{j(i)}} < \epsilon$. This again means that $f_n|_{\mathbb{K}B_i} : \mathbb{K}B_i \to \mathbb{K}B'_{j(i)}$ is a Cauchy sequence of continuous linear operators into a Banach space. By standard functional analysis, there are limiting maps $f_i : \mathbb{K}B_i \to \mathbb{K}B'_{j(i)}$ that are the pointwise and uniform limit of $f_n|_{B_i}$. The f_i are mutual extensions since if $x \in \mathbb{K}B_{i_1} \cap \mathbb{K}B_{i_2}$, then $f_{i_1}(x) = \lim_n f_n(x) = f_{i_2}(x)$. So denote the limiting map by f which restricts to f_i on $\mathbb{K}B_i$. Finally, f_n converges to f in the $\|\cdot\|_{D_j}$-norm: Choose N large enough such that for $n, m \geq N$ we have $\|f_n - f_m\|_{D_j} < \epsilon$. For any $x \in B_i$ we get

$$\|f_n(x) - f(x)\|_{B'_{j(i)}} = \lim_m \|f_n(x) - f_m(x)\|_{B'_{j(i)}} < \epsilon. \tag{5.33}$$

Hence $(f_n - f)(B_i) \subset \epsilon B'_{j(i)}$ for all i. But this is equivalent to $\|f_n - f\|_{D_j} \leq \epsilon$. $\square$

From now on we will work with complete convex bornological spaces F if not stated otherwise.

Lemma 5.31 (Universal Property)**.** *Every bounded multilinear map $f : F_1 \times F_2 \times \cdots \times F_k \to F$ factors uniquely through a map $F_1\hat{\otimes}\ldots\hat{\otimes}F_k \to F$. In particular both bracketings fulfill the universal property, so the tensor product is associative with unique associator isomorphisms:*

$$\alpha_{F_1F_2F_3} : F_1\hat{\otimes}(F_2\hat{\otimes}F_3) \xrightarrow{\cong} (F_1\hat{\otimes}F_2)\hat{\otimes}F_3. \tag{5.34}$$

The following is Theorem 1.91 in [Mey07].

Theorem 5.32. *Let V, W be nuclear Fréchet spaces. Then the canonical morphism*

$$\mathrm{vN}(V)\hat{\otimes}\,\mathrm{vN}(W) \to \mathrm{vN}(V\hat{\otimes}_\pi W) \tag{5.35}$$

is an isomorphism.
More generally, let V, W be nuclear LF spaces. Then the canonical morphism

$$\mathrm{vN}(V)\hat{\otimes}\,\mathrm{vN}(W) \to \mathrm{vN}(V\hat{\otimes}_\iota W) \tag{5.36}$$

is an isomorphism.

Remark 5.33. The second statement indeed generalizes the first, since inductive and projective tensor product agree on Fréchet spaces: Separately continuous maps are automatically jointly continuous.
Working with the *precompact* bornology instead of vN the first statement of the theorem holds for all Fréchet spaces.

Corollary 5.34. *We have the following isomorphisms of bornological spaces, where we view the function algebras with their canonical locally convex structure as Fréchet and LF spaces respectively equipped with the* vN *bornology:*

$$C^\infty(M)\hat{\otimes}C^\infty(N) \cong C^\infty(M\times N) \quad \textit{and} \quad C^\infty_c(M)\hat{\otimes}C^\infty_c(N) \cong C^\infty_c(M\times N)\,. \tag{5.37}$$

To summarize:

Proposition 5.35. *The category* CBorn *together with the complete bornological tensor product* $\hat{\otimes}$ *is a cartesian closed* $\mathbb{K}$*-linear symmetric monoidal category.*
It admits all kernels, cokernels, direct sums and products. Hence it is a bicomplete, i.e. has all limits and colimits.
The functor vN *is a monoidal functor on* $(\mathsf{Frechet}^{nuclear}, \hat{\otimes}_\pi)$.

Let us recall the definition of symmetric monoidal category. A *monoidal category* C is a category equipped with a (tensor) product $\otimes : \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ and a unit $\mathbb{1}\in\mathsf{C}$ as well as natural isomorphisms

$$\alpha_{A,B,C} : (A\otimes B)\otimes C \xrightarrow{\cong} A\otimes(B\otimes C), \quad \lambda_A : \mathbb{1}\otimes A \xrightarrow{\cong} A, \quad \rho_A : A\otimes\mathbb{1}\xrightarrow{\cong} A \tag{5.38}$$

called associator, left and right unit constraint. They are required to fulfill the pentagon and triangle coherence relations:

$$\begin{tikzcd}
& ((A\otimes B)\otimes C)\otimes D \arrow[dl, "\alpha_{ABC}\otimes\mathrm{id}_D"'] \arrow[dr, "\alpha_{A\otimes B,C,D}"] & \\
(A\otimes(B\otimes C))\otimes D \arrow[d, "\alpha_{A,B\otimes C,D}"'] & & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\alpha_{A,B,C\otimes D}"] \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "\mathrm{id}_A\otimes\alpha_{B,C,D}"] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd} \tag{5.39}$$

$$\begin{tikzcd}
(A\otimes\mathbb{1})\otimes B \arrow[rr, "\alpha_{A,\mathbb{1},B}"] \arrow[dr, "\rho_A\otimes\mathrm{id}_B"'] & & A\otimes(\mathbb{1}\otimes B) \arrow[dl, "\mathrm{id}_A\otimes\lambda_B"] \\
& A\otimes B &
\end{tikzcd} \tag{5.40}$$

A *braiding* on a monoidal category is a natural isomorphism

$$\gamma_{AB} : A\otimes B \xrightarrow{\cong} B\otimes A \tag{5.41}$$

satisfying the hexagon identities

$$\begin{tikzcd}
& A\otimes(B\otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B\otimes C)\otimes A \arrow[dr, "\alpha_{BCA}"] & \\
(A\otimes B)\otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\gamma_{AB}\otimes\mathrm{id}_C"'] & & & B\otimes(C\otimes A) \\
& (B\otimes A)\otimes C \arrow[r, "\alpha_{B,A,C}"'] & B\otimes(A\otimes C) \arrow[ur, "\mathrm{id}_B\otimes\gamma_{AC}"'] &
\end{tikzcd} \tag{5.42}$$

$$\begin{tikzcd}[column sep=small]
& (A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}_{A,B,C}"] \arrow[dr, "\mathrm{id}_A\otimes\gamma_{BC}"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[ur, "\gamma_{AC}\otimes\mathrm{id}_B"'] &
\end{tikzcd} \quad . \tag{5.43}$$

The braiding is called *symmetric* if $\gamma_{AB}\gamma_{BA} = \mathrm{id}_{A\otimes B}$. If it is symmetric, it suffices to show only one of the hexagon identities.

Proof of Proposition 5.35. We fix a choice of completed tensor product satisfying the universal property. It is functorial by this precise universal property. All bracketings are uniquely isomorphic by the universal property making all coherence relation diagrams automatically commutative. The unit is of course the bornological space $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Similarly the unit constraints are isomorphisms since bilinear maps out of $\mathbb{K} \times A$ are the same linear maps out of A. The braiding is the unique isomorphism $A\hat{\otimes}B \to B\hat{\otimes}A$. Again, the hexagon diagrams are commutative by the universal property. Similarly, symmetry of the braiding is another consequence of the universal property. $\square$

5.4 Algebras in symmetric monoidal Categories

We quote the *microcosm principle*:

> [Baez-Dolan] Certain algebraic structures can be defined in any category equipped with a categorified version of the same structure.

We want to abstain from ad-hoc definitions and develop an abstract framework that allows more flexibility. Convolution algebras should be algebras in some monoidal category. This is problematic in locally convex spaces since this category does not have a good monoidal product and, for the only good candidate $\hat{\otimes}_\pi$, the convolution algebras we consider are *not* algebras in the following sense, since convolution is only separately continuous.

Definition 5.36. An *algebra* in a monoidal category $(\mathsf{C}, \otimes, \mathbb{1})$ is an object A together with a multiplication $\mu : A \otimes A \to A$ satisfying the associativity relation:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha_{A,A,A}"] \arrow[d, "\mu\otimes\mathrm{id}_A"] & A \otimes (A \otimes A) \arrow[r, "\mathrm{id}_A\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[rr, "\mu"] & & A
\end{tikzcd} \tag{5.44}$$

A *unital* algebra is an algebra together with a unit $\eta : \mathbb{1} \to A$ satisfying the left and right unit relation:

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "\eta\otimes\mathrm{id}_A"] \arrow[dr, "\lambda_A"] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{1} \arrow[l, "\mathrm{id}_A\otimes\eta"'] \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd} \tag{5.45}$$

Often, unital algebras are referred to as monoid objects. We are using different terminology to differentiate between unital and nonunital algebras. In the presence of a symmetric monoidal structure we can define a *commutative* algebra by requiring the commutativity of the following diagram:

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\gamma_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd} \tag{5.46}$$

Definition 5.37. A *morphism* $f : A \to B$ of algebras in C is required to commute with μ:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu_A"] \arrow[d, "f\otimes f"] & A \arrow[d, "f"] \\
B \otimes B \arrow[r, "\mu_B"] & B
\end{tikzcd} \tag{5.47}$$

A *unital* morphism between unital algebras additionally satisfies $\eta_B = f \circ \eta_A$.
We denote the category of algebras in C by $\mathrm{Alg}(\mathsf{C})$ and the category of unital algebras and unital homomorphisms by $\mathrm{Alg}_+(\mathsf{C})$.

Remark 5.38. We can take an even more abstract viewpoint. Unital algebras are monoidal functors from the monoidal category Alg_+ to C. A unital morphism of unital algebras is a monoidal natural transformation. From this it easily follows that a monoidal functor $\mathsf{C} \to \mathsf{D}$ induces a functor $\mathrm{Alg}_+(\mathsf{C}) \to \mathrm{Alg}_+(\mathsf{D})$

Definition 5.39. Let $A \in \mathrm{Alg}(\mathsf{C})$. A *left A-module* is an object M in C with a scalar multiplication $\mu_M : A \otimes M \to M$. It should be associative in the following sense:

$$\begin{tikzcd}
(A \otimes A) \otimes M \arrow[r, "\alpha_{A,A,M}"] \arrow[d, "\mu \otimes \mathrm{id}_M"] & A \otimes (A \otimes M) \arrow[r, "\mathrm{id}_A \otimes \mu_M"] & A \otimes M \arrow[d, "\mu_M"] \\
A \otimes M \arrow[rr, "\mu_M"] & & M
\end{tikzcd} \tag{5.48}$$

A *unital* module over a unital algebra satisfies additionally $\mu_M \circ (\eta \otimes \mathrm{id}_M) \circ \lambda_M^{-1} = \mathrm{id}_M$. A morphism of left A-modules is a map $f : M \to N$ such that $f \circ \mu_M = \mu_N \circ (\mathrm{id}_A \otimes f)$. We say that f is A-linear.
We denote the category of left modules over A by $\mathrm{Mod}(A)$ and the category of unital left modules over A by $\mathrm{Mod}_+(A)$.

Similarly, one defines right modules. Bimodules are equipped with commuting left and right actions.

Definition 5.40. If $A \in \mathrm{Alg}(\mathsf{C})$ is an algebra in a symmetric monoidal category, we can define A^{op} with the reverse multiplication $\bar{\mu} = \mu \circ \gamma_{A,A}$.

Now, (unital) right modules are just left modules over A^{op}. It is also easy to see that unital A-B-bimodules are just left modules over $A \otimes B^{\mathrm{op}}$. For the latter identification we really need unitality of the modules and algebras. We denote this bimodule category by $\mathrm{Mod}_+(A, B) = \mathrm{Mod}_+(A \otimes B^{\mathrm{op}})$.

Example 5.41. There is always the initial unital algebra $\mathbb{1}$ with multiplication induced from either of the unit constraints. This is the analogue of the base field. We have $\mathrm{Mod}_+(\mathbb{1}) \cong \mathsf{C}$.
Similarly, the initial nonunital algebra is 0 with $\mathrm{Mod}(0) \cong \mathsf{C}$.

Suppose C admits finite direct sums. The following shows how to freely adjoin units to algebras and how to make modules unital.

Lemma 5.42 (Adjunction of a free unit)**.** *There is an adjunction:*

$$\begin{tikzcd}
{}_+ : \mathrm{Alg}(\mathsf{C}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_+(\mathsf{C}) \arrow[l, shift left, hook']
\end{tikzcd} \tag{5.49}$$

Here A_+ is the coproduct $A \oplus \mathbb{1}$ in C with multiplication as in the diagram:

$$\begin{tikzcd}
(A \oplus \mathbb{1}) \otimes (A \oplus \mathbb{1}) \arrow[r, "\cong"] \arrow[d] & (A \otimes A) \oplus (A \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes A) \oplus \mathbb{1} \otimes \mathbb{1} \arrow[d, "\mu \oplus \rho \oplus \lambda \oplus \lambda"] \\
A \oplus \mathbb{1} & (A \oplus A \oplus A) \oplus \mathbb{1} \arrow[l]
\end{tikzcd} \tag{5.50}$$

This means that any algebra homomorphism $A \to B$ to a unital algebra B automatically extends a unital homomorphism $A_+ \to B$.
If A was unital already then $A_+ \cong A \oplus \mathbb{1}$ as algebras. We then have a canonical isomorphism of categories induced by the adjunction unit $A \to A_+$.

$$\mathrm{Mod}(A) \cong \mathrm{Mod}_+(A_+). \tag{5.51}$$

With the previous notion of bimodules over not necessarily unital algebras A and B, we now see $\mathrm{Mod}(A, B) = \mathrm{Mod}_+(A_+ \otimes B_+^{\mathrm{op}})$.

Lemma 5.43 (Free-Forgetful adjunction)**.** *Let A be a unital algebra in C. The forgetful functor $U : \mathrm{Mod}_+(A) \to \mathsf{C}$ has a left adjoint F.*

$$F : \mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Mod}_+(A) : U \tag{5.52}$$

$$F(C) = A \otimes C$$

We call the module $F(C)$ the free (unital) A-module generated by C.
If A is nonunital we still get a version of the free-forgetful adjunction.

$$F : \mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Mod}_+(A_+) \cong \mathrm{Mod}(A) : U \tag{5.53}$$

In this case, we call $A_+ \otimes C$ the free A-module generated by C. Similarly, the free A-B-bimodule is $A_+ \otimes B_+^{\mathrm{op}} \otimes C$.

The proofs of the above lemmata are easy and less enlightening than the statements themselves.

5.4.1 Balanced Tensor Products and Smooth Modules

Recall that by an *additive* category we mean a category enriched over Ab admitting all finite products. It follows from the axioms that finite products and coproducts agree. A *pre-abelian* category is additive and has all kernels and cokernels, hence all finite limits and colimits.

This subsection quickly discusses content that can be found with proofs and in more detail in [Mey07, A.2.6]. Self-induced algebras and smooth modules were introduced in [Mey11] and [Grø96]. Fix a pre-abelian symmetric monoidal category C. Given algebras $A, B, C \in \mathsf{C}$ as well as $M \in \mathrm{Mod}(A, B)$ and $N \in \mathrm{Mod}(B, C)$ we can form the *balanced* tensor product

$$M \otimes_B N = \mathrm{coker}\left(M \otimes B \otimes N \xrightarrow{\mu_M \otimes \mathrm{id}_N - \mathrm{id} \otimes \mu_N} M \otimes N\right). \tag{5.54}$$

In terms of elementary tensors, the map is given by $m \otimes b \otimes n \mapsto m.b \otimes n - m \otimes b.n$. Similarly, the balanced internal $\underline{\mathrm{Hom}}_A$ is given as

$$\underline{\mathrm{Hom}}_A(M, N) = \ker\left(\underline{\mathrm{Hom}}(M, N) \xrightarrow{\mu_M^* - \mu_{N*}} \underline{\mathrm{Hom}}(A \otimes M, N)\right). \tag{5.55}$$

The maps in this kernel are just those satisfying $f(a.m) = a.f(m)$. There is the following general adjointness relation.

$$\mathrm{Hom}_{A,C}(M \otimes_B N, X) \cong \mathrm{Hom}_{A,B}(M, \underline{\mathrm{Hom}}_C(N, X)). \tag{5.56}$$

Definition 5.44. An algebra A is called *self-induced* if multiplication induces an isomorphism $A \otimes_A A \cong A$.
If A is self-induced, $M \in \mathrm{Mod}(A)$ is called *smooth* if scalar multiplication induces an isomorphism $A \otimes_A M \cong M$. Denote the resulting full subcategory of smooth modules by $\mathrm{SMod}(A)$.

For a self-induced algebra A there is a *smoothening* functor

$$\begin{aligned} S = S_A : \mathrm{Mod}(A) &\to \mathrm{SMod}(A) \\ M &\mapsto A \otimes_A M \end{aligned} \tag{5.57}$$

which is right adjoint to the inclusion $\mathrm{SMod}(A) \to \mathrm{Mod}(A)$.

Lemma 5.45. *1. S is idempotent.*

2. S is right adjoint to the inclusion $\mathrm{SMod}(A) \to \mathrm{Mod}(A)$*. Scalar multiplication is the adjunction unit $SM \to M$.*

3. S is left adjoint to the roughening *functor* $R = \underline{\mathrm{Hom}}_A(A, -)$.

Proof. First note that $A \otimes_A \otimes_A M \cong A \otimes_A M$ by the associativity isomorphisms.
Scalar multiplication $SN = A \otimes_A N \to N$ induces a natural isomorphism

$$\mathrm{Hom}_A(M, SN) \xrightarrow{\cong} \mathrm{Hom}_A(M, N) \tag{5.58}$$

where the inverse is given by applying the functor S and using $SM \cong M$.

$$\begin{tikzcd} SM \arrow[r] \arrow[d, "\cong"] & SN \arrow[d] \\ M \arrow[r] & N \end{tikzcd} \tag{5.59}$$

The last claim follows from the more general Equation (5.56). □

Any A-B-bimodule M induces $M \otimes_B - : \mathrm{Mod}(B) \to \mathrm{Mod}(A)$. If M is smooth as a left A-module it maps $\mathrm{Mod}(B) \to \mathrm{SMod}(A)$ and consequently induces a functor between smooth modules:

$$A \otimes_A (M \otimes_B N) \cong (A \otimes_A M) \otimes_B N \cong M \otimes_B N \tag{5.60}$$

There is a natural isomorphism $M \otimes_B - \cong S_B(M) \otimes_B - : \mathrm{SMod}(B) \to \mathrm{SMod}(A)$:

$$M \otimes_B N \cong M \otimes_B (B \otimes_B N) \cong (M \otimes_B B) \otimes_B N \tag{5.61}$$

Hence, we will assume that M is smooth as a left A- and as a right B-module.
A C-*functor* is a functor Φ for which naturally $\Phi(M \otimes N) \cong \Phi(M) \otimes N$ and satisfying some coherence relations. The following is Proposition 5.6 in [Mey11].

Proposition 5.46 (Eilenberg-Watts)**.** *A functor* $\Phi : \mathrm{SMod}(B) \to \mathrm{SMod}(A)$ *is of the form $M \otimes_B -$ if and only if it is a C-functor and preserves cokernels.* $M \in \mathrm{SMod}(A, B)$ *is unique up to natural isomorphism.*

Proof. Let Φ satisfy the hypotheses. Then $\Phi(B) \in \mathrm{SMod}(A)$ is also a right B-module via $\Phi(B) \otimes B \cong \Phi(B \otimes B) \xrightarrow{\Phi(\mu_B)} \Phi(B)$. We now show that there is a natural isomorphism $\Phi(N) \cong \Phi(B) \otimes_B N$. Spelling out the balanced tensor product, $N \in \mathrm{SMod}(B)$ means that the multiplication induces

$$\mathrm{coker}(B \otimes B \otimes N \to B \otimes N) \xrightarrow{\cong} N\,. \tag{5.62}$$

Hence there is a chain of natural isomorphisms

$$\begin{aligned} \Phi(N) \cong \Phi(\mathrm{coker}(B \otimes B \otimes N \to B \otimes N)) &\cong \mathrm{coker}(\Phi(B \otimes B \otimes N) \to \Phi(B \otimes N)) \\ &\cong \mathrm{coker}(\Phi(B) \otimes B \otimes N \to \Phi(B) \otimes N) \\ &\cong \Phi(B) \otimes_B N\,. \end{aligned} \tag{5.63}$$

By construction, $M = \Phi(B)$ is unique up to natural isomorphism.
Conversely, it is easy to see that $M \otimes_B -$ is a C-functor. It preserves cokernels since it is a left adjoint to $S\underline{\mathrm{Hom}}_A(M, -)$. □

The following provides a characterisation of unitality in module theoretic terms. It appears in [Mey11, Prop. 3.5].

Lemma 5.47. *An algebra A is rough as a left A-module, i.e. $A \cong \underline{\mathrm{Hom}}_A(A, A)$ via the natural map if and only if A is unital.*

Proof. If A is unital, the natural map $A \to \underline{\mathrm{Hom}}_A(A, A)$ has an inverse by looking at the image of the unit. If conversely this natural map is an isomorphism, we have

$$\mathrm{Hom}(\mathbb{1}, A) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}_A(A, A)) \cong \mathrm{Hom}_A(A, A)\,. \tag{5.64}$$

The map id_A on the right hand side is thereby associated to a map $\eta : \mathbb{1} \to A$ that thus fulfills $\mathrm{id}_A = \mu \circ (\mathrm{id}_A \otimes \eta)$. We hence have a right unit. To see that it is also a left unit we can compose $\mu \circ (\eta \otimes \mathrm{id}_A) : A \to A$ with $A \to \underline{\mathrm{Hom}}_A(A, A)$. The composite is just $A \to \underline{\mathrm{Hom}}_A(A, A)$ since η is a right unit using associativity of the multiplication. Hence A indeed has a unit. □

5.4.2 The Morita Bicategory of smooth Algebras

For classical unital k-algebras there is a 2-category $\mathsf{Alg}_k^{\mathrm{bi}}$ with objects unital k-algebras and 1-morphisms A-B-bimodules (regarded as a morphism $B \to A$) and as 2-morphisms bimodule homomorphisms. Composition is given by tensoring over the intermediate algebra. The classical Eilenberg-Watts theorem shows that this is equivalent to the 2-category with objects $\mathrm{Mod}(A)$ with 1-morphisms the colimit-preserving functors and natural transformations as 2-morphisms.
We will now set up an analogue of this for self-induced algebras and smooth modules. We start with a pre-abelian symmetric monoidal category C. We will define a 2-category $\mathsf{SAlg}^{\mathrm{bi}}(\mathsf{C})$ as follows: The objects are self-induced algebras. The 1-morphisms are smooth bimodules, i.e. $\mathsf{SAlg}^{\mathrm{bi}}(A, B) = \mathrm{SMod}(B, A)$. The 2-morphisms are bimodule maps.

Proposition 5.48. $\mathsf{SAlg}^{\mathrm{bi}}(\mathsf{C})$ *is a weak 2-category.*

Proof. We have already seen that $\mathrm{SMod}(A, B)$ is a category. Horizontal composition is given by tensoring over the intermediate algebra, that is $\otimes_B : \mathrm{SMod}(A, B) \times \mathrm{SMod}(B, C) \to \mathrm{SMod}(A, C)$. For each object A we can regard A as an A-bimodule in $\mathrm{SMod}(A, A)$ which acts as an identity up to natural isomorphism $A \otimes_A M \cong M$ and $N \otimes_A A \cong N$ where the natural isomorphisms are given by scalar multiplication. Finally, there is an associator $(M \otimes_A N) \otimes_B O \to M \otimes_A (N \otimes_B O)$. It is straightforward to verify the pentagon relation for the five different bracketings of a composition of four bimodules M, N, O, P. □

Definition 5.49. Two self-induced algebras $A, B \in \mathrm{Alg}(\mathsf{C})$ are called *Morita equivalent* if they are isomorphic in $\mathsf{SAlg}^{\mathrm{bi}}(\mathsf{C})$, that is if there is a smooth A-B-bimodule P and a smooth B-A-bimodule Q such that

$$P \otimes_B Q \cong A \quad \text{and} \quad Q \otimes_A P \cong B \tag{5.65}$$

as bimodules over A and B respectively.

Remark 5.50. In the above situation we necessarily have

$$Q \cong S\underline{\mathrm{Hom}}_A(P, A)\,. \tag{5.66}$$

This is a consequence of the following: $P \otimes_B - : \mathrm{SMod}(B) \to \mathrm{SMod}(A)$ is a functor of module categories and always has a right adjoint:

$$\mathrm{Hom}_A(P \otimes_B M, N) \cong \mathrm{Hom}_B(M, \underline{\mathrm{Hom}}_A(P, N)) \cong \mathrm{Hom}_B(M, S\underline{\mathrm{Hom}}_A(P, N)) \tag{5.67}$$

Since $Q \otimes_A -$ is the inverse up to natural isomorphism it is also a right adjoint. Hence, $Q \otimes_A - \cong S\underline{\mathrm{Hom}}_A(P, -)$ naturally by uniqueness of adjoints. Plugging in A yields the claim.

5.4.3 Homological Algebra

We still work with a symmetric monoidal pre-abelian category C.

Remark 5.51. All categories we treated so far are even *quasi-abelian*. A proof for Born is in [PS01]. There are natural model structures on quasi-abelian categories. However, we will *not* use this notion or the resulting theory in the sequel. The structure of exact category that one gets this way even differs from our choice, c.f. [Mey04]. For us, it is important that free modules are projective.

Fix a pre-abelian symmetric monoidal category C and $A \in \mathrm{Alg}(\mathsf{C})$.

Definition 5.52. A *strict* epimorphism of A-modules $M \twoheadrightarrow N$ is a homomorphism that admits a section $N \to M$ which is not necessarily A-linear.
An A-module P is *projective* if it lifts along all strict epimorphisms, i.e. for any $P \to N$ and $M \twoheadrightarrow N$ there is a lift making the diagram commute:

$$\begin{tikzcd} & M \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dashed] & N \end{tikzcd} \tag{5.68}$$

Example 5.53. Free modules are projective. Let a lifting problem be given:

$$\begin{tikzcd} & M \arrow[d, two heads, "\pi"] \\ F(C) \arrow[r, "f"] & N \end{tikzcd} \tag{5.69}$$

Let $\sigma : N \to M$ be a section. $C \to F(C)$ is just the adjunction counit. Then the map $C \to M$ induced by commutativity will not be A-linear but we can use the adjunction to get an A-linear lift $F(C) \to M$:

$$\begin{tikzcd} & & M \arrow[d, two heads, "\pi"'] \\ C \arrow[r] \arrow[urr] & F(C) \arrow[r, "f"] \arrow[ur, dotted, "\exists!"] & N \arrow[u, bend right, "\sigma"'] \end{tikzcd} \tag{5.70}$$

□

Lemma 5.54 (Classification of projective modules)**.** *Let P be a module over A. The following are equivalent:*

1. *P is projective.*

2. *Every strict epimorphism $M \twoheadrightarrow P$ splits.*

3. P is a direct summand of a free module.

Proof. If P is projective and $M \twoheadrightarrow P$ then there exists a lift of the identity $\mathrm{id} : P \to P$. This is the desired splitting.
Consider the identity of P. Using the adjunction unit it fits into the following diagram where we can see P as a direct summand of $F(P)$ explicitly:

$$\begin{tikzcd} & F(P) \arrow[d, two heads] \\ P \arrow[ur, hook] \arrow[r, equal] & P \end{tikzcd} \tag{5.71}$$

Any direct summand E of a projective module Q is projective by the following diagram.

$$\begin{tikzcd} & & & M \arrow[d, two heads] \\ E \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & Q \arrow[r] \arrow[urr, dashed] & E \arrow[r] & N \end{tikzcd} \tag{5.72}$$

Since free modules are projective the claim follows. $\square$

For the purposes of homological algebra in arbitrary categories we need to replace exactness by contractibility as follows:

Definition 5.55. A projective (resp. free) *resolution* of an A-module M is a resolution of M by a chain complex of projective (resp. free) A-modules $P_i, i \in \mathbb{N}$ such that the augmented complex is contractible via a chain homotopy $s_i : P_i \to P_{i+1}$, i.e.

$$d_{i+1}s_i + s_{i-1}d_i = \mathrm{id}\,. \tag{5.73}$$

$$\begin{tikzcd} 0 & M \arrow[l] \arrow[r, bend left, "s_{-1}"] & P_0 \arrow[l, "d_0"] \arrow[r, bend left, "s_0"] & P_1 \arrow[l, "d_1"] \arrow[r, bend left, "s_1"] & P_2 \arrow[l, "d_2"] \arrow[r, bend left, "s_2"] & \dots \arrow[l] \end{tikzcd}$$

The homotopies are *not* A-linear in general, but they need to be morphisms in the ambient category.

Consequently, a sequence $X \rightarrowtail Y \twoheadrightarrow Z$ is short exact if and only if the right map is split (not necessarily A-linear) and X is the kernel of the right map. This gives us an exact category structure.

Lemma 5.56 (Existence of Free Resolutions)**.** *Any A-module M admits a free (in particular projective) resolution.*

Proof. This is an inductive construction as in the following diagram where we define P_i to be free/projective such that $P_i \twoheadrightarrow \ker(d_{i-1})$ is a strict epimorphism.

$$\begin{tikzcd} & & & \ker(d_0) \arrow[dl, hook] & & & \dots \arrow[dl, hook] \\ 0 & M \arrow[l] & P_0 \arrow[l, two heads, "d_0"'] & & P_1 \arrow[ll, "d_1"'] \arrow[ul, two heads] & & P_2 \arrow[ll, "d_2"'] \arrow[dl, two heads] & \dots \arrow[l] \\ & & & & & \ker(d_1) \arrow[ul, hook] \end{tikzcd} \tag{5.74}$$

Let $s_{-1} : M \to P_0$ be a section. Then $d_0 s_{-1} = \mathrm{id}$, which is Equation (5.73) for $i = -1$. Now proceed inductively. The map $p_0 = \mathrm{id} - s_{-1}d_0 : P_0 \to P_0$ is a projection onto $\ker(d_0)$. Hence, $\ker(d_0)$ is a direct summand of P_0. Let σ_0 be a section of $d_1 : P_1 \twoheadrightarrow \ker(d_0)$. Extend this by zero to a map $s_0 : P_0 \to P_1$. In formulas this reads

$$s_0 = \sigma_0 \circ (\mathrm{id} - s_{-1}d_0). \tag{5.75}$$

Furthermore

$$d_1 s_0 + s_{-1}d_0 = d_1 \circ \sigma_0 \circ (\mathrm{id} - s_{-1}d_0) + s_{-1}d_0 = \mathrm{id} \tag{5.76}$$

The induction proceeds in the same fashion. Existence of the s_i crucially depends on d_{i+1} to be a strict epimorphism onto $\ker(d_i)$, i.e. admitting a section. □

Lemma 5.57. *Let $f : M \to N$ be a homomorphism of modules over A. Let $P_\bullet \to M, Q_\bullet \to N$ be projective resolutions. Then there is an extension of f to a chain map of augmented complexes. Any two such extensions are chain homotopic.*

Proof. Since $d_0 : Q_0 \to N$ is a strict epimorphism and P_0 is projective, $f \circ d_0$ lifts to a map $f_0 : P_0 \to Q_0$.
Since $d_1 : Q_1 \to \ker(d_0)$ is a strict epimorphism by Equation (5.73) and P_1 is projective $f_0 \circ d_1$ lifts along d_1 to a map $f_1 : P_1 \to Q_1$. Now continue this inductively.

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "d_0"'] \arrow[d, dotted, "f_0"] & P_1 \arrow[l, "d_1"'] \arrow[d, dotted, "f_1"] & P_2 \arrow[l, "d_2"'] \arrow[d, dotted, "f_2"] & \dots \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l, "d_0"] & Q_1 \arrow[l, "d_1"] & Q_2 \arrow[l, "d_2"] & \dots \arrow[l]
\end{tikzcd} \tag{5.77}$$

Suppose we are given two chain maps f_i, g_i extending f.

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "d_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & P_1 \arrow[l, "d_1"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & P_2 \arrow[l, "d_2"'] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "g_2"] & \dots \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l, "d_0"] & Q_1 \arrow[l, "d_1"] & Q_2 \arrow[l, "d_2"] & \dots \arrow[l]
\end{tikzcd} \tag{5.78}$$

By commutativity, $f_0 - g_0$ takes values in $\ker(d_0)$ and $d_1 : Q_1 \to \ker(d_0)$ is a strict epimorphism. Hence, their difference lifts to a map $h_0 : P_0 \to Q_1$ such that

$$d_1 h_0 = f_0 - g_0. \tag{5.79}$$

Going further, $f_1 - g_1 - h_0 d_1$ takes values in $\ker(d_1)$ since

$$d_1 \circ (f_1 - g_1 - h_0 d_1) = (f_0 - g_0) \circ d_1 - d_1 h_0 \circ d_1 = 0. \tag{5.80}$$

Again we get a lift $h_1 : P_1 \to Q_2$ that then satisfies

$$d_2 h_1 + h_0 d_1 = f_1 - g_1 \tag{5.81}$$

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "d_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \arrow[dr, dotted, "h_0"] & P_1 \arrow[l, "d_1"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] \arrow[dr, dotted, "h_1"] & P_2 \arrow[l, "d_2"'] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "g_2"] & \dots \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l, "d_0"] & Q_1 \arrow[l, "d_1"] & Q_2 \arrow[l, "d_2"] & \dots \arrow[l]
\end{tikzcd} \tag{5.82}$$

□

Corollary 5.58 (Whitehead theorem)**.** *Projective resolutions are unique up to chain homotopy equivalence.*

Denote the category of chain complexes of A-modules with morphisms the homotopy classes of chain maps by $\mathrm{Ho}(A)$. The above corollary says that taking projective resolutions is a functor $\mathrm{Mod}(A) \to \mathrm{Ho}(A)$. Denote $\mathrm{Ho} = \mathrm{Ho}(\mathbb{1})$.

5.4.4 Total Derived Functors

We want to define here the derived functors $\mathbb{R}\operatorname{Hom}_A, \otimes^{\mathbb{L}}_A$. For our purposes their domain need not be the entire derived category. The relevant theory for this is covered in detail in [Mey04]. By the Whitehead theorem, taking projective resolutions provides a functor

$$\operatorname{Mod}(A) \to \operatorname{Ho}(A) \tag{5.83}$$

Hence, define for $M, N \in \operatorname{Mod}(A)$ and $P(M) \to M$ a projective resolution:

$$\mathbb{R}\operatorname{Hom}_A(M, N) := \operatorname{Hom}_A(P(M), N) \tag{5.84}$$

This is a chain complex and well-defined up to isomorphism in Ho. Similarly for $M \in \operatorname{Mod}(A^{\mathrm{op}})$ and $N \in \operatorname{Mod}(A)$ define:

$$M \otimes^{\mathbb{L}}_A N = P(M) \otimes_A N \tag{5.85}$$

By homotopy equivalence of projective resolutions we get functors:

$$\mathbb{R}\operatorname{Hom}_A : \operatorname{Mod}(A)^{\mathrm{op}} \times \operatorname{Mod}(A) \to \operatorname{Ho} \tag{5.86}$$
$$\otimes^{\mathbb{L}}_A : \operatorname{Mod}(A^{\mathrm{op}}) \times \operatorname{Mod}(A) \to \operatorname{Ho} \tag{5.87}$$

If there is a forgetful functor $\mathsf{C} \to \mathsf{Vect}$ or to Ab we can also take cohomology via

$$\operatorname{Ext}_A(M, N) := H^n(\mathbb{R}\operatorname{Hom}_A(M, N)) \tag{5.88}$$
$$\operatorname{Tor}_A(M, N) := H_n(M \otimes^{\mathbb{L}}_A N) \tag{5.89}$$

Since (co)homology is homotopy invariant, these are well-defined abelian groups/vector spaces up to natural isomorphism. For example, for CBorn, cohomology is taken in Vect.

Remark 5.59. There is a problem one wants to forego by not taking cohomology. One point is losing data. More important is that the cohomology depends on whether we quotient out the image $\operatorname{Im}(d)$ or its closure. There is no problem if the image is closed, e.g. in the case of contractible complexes.

5.4.5 Smoothly projective Modules and Quasi-Unitality

Definition 5.60. An algebra A is called *quasi-unital* if the multiplication $A \otimes A \to A$ admits sections by a left and by a right A-module homomorphism.

Clearly, this also provides section of the multiplications $A \otimes A_+ \to A$ and $A_+ \otimes A \to A$. Note that A is quasi-unital if and only if it is projective as a left and a right module over itself.

Example 5.61.

1. Any unital algebra A is quasi-unital via the sections $a \mapsto 1 \otimes a$ and $a \mapsto a \otimes 1$.

2. The bornological algebra $C_c^\infty(M)$ with pointwise multiplication is quasi-unital. To see this, choose a partition of unity $\sum \lambda_i^2 = 1$. Then $C_c^\infty(M) \otimes C_c^\infty(M) \cong C_c^\infty(M \times M)$ and under this identification multiplication is restriction to the diagonal.

$$\sigma(f)(x, y) = \sum_{i,j} \lambda_i(x) f(x) \lambda_j(y)$$

provides a left $C_c^\infty(M)$-linear section of the multiplication.

Remark 5.62. Our definition of quasi-unitality does not include approximate units as the one in [Mey04] for example. They do exist in the groupoid case, but were not important to any of the proofs, so we chose to omit it.

Note, that if A is quasi-unital, it is in particular self-induced. For the following let A, B be self-induced algebras in C.

Definition 5.63. A module $P \in \mathrm{Mod}(A)$ is called *smoothly projective* if it has the right lifting property of Diagram 5.68 against all strict epimorphisms of smooth modules.

There is one very special class of projective modules, namely the free ones. We concluded above that $A \otimes M$ is not free and does not have a universal property in general. We can remedy this situation in the setting of smooth modules.

Lemma 5.64. *$S(A_+) = A$. Moreover, $S(A_+ \otimes M) \cong A \otimes M$ and we get a kind of free-forgetful adjunction for $N \in R\,\mathrm{Mod}(A)$, i.e. $N \cong \mathrm{Hom}_A(A, N)$:*

$$\mathrm{Hom}_A(A \otimes M, N) \cong \mathrm{Hom}(M, N) \tag{5.90}$$

Proof. We have the following commutative diagram:

$$\begin{tikzcd} & A & \\ A \otimes_A A \arrow[ur, "\mu"] \arrow[rr] \arrow[ur, phantom, "\cong" near end, shift right=3] & & A \otimes_A A_+ \arrow[ul, "\mu"'] \end{tikzcd} \tag{5.91}$$

The lower map is an isomorphism with inverse given by the roof: By commutativity, the composite is the identity on $A \otimes_A A$. For $A \otimes_A A_+$, we can compute the composition on elements $a \otimes (b, \lambda)$. Going around the triangle, let $a_{(1)} \otimes a_{(2)} \in A \otimes_A A$ with $\mu(a_{(1)} \otimes a_{(2)}) = ab + \lambda a$. Then the inclusion at the bottom maps to

$$\begin{aligned} a_{(1)} \otimes (a_{(2)}, 0) &= a_{(1)}a_{(2)} \otimes (0,1) = ab + \lambda a \otimes (0,1) \\ &= a \otimes (b, 0) + a \otimes (0, \lambda) = a \otimes (b, \lambda)\,. \end{aligned} \tag{5.92}$$

Hence, going around the triangle is indeed the identity and $S(A_+) \cong A$. The rest of the statement follows formally from the fact that S is a left adjoint to R. □

Lemma 5.65. *If A is quasi-unital, then a module is smoothly projective if and only if it is projective.*

Proof. Let $P \in \mathrm{SMod}(A)$ be smoothly projective. Then $A \otimes P \twoheadrightarrow P$ splits and hence P is a direct summand of the smoothly free module $A \otimes P$. But A is a direct summand of $A_+ \otimes A$. Hence, P is a direct summand of the free module $A_+ \otimes A \otimes P$. This shows projectivity. □

Lemma 5.66. *Let A, B be quasi-unital algebras. Let $Q \in \mathrm{SMod}(A, B)$. Then $Q \otimes_B -$ preserves smoothly projective modules if and only if Q is smoothly projective as a left A-module.*

Proof. Since $Q \otimes_B B \cong Q$ this is a necessary condition. Let now Q be smoothly projective. Q is a direct summand of $A \otimes Q$ as an A-module. Any smoothly projective B-module P is a direct summand of $B \otimes P$. Then $Q \otimes_B P$ is a direct summand of $Q \otimes_B B \otimes P \cong Q \otimes P$. This again is a direct summand of the smoothly free module $A \otimes Q \otimes P$. Hence, $Q \otimes_B P$ is projective. □

For arbitrary bimodules $P \in \mathrm{SMod}(A, B)$ it is not clear that $Q \otimes_B -$ preserves the exact category structure, i.e. that it maps strict epimorphisms to strict epimorphisms and kernels to kernels.

Lemma 5.67. *Let A, B be quasi-unital algebras.*

1. *If $P \in \mathrm{SMod}(A, B)$ is projective as a right B-module, $P \otimes_B -$ maps strict epimorphisms to strict epimorphisms.*

2. *If $P \in \mathrm{SMod}(A, B)$ and $Q \in \mathrm{SMod}(B, A)$ induce a Morita equivalence between A and B then $P \otimes_B -$ maps kernels to kernels.*

Proof. Let P be projective as a right B-module. Then P is a direct summand of $P \otimes B$. Let $f : M \twoheadrightarrow N$ be a strict epimorphism and $s : N \to M$ a linear section. Going around the following diagram gives a linear section of $\mathrm{id} \otimes f : P \otimes_B M \to P \otimes_B N$.

$$\begin{tikzcd}
P \otimes M \arrow[d, "\mathrm{id}\otimes f"] & P \otimes B \otimes_B M \arrow[l, "\mathrm{id}\otimes\mu_M"', "\cong"] \arrow[r] & P \otimes_B M \arrow[d, "\mathrm{id}\otimes f"] \\
P \otimes N \arrow[u, bend left, "\mathrm{id}\otimes s"] & P \otimes B \otimes_B N \arrow[l, "\mathrm{id}\otimes\mu_N"', "\cong"] \arrow[r] & P \otimes_B N \arrow[l, bend right]
\end{tikzcd} \tag{5.93}$$

Hence, $P \otimes_B -$ preserves strict epimorphisms.
Let now P, Q be a Morita equivalence. Then $P \otimes_B -$ is an equivalence of categories. Let $f : M \to N$ be a homomorphism of B-modules. Then $\ker(f)$ is the limit or pullback of the diagram $M \to N \leftarrow 0$. Since P preserves limits, $P \otimes_B \ker(f) \cong \ker(\mathrm{id} \otimes f)$. $\square$

Lemma 5.68. *Let A, B be quasi-unital. Let $P \in \mathrm{SMod}(A, B)$ and $Q \in \mathrm{SMod}(B, A)$ be bimodules inducing a Morita equivalence between A and B. Assume additionally that P, Q are projective as right modules. Then P, Q map projective resolutions to projective resolutions.*

Proof. Note that under these conditions, P maps projectives to projectives, kernels to kernels and strict epimorphisms to strict epimorphism. This means it preserves admissible short exact sequences. A resolution is a resolution if and only if the constituent short sequences are admissible short exact sequences. Hence, P preserves resolutions. $\square$

In the previous situation we say that the quasi-unital algebras A, B are *projectively Morita-equivalent.*

Remark 5.69. We do not know whether we could have phrased all of the homological algebra solely in SMod. The principal problem is that kernels of maps are not necessarily smooth. Hence, we need to phrase lifting properties with respect to all modules, not just the smooth ones to obtain Whitehead's theorem.

5.4.6 The Bar Complex and Quasi-Unitality

The usual proof that the bar complex is acylic crucially uses the unit of the algebra. There is the notion of H-unitality which means that the bar complex is exact. We are not content with this, since this does not yield a continuous [11] contraction and hence not a resolution in our sense. Also, it is not clear in general whether the constituent modules are projective.

[11] Of course continuity does not make sense in general. We just have to keep in mind that the contraction needs to consist of morphisms in the ambient category C.

Lemma 5.70. *If A is quasi-unital, then the following Bar complex $(\mathrm{Bar}(A), b')$ is a projective resolution of A as an A-bimodule:*

$$\ldots \to (A \otimes A^{\mathrm{op}}) \otimes A^{\otimes n} \overset{b'}{\to} (A \otimes A^{\mathrm{op}}) \otimes A^{\otimes n-1} \overset{b'}{\to} \ldots \overset{b'}{\to} (A \otimes A^{\mathrm{op}}) \otimes A \to A \otimes A^{\mathrm{op}} \to 0 \tag{5.94}$$

where

$$\begin{aligned} b'(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = aa_1 \otimes b \otimes a_2 \otimes \cdots \otimes a_n \\ + \sum_{i=1}^{n-1} (-1)^i a \otimes b \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^n a \otimes a_n b \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned} \tag{5.95}$$

and the augmentation $\mu : A \otimes A^{\mathrm{op}} \to A$ is multiplication.
If we rearrange the factors of the tensor product to $A \otimes A^{\otimes n} \otimes A$ then the differential takes the easier form

$$b'(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \tag{5.96}$$

Proof. Note that the maps b', μ are A-bimodule homomorphisms. We also get that $A \otimes M \otimes A^{\mathrm{op}}$ is always a projective A-bimodule since it is free if A has a unit or a direct summand of the free module $A_+ \otimes A \otimes M \otimes A \otimes A_+$ via the sections of the multiplication. We are left to show the existence of a contraction of the augmented complex.
Let $\sigma : A \to A \otimes A$ be a section of the multiplication μ that is also a right A-module homomorphism. Define $s_{-1} = \sigma$. Then $\mu s_{-1} = \mathrm{id}$. Now define $s_n : A^{\otimes n+2} \to A^{\otimes n+3}$ by

$$s_n(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = a_{(1)} \otimes b \otimes a_{(2)} \otimes a_1 \otimes \cdots \otimes a_n. \tag{5.97}$$

Here we have used a Sweedler-like notation to write $\sigma(a) = \sum a_{(1)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}$. Note that this in general will neither be a sum nor a finite sum due to the nature of completed tensor products. It is just a notational convenience.
Then we arrive at

$$\begin{aligned} s_{n-1} b'(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = (aa_1)_{(1)} \otimes b \otimes (aa_1)_{(2)} \otimes a_2 \otimes \cdots \otimes a_n \\ + \sum_{i=1}^{n-1} (-1)^i a_{(1)} \otimes b \otimes a_{(2)} \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^n a_{(1)} \otimes a_n b \otimes a_{(2)} \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned} \tag{5.98}$$

$$\begin{aligned} b' s_n(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = a_{(1)} a_{(2)} \otimes b \otimes a_1 \otimes \cdots \otimes a_n - a_{(1)} \otimes b \otimes a_{(2)} a_1 \otimes \cdots \otimes a_n \\ + \sum_{i=2}^{n} (-1)^i a_{(1)} \otimes b \otimes a_{(2)} \otimes a_1 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n \\ + (-1)^{n+1} a_{(1)} \otimes a_n b \otimes a_{(2)} \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned} \tag{5.99}$$

Then $b's + sb' = \mathrm{id}$ using the identities $a_{(1)} a_{(2)} = \mu(\sigma(a)) = a$ and $(aa_1)_{(1)} \otimes (aa_1)_{(2)} = \sigma(aa_1) = \sigma(a) a_1 = a_{(1)} \otimes a_{(2)} a_1$. □

To rephrase the lemma, if A is quasi-unital, then the following augmented bar complex is contractible:

$$\ldots \longrightarrow A^{\otimes n} \xrightarrow{b'} \ldots \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \longrightarrow 0 \tag{5.100}$$

We call this property *strong H-unitality.* H-unitality requires this complex to only be exact in the algebraic sense. (for categories with a forgetful functor to Ab.)

5.4.7 Strong H-unitality for Convolution Algebras

The following is a strengthening of the H-unitality result in [CM01]. Their result states that for general Lie groupoids, the convolution algebra is H-unital.

Proposition 5.71. *For a proper Lie groupoid G, the smooth bornological groupoid convolution algebra $\mathcal{A}_G$ is quasi-unital, i.e. especially strongly H-unital.*

Let us also record the following corollary.

Corollary 5.72. *For a proper Lie groupoid G, the bar complex* $\mathrm{Bar}(\mathcal{A}_G)$ *is a projective resolution as a bimodule over itself.*

The place where the proof breaks down generally is the existence of a special function λ. This cannot be fixed and exists if and only if G is proper.

Proof. We will only construct a left A-linear section of the multiplication. The opposite section is constructed completely analogously. The proof involves two steps, since the multiplication can be viewed as the composition of left A-module homomorphisms:

$$C_c^\infty(G \times G) \xrightarrow{\iota^*} C_c^\infty(G^{(2)}) \xrightarrow{\mu} C_c^\infty(G) = \mathcal{A}_G \tag{5.101}$$

Here we identified $\mathcal{A}_G \otimes \mathcal{A}_G \cong C_c^\infty(G \times G)$ as bornological spaces. The algebra multiplication is just restriction along the inclusion $\iota : G^{(2)} \hookrightarrow G^2$ followed by the fiber integration given by:

$$\begin{aligned} \mu : C_c^\infty(G^{(2)}) \longrightarrow & C_c^\infty(G) \\ F \longmapsto & \left[g \mapsto \int_{t(h)=g} F(h, h^{-1}g) \mathrm{d}h \right]. \end{aligned} \tag{5.102}$$

Now choose a function $\lambda \in C^\infty(G)$ with compact support in the fiber direction such that

$$\int_{t^{-1}(t(g))} \lambda(g^{-1}h) \mathrm{d}h = 1 \quad \forall g \in G. \tag{5.103}$$

Phrasing the support condition properly, we demand that $t : \mathrm{supp}(\lambda) \to G^{(0)}$ is a proper map. Such a function λ is only guaranteed to exist for proper groupoids. (Proposition 8.1) Then define the map $\sigma \colon \mathcal{A}_G \to C_c^\infty(G^{(2)})$ by

$$\sigma(f)(g, h) = f(gh)\lambda(h^{-1}). \tag{5.104}$$

Indeed, the function $\sigma(f)$ has compact support in $t(\mathrm{supp} f) \times_{G^{(0)}} (t^{-1}(s(\mathrm{supp}(f))) \cap \mathrm{supp}(\lambda))^{-1}$ and σ is a left $\mathcal{A}_G$-linear section for μ:

$$(\mu(\sigma(f)))(g) = \int_{t(h)=g} f(hh^{-1}g)\lambda(g^{-1}h) \mathrm{d}h = f(g) \tag{5.105}$$

$$\begin{aligned} \sigma(\alpha * f)(g, h) &= \int \alpha(k) f(k^{-1}gh)\lambda(h^{-1}) \mathrm{d}k \\ &= (\alpha.\sigma(f))(g, h) \end{aligned} \tag{5.106}$$

The second step is to find a section of $\iota^* \colon C_c^\infty(G \times G) \to C_c^\infty(G^{(2)})$. This is patched together from local retractions. Consider $M \times_M G \hookrightarrow M \times G$.

$$\begin{tikzcd} & G \arrow[d, "t"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd} \tag{5.107}$$

By Lemma 5.73 we can find for each $(x, g) \in M \times_M G$ an open neighbourhood $U \subset M \times G$ and a retraction $r : U \to U \cap (M \times_M G)$ making the following diagram commute:

$$\begin{tikzcd} M \times G \supset U \arrow[rr, "r"] \arrow[dr, "\mathrm{pr}_M"'] & & M \times_M G \arrow[dl, "\mathrm{pr}_M"] \\ & M & \end{tikzcd} \tag{5.108}$$

Now cover $M \times_M G$ by a locally finite covering $\{U_i\}_{i\in\mathbb{N}}$ with a subordinate partition of unity $\{\chi_i\}_{i\in\mathbb{N}}$ such that each U_i admits a local retraction r_i as above. By Equation (5.108) we can write $r_i(m, g) = (m, \tilde{r}_i(m, g))$ and $t(\tilde{r}_i(m, g)) = m$. Since r_i is a retraction, $\tilde{r}_i(m, h) = h$ if and only if $m = t(h)$. Now define a section of the restriction by

$$\sigma(F)(g, h) = \sum_i F\left(g, \tilde{r}_i(s(g), h)\right) \chi_i(s(g), h). \tag{5.109}$$

The support of each summand is compact since it is contained in $\mathrm{pr}_1(\mathrm{supp}F) \times \mathrm{pr}_2(\mathrm{supp}\chi_i)$. Only finitely many summands occur for each F, since $\mathrm{pr}_2(\mathrm{supp}f)$ intersects only finitely many U_i. Hence, $\sigma(F)$ has compact support indeed. The section is also continuous (bounded) since it is locally a finite sum. Clearly, $\iota^* \circ \sigma = \mathrm{id}$ since $\tilde{r}_i \circ \iota(g, h) = h$ and the χ_i then sum to one. Finally, it is a left $\mathcal{A}_G$-module homomorphism:

$$\begin{aligned} (\alpha.\sigma(F))(g, h) &= \int \alpha(k) \sum_i F\left(k^{-1}g, \tilde{r}_i(s(k^{-1}g), h)\right) \chi_i\left(s(k^{-1}g), h\right) \mathrm{d}k \\ &= \sum_i \int \alpha(k) F\left(k^{-1}g, \tilde{r}_i(s(g), h)\right) \mathrm{d}k \cdot \chi_i\left(s(g), h\right) \\ &= \sigma(\alpha.F)(g, h), \quad \forall \alpha \in \mathcal{A}_G \quad \square \end{aligned} \tag{5.110}$$

Lemma 5.73. *Let $t : N \to Z$ be a submersion and $f : M \to Z$ be an arbitrary smooth map. Then there are local retractions to $M \times_Z N$. That is, for every point $(m, n) \in M \times_Z N$ there is an open neighbourhood U in $M \times N$ and a retraction $r : U \to U \cap (M \times_Z N)$ making the following diagram commute:*

$$\begin{tikzcd} M \times N \supset U \arrow[rr, "r"] \arrow[dr, "\mathrm{pr}_M"'] & & M \times_Z N \arrow[dl, "\mathrm{pr}_M"] \\ & M & \end{tikzcd} \tag{5.111}$$

Proof. Fix $(m, n) \in M \times_Z N$. Then there are local charts on a neighbourhood W of $n \in N$ and $t(n) \in Z$ such that t is just $\mathbb{R}^{p+q} \ni (x_0, x_1) \mapsto x_0 \in \mathbb{R}^p$. Since t is an open map, $V := f^{-1}(t(V))$ is open. We can now define $U := V \times W \to V \times_Z W$ by $(y, x_0, x_1) \mapsto (y, f(y), x_1)$. This is a retraction by construction and also makes the above diagram commute. $\square$

5.4.8 Bar complex for self-induced Algebras

Let A be self-induced. Then $A \otimes A^{\mathrm{op}}$ is also self-induced. The enhanced Bar complex $\mathrm{Bar}_+(A) = \big((A_+ \otimes A_+^{\mathrm{op}}) \otimes A^{\otimes n}, b'\big)$ is a free A_+-bimodule resolution of A_+ with augmentation given by multiplication. The proof for this is almost verbatim as in Lemma 5.70. One uses the section

$$s\left((a+\lambda) \otimes (b+\mu) \otimes a_1 \otimes \cdots \otimes a_n\right) = 1 \otimes (b+\mu) \otimes a \otimes a_1 \otimes \cdots \otimes a_n\,. \tag{5.112}$$

See also [Mey07, A.5.2] for a different construction of the Bar complexes.

Lemma 5.74. *The smoothening of the enhanced bar complex is the ordinary bar complex.*

$$S\,\mathrm{Bar}_+(A) \cong \mathrm{Bar}(A) \tag{5.113}$$

In particular, $\mathrm{Bar}(A)$ *is a smoothly projective (even smoothly free) resolution of* A *as an* A*-bimodule.*

As shown before, if A is quasi-unital additionally, then $\mathrm{Bar}(A)$ is even a projective resolution.

Proof. The smoothening here is with respect to $A \otimes A^{\mathrm{op}}$. Degreewise we obtain $S(A_+ \otimes A^{\otimes n} \otimes A_+) \cong A \otimes A^{\otimes n} \otimes A$ with the isomorphism induced by the multiplication $A_+ \otimes_A A \xrightarrow{\cong} A$. An easy computation shows commutativity of the following diagrams:

$$\begin{array}{ccc} A \otimes_A A_+ \otimes A^{\otimes n} \otimes A_+ \otimes_A A & \xrightarrow{\mathrm{id}_A \otimes b' \otimes \mathrm{id}_A} & A \otimes_A A_+ \otimes A^{\otimes n-1} \otimes A_+ \otimes_A A \\ \Big\downarrow{\scriptstyle \mu \otimes \mathrm{id}_A^{\otimes n} \otimes \mu} & & \Big\downarrow{\scriptstyle \mu \otimes \mathrm{id}_A^{\otimes n-1} \otimes \mu} \\ A \otimes A^{\otimes n} \otimes A & \xrightarrow{\quad b' \quad} & A \otimes A^{\otimes n-1} \otimes A \end{array} \tag{5.114}$$

This is the desired natural isomorphism of chain complexes $S\,\mathrm{Bar}_+(A) \to \mathrm{Bar}(A)$. $\square$

5.5 Hochschild Cohomology

The Hochschild cochain complex of an algebra A with values in a bimodule M is the complex

$$C^n(A,M) = \mathrm{Hom}(A^{\otimes n}, M) \tag{5.115}$$

with differential

$$\begin{aligned} df(a_1 \otimes \dots a_{n+1}) = a_1 f(a_2 \otimes \dots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}) \\ + (-1)^{n+1} f(a_1 \otimes \dots \otimes a_n) a_{n+1}. \end{aligned} \tag{5.116}$$

When we are working in a cartesian closed (symmetric monoidal pre-abelian) category we want to be working instead with

$$C^n(A,M) = \underline{\mathrm{Hom}}(A^{\otimes n}, M)\,. \tag{5.117}$$

This ensures that we do not lose information (e.g. we can still talk about smooth families of cochains when working in CBorn). In the particular case of CBorn the cocycles are identified with k-multilinear bounded maps $A^{\times k} \to M$. Note that in particular we consider only continuous or bounded cocycles. For Vect this reproduces the standard definition of Hochschild cohomology. When we say "Hochschild cohomology of A" we usually mean $H^*(A,A)$ as opposed to the functor $H^*(A,A^*)$. The former is more interesting for deformation theory. The latter is often used in cyclic theory.

We can now formulate a purely algebraic question.

Question 5.75. Is Hochschild cohomology Morita invariant? That is, let $A, B \in \mathsf{SAlg}^{\mathrm{bi}}(\mathsf{C})$ be Morita equivalent self-induced algebras. Is it true that $H^*(A,A) \cong H^*(B,B)$ are isomorphic or that $C^*(A,M) \simeq C^*(A,N)$ are homotopy equivalent?

5.5.1 The Case of unital Algebras

For the moment we want to assume unitality of A. Then we can consider the Hochschild cohomology complex $C^*(A,M)$. It is easy to use now the free-forgetful adjunction:

$$\begin{aligned} \mathrm{Hom}(A^{\otimes n}, M) &\cong \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A \otimes A^{\mathrm{op}} \otimes A^{\otimes n}, M) \\ &\cong \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(\mathrm{Bar}(A)_n, M) \end{aligned} \tag{5.118}$$

As we compute below, the differentials agree. Since $\mathrm{Bar}(A)$ is a projective resolution, the cohomology of the complexes computes the derived functor Ext. To phrase this differently - without taking cohomology of $\mathbb{R}\,\mathrm{Hom}$ - *any* projective resolution is homotopy-equivalent to $\mathrm{Bar}(A)$ and hence can be used instead for computation of $\mathbb{R}\,\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A,M)$. We record:

Proposition 5.76. *For unital algebras A, we have $C^n(A,M) \simeq \mathbb{R}\,\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A,M)$ and $H^n(A,M) \cong \mathrm{Ext}^n_{A \otimes A^{\mathrm{op}}}(A,M)$.*

5.5.2 The Case of nonunital Algebras

We have to make a couple of new definitions here. From now on we do not assume A to have a unit. Recall the enhanced bar complex $\mathrm{Bar}_+(A)$ which is given by $A_+ \otimes A^{\otimes n} \otimes A_+$ in degree n. Using the forgetful free adjunction again we arrive at:

$$\mathrm{Hom}(A^{\otimes n}, M) \cong \mathrm{Hom}_{A_+ \otimes A_+^{\mathrm{op}}}(A_+ \otimes A_+^{\mathrm{op}} \otimes A^{\otimes n}, M) \cong \mathrm{Hom}_{A_+ \otimes A_+^{\mathrm{op}}}(\mathrm{Bar}_+(A), M) \tag{5.119}$$

A short computation shows that the differentials agree: For $f : A^{\otimes n} \to M$ denote by $\hat{f} : A_+ \otimes A^{\otimes n} \otimes A_+ \to M$ the A_+-bilinear map from the adjunction that is given by

$$\hat{f}(a_0, \ldots, a_{n+1}) = a_0 f(a_1, \ldots, a_n) a_{n+1} \,. \tag{5.120}$$

$$\begin{aligned}
\widehat{df}(a_0, \ldots, a_{n+2}) &= a_0 df(a_1, \ldots, a_{n+1}) a_{n+2} \\
&= a_0 a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) a_{n+2} \\
&\quad + \sum_{i=1}^{n} (-1)^i a_0 f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) a_{n+2} \\
&\quad + (-1)^{n+1} a_0 f(a_1 \otimes \cdots \otimes a_n) a_{n+1} a_{n+2} \\
&= \hat{f}(a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2}) \\
&\quad + \sum_{i=1}^{n} (-1)^i \hat{f}(a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \otimes a_{n+2}) \\
&\quad + (-1)^{n+1} \hat{f}(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} a_{n+2}) \\
&= \hat{f}(b'(a_0 \otimes \cdots \otimes a_{n+2}))
\end{aligned} \tag{5.121}$$

Hence, Hochschild cohomolog computes the derived functors $\mathbb{R}\operatorname{Hom}$ or Ext.

Proposition 5.77. *For nonunital A we have*

$$C^*(A, M) \cong \mathbb{R}\operatorname{Hom}_{A_+ \otimes A_+^{\mathrm{op}}}(A_+, M) \tag{5.122}$$

$$H^n(A, M) \cong \operatorname{Ext}^n_{A_+ \otimes A_+^{\mathrm{op}}}(A_+, M) \tag{5.123}$$

To relate this to the previously discussed unital algebras A, note that $A_+ \cong A \oplus \mathbb{1}$ as algebras and as A_+-bimodules. Therefore $\mathbb{1}$ is a projective A_+-bimodule if A is unital. Also free unital A-bimodules are still projective as A_+-bimodules. Hence $\operatorname{Bar}(A) \oplus \mathbb{1}$ is a projective resolution of A_+ by bimodules, where we interpret $\mathbb{1}$ as a complex concentrated in degree zero. As such it must be homotopy equivalent to $\operatorname{Bar}_+(A)$. Also having a unit enforces

$$\operatorname{Hom}_{A_+ \otimes A_+^{\mathrm{op}}}(\mathbb{1}, M) = 0, \tag{5.124}$$

since we are dealing with the trivial A-action on $\mathbb{1}$ and $f(\lambda) = f(\lambda) \cdot 1 = f(\lambda \cdot 1) = f(0) = 0$. Hence

$$\begin{aligned}
\underline{\operatorname{Hom}}_{A_+ \otimes A_+^{\mathrm{op}}}(\operatorname{Bar}_+(A), M) &\simeq \underline{\operatorname{Hom}}_{A_+ \otimes A_+^{\mathrm{op}}}(\operatorname{Bar}(A), M) \oplus \underline{\operatorname{Hom}}_{A_+ \otimes A_+^{\mathrm{op}}}(\mathbb{1}, M) \\
&\cong \underline{\operatorname{Hom}}_{A_+ \otimes A_+^{\mathrm{op}}}(\operatorname{Bar}(A), M) \,.
\end{aligned} \tag{5.125}$$

Henceforth we will drop the $+$-subscript in the Hom-sets, i.e. we will write $\operatorname{Hom}_A = \operatorname{Hom}_{A_+}$. This is consistent with the fact that A-module homomorphisms are the same as A_+-module homomorphisms.

5.5.3 The Case of self-induced and quasi-unital Algebras

Similarly to having a unit, being self-induced or even having a quasi-unit also further simplifies matters.

Lemma 5.78. *If A is self-induced and $M \cong RM$ is a rough bimodule, we have*

$$C^*(A, M) \cong \underline{\operatorname{Hom}}_{A \otimes A^{\mathrm{op}}}(\operatorname{Bar}(A), M) = \mathbb{R}\operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(A, M) \tag{5.126}$$

Proof. We only use natural isomorphisms discussed above and the adjointness of smoothening and roughening:

$$\begin{aligned} C^*(A, M) &\cong \underline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}(\mathrm{Bar}_+(A), RM) \\ &\cong \underline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}(S\,\mathrm{Bar}_+(A), M) \\ &\cong \underline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}(\mathrm{Bar}(A), M)\,. \quad \square \end{aligned} \tag{5.127}$$

Theorem 5.79. *Let A, B be projectively Morita equivalent quasi-unital algebras. Let $M \cong RM$ be a rough A-bimodule such that also $Q \otimes_A M \otimes_A P$ is a rough B-bimodule. Let $P \in \mathrm{SMod}(A, B)$ and $Q \in \mathrm{SMod}(B, A)$ be the projective bimodules inducing the Morita equivalence $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$. Then their Hochschild cochain complexes with values in the rough bimodule are homotopy-equivalent.*

$$C^*(A, M) \simeq C^*(B, Q \otimes_A M \otimes_A P) \tag{5.128}$$

In particular, their Hochschild cohomology with values in M is isomorphic.

A similar theorem can be found - purely algebraically and for Hochschild *homology* - in [Lod98, Theorem 1.2.7].

Proof. Under the assumptions above $\mathrm{Bar}(A)$ and $\mathrm{Bar}(B)$ are projective bimodule resolutions of A and B respectively by Lemma 5.70 and we know that $P \otimes_B \mathrm{Bar}(B) \otimes_B Q$ is a projective resolution of A as an A-bimodule by Lemma 5.68 and hence is homotopy-equivalent to $\mathrm{Bar}(A)$.

$$\begin{aligned} \underline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}(\mathrm{Bar}(A), M) &\simeq \underline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}(P \otimes_B \mathrm{Bar}(B) \otimes_B Q, M) \\ &\cong \underline{\mathrm{Hom}}_{B\otimes B^{\mathrm{op}}}(\mathrm{Bar}(B), Q \otimes_A M \otimes_A P) \end{aligned} \tag{5.129}$$

We used the adjointness $\underline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}(P \otimes_B K \otimes_B Q, N) \cong \underline{\mathrm{Hom}}_{B\otimes B^{\mathrm{op}}}(M, Q \otimes_A L \otimes_A P)$ that holds for smooth bimodules. Finally, we know that since A is a smooth A-bimodule and A is self-induced we can use the above lemma. All these natural isomorphisms together yield $C^*(A, M) \simeq C^*(B, Q \otimes_A M \otimes_A P)$. $\square$

If A, B have units, then *all* bimodules are rough and smooth. In particular, for $M = A$, we get $C^*(A, A) \simeq C^*(B, B)$. However, we have shown in Lemma 5.47 that A is rough as an A-bimodule if and only if it has a unit. Hence, the previous result does *not* apply to $C^*(A, A)$ in this case.

5.5.4 Low Degrees of $H^*(A, A)$

This section revisits classical well-known results on Hochschild cohomology.

We first consider degree zero. There is a natural isomorphism $C^0(A, A) = \mathrm{Hom}(\mathbb{1}, A) \cong A$ induced by the free-forgetful adjunction by which element $a \in A$ (a map from the terminal object to A). The differential of this is the commutator $[a, -]$ as an operation $A \to A$. This differential vanishes if and only if a commutes with all other elements, i.e. lies in the center $Z(A)$.

$$H^0(A, A) \cong Z(A)\,. \tag{5.130}$$

Now consider degree one. A cochain in $C^1(A, A)$ is a morphism $F : A \to A$. It is a cocycle if and only if

$$df(a, b) = af(b) - f(ab) + f(a)b = 0 \quad \forall a, b \in A\,. \tag{5.131}$$

Hence, cocycles are the derivations $\mathrm{Der}(A)$. Note that these derivations need to be morphisms in the ambient category, i.e. bounded, continuous, etc.. Sometimes this is automatic, e.g. by locality from the Leibniz rule. Degree 1 coboundaries are commutators, which we refer to as *inner derivations* $\mathrm{InnDer}(A)$. All in all, the degree 1 Hochschild cohomology are *outer derivations.*

$$H^1(A, A) \cong \mathrm{OutDer}(A) \; = \mathrm{Der}(A)/\,\mathrm{InnDer}(A)\,. \tag{5.132}$$

For degree 2 we need some definitions.

Definition 5.80. An *extension* E of an algebra A consists of a short exact sequence of A-bimodules, i.e. $I \cong \ker(\pi)$ and $E \twoheadrightarrow A$ is a strict epimorphism and an algebra homomorphism.

$$I \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} A \tag{5.133}$$

If there is a section σ which is an algebra homomorphism then the extension is called split. The extension is called *square-zero* if $I \cdot I = 0$.

Fix a square-zero extension. Note first that I becomes an A-bimodule by multiplication in E by any chosen section $\sigma : A \to E$. Since the extension is square-zero, this is independent of the choice of section. Furthermore, corresponding to any section σ we have an associated cocycle in $C^2(A, I)$:

$$\begin{aligned} &c : A \otimes A \to I \\ &c = \iota^{-1} \circ (\mu_E \circ (\sigma \otimes \sigma) - \sigma \circ \mu_A) \end{aligned} \tag{5.134}$$

Here we use that the factored map $\iota : I \to \ker(\pi)$ is an isomorphism and that the difference in parentheses indeed factors through the kernel. If σ' is a different section, then $\alpha = \iota^{-1} \circ (\sigma - \sigma')$ defines a map $\alpha : A \to I$ with

$$\begin{aligned} c - c' &= \iota^{-1} \circ \big(\mu_E(\sigma \otimes \sigma - \sigma' \otimes \sigma') - (\sigma - \sigma') \circ \mu_A\big) \\ &= \iota^{-1} \circ \big(\mu_E(\sigma \otimes (\sigma - \sigma') + \mu_E((\sigma - \sigma') \otimes \sigma')) - (\sigma - \sigma') \circ \mu_A\big) \\ &= \mu_I \circ (\mathrm{id}_A \otimes \alpha) - \alpha \circ \mu_A + \mu_I \circ (\alpha \otimes \mathrm{id}_A) = d\alpha\,. \end{aligned} \tag{5.135}$$

We record:

Lemma 5.81. *Any two extensions are isomorphic if and only if their cocycles are in the same cohomology class in $H^2(A, I)$. Hence, Hochschild cohomology classifies square-zero algebra extensions.*

5.5.5 Smooth Deformations of Multiplication

If $A \in \mathsf{CBorn}$, we say that $\{\mu_t : t \in (-\epsilon, \epsilon)\}$ is a smooth family of multiplications on A, if (A, μ_t) is an algebra for all t and if the map

$$\mu : t \mapsto \underline{\mathrm{Hom}}(A \otimes A, A) \tag{5.136}$$

is a smooth curve. We let $c = \frac{d}{dt}|_{t=0}\mu_t : A \otimes A \to A$. [12] The chain rule, applied to the associativity constraint, together with $\mu_0(x, y) = xy$, yields:

$$\begin{aligned} 0 = \left.\frac{d}{dt}\right|_{t=0} &\mu_t(x \otimes \mu_t(y, z)) - \mu_t(\mu_t(x, y) \otimes z) \\ = xc(y \otimes z) + c(x \otimes yz) - c(x \otimes y)z &- c(xy \otimes z) = dc(x \otimes y \otimes z). \end{aligned} \tag{5.137}$$

Two smooth families of multiplications μ_t, ν_t are equivalent if there is a smooth family of isomorphisms $\phi_t : A \to A$ such that $\phi_0 = \mathrm{id}$ and

$$\mu_t \circ (\phi_t \otimes \mathrm{id}) = \nu_t \circ (\mathrm{id} \otimes \phi_t). \tag{5.138}$$

We let $\alpha = \frac{d}{dt}|_{t=0}\phi_t$. Then the chain rule, together with $\phi_0 = \mathrm{id}$, yields:

$$(c_\mu - c_\nu)(x \otimes y) = x\alpha(y) - \alpha(x)y = (d\alpha)(x \otimes y) \tag{5.139}$$

All in all we arrive at the following proposition.

Proposition 5.82. *Any smooth deformation μ_t of a bornological algebra defines an infinitesimal cocycle c_μ. If two deformations are equivalent, their cocycles are cohomologous.*

By virtue of Lemma 5.81, any smooth deformation hence defines a square zero extension.

$$A \to A \oplus A\hbar \twoheadrightarrow A \tag{5.140}$$

The multiplication is explicitly given by

$$(x_1 + y_1\hbar)(x_2 + y_2\hbar) = \left(x_1x_2 + \left.\frac{d}{dt}\right|_{t=0} \mu_t(x_1, x_2)\right) + (x_1y_2 + y_1x_2)\hbar\,. \tag{5.141}$$

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[12] As noted in [KM97, Ch.1.1], the concept of a smooth curve in a locally convex space only depends on the bornology. That is, the notion of smoothness and the derivative are the ordinary notions in locally convex spaces, where we e.g. use the bornivorous topology.

6 Convolution is a 2-Functor

This section aims to establish the following: Morita equivalent groupoids have Morita equivalent convolution algebras. The first subsection is a kind of sanity check. The statement should at least be true for manifolds regarded as trivial groupoids and the Morita equivalent Čech groupoid associated to an open cover. We write down explicit bimodules that establish the Morita equivalence. This is an instance of the general 2-functoriality which is proven in the subsequent subsection. Hence, the reader may skip the next subsection and may only review it for context.

6.1 Morita Equivalence for the Čech Algebra

Any manifold M is Morita equivalent as a groupoid to the Čech groupoid associated to an open cover $(U_i)_{i\in I}$ of M. This is, since the Čech groupoid is the groupoid pullback along the surjective submersion $\pi : \sqcup_i U_i \to M$. The associated biprincipal bibundle realizing this in the 2-category of groupoids and bibundles is the following:

$$\begin{tikzcd}
\sqcup_{i,j}U_{ij} \arrow[d, shift left] \arrow[d, shift right] & \sqcup_i U_i \arrow[dl, equal] \arrow[dr, "\pi"] & M \arrow[d, shift left] \arrow[d, shift right] \\
\sqcup_i U_i & & M
\end{tikzcd} \tag{6.1}$$

The convolution algebras are $C_c^\infty(M)$ with the pointwise product on the right and a subalgebra $\mathcal{A}_{\text{Čech}}$ of the matrix algebra of $C_c^\infty(M)$ on the right. For $\Phi = (\Phi_{i,j})$ and $\Psi = (\Psi_{ij})$ we have

$$(\Phi * \Psi)_{ij}(x) = \sum_k \Phi_{ik}(x)\Psi_{kj}(x)\,. \tag{6.2}$$

The biprincipal bibundle induces an $\mathcal{A}_{\text{Čech}}$-$C_c^\infty(M)$-bimodule. It is $C_c^\infty(\sqcup U_i) = \bigoplus_i C_c^\infty(U_i)$ with the actions

$$(\Phi.\varphi)_i(x) = \sum_j \Phi_{ij}(x)\varphi_j(x)\,, \tag{6.3}$$

$$(\varphi.f)_i(x) = \varphi_i(x)f(x) = (\varphi\pi^*f)(x)\,. \tag{6.4}$$

The opposite bimodule with the same underlying vector space comes with the corresponding $\mathcal{A}_{\text{Čech}}$-action by right matrix multiplication and the same action by $C_c^\infty(M)$ by commutativity.

There are natural bilinear pairings given on elementary tensors by:

$$\begin{aligned} \langle -,-\rangle_{\mathcal{A}_{\text{Čech}}} : P \otimes P^{\text{op}} &\longrightarrow \mathcal{A}_{\text{Čech}} \\ (\varphi \otimes \psi) &\longmapsto \varphi_i(x)\psi_j(x) \end{aligned} \tag{6.5}$$

$$\begin{aligned} \langle -,-\rangle_{C_c^\infty(M)} : P^{\text{op}} \otimes P &\longrightarrow C_c^\infty(M) \\ (\phi \otimes \psi) &\longmapsto \sum_i \phi_i(x)\psi_i(x) \end{aligned} \tag{6.6}$$

As a bornological space we can compute $P \otimes P \cong \bigoplus_{i,j} C_c^\infty(U_i \times U_j)$. The above pairings extend to this completion via

$$\bigoplus_{i,j} \Delta_{i,j}^* : \bigoplus_{i,j} C_c^\infty(U_i \times U_j) \longrightarrow \bigoplus_{i,j} C_c^\infty(U_{ij}) = \mathcal{A}_{\text{Čech}} \tag{6.7}$$

and

$$\bigoplus_{i,j} C_c^\infty(U_i \times U_j) \xrightarrow{\Delta^*} \bigoplus_i C_c^\infty(U_i) \xrightarrow{Tr} C_c^\infty(M) \quad . \tag{6.8}$$

Here $\Delta_{i,j} : U_{ij} \hookrightarrow U_i \times U_j$ is the diagonal. Importantly, it is a proper map. In the second diagram, $\Delta^* = \oplus_i \Delta_{i,i}^*$, which disregards all off-diagonal entries, and $Tr((f_i)) = \sum_i f_i$.

Proposition 6.1. *The pairings induce isomorphisms of bornological bimodules*

$$P \otimes_{C_c^\infty(M)} P^{\mathrm{op}} \longrightarrow \mathcal{A}_{\check{C}ech} \tag{6.9}$$

$$P^{\mathrm{op}} \otimes_{\mathcal{A}_{\check{C}ech}} P \longrightarrow C_c^\infty(M) \tag{6.10}$$

Hence, P, P^{op} constitute bornological Morita equivalence bimodules.

The proof is an immediate consequence of the theorem in the next subsection: We only need to prove functoriality to show that the composition of bimodules is the bimodule associated to the composition. The composite of P and P^{op} is the identity and is mapped to the identity bimodule. Let us apply this to M and the open cover consisting of n open sets, each of which are equal to M. Then, $\mathcal{A}_{\check{\mathrm{C}}\mathrm{ech}}$ is isomorphic to the $n \times n$-matrices over $C_c^\infty(M)$ and hence $C_c^\infty(M)$ and $\mathrm{Mat_{n\times n}}(\mathrm{C_c^\infty(M)})$ are Morita equivalent algebras in $\mathsf{SAlg}^{\mathrm{bi}}(\mathsf{CBorn})$. This should be familiar for $M = pt$. In this case $\mathbb{C}$ and $\mathrm{Mat_{n\times n}}(\mathbb{C})$ are Morita equivalent. The bimodule is $\mathbb{C}^n$ with the obvious actions.

6.2 Functoriality for Groupoid Bibundles

We work with groupoids and fixed left Haar systems here. Any right principal X-Y-bibundle Q defines a bornological $\mathcal{A}_X$-$\mathcal{A}_Y$-bimodule $\mathcal{M}_Q$.

$$\begin{tikzcd}
X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & & Q \arrow[dll, "l_Q"] \arrow[drr, "r_Q"'] & & Y_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
X_0 & & & & Y_0
\end{tikzcd} \tag{6.11}$$

As a bornological vector space it is given by $\mathcal{M}_Q = C_c^\infty(Q)$ with the actions

$$f.\varphi(q) = \int_{t^{-1}(l(q))} f(x)\varphi(x^{-1}.q)\mathrm{d}x \tag{6.12}$$

$$\varphi.g(q) = \int_{t^{-1}(r(q))} \varphi(q.y)g(y^{-1})\mathrm{d}y \tag{6.13}$$

We now try to relate the composition of bibundles to the composition of bimodules.

$$\begin{tikzcd}
& & & & Q \circ R \arrow[dllll, bend right, "l_{Q\circ R}"'] \arrow[drrrr, bend left, "r_{Q\circ R}"] & & & & \\
X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & & Q \arrow[dll, "l_Q"] \arrow[drr, "r_Q"'] & & Y_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & & R \arrow[dll, "l_R"] \arrow[drr, "r_R"'] & & Z_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
X_0 & & & & Y_0 & & & & Z_0
\end{tikzcd} \tag{6.14}$$

Proposition 6.2. *There is a map of $\mathcal{A}_X$-$\mathcal{A}_Z$-bimodules:*

$$\begin{aligned} U : \mathcal{M}_Q \otimes_{\mathcal{A}_Y} \mathcal{M}_R &\longrightarrow \mathcal{M}_{Q\circ R} \\ \varphi \otimes \psi &\longmapsto \left[[q,r] \mapsto \int \varphi(q.y)\psi(y^{-1}.r)\mathrm{d}y \right] \end{aligned} \tag{6.15}$$

If Y is proper, U is an isomorphism of bornological $\mathcal{A}_X$-$\mathcal{A}_Z$-bimodules.

The integral on the right hand side is over $t^{-1}(r_Q(q))$ and this is the only possible domain where the expression makes sense. It is easy to check that this is independent of representative $[q,r] = [q.y, y^{-1}.r]$ by invariance of the integral and that indeed $U(\varphi.f \otimes \psi) = U(\varphi \otimes f.\psi)$ and that U intertwines the actions.

Our method to prove the isomorphism will be to first dissect the involved maps and then to see that we have a (not necessarily bilinear) section. Then we identify the kernel of U on the bornological tensor product with the completed image of the elementary balancing tensors $\alpha.y \otimes \beta - \alpha \otimes y.\beta$. That is, we show algebraic exactness of the following complex of bornological maps:

$$\begin{tikzcd} C_c^\infty(Q \times Y \times R) \arrow[r, "I"] & C_c^\infty(Q \times R) \arrow[r, "U"] & C_c^\infty(Q \circ R) \arrow[r] \arrow[l, bend right, dotted] & 0 \end{tikzcd} \tag{6.16}$$

Because we have exhibited a section, U factors to a bounded isomorphism of bimodules on the balanced tensor product $\operatorname{coker}(I)$.
The bornological tensor product $\mathcal{M}_Q \otimes \mathcal{M}_R$ can be identified with $C_c^\infty(Q \times R)$ and U extends to the map

$$U(\Phi)([q,r]) = \int_{\pi^{-1}([q,r])} \Phi(q.y, y^{-1}.r)\mathrm{d}y\,, \tag{6.17}$$

which factors as restriction and a fiber integral :

$$C_c^\infty(Q \times R) \xrightarrow{\iota^*} C_c^\infty(Q \times_{Y_0} R) \xrightarrow{\pi_*} C_c^\infty(Q \circ R) \tag{6.18}$$

Using Lemma 5.73 again, we see that the first map is a strict epimorphism, i.e. admits a section. If Y is a proper groupoid we can exhibit a section of the second map:

$$\sigma(\phi)(q,r) = \phi([q,r])\Lambda(q,r) \tag{6.19}$$

Here $\Lambda \in C^\infty(Q \times R)$ is a function which fulfills $\pi : \operatorname{supp}(\Lambda) \to Q \circ R$ is proper and

$$\int \Lambda(q.y, y^{-1}.r)\mathrm{d}y = 1\,. \tag{6.20}$$

Proof that Λ exists. Let $\lambda \in C^\infty(Y)$ be as in 8.1 such that $t : \operatorname{supp}(\lambda) \to Y_0$ is proper and

$$\int_{t^{-1}(s(y'))} \lambda(y'y)\mathrm{d}y = 1 \qquad \forall y' \in Y\,. \tag{6.21}$$

$\pi : Q \times_{Y_0} R \to Q \circ R$ is a principal Y-bundle with the diagonal action. Consider a cover $\{U_i\}$ of $Q \circ R$ over which it trivializes with a subordinate partition of unity $\{\chi_i\}$. Then $U_i \times_{Y_0} Y \to \pi^{-1}(U_i) \subset Q \times_{Y_0} R$ is an isomorphism. Denote the inverse Y-equivariant map by $y_i : \pi^{-1}(U_i) \to Y$ and define

$$\Lambda(q,r) = \sum_i \chi_i([q,r])\lambda(y_i(q,r))\,. \tag{6.22}$$

Let $\beta : U_i \to Y_0$ be the canonical map sending $U_i \times_{Y_0} Y_0 \to Q \times_{Y_0} R \to Y_0$. If $K \subset Q \circ R$ is compact, it will be covered by only finitely many U_i for which $\chi_i \neq 0$. In $\pi^{-1}(U_i) \cong U_i \times_{Y_0} Y$, $\chi_i(u)\lambda(y)$ has support in $\operatorname{supp}(\chi_i) \times_{Y_0} \big(t^{-1}(\beta(K \cap \operatorname{supp}(\chi_i)) \cap \operatorname{supp}(\lambda))\big)$ which is compact. This shows properness of $\pi : \operatorname{supp}(\Lambda) \to Q \circ R$. Furthermore

$$\begin{aligned} \int \sum_i \chi_i([q.y, y^{-1}.r])\lambda(y_i(q.y, y^{-1}.r))\mathrm{d}y &= \int \sum_i \chi_i([q,r])\lambda(y_i(q,r)y)\mathrm{d}y \\ &= \sum_i \chi_i \int \lambda(y_i y)\mathrm{d}y = 1\,. \quad \square \end{aligned} \tag{6.23}$$

Proof of Proposition 6.2. We already know that there is a section of U. We are left to show injectivity of U. For this we need to identify the kernel of U in $C_c^\infty(Q \times R)$ with the image of

$$\begin{aligned} I : C_c^\infty(Q \times Y \times R) &\longrightarrow C_c^\infty(Q \times R) \\ \Psi &\longmapsto \left[(q,r) \mapsto \int \Psi(q.y, y^{-1}, r)\mathrm{d}y - \int \Psi(q, y', y'^{-1}.r)\mathrm{d}y'\right] . \end{aligned} \tag{6.24}$$

We proceed in two steps. First, let $\Phi \in \ker(C_c^\infty(Q \times R) \to C_c^\infty(Q \times_{Y_0} R))$ be a function vanishing on the submanifold. We show that this is already in the image of Equation (6.24). For this choose $\Psi \in C_c^\infty(Q \times Y \times R)$ with

$$\Psi(q,y,r) = \begin{cases} \Phi(qy, r)\lambda(y^{-1}) & (q,y,r) \in Q \times_{Y_0} Y \times R \\ 0 & (q,y,r) \in Q \times Y \times_{Y_0} R \end{cases}, \tag{6.25}$$

where λ is as in Equation (6.21). This exists since $\Phi = 0$ on $Q \times_{Y_0} Y \times_{Y_0} R$. Then indeed

$$\int \Psi(q.y, y^{-1}, r)\mathrm{d}y - \int \Psi(q, y', y'^{-1}.r)\mathrm{d}y' = \int \Phi(q,r)\lambda(y)\mathrm{d}y - 0 = \Phi(q,r) . \tag{6.26}$$

The following Lemma 6.3 shows

$$\begin{aligned} &\ker\left(\pi_* : C_c^\infty(Q \times_{Y_0} R) \to C_c^\infty(Q \circ R)\right) \\ &\quad = \operatorname{Im}\left((\pi_1)_* - (\pi_2)_* : C_c^\infty(Q \times_{Y_0} R \times_{Y_0} Q \times_{Y_0} R) \to C_c^\infty(Q \times_{Y_0} R)\right) , \end{aligned} \tag{6.27}$$

so we now define a map

$$\begin{aligned} \hat{} : C_c^\infty((Q \times_{Y_0} R \times_{Y_0} Q \times_{Y_0} R) &\longrightarrow C_c^\infty(Q \times_{Y_0} Y \times_{Y_0} R) \\ \Xi &\longmapsto \left[\hat{\Xi}(q,y,r) = \Xi(q, y.r, q.y, r)\right] \end{aligned} \tag{6.28}$$

By choosing a section of the restriction to the submanifold $Q \times_{Y_0} Y \times_{Y_0} R \subset Q \times Y \times R$, we get a composite

$$C_c^\infty(Q \times_{Y_0} R \times_{Y_0} Q \times_{Y_0} R) \to C_c^\infty(Q \times Y \times R) , \tag{6.29}$$

which satisfies $I(\hat{\Xi})|_{Q \times_{Y_0} R} = ((\pi_1)_* - (\pi_2)_*)\Xi$, i.e. makes the diagram below commute:

$$\begin{aligned} \left[I(\hat{\Xi})|_{Q \times_{Y_0} R}\right](q,r) &= \int_{t^{-1}(r_Q(q))} \hat{\Xi}(q.y, y^{-1}, r)\mathrm{d}y - \int_{t^{-1}(l_R(r))} \hat{\Xi}(q, y', y'^{-1}.r)\mathrm{d}y' \\ &= \int_{t^{-1}(r_Q(q))} \Xi(q.y, y^{-1}.r, q, r) - \Xi(q, r, q.y, y^{-1}.r)\mathrm{d}y \\ &= \left[((\pi_1)_* - (\pi_2)_*)\Xi\right](q,r) \end{aligned} \tag{6.30}$$

$$\begin{tikzcd} & & & 0 & 0 \\ & & & C_c^\infty(Q \circ R) \arrow[u] \arrow[ur] & \\ 0 \arrow[r] & \ker \arrow[r] \arrow[d, hook] & C_c^\infty(Q \times R) \arrow[r] \arrow[ur, "U"] & C_c^\infty(Q \times_{Y_0} R) \arrow[r] \arrow[u, "\pi_*"'] & 0 \\ & \operatorname{Im}(I) \arrow[ur] & & C_c^\infty(Q \times_{Y_0} R \times_{Y_0} Q \times_{Y_0} R) \arrow[ll] \arrow[u, "(\pi_1)_* - (\pi_2)_*"'] & \end{tikzcd} \tag{6.31}$$

We have seen that the vertical and horizontal sequences are exact. A diagram chase finishes the proof that $\ker(U) = \operatorname{Im}(I)$. ☐

Lemma 6.3 (Mayer-Vietoris for groupoid principal bundles)**.** *Let Y be a proper groupoid. For any principal Y-bundle $\pi : P \to B$, the following is exact:*

$$C_c^\infty(P \times_B P) \xrightarrow{(\pi_1)_* - (\pi_2)_*} C_c^\infty(P) \xrightarrow{\pi_*} C_c^\infty(B) \longrightarrow 0 \tag{6.32}$$

Proof. We have already seen, using a function Λ, that π_* has a section and is hence surjective. For the construction of Λ we only used properness of Y and the principality of the bundle π. Now let $f \in \ker(\pi_*)$. Define $F(p_1, p_2) = \Lambda(p_1) f(p_2) \in C_c^\infty(P \times_B P)$.

$$\begin{aligned} ((\pi_1)_* - (\pi_2)_*)F(p) &= \int F(p.y, p) - F(p, p.y)\mathrm{d}y \\ &= \int \Lambda(p.y)\mathrm{d}y f(p) - \int f(p.y)\mathrm{d}y \Lambda(p) \\ &= f(p) - 0 \end{aligned} \tag{6.33}$$

Hence, $f \in \mathrm{im}((\pi_1)_* - (\pi_2)_*)$. □

For the special case of Y being the trivial groupoid associated to a manifold M and U_i being an open cover of M the space $\bigsqcup_i U_i$ is a principal Y-bundle. The naming of the previous lemma becomes clearer in this situation.

Lemma 6.4 (Generalized Mayer-Vietoris for compact support)**.** *The following sequence is exact:*

$$\bigoplus_{i_0 < \cdots < i_{p+1}} C_c^\infty(U_{i_0 \ldots i_{p+1}}) \xrightarrow{\delta} \bigoplus_{i_0 < \cdots < i_p} C_c^\infty(U_{i_0 \ldots i_p}) \longrightarrow \ldots \longrightarrow \bigoplus_i C_c^\infty(U_i) \xrightarrow{\delta = Tr} C_c^\infty(M) \longrightarrow 0 \tag{6.34}$$

Here $(\delta f)_{i_0 \ldots i_p} = \sum_i f_{i i_0 \ldots i_p}$ with the convention that $f_{\ldots i \ldots j \ldots} = -f_{\ldots j \ldots i \ldots}$.

Lemma 6.5. *Let X be a proper groupoid. The $\mathcal{A}_X$-module $\mathcal{M}_Q$ is smooth.*

Proof. We need to show that left multiplication

$$\mathcal{A}_X \otimes_{\mathcal{A}_X} \mathcal{M}_Q \to \mathcal{M}_Q \tag{6.35}$$

is an isomorphism. This follows from Proposition 6.2 with the identity X-X-bibundle for which $\mathcal{A}_X = \mathcal{M}_X$ as $\mathcal{A}_X$-bimodules.

$$\begin{tikzcd} X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & X_1 \arrow[dl, "t"] \arrow[dr, "s"'] & X_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ X_0 & & X_0 \end{tikzcd} \tag{6.36}$$

□

Theorem 6.6. *The assignment of the bornological convolution algebra to a proper Lie groupoid is a weak 2-functor:*

$$\begin{aligned} \mathsf{GrpdBiBun}^{\mathrm{proper}} &\longrightarrow \mathsf{SAlg}^{\mathrm{bi}}(\mathsf{CBorn}) \\ X &\longmapsto \mathcal{A}_X \\ {}_X P_Y &\longmapsto \mathcal{M}_P \end{aligned} \tag{6.37}$$

The module $\mathcal{M}_P$ is projective as a right Y-module. Hence, Morita equivalent proper Lie groupoids are mapped to projectively Morita equivalent quasi-unital bornological algebras.

Proof. We have already defined the 2-functor on objects and on morphisms. Part of the weak 2-functor is also the isomorphism $U : \mathcal{M}_P \otimes_{\mathcal{A}_Y} \mathcal{M}_Q \xrightarrow{\cong} \mathcal{M}_{P\circ Q}$. We are left to define the action on 2-morphisms, to show naturality of U and to verify commutativity of the coherence diagrams.
The category of morphisms between X and Y in the bibundle category consists of bibundles and biequivariant intertwiners. The biequivariant intertwiners are automatically isomorphisms. Hence, given such a $\Phi : P \to P'$, we can define $\Phi_* : \mathcal{M}_P \to \mathcal{M}_{P'}$ via $f \mapsto (\Phi^{-1})^* f$. The map Φ_* is easily seen to be an (iso)morphism of $\mathcal{A}_X$-$\mathcal{A}_Y$-bimodules. It also immediately follows that $\Phi_* \circ \Phi'_* = (\Phi \circ \Phi')_*$ and $\mathrm{id}_{P*} = \mathrm{id}_{\mathcal{M}_P}$. This verifies that the 2-functor is in particular a 1-functor on the 1-category of bibundle morphism between two groupoids X, Y.
Now to show naturality of U we let $\Phi : P \to P'$ and $\Psi : Q \to Q'$. Then there is a map $\Phi \circ \Psi : P \circ Q \to P' \circ Q'$ mapping $[p, q] \mapsto [\Phi(p), \Psi(q)]$. The following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_P \otimes_{\mathcal{A}_Y} \mathcal{M}_Q \arrow[r, "U"] \arrow[d, "\Phi_*\otimes\Psi_*"] & \mathcal{M}_{P\circ Q} \arrow[d, "(\Phi\circ\Psi)_*"] \\
\mathcal{M}_{P'} \otimes_{\mathcal{A}_Y} \mathcal{M}_{Q'} \arrow[r, "U"] & \mathcal{M}_{P'\circ Q'}
\end{tikzcd} \tag{6.38}$$

$$\begin{aligned}
\left[U((\Phi_* \otimes \Psi_*)(\alpha \otimes \beta))\right]([p, q]) &= \int \alpha(\Phi^{-1}(p.y))\beta(\Psi^{-1}(y^{-1}.q))\mathrm{d}y \\
&= \int \alpha(\Phi^{-1}(p).y)\beta(y^{-1}.\Psi^{-1}(q))\mathrm{d}y \\
&= U(\alpha \otimes \beta)([\Phi^{-1}(p), \Psi^{-1}(q)]) \\
&= \left[(\Phi \circ \Psi)_*(U(\alpha \otimes \beta))\right]([p, q])\,.
\end{aligned} \tag{6.39}$$

This shows the naturality of U, i.e. that U is a bifunctor. We are left to show coherence. For a weak 2-functor F these are the following diagrams built from the structure maps.

$$\begin{tikzcd}
 & \mathrm{id} \circ F(f) \arrow[dl] \arrow[dr] & \\
F(\mathrm{id}) \circ F(f) \arrow[dr] & & F(f) \\
 & F(\mathrm{id} \circ f) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(f) \circ \mathrm{id} \arrow[dl] \arrow[dr] & \\
F(f) \circ F(\mathrm{id}) \arrow[dr] & & F(f) \\
 & F(f \circ \mathrm{id}) \arrow[ur] &
\end{tikzcd} \tag{6.40}$$

$$\begin{tikzcd}
(F(f) \circ F(g)) \circ F(h) \arrow[r] \arrow[d] & F(f) \circ (F(g) \circ F(h)) \arrow[d] \\
F(fg) \circ F(h) \arrow[d] & F(f) \circ F(gh) \arrow[d] \\
F((fg)h) \arrow[r] & F(f(gh))
\end{tikzcd} \tag{6.41}$$

For Y regarded as the identity Y-Y-bibundle we can identify $\mathcal{M}_Y$ with the smooth $\mathcal{A}_Y$ identity-bimodule $\mathcal{A}_Y$. Both are just $C_c^\infty(Y)$ as a bornological vector space with the same left and right action by convolution. The left identity coherence diagram then amounts to showing that $U : \mathcal{M}_P \otimes_{\mathcal{A}_Y} \mathcal{M}_Y \to \mathcal{M}_{P\circ Y} \to \mathcal{M}_P$ agrees with scalar multiplication

$\mathcal{M}_P \otimes_{\mathcal{A}_Y} \mathcal{A}_Y \to \mathcal{M}_P$. This follows from the following calculation:

$$\begin{aligned} U(\alpha \otimes f)([p, \mathrm{id}]) &= \int \alpha(py) f(y^{-1}) \mathrm{d}y \\ &= \alpha.f(p)\,. \end{aligned} \tag{6.42}$$

The right identity coherence is proved analogously. The last coherence we have to check is that the different ways of bracketing $\mathcal{M}_P \otimes_{\mathcal{A}_Y} \mathcal{M}_Q \otimes_{\mathcal{A}_Z} \mathcal{M}_R \to \mathcal{M}_{P \circ Q \circ R}$ agree. We calculate:

$$\begin{aligned} U(U(\alpha \otimes \beta) \otimes \gamma)([[p, q], r]) &= \int U(\alpha \otimes \beta)([p, q.z]) \gamma(z^{-1}.r) \mathrm{d}z \\ &= \int \int \alpha(p.y) \beta(y^{-1}.q.z) \gamma(z^{-1}.r) \mathrm{d}y \mathrm{d}z \\ U(\alpha \otimes U(\beta \otimes \gamma))([p, [q, r]]) &= \int \alpha(p.y) U(\beta \otimes \gamma)([y^{-1}.q, r]) \mathrm{d}y \\ &= \int \int \alpha(p.y) \beta(y^{-1}.q.z) \gamma(z^{-1}.r) \mathrm{d}z \mathrm{d}y \end{aligned} \tag{6.43}$$

The hexagon identity follows from these and the isomorphism $P \circ (Q \circ R) \to (P \circ Q) \circ R$ where $[p, [q, r]] \mapsto [[p, q], r]$. $\square$

Remark 6.7. If we knew that U is an isomorphism on a larger class of groupoids, the exact same proof would show 2-functoriality for this larger class and not only for proper groupoids. There is a notion of *lax 2-functor* that does not require the natural transformation $F(f) \circ F(g) \to F(f \circ g)$ to be an isomorphism. This is not useful for Morita theory.

Left principal X-bundles over a manifold M are the same as X-M-bibundles, when we regard M as a trivial groupoid. Hence, $\mathcal{M}_P$ is also defined for just one-sided principal bundles.

Lemma 6.8. *For any left principal X-bundle P, the induced $\mathcal{A}_X$-module $\mathcal{M}_P$ is smoothly projective.*
If X, Y are proper and if P is biprincipal, the module $\mathcal{M}_P$ is projective as a left $\mathcal{A}_X$-module and as right $\mathcal{A}_Y$-module.

Proof. Let X be a groupoid and P a left principal X-bundle. We show that the left $\mathcal{A}_X$-module $\mathcal{M}_P = C_c^\infty(P)$ is a direct summand of a smoothly free module of the form $\mathcal{A}_X \otimes B$.
Similarly to the usual Serre-Swan theorem, we will reduce to trivial bundles. A trivial left principal X-bundle over a manifold M is given by pulling back the trivial bundle $X \to X_0$ along any smooth map $f : M \to X^{(0)}$:

$$\begin{tikzcd} X \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & X \times^{s,f}_{X_0} M \arrow[dl] \arrow[d] \\ X_0 & M \arrow[l, "f"] \end{tikzcd} \tag{6.44}$$

The left action on this principal bundle is $x.(y, m) = (xy, m)$. Hence, the induced action on $\mathcal{M}_Y = C_c^\infty(X \times_{X_0} M)$ is:

$$f.\varphi(y, m) = \int f(x) \varphi(x^{-1}y, m) \mathrm{d}y\,, \tag{6.45}$$

and it only acts on the first variable. Thus, the restriction $\iota^* : C_c^\infty(X \times M) \to C_c^\infty(X \times_{X_0} M)$ is $\mathcal{A}_X$-linear:

$$C_c^\infty(X \times M) \cong C_c^\infty(X) \otimes C_c^\infty(M) \xrightarrow{\iota^*} C_c^\infty(X \times_{X_0} M) \tag{6.46}$$

There is an $\mathcal{A}_X$-linear section of this map. This is completely analogous to Proposition 5.71 and uses the local retraction technique. We choose a locally finite countable covering $\{U_i\}$ of $X_0 \times_{X_0} M \subset X_0 \times M$ and retractions $r_i : U_i \to U_i \cap (X_0 \times_{X_0} M)$ of the form $r_i(x,m) = (x, \tilde{r}_i(x,m))$ as in Equation (5.108). Then the desired $\mathcal{A}_X$-linear section is

$$\sigma(F)(x,m) = \sum_i F(x, \tilde{r}_i(x,m)) \chi_i(s(x), \tilde{r}_i(x,m)) \,. \tag{6.47}$$

The verification of $\mathcal{A}_X$-linearity is completely analogous.
All in all, any such trivial bundle over M is a direct summand of $C_c^\infty(X) \otimes C_c^\infty(M)$.
If P is now any left principal X-bundle over M it will only trivialise over an open cover $\{U_i\}$ of M.

$$\begin{tikzcd} X \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & P \arrow[dl] \arrow[d] & \bigsqcup P|_{U_i} \arrow[l] \arrow[d] \\ X_0 & M & \bigsqcup_i U_i \arrow[l] \arrow[u, dotted, bend right] \end{tikzcd} \tag{6.48}$$

In the depicted diagram the right bundle is trivial and hence $\mathcal{M}_{\bigsqcup_i P|_{U_i}} = \bigoplus_i \mathcal{M}_{P|_{U_i}}$ is smoothly projective. The inclusions $(\iota_i)_* : C_c^\infty(P|_{U_i}) \hookrightarrow C_c^\infty(P)$ induce the addition map $\bigoplus_i \mathcal{M}_{P|_{U_i}} \to \mathcal{M}_P$. A partition of unity $\{\chi_i\}$ subordinate to $\{U_i\}$ induces a section of this addition via

$$\begin{aligned} C_c^\infty(P) &\to \oplus_i C_c^\infty(P|_{U_i}) \\ \varphi &\mapsto ((\chi_i \circ \pi)\varphi) \,. \end{aligned} \tag{6.49}$$

These are maps of $\mathcal{A}_X$-modules since

$$((\iota_i)_* f.\varphi)(p) = \int f(x)\varphi(x^{-1}.p)\mathrm{d}x = (f.(\iota_i)_*\varphi)(p) \tag{6.50}$$

and

$$\begin{aligned} f.(\chi_i \circ \pi)\varphi(p) &= \int f(x)\chi_i(\pi(x^{-1}.p))\varphi(x^{-1}.p)\mathrm{d}x \\ &= \int f(x)\varphi(x^{-1}.p)\mathrm{d}x \chi_i(\pi(p)) \\ &= (f.\varphi)(p)(\chi_i \circ \pi)(p) \end{aligned} \tag{6.51}$$

All in all, $C_c^\infty(P) = \mathcal{M}_P$ is a direct summand of $\bigoplus_i C_c^\infty(P|_{U_i}) = \mathcal{M}_{\bigsqcup_i P|_{U_i}}$. We have already seen that the latter is smoothly projective and hence $\mathcal{M}_P$ is smoothly projective, i.e. a direct summand of $\mathcal{A}_X \otimes B$. If $\mathcal{A}_X$ is quasi-unital, then this is equivalent to $\mathcal{M}_P$ being projective and this is the case for all proper X. □

Example 6.9. Let $G \rightrightarrows M$ be a transitive proper groupoid. Then G is Morita equivalent to its compact isotropy Lie group $G_x^x = s^{-1} \cap t^{-1}(x)$ at any point via the bibundle

$$\begin{tikzcd} G \arrow[d, shift right] \arrow[d, shift left] & s^{-1}(x) \arrow[dl, "t"'] \arrow[dr] & G_x^x \arrow[d, shift right] \arrow[d, shift left] \\ M & & \{x\} \end{tikzcd} \tag{6.52}$$

Hence, $\mathcal{A}_G$ and $\mathcal{A}_{G_x^x}$ are Morita equivalent.

Example 6.10. A special case of the following is the free action of G on itself. This yields the action groupoid we will conveniently name EG since its geometric realization has a free G-action and is contractible. Then, the isotropy group is trivial and $\mathcal{A}_{EG} \simeq \mathbb{C}$.

Example 6.11. Let M be a manifold with open cover $\{U_i\}$. The canonical groupoid homomorphism from the Čech groupoid $\bigsqcup U_{ij} \rightrightarrows \bigsqcup_i U_i$ to $M \rightrightarrows M$ is a Morita morphism. The convolution algebras $\mathcal{A}_{\text{Čech}}$ and $C_c^\infty(M)$ are Morita equivalent.

The previous lemma shows that in the situation of Morita equivalent proper Lie groupoids, the bimodules giving the Morita equivalence of convolution algebras even constitute a projective Morita equivalence. The previous reasoning can be summarized in a table:

$$
\begin{tikzcd}
\boxed{\text{Morita equivalent proper Lie groupoids } G,\ H} \arrow[d, "\text{2-functoriality, Theorem 6.6}"] \\
\boxed{\text{projectively Morita equivalent, quasi unital convolution algebras } \mathcal{A}_G,\ \mathcal{A}_H} \arrow[d, "\begin{array}{l}\text{homotopy equivalence of projective}\\ \text{resolutions and Lemma 5.68}\end{array}"] \\
\boxed{\text{Morita equivalent bar complex resolutions } \mathrm{Bar}(\mathcal{A}_G) \text{ and } \mathrm{Bar}(\mathcal{A}_H)} \arrow[d] \\
\boxed{\text{Naturally homotopy equivalent derived functors } \mathbb{R}\operatorname{Hom}_{A\otimes A^{\mathrm{op}}}(A,-)} \arrow[d, dotted, "\text{e.g. for rough bimodules, Theorem 5.79}"] \\
\boxed{\text{homotopy equivalent Hochschild cochain complexes}}
\end{tikzcd}
$$

The following is an example of how this is useful in practice. This was actually the main motivation for developing an abstract formalism.

Example 6.12. Suppose Γ is a finite group acting on the compact manifold M. Assume furthermore that the action is free. Then M/Γ exists. The action groupoid $\Gamma \ltimes M \rightrightarrows M$ has a unital convolution algebra $\mathcal{A}_{\Gamma\ltimes M}$ with unit $\delta_e(\gamma, m) = \begin{cases} 1 & \gamma = e \\ 0 & \text{else} \end{cases}$. The action groupoid is Morita equivalent to the manifold $M/\Gamma \rightrightarrows M/\Gamma$. The latter also has a unital convolution algebra. Hence, combining all our results so far, we obtain a homotopy equivalence of the Hochschild cochain complexes $C^*(\mathcal{A}_{\Gamma\ltimes M}, \mathcal{A}_{\Gamma\ltimes M})$ and $C^*(C^\infty(M/\Gamma), C^\infty(M/\Gamma))$. We will show that the latter is homotopy equivalent to the cochain complex of multivector fields with zero differential in the following chapter. All in all:

$$
(C^*(\mathcal{A}_{\Gamma\ltimes M}, \mathcal{A}_{\Gamma\ltimes M}), d) \simeq (\mathfrak{X}^*(M/\Gamma), 0)\,. \tag{6.53}
$$

It would be great to extend this to Γ any compact Lie group. The problem arises only through nonunitality and the limitations of Theorem 5.79.

7 Applications

This section explores how to compute the Hochschild cohomology of special convolution algebras using the underlying bornological theory of the previous chapters. We first prove a strong formality version of the HKR theorem in Section 7.2 computing Hochschild cohomology for the algebra of smooth functions, partially based on [Con85]. Later, in Section 7.3, based on [NPPT06], we will do this equivariantly and compute the Hochschild cohomology for the convolution algebra of a proper étale action groupoid. In the case where the quotient by the group action exists, we then have two ways to compute Hochschild cohomology. Using the result on 2-functoriality and the result on the Hochschild cohomology of Morita equivalent algebras plus the HKR theorem. We see that both methods agree. Finally, we can also explicitly characterise the image of deformation cohomology inside the Hochschild cohomology under the map Φ in this case.

7.1 Koszul Resolution

We start with an algebraic example to motivate the smooth version later. Consider the polynomial algebra $\mathbb{C}[X]$. Then we have a projective resolution as a $\mathbb{C}[X, Y]$-module.

$$0 \longleftarrow \mathbb{C}[X] \xleftarrow{\Delta^*} \mathbb{C}[X, Y] \xleftarrow{-\cdot(X-Y)} \mathbb{C}[X, Y] \longleftarrow 0 \tag{7.1}$$

Here Δ^* is given by setting $X = Y$. By Taylor's theorem, a function vanishing for $X = Y$ is in the image of multiplication by $(X - Y)$, a function that vanishes only on the diagonal. This so-called Koszul resolution generalizes well, even to the smooth/bornological setting.

7.2 Smooth Koszul Resolution and HKR

This paragraph is a more detailed version of Connes' work [Con85]. Consider a compact manifold M. We will introduce a projective resolution of the complex Fréchet algebra $C^\infty(M)$ as a $C^\infty(M)$-bimodule, i.e. as a $C^\infty(M) \otimes C^\infty(M)^{\mathrm{op}}$-module. [13] Note that $C^\infty(M)$ is commutative and hence $C^\infty(M)^{\mathrm{op}} = C^\infty(M)$. Furthermore we can identify $C^\infty(M \times M) \cong C^\infty(M) \otimes C^\infty(M)$. The action of $C^\infty(M \times M)$ on $C^\infty(M)$ is given by multiplication by the pullback along the diagonal $\Delta : M \to M \times M$ as we can calculate on elementary tensors:

$$f_1 \otimes f_2 . g = f_1 g f_2 = (f_1 f_2) g = \Delta^*(f_1 \otimes f_2) g \qquad \forall f_1, f_2 \in C^\infty(M), g \in C^\infty(M \times M) \tag{7.2}$$

What is hence needed is a projective resolution of $C^\infty(M)$ as a $C^\infty(M \times M)$-module acting via pullback along the diagonal embedding. We will slightly generalize this situation to an arbitrary submanifold $S \subset M$. We now start our setup:

Definition 7.1. Let $V \to M$ be a vector bundle with local coordinates (x^i, v^i). The *Euler vector field* of V is $E_v = v^i \left.\frac{\partial}{\partial v^i}\right|_v$.

The flow of the Euler vector field V is $\Phi_s(v) = e^s v$. This in particular shows why E is well-defined, regardless of the choice of coordinates. For convenience we will also consider the flow $\Psi_s(v) = sv$.

Let $\iota : S \hookrightarrow M$ be a non-empty embedded submanifold, M connected. Then, the restriction $\iota^* : C^\infty(M) \to C^\infty(S)$ is a homomorphism of Fréchet algebras making $C^\infty(S)$

[13] Here, $\otimes$ is the complete bornological tensor product which agrees with the projective tensor product $\hat{\otimes}_\pi$.

into a module over $C^\infty(M)$. To have a projective resolution of this module over $C^\infty(M)$ we use some geometric constructions. Let $\theta : NS \to M$ be a tubular neighbourhood embedding for the normal bundle of S. Let $\tilde{U}$ be a star-shaped neighbourhood of the zero section that is mapped diffeomorphically by θ to an open subset $U \subset M$ containing S. Also fix a smooth bump function $\chi : M \to [0,1]$ that is identically 1 in a neighbourhood of S and $\mathrm{supp}(\chi) \subset U$. For $V = \mathrm{supp}(\chi)^c$, we then have $U \cup V = M$ and $\{\chi, 1-\chi\}$ is a partition of unity subordinate to this cover.

Proposition 7.2. *There is a complex Euler-like vector field X on M such that we have a projective resolution of locally convex or bornological $C^\infty(M)$-modules:*

$$C^\infty(S) \xleftarrow{\iota^*} C^\infty(M) \xleftarrow{i_X} \Omega^1(M) \xleftarrow{i_X} \Omega^2(M) \longleftarrow \dots \tag{7.3}$$

The zero locus of X is precisely S.

Proof. We will need to construct X. Note also that, as in the proof of *Lemma* 5.56, it is enough to construct continuous sections $\ker(d_i) \to P_i$. Firstly, we do have a continuous section of ι^* by the tubular neigbourhood since it induces a retraction $r : U \to S$:

$$\begin{aligned} s_{-1} : C^\infty(S) \to C^\infty(U) \to C^\infty(M) \\ f \mapsto r^* f \mapsto \chi r^* f \end{aligned} \tag{7.4}$$

Now let E be the Euler vector field of the normal bundle NS. Define X_1 on U to be θ-related to E, i.e. $(X_1)_{\theta(v)} = d\theta_v(E_v)$. Let X_2 be a real vector field on M that only vanishes on S. Such a vector field always exists. Define $X = \chi X_1 + i(1-\chi)X_2$. This vector field only vanishes on S. We can also find a 1-form α supported in V such that $\alpha(X) = \alpha(iX_2) = 1$ on U^c.
We proceed by treating U, V separately by the partition of unity. If $\omega \in \ker(i_X)$ and $\mathrm{supp}(\omega) \subset V$ then

$$i_X(\alpha \wedge \omega) = (i_X\alpha) \wedge \omega - \alpha \wedge i_X\omega = \omega\,. \tag{7.5}$$

Now let $f \in \ker(\iota^*)$, which is a function vanishing on the submanifold S. If $\mathrm{supp}(f) \subset U$, we can argue about $\theta^* f \in C^\infty(\tilde{U}) \subset C^\infty(NS)$. Then using $d\Psi_s(E_v) = sE_{sv}$ we get:

$$\begin{aligned} f(v) &= \int_0^1 \frac{\partial}{\partial t}\Big|_{t=s} f(tv)\mathrm{d}s \\ &= \int_0^1 df_{sv}(E_{sv})\mathrm{d}s = \int_0^1 df_{\Psi_s(v)}(\frac{1}{s} d\Psi_s(E_v))\mathrm{d}s \\ &= i_E \int_0^1 \frac{1}{s}(\Psi_s^* df)_v \mathrm{d}s\,. \end{aligned} \tag{7.6}$$

Similarly for a k-form $\omega \in \Omega^k(NS)$ that is in $\ker(i_E)$ we can calculate

$$\begin{aligned} i_E \int_0^1 \frac{1}{s}\Psi_s^* d\omega \mathrm{d}s &= \int_{-\infty}^0 \Psi_{e^s}^* i_E d\omega \mathrm{d}s = \int_{-\infty}^0 \Phi_s^* \mathcal{L}_E \omega \mathrm{d}s \\ &= \int_{-\infty}^0 \Phi_s^* \frac{\partial}{\partial t}\Big|_{t=0} \Phi_t^* \omega \mathrm{d}s = \int_{-\infty}^0 \frac{\partial}{\partial t}\Big|_{t=s} \Phi_t^* \omega \mathrm{d}s \\ &= \Phi_0^* \omega = \omega\,, \end{aligned} \tag{7.7}$$

where we used that $\Psi_{e^s} = \Phi_s$, $d\Phi_s(E) = E$, Cartan's magic formula $\mathcal{L}_E\omega = i_E d\omega + d i_E \omega = i_E d\omega$ and the definition of the Lie derivative. Note that $\Phi_{-\infty} = \Psi_0$ is multiplication by zero and hence has zero differential making the pullback trivial. Also, the original integral on the left hand side is well-defined and depends continuously on ω since the differential $d\Psi_s$ is multiplication by s, thus making it non-singular.

Putting everything together for $k \geq 0$ we can define

$$\begin{aligned} s_k : \ker(i_X) \subset \Omega^k(M) &\to \Omega^{k+1}(M) \\ \omega &\mapsto \int_0^1 \frac{1}{s}(\theta\Psi_s\theta^{-1})^*(\chi\omega)\mathrm{d}s + (1-\chi)\alpha \wedge \omega\,. \end{aligned} \tag{7.8}$$

The previous calculations show that it is indeed a continuous or, equivalently, bounded section of i_X. Since by Lemma 7.8 modules of sections of bundles are projective, we are done. □

In the following particular case, we can choose a special tubular neighbourhood embedding allowing us to add a condition on the Euler-like vector field X.

Corollary 7.3. *The diagonal embedding $\Delta : M \to M \times M$ induces a resolution of $C^\infty(M)$ as a $C^\infty(M \times M)$-module. There is a vector field X whose zero locus is precisely the diagonal such that the following is a projective resolution:*

$$C^\infty(M) \xleftarrow{\Delta^*} C^\infty(M \times M) \xleftarrow{i_X} \Omega^1(M \times M) \xleftarrow{i_X} \dots \xleftarrow{i_X} \Omega^{2n}(M \times M) \longleftarrow 0 \tag{7.9}$$

Moreover, we can choose X vertical with respect to $\mathrm{pr}_1 : M \times M \to M$.

A slightly more careful analysis of $M \hookrightarrow M \times M$ gives a projective resolution of length n instead of $2n$ by only considering vertical forms. For this, let $E_k = \mathrm{pr}_2^*(\bigwedge^k T^*_{\mathbb{C}}M)$ be the pullback of the k-th exterior power of the complexified tangent bundle along the projection onto the second factor of $M \times M$. Sections of E_k are precisely k-forms on $M \times M$ that vanish upon insertion of a horizontal vector, that is one in the first component of $T_{(m,n)}(M \times M) \cong T_mM \oplus T_nM$.
We also write $\Gamma(E_k) = \Omega^k_{\mathrm{ver}}(M \times M)$. The exact same proof as above shows that the following is a projective resolution:

$$C^\infty(M) \xleftarrow{\Delta^*} C^\infty(M \times M) \xleftarrow{i_X} \Omega^1_{\mathrm{ver}}(M^2) \xleftarrow{i_X} \dots \xleftarrow{i_X} \Omega^n_{\mathrm{ver}}(M^2) \longleftarrow 0 \tag{7.10}$$

All of the preceding theory is used to prove the smooth, locally convex, resp. bornological version of the HKR-theorem. It appeared first in Connes' paper on noncommutative differential geometry sparking the interest in locally convex Hochschild cohomology.

Theorem 7.4 (Hochschild-Kostant-Rosenberg Theorem)**.** *The Hochschild cohomology of the algebra of smooth functions is given by multivector fields:*

$$H^*(C^\infty(M), C^\infty(M)) \cong \mathfrak{X}^k(M) = \Gamma(\wedge^k TM) \tag{7.11}$$

This holds if we regard $C^\infty(M)$ as a locally convex Fréchet algebra or a bornological algebra.
More precisely, there is even a chain homotopy equivalence between multivector fields with the zero differential and the Hochschild cochain complex.

Remark 7.5.

1. Recall that H^1 always computes outer derivations. For commutative algebras there are no inner derivations. Hence, for commutative algebras H^1 consists of all derivations, which are classically identified with vector fields for the algebra $C^\infty(M)$.

2. Note that Hochschild cohomology and vector fields share a limited functoriality.

3. The HKR isomorphism is not a map of dgla's, even though there are the natural Gerstenhaber and Schouten-Nijenhuis Lie brackets. Kontsevich formality shows however that it can be made into an L_∞-algebra isomorphism. [Kon03]

Proof. Recall that any two projective resolutions are chain homotopy equivalent. We will work with the one constructed in Equation (7.10). Hochschild cohomology of $C^\infty(M)$ can then be computed up to unique isomorphism by the cohomology of the cochain complex $\mathrm{Hom}_{C^\infty(M\times M)}(\Omega^k_{\mathrm{ver}}(M\times M), C^\infty(M))$.
Note that naturally we can identify

$$\mathrm{Hom}_{C^\infty(M)}(\Omega^k(M), C^\infty(M)) \cong \mathrm{Hom}_{\mathrm{Bun}}(\wedge^k T^*_{\mathbb{C}}M, \mathbb{C}) \cong \Gamma((\wedge^k T^*_{\mathbb{C}}M)^*) \cong \Gamma(\wedge^k T_{\mathbb{C}}M)\,. \tag{7.12}$$

We get chain maps

$$\begin{tikzcd} \dots \arrow[r] & \underline{\mathrm{Hom}}_{C^\infty(M)}(\Omega^k(M), C^\infty(M)) \arrow[r, "0"] \arrow[loop above, "\mathrm{id}"] \arrow[d, shift left, "(-)\circ\Delta^*"] & \underline{\mathrm{Hom}}_{C^\infty(M)}(\Omega^{k+1}(M), C^\infty(M)) \arrow[r] \arrow[loop above, "\mathrm{id}"] \arrow[d, shift left, "(-)\circ\Delta^*"] & \dots \\ \dots \arrow[r] & \underline{\mathrm{Hom}}_{C^\infty(M\times M)}(\Omega^k_{\mathrm{ver}}(M\times M), C^\infty(M)) \arrow[r, "i_X^*"] \arrow[u, shift left, "(-)\circ\mathrm{pr}_1^*"] & \underline{\mathrm{Hom}}_{C^\infty(M\times M)}(\Omega^{k+1}_{\mathrm{ver}}(M\times M), C^\infty(M)) \arrow[r] \arrow[u, shift left, "(-)\circ\mathrm{pr}_1^*"] & \dots \end{tikzcd} \tag{7.13}$$

The precomposition with the pullback Δ^* is indeed well-defined since $\psi(\Delta^*(f\omega)) = \psi((f\circ\Delta)\cdot\Delta^*\omega) = (f\circ\Delta)\cdot\psi(\Delta^*\omega)$, showing $C^\infty(M\times M)$-linearity. It is also a cochain map since X vanishes on the diagonal and hence $\psi(\Delta^* i_X\omega) = 0$.

The precomposition with the pullback pr_1^* also yields a well-defined $C^\infty(M)$-linear map since

$$\phi(\mathrm{pr}_1^*(g\eta)) = \phi((g\circ\mathrm{pr}_1)\,\mathrm{pr}_1^*\eta) = (g\circ\mathrm{pr}_1).\phi(\mathrm{pr}_1^*\eta) = (g\circ\mathrm{pr}_1\circ\Delta)\phi(\mathrm{pr}_1^*\eta) = g\phi(\mathrm{pr}_1^*\eta)\,. \tag{7.14}$$

To prove that it is a cochain map we need $i_X\circ\mathrm{pr}_1^* = 0$. This is a consequence of $d\,\mathrm{pr}_1(X) = 0$, i.e. X being vertical.

By the following lemma, $(\Delta^*)^*$ is bijective in every degree since

$$\underline{\mathrm{Hom}}_{C^\infty(M\times M)}(\Gamma(\mathrm{pr}_2^*\bigwedge^k T^*_{\mathbb{C}}M), C^\infty(M)) \cong \underline{\mathrm{Hom}}_{C^\infty(M)}(\Gamma(\Delta^*\,\mathrm{pr}_2^*\bigwedge^k T^*_{\mathbb{C}}M), C^\infty(M)) \tag{7.15}$$

and $\Delta^*\,\mathrm{pr}_2^*\bigwedge^k T^*_{\mathbb{C}}M = \bigwedge^k T^*_{\mathbb{C}}M$. By exhibiting a left inverse, we have found an inverse. To conclude, we have hence constructed a chain homotopy equivalence from multivector fields to the Hochschild cohomology of $C^\infty(M)$. □

Lemma 7.6. *Let $\iota : S \hookrightarrow M$ be an embedding. Let E be a vector bundle over M and F a vector bundle over S. Then, there are natural vector space isomorphisms*

$$\mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(F)) \cong \mathrm{Hom}_{C^\infty(S)}(\Gamma(\iota^*E), \Gamma(F)) \cong \mathrm{Hom}_{\mathrm{Bun}}(\iota^*E, F)\,. \tag{7.16}$$

Thus, all C^∞-linear maps are induced by vector bundle maps and all C^∞-linear maps are automatically continuous resp. bornological.

Proof. There is a natural vector bundle map $\iota^*E \to E$ covering ι. The pullback of sections induces a map $\iota^* : \Gamma(E) \to \Gamma(\iota^*E)$ which is $C^\infty(M)$-linear. Precomposition by ι^* yields the map

$$(\iota^*)^* : \mathrm{Hom}_{C^\infty(S)}(\Gamma(\iota^*E), \Gamma(F) \to \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(F))\,. \tag{7.17}$$

We will show that this is an isomorphism by exhibiting an inverse. For this let $\Phi : \Gamma(E) \to \Gamma(F)$ be $C^\infty(M)$-linear. The first step is showing locality around S. That is, suppose

σ vanishes in a neighbourhood $U \subset M$ of a point $p \in S$. Let χ be a cutoff function in $C^\infty(M)$ vanishing in a slightly smaller neighbourhood such that $\chi\sigma = \sigma$. Then clearly $\Phi(\sigma) = (\chi \circ \iota)\Phi(\sigma)$ must vanish in a neighbourhood in S of p. Hence, we can compute Φ on any local section of E in a neighbourhood of S by an arbitrary extension to a global section. The same argument shows that Φ only depends on the values of σ in an arbitrarily small neighbourhood of S by using a cutoff. Now let σ vanish only at a point $p \in S$. Switching to a local frame $\sigma = \sigma^i e_i$ and each σ^i vanishes at p. We can now apply Φ and use local $C^\infty(M)$-linearity to see that $\Phi(\sigma) = (\sigma^i \circ \iota)\Phi(e_i)$ vanishes at p. We can hence evaluate Φ pointwise on $\iota^* E$ by $\Phi(\sigma(p)) = \Phi(\sigma)(p)$. This is a vector bundle map $\iota^* E \to F$ which induces a map between sections. This construction of pointwise evaluation is clearly inverse to the natural maps from right to left. □

Lemma 7.7. *Every connected compact manifold M admits a vector field vanishing only at one point.*

Proof. By transversality, there is a vector field with discrete isolated singularities. This will be a finite set by compactness. The first step is to show that there is a diffeomorphism $M \to M$ under which all singularities are mapped to an arbitrary embedded disk. For this, connect all singularities by paths and choose a spanning tree. A tubular neighbourhood of this will be a disk: We can even cook up a diffeotopy to a disk around the root singularity of the tree. For this we choose a vector field supported on some open neighbourhood of the tree pointing towards the root vertex. For example one could extend the velocity vector field of the spanning tree. The flow of this vector field will be a diffeotopy that in some finite time flows all singularities into a disk around the root vertex. Hence we get a smooth vector field X_1 with singularities in some embedded disk D_{r_0} inside of a slightly larger embedded disk D. In the interior we tweak the vector field a bit by “Alexander's tric”. In polar coordinates:

$$X_2(r,\theta) = \begin{cases} \chi(r)X(r_0,\theta) & r < r_0 \\ X_1(r,\theta) & r \geq r_0 \end{cases}. \tag{7.18}$$

Here χ needs to be a function vanishing at zero whose derivative also vanishes at zero and which is constantly one for $r \geq r_0$. The resulting vector field is continuous, has a single singularity at one point only and is smooth in a neighbourhood of this point. Whitney approximation then yields a smooth vector field X_3 coinciding with X_2 near this point and arbitrarily close to X_2 everywhere else. For instance, by compactness $\|X_2\| > \epsilon$ outside D_{r_0} and and choosing X_3 to agree with X_2 on D_{r_0} as well as $\|X_2 - X_3\| < \frac{\epsilon}{2}$ gives the desired properties. □

Lemma 7.8. *Let $V \to M$ be a vector bundle. Then $\Gamma(V)$ is a finitely generated projective module over the unital algebra $C^\infty(M)$ considered as an algebra in* Vect, CBorn *and Fréchet spaces. $\Gamma_c(V)$ is finitely generated projective over the algebra $C_c^\infty(M)$ considered as an algebra in* CBorn.

Proof. By standard arguments, V is a direct summand of a trivial bundle. Hence $\Gamma(V)$ (resp. $\Gamma_c(V)$)is a direct summand of $C^\infty(M)^n$ (resp. $C_c^\infty(M)^n$). Both of these are direct sums of projective modules and hence projective. □

7.3 Hochschild Cohomology of proper étale action Groupoids

In [NPPT06] the Hochschild cohomology of the convolution algebra of proper étale groupoids is computed. In loc. cit. the authors use a variety of techniques. A key computation is the computation of the Hochschild cohomology of action groupoids for actions of finite groups. The rationale is that a proper étale groupoid $G \rightrightarrows M$ is an atlas for the orbifold M/G and orbifolds are locally quotients of euclidean space by finite groups. Hence, using an appropriate sheaf cohomology the problem is reduced to this local situation. Due to lack of space, we do not treat this global case here. We fix a mistake in the definition of a cochain map and we provide an original proof for the cohomology of a dual Koszul complex in **??**.

Our setup consists of a finite group Γ acting on a compact manifold M. We write $A = C^\infty(M)$ and we write

$$A \rtimes \Gamma := \mathcal{A}_{\Gamma \ltimes M} = C^\infty(\Gamma \times M) \tag{7.19}$$

for the convolution algebra of the action groupoid. The convolution of $f_1, f_2 \in C^\infty(\Gamma \times M)$ is given by

$$f_1 * f_2(\gamma, x) = \sum_{\gamma_1\gamma_2 = \gamma} f_1(\gamma_1, \gamma_2.x) f_2(\gamma_2, x)\,. \tag{7.20}$$

There are the special functions in $A \rtimes \Gamma$

$$\delta_\gamma(\gamma', p) = \begin{cases} 1 & \gamma = \gamma' \\ 0 & \text{else} \end{cases}\,, \tag{7.21}$$

which satisfy $\delta_{\gamma_1}\delta_{\gamma_2} = \delta_{\gamma_1\gamma_2}$ and $\delta_\gamma * f(\gamma', x) = f(\gamma^{-1}\gamma', x)$. Any $a \in A \rtimes \Gamma$ can be written as

$$a = \sum_{\gamma \in \Gamma} f_\gamma \delta_\gamma \tag{7.22}$$

with $f_\gamma \in A$. There is an action of Γ on A via

$$\gamma.f(p) = f(\gamma^{-1}.p)\,. \tag{7.23}$$

For $f_1, f_2 \in A$ and $\gamma_1, \gamma_2 \in \Gamma$ we have the important crossed product formula

$$f_1\delta_{\gamma_1} * f_2\delta_{\gamma_2} = f_1(\gamma_2.f_2)\delta_{\gamma_1\gamma_2}\,. \tag{7.24}$$

This determines $*$ completely!

The idea is now to separate the Hochschild cohomology $C^*(A \rtimes \Gamma, A \rtimes \Gamma)$ into a part group cohomology and a part belonging to a manifold.

Lemma 7.9. *We have $C^*(A \rtimes \Gamma, A \rtimes \Gamma) \cong C^*_\Gamma$, where*

$$C^k_\Gamma = \mathrm{Hom}(\mathbb{C}\Gamma^k, \mathrm{Hom}(A^{\otimes k}, A \rtimes \Gamma))\,. \tag{7.25}$$

To a degree k cochain $\Phi : (A \rtimes \Gamma)^{\otimes k} \to A \rtimes \Gamma$ we associate

$$\begin{aligned} \hat{\Phi} : \mathbb{C}\Gamma^k &\longrightarrow \mathrm{Hom}(A^{\otimes k}, A \rtimes \Gamma) \\ \hat{\Phi}(\gamma_1, \ldots, \gamma_k)(f_1, \ldots, f_k) &= \Phi\left(f_1\delta_{\gamma_1}, \gamma_1^{-1}.f_2\delta_{\gamma_2}, \ldots, \gamma_{k-1}^{-1}\cdots\gamma_1^{-1}.f_k\delta_{\gamma_k}\right) * \delta_{(\gamma_1\cdots\gamma_k)^{-1}}\,. \end{aligned} \tag{7.26}$$

Remark 7.10. In [NPPT06, Prop. 3.7], the cochain map is different. Their map does *not* work and 7.26 is the correction. Fortunately, everything else still holds.

The last lemma is only a twisted version of the tensor-hom-adjunction. Let us compute the differential that is induced on the right hand side. This heavily uses 7.24. The inner face maps are easy:

$$\begin{aligned}\widehat{d_1\Phi}(\gamma_1,\gamma_2)(f_1,f_2) &= d_1\Phi(f_1\delta_{\gamma_1},\gamma_1^{-1}.f_2\delta_{\gamma_2})*\delta_{(\gamma_1\gamma_2)^{-1}}\\ &= \Phi(f_1\delta_{\gamma_1}*\gamma_1^{-1}.f_2)*\delta_{(\gamma_1\gamma_2)^{-1}}\\ &= \Phi(f_1f_2\delta_{\gamma_1\gamma_2})*\delta_{(\gamma_1\gamma_2)^{-1}}\\ &= \hat{\Phi}(\gamma_1\gamma_2)(f_1f_2)\,.\end{aligned}\tag{7.27}$$

And the parts that cause trouble in loc.cit.:

$$\begin{aligned}&\widehat{d_0\Phi}(\gamma_1,\ldots,\gamma_{k+1})(f_1,\ldots,f_{k+1}) = d_0\Phi(f_1\delta_{\gamma_1},\ldots,\gamma_k^{-1}\ldots\gamma_1^{-1}.f_{k+1}\delta_{\gamma_{k+1}})*\delta_{(\gamma_1\ldots\gamma_{k+1})^{-1}}\\ &\quad= f_1\delta_{\gamma_1}*\Phi(\gamma_1^{-1}.f_2\delta_{\gamma_2},\ldots,\gamma_{k+1}^{-1}\ldots\gamma_1^{-1}.f_{k+1}\delta_{\gamma_{k+1}})*\delta_{(\gamma_2\ldots\gamma_{k+1})^{-1}}*\delta_{\gamma_1^{-1}}\\ &\quad= f_1\delta_e*\delta_{\gamma_1}*\hat{\Phi}(\gamma_1^{-1}.f_1,\ldots,\gamma_1^{-1}f_{k+1})*\delta_{\gamma_1^{-1}}\\ &\quad= f_1\delta_e*(\gamma_1.\hat{\Phi})(\gamma_2,\ldots,\gamma_{k+1})(f_2,\ldots,f_{k+1})\,.\end{aligned}\tag{7.28}$$

$$\begin{aligned}&\widehat{d_{k+1}\Phi}(\gamma_1,\ldots,\gamma_{k+1})(f_1,\ldots,f_{k+1}) = d_{k+1}\Phi(f_1\delta_{\gamma_1},\ldots,\gamma_k^{-1}\ldots\gamma_1^{-1}.f_{k+1}\delta_{\gamma_{k+1}})*\delta_{(\gamma_1\ldots\gamma_{k+1})^{-1}}\\ &\quad= \Phi(f_1\delta_{\gamma_1},\ldots,\gamma_{k-1}^{-1}\ldots\gamma_1^{-1}.f_k\delta_{\gamma_k})*\gamma_k^{-1}\ldots\gamma_1^{-1}.f_{k+1}\delta_{\gamma_{k+1}}*\delta_{(\gamma_1\ldots\gamma_{k+1})^{-1}}\\ &\quad= \Phi(f_1\delta_{\gamma_1},\ldots,\gamma_{k-1}^{-1}\ldots\gamma_1^{-1}.f_k\delta_{\gamma_k})*\delta_{(\gamma_1\ldots\gamma_k)^{-1}}*f_{k+1}\delta_e\\ &\quad= \hat{\Phi}(\gamma_1,\ldots,\gamma_k)(f_1,\ldots,f_k)*f_{k+1}\delta_e\,.\end{aligned}\tag{7.29}$$

From here, one separates the differential into a horizontal and a vertical part to get a cohomological double complex, or rather a bicosimplical complex. We need the Γ-action on $\mathrm{Hom}(A^{\otimes n}, A\rtimes\Gamma)$ that is given by

$$(\gamma.\Phi)(f_1,\ldots,f_n) = \delta_\gamma*\Phi(\gamma^{-1}.f_1,\ldots,\gamma^{-1}.f_n)*\delta_{\gamma^{-1}}\,.\tag{7.30}$$

Let $C_\Gamma^{*,*}$ be the complex $C_\Gamma^{m,n} = \mathrm{Hom}(\mathbb{C}\Gamma^m, \mathrm{Hom}(A^{\otimes n}, A\rtimes\Gamma))$. The structure maps are

$$\begin{gathered}d_v^i : C_\Gamma^{m,n} \longrightarrow C_\Gamma^{m+1,n}\\ d_v^i\Psi(\gamma_1,\ldots,\gamma_{m+1}) = \begin{cases}\gamma_1.\Psi(\gamma_2,\ldots,\gamma_{m+1}) & i=0\\ \Psi(\gamma_1,\ldots,\gamma_i\gamma_{i+1},\ldots,\gamma_{m+1}) & 1\le i\le m\\ \Psi(\gamma_1,\ldots,\gamma_m) & i=m+1\end{cases}\end{gathered}\tag{7.31}$$

$$\begin{gathered}d_h^i : C_\Gamma^{m,n} \longrightarrow C_\Gamma^{m,n+1}\\ d_h^i\Psi(\gamma_1,\ldots,\gamma_m)(f_1,\ldots,f_n) = \begin{cases}f_1\delta_e*\Psi(\gamma_1,\ldots,\gamma_m)(f_2,\ldots,f_n) & i=0\\ \Psi(\gamma_1,\ldots,\gamma_m)(f_1,\ldots,f_if_{i+1},\ldots,f_n) & 1\le i\le n\\ \Psi(\gamma_1,\ldots,\gamma_m)(f_1,\ldots,f_n)*f_{n+1}\delta_e & i=n+1\end{cases}\end{gathered}\tag{7.32}$$

$$\begin{gathered}s_v^i : C_\Gamma^{m,n} \longrightarrow C_\Gamma^{m-1,n} \qquad \forall 0\le i\le m-1\\ s_v^i\Psi(\gamma_1,\ldots,\gamma_{m-1}) = \Psi(\gamma_1,\ldots,\gamma_i,e,\gamma_{i+1},\ldots,\gamma_{m-1})\end{gathered}\tag{7.33}$$

$$\begin{gathered}s_h^i : C_\Gamma^{m,n} \longrightarrow C_\Gamma^{m,n-1} \qquad \forall 0\le i\le n-1\\ s_v^i\Psi(\gamma_1,\ldots,\gamma_m)(f_1,\ldots,f_{n-1}) = \Psi(\gamma_1,\ldots,\gamma_m)(f_1,\ldots,f_{i-1},1,f_i,\ldots,f_{n-1})\end{gathered}\tag{7.34}$$

The horizontal face and degeneracy maps (d_h^i and s_h^i) commute with all vertical face and degeneracy maps (d_v^j and s_v^j). Furthermore, the usual cosimplicial identities hold for the vertical and horizontal maps individually. These identities show that $C_\Gamma^{*,*}$ is a bicosimplicial vector space. The Dold-Kan correspondence for bicosimplicial abelian groups yields a double complex with horizontal and vertical differentials:

$$d_v = \sum_{i=0}^{m+1} (-1)^i d_v^i : C_\Gamma^{m,n} \to C_\Gamma^{m+1,n} \quad , \quad d_h = \sum_{j=0}^{n+1} (-1)^{m+j} d_h^j : C_\Gamma^{m,n} \to C_\Gamma^{m,n+1} \tag{7.35}$$

The total complex is $\mathrm{Tot}(C_\Gamma^{*,*})^k = \bigoplus_{m+n=k} C_\Gamma^{m,n}$ with differential $d_h + d_v$. The diagonal is the cosimplicial vector space $\mathrm{diag}(C_\Gamma^{*,*})^k = C_\Gamma^{k,k} = C_\Gamma^k$ with face maps $d_h^i d_v^i = d_v^i d_h^i$ and degeneracies $s_h^i s_v^i = s_v^i s_h^i$.

Lemma 7.11. *We have* $C^*(A \rtimes \Gamma, A \rtimes \Gamma) \cong \mathrm{diag}(C_\Gamma^{*,*}) = C_\Gamma^*$ *via* $\Phi \mapsto \hat{\Phi}$ *as in Equation* (7.26)*.*

Proof. This is just showing that $\widehat{d_i \Phi} = d_h^i d_v^i \hat{\Phi}$, which is immediate using the identities Equations (7.27) to (7.29). ☐

The bicosimplicial version of the Eilenberg-Zilber theorem now equates the cohomology of the diagonal with the cohomology of the total complex. The explicit maps use the degeneracies and the reader is referred to [GM04, Appendix A.2]. The bisimplicial version for homology is also in [Wei94, Theorem 8.51].

$$H^*(\mathrm{Tot}(C_\Gamma^{*,*})) \cong H^*(\mathrm{diag}(C_\Gamma^{*,*}) \cong H^*(A \rtimes \Gamma, A \rtimes \Gamma) \tag{7.36}$$

Hence, the computation of the Hochschild cohomology can be accomplished by computing cohomology of the double complex. The next step is to use the spectral sequence associated the first quadrant double complex.

$$E_2^{pq} = H_v^p H_h^q (C^{*,*}) \Rightarrow H^{p+q}(\mathrm{Tot}(C_\Gamma^{*,*})) \tag{7.37}$$

Note that d_h is precisely the Hochschild differential of the complex $C^*(A, A \rtimes \Gamma)$. Hence, $H_h(C^{*,*})^{pq} = \mathrm{Hom}(\mathbb{C}\Gamma^p, H^q(A, A \rtimes \Gamma))$. The differential d_v is precisely the group cohomology differential for the induced Γ action on $H^*(A, A \rtimes \Gamma)$. Since Γ is finite, its group cohomology with values in a vector space vanishes above degree zero. (c.f. [Wei94, 6.5.8]) The proof is similar to the vanishing of deformation cohomology for proper groupoids. The degree zero part of group cohomology always computes Γ-invariants. The spectral sequence collapses consequently:

$$E_2^{pq} = \begin{cases} H^q(A, A \rtimes \Gamma)^\Gamma & p = 0 \\ 0 & \text{else} \end{cases} \tag{7.38}$$

We can hence conclude that $E_\infty^{pq} = E_2^{pq}$ and all extension problems are trivial. In conclusion:

$$H^k(\mathrm{Tot}(C^{*,*})) \cong H^k(A, A \rtimes \Gamma)^\Gamma \tag{7.39}$$

It remains to analyse the right hand side. Recall that $A = C^\infty(M)$ and we do have a good projective resolution of $C^\infty(M)$ as a bimodule over itself via Equation (7.10). Note that the A-bimodule structure on $A \rtimes \Gamma$ is given by

$$f.a.g = f\delta_e * \sum_{\gamma \in \Gamma} f_\gamma \delta_\gamma * g\delta_e = \sum_{\gamma \in \Gamma} (f f_\gamma \gamma.g)\delta_\gamma \,. \tag{7.40}$$

Hence, $A \rtimes \Gamma$ decomposes into a sum of bimodules

$$\begin{aligned}\bigoplus_{\gamma\in\Gamma} A_\gamma &\cong A \rtimes \Gamma \\ (f_\gamma) &\mapsto \sum_{\gamma\in\Gamma} f_\gamma \delta_\gamma\,,\end{aligned} \tag{7.41}$$

where $A_\gamma = A = C^\infty(M)$ as a space but with the A-bimodule structure

$$f.f_\gamma.g(x) = f(x)f_\gamma(x)\gamma.g(x) = f \otimes g(x, \gamma^{-1}.x)f_\gamma(x)\,. \tag{7.42}$$

Identifying A-bimodules with $C^\infty(M \times M)$-modules, we get that $F \in C^\infty(M \times M)$ acts on $f_\gamma \in A_\gamma$ by

$$F.f_\gamma = (\Delta_\gamma^* F)f\,, \tag{7.43}$$

where

$$\begin{aligned}\Delta_\gamma : M &\to M \times M \\ x &\mapsto (x, \gamma^{-1}x)\,.\end{aligned} \tag{7.44}$$

Hence, if we consider the embedding $\Delta_\gamma : M \to M \times M$, then A_γ can be identified with the smooth functions on this submanifold and we can apply Lemma 7.6 to compute:

$$\mathrm{Hom}_{C^\infty(M\times M)}(\Omega^k_{\mathrm{hor}}(M \times M), A_\gamma) \cong \mathrm{Hom}_{C^\infty(M)}(\Omega^k(M), C^\infty(M)) \cong \mathfrak{X}^k(M) \tag{7.45}$$

The isomorphism is induced by pairing a multivector field $Y \in \mathfrak{X}^k(M)$ with $\Delta_\gamma^*\omega$ for $\omega \in \Omega^k_{\mathrm{hor}}(M \times M)$ by the natural pairing $\langle \Delta_\gamma^*\omega, Y\rangle$. Recall that the differential on the right hand side is precomposition with the insertion $i_{\tilde{X}}$, where $\tilde{X}$ was a special Euler-like vector field. If we write $\kappa = d\,\mathrm{pr}_1(\tilde{X})$ along $\Delta_\gamma(M)$, then we can compute:

$$\langle \Delta_\gamma^*(i_X\omega), Y\rangle = \langle i_\kappa \Delta_\gamma^*\omega, Y\rangle = \langle \Delta_\gamma^*\omega, \kappa \wedge Y\rangle \tag{7.46}$$

Here we used also that ω is horizontal, so that it vanishes on $X - d\Delta_\gamma(\kappa)$. The above calculation shows that

$$(C^*(A, A_\gamma), d) \simeq (\mathrm{Hom}_{C^\infty(M\times M)}(\Omega^*_{\mathrm{hor}}(M \times M), i_X{}^*) \cong (\mathfrak{X}^*(M), \kappa \wedge -) \tag{7.47}$$

Determining the cohomology of the dual Koszul complex $\mathfrak{X}^*(M)$ with differential $\kappa \wedge -$ now uses a localisation and linearisation argument. Note first that κ only vanishes on the fixed point space M^γ since Δ_γ maps this onto the diagonal where $\tilde{X}$ vanishes. So, let $X \in \mathfrak{X}^k(M)$ with $\kappa \wedge X = 0$. Then, on $M \setminus M^\gamma$ there is a 1-form ω with $i_\omega\kappa = 1$. On $M \setminus M^\gamma$ we have:

$$0 = i_\omega(\kappa \wedge X) = X - \kappa \wedge i_\omega X\,. \tag{7.48}$$

So, if φ is a cutoff function that is supported in any neighbourhood of M^γ, then $(1 - \varphi)X = \kappa \wedge i_\omega(1 - \varphi)X$ and hence X is cohomologous to φX.

We now proceed with the linearisation around M^γ. Choose a Γ-invariant Riemannian metric on M. This is always possible by averaging. Let $p \in M^\gamma$. For $v \in T_pM$, $\gamma.\exp(tv)$ is a geodesic with initial velocity $d\gamma_p(v)$ and hence

$$\gamma.\exp(tv) = \exp(d\gamma(tv))\,. \tag{7.49}$$

That is, around any fixed point exp provides a local diffeomorphism $T_pM \to M$ under which the action of γ linearises to an action by a linear isometry on a vector space. In particular, M^γ is locally euclidean as the local image of the eigenvalue 1 subspace of $d\gamma$. The dimension of M^γ might jump globally.

Choose a locally finite covering U_i of M^γ such that $\exp_{p_i} : \tilde{U}_i \to U_i$ is a diffeomorphism and a subordinate partition of unity χ_i. Let $\kappa \wedge X = 0$. We may assume that X vanishes outside $\bigcup U_i$ by the above localisation. Now, $\kappa \wedge X = 0$ if and only if $\kappa \wedge \chi_i X = 0 \forall i$ and $X = \kappa \wedge Y$ if and only if $\chi_i X = \kappa \wedge Y_i \forall i$. That is, we can localise further to $\mathrm{supp} X \subset U_i$. We can now linearise to T_pM and express everything there. Write $T_pM = (T_pM)^\gamma \oplus ((T_pM)^\gamma)^\perp$. Choose an orthonormal basis $e_1, \ldots, e_{l_\gamma}$ of the eigenvalue 1 space $(T_pM)^\gamma$ of $d\gamma$ and an orthonormal basis $e_{l(\gamma)+1} \ldots, e_n$ of the orthogonal complement. Denote the corresponding coordinate functions by x^i. From the construction of κ [14] it is given by

$$\kappa_q = \sum_{i=1}^{n} (x^i(q) - x^i(\gamma^{-1}.q)) \frac{\partial}{\partial x^i} = \sum_{i=l_\gamma+1}^{n} (x^i(q) - x^i(\gamma^{-1}.q)) \frac{\partial}{\partial x^i} \,. \tag{7.50}$$

On $(T_pM^\gamma)^\perp$, the linear map $q \mapsto (1-\gamma^{-1})q$ is invertible. So we can assume

$$\kappa^i = \begin{cases} 0 & i \leq l_\gamma \\ x^i & \text{else} \end{cases} \,. \tag{7.51}$$

We have reduced the situation now to $M = \mathbb{R}^n$ with a linear isometric Γ-action where $M^\gamma = \mathbb{R}^{l_\gamma} \times \{0\}$ and we want to determine the cohomology of

$$\ldots \longrightarrow C^\infty(\mathbb{R}^n) \otimes \bigwedge\nolimits^k (e_1, \ldots, e_n) \xrightarrow{\kappa^i e_i \wedge -} C^\infty(\mathbb{R}^n) \otimes \bigwedge\nolimits^{k+1} (e_1, \ldots, e_n) \longrightarrow \ldots \tag{7.52}$$

The differential does not act on $e_1, \ldots, e_{l_\gamma}$. The correct way to phrase this is that $\bigwedge^k(e_1, \ldots, e_n) \cong \bigoplus_{p+q=k} \bigwedge^p(e_1, \ldots, e_{l_\gamma}) \otimes \bigwedge^q(e_{l_\gamma+1}, \ldots, e_n)$ and that under this isomorphism the differential is the total differential of a double complex where the p-differential is zero. The cohomology becomes:

$$\begin{aligned} H^k\left(C^\infty(\mathbb{R}^n) \otimes \bigwedge(e_1, \ldots, e_n)\right) &\cong \bigoplus_{p+q=k} \bigwedge\nolimits^p (e_1, \ldots, e_{l_\gamma}) \otimes H^q \left(C^\infty(\mathbb{R}^n) \otimes \bigwedge(e_{l_\gamma+1}, \ldots, e_n)\right) \\ &\cong \bigwedge\nolimits^{k-n+l_\gamma} (e_1, \ldots, e_{l_\gamma}) \otimes C^\infty(\mathbb{R}^{l_\gamma}) \\ &\cong \Gamma\left(M^\gamma, \bigwedge\nolimits^{k-n+l_\gamma} TM^\gamma\right) \end{aligned} \tag{7.53}$$

Here we used that by Lemma 7.12 only the term for $q = n - l_\gamma$ is nonzero. The isomorphism is given by disregarding all components except those in $\bigwedge^{k-n+l_\gamma}(e_1, \ldots, e_{l_\gamma}) \otimes e_{l_\gamma+1} \wedge \cdots \wedge e_n$ and restricting to $\mathbb{R}^{l_\gamma} = M^\gamma$. The inverse isomorphism is induced by pullback along $\mathbb{R}^n \to \mathbb{R}^{l_\gamma}$ and multiplication by $e_{l_\gamma+1} \wedge \cdots \wedge e_n$. This is just a sort of extension of the multivector field. Note that *any* extension suffices, since this will be a right inverse of restriction and hence automatically also a left inverse.

Lemma 7.12. *Let $\kappa^i = 0$ for $i \leq l_\gamma$ and $\kappa^i = x^i$ else. The cohomology of the dual Koszul complex for $\mathbb{R}^n$ with differential $\kappa^i e_i \wedge -$ is:*

$$H^k(C^\infty(\mathbb{R}^n) \otimes \bigwedge(e_1, \ldots, e_n), \kappa^i e_i \wedge -) \cong \begin{cases} C^\infty(\mathbb{R}^{l_\gamma}) & k = n - l_\gamma \\ 0 & \textit{else} \end{cases} \tag{7.54}$$

[14] κ is constructed from $\tilde{X}$, which we denoted by X in Prop. 7.2. Around the diagonal, X was the Euler vector field of the normal bundle for a tubular neighbourhood embedding. We choose this embedding to be induced by the exponential map of the Γ-invariant Riemannian metric.

Proof. For simplicity, we assume $l_\gamma = 0$. The proof still works more generally by adding a parameter globally or noting that $C^\infty(\mathbb{R}^{l_\gamma}) \otimes -$ is exact. We start with $k = 0$. A smooth function f for which $x^i f = 0$ everywhere must vanish identically. The case $k = 1$ is the most interesting. Let $X = f^i e_i$ with $\kappa \wedge X = 0$. Rewriting this in the basis $e_i \wedge e_j$ this translates to $x^i f^j = x^j f^i$. To show that this is exact, we need to write $f^i = f x^i$ for some smooth function f. Differentiating with respect to x^j we get:

$$f^i + x^j \partial_j f^i = x^i \partial_j f^i \tag{7.55}$$

Using this we can calculate:

$$\begin{aligned} f^i(x) = \int_0^1 \frac{d}{dt} t^n f(tx)\mathrm{d}t &= \int_0^1 n t^{n-1} f^i(tx) + t^n \sum_{j=1}^n \partial_j f^i(tx) x^j \mathrm{d}t \\ &= \int_0^1 t^{n-1} \sum_{j=1}^n (f^i + \partial_j f^i x^j)(tx)\mathrm{d}t \\ &= \sum_{j=1}^n \int_0^1 t^{n-1} (x^i \partial_j f^j)(tx)\mathrm{d}t \\ &= \left(\sum_{j=1}^n \int_0^1 t^n \partial_j f^j(tx)\mathrm{d}t \right) x^i \end{aligned} \tag{7.56}$$

This shows that also the first cohomology vanishes. Let $k < n$ be arbitrary now. Similarly, for $X = X^I e_I$ with $\kappa \wedge X = 0$ and $I = (i_1 < \cdots < i_k)$ running through ordered multiindices, we have $X = \kappa \wedge Y$ where

$$Y^{(j_2,\ldots,j_k)} = \sum_{j_1=1}^n \int_0^1 t^n \partial_{j_1} X^{(j_1,\ldots,j_k)}(tx)\mathrm{d}t\,. \tag{7.57}$$

For degree n we note that the condition $\kappa \wedge X = 0$ is void. Writing $Y = (-1)^i Y^i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$ we have $\kappa \wedge Y = (\sum_i x^i Y^i) e_1 \wedge \cdots \wedge e_n$. By Taylor's theorem the image of this are precisely all functions in $C^\infty(M)$ that vanish at the origin. Evaluation at 0 hence gives an isomorphism $H^n(C^\infty(\mathbb{R}^n) \otimes \bigwedge(e_1, \ldots, e_n)) \cong \mathbb{C}$. □

Remark 7.13. The apparent remarkable duality to the proof of the Poincaré Lemma is fully realized by applying the Fourier transform. Upon doing this the multiplication by x^i turns into differentiation and $x^i f^j - x^j f^i = 0$ becomes $d(\sum_i \hat{f}^i dx^i) = 0$.

Note that the dimension of the fixed point set M^γ might jump from point to point. However, as we have seen it is locally constant and each component M^γ_α is a manifold. Since the cohomology of the global dual Koszul complex depends on $\dim(M^\gamma_\alpha) = l_\gamma$, we need to take each component individually into account. Our localisation argument above now produces:

Proposition 7.14. *The cohomology of the dual Koszul complex is*

$$H^k(\mathfrak{X}^*(M), \kappa \wedge -) \cong \bigoplus_{M^\gamma_\alpha \in \mathrm{Comp}(M^\gamma)} \Gamma\left(M^\gamma_\alpha, \bigwedge^{k-\dim(M)+\dim(M^\gamma_\alpha)} TM^\gamma_\alpha \right). \tag{7.58}$$

Hence, also the Hochschild cohomology of the $C^\infty(M)$-bimodule A_γ is given by

$$H^k(A, A_\gamma) \cong \bigoplus_{M^\gamma_\alpha \in \mathrm{Comp}(M^\gamma)} \Gamma\left(M^\gamma_\alpha, \bigwedge^{k-\dim(M)+\dim(M^\gamma_\alpha)} TM^\gamma_\alpha \right). \tag{7.59}$$

Let $Y_\alpha \in \Gamma\left(M, \bigwedge^{\operatorname{codim} M^\gamma_\alpha}\right)$ *such that the restriction* $Y_\alpha|_{M^\gamma_\alpha}$ *is in* $\Gamma\left(\bigwedge^{\operatorname{codim}(M^\gamma_\alpha)}(TM^\gamma_\alpha)^\perp\right)$. *Let* $X \in \Gamma\left(M^\gamma_\alpha, \bigwedge^{k-\operatorname{codim}(M^\gamma_\alpha)} TM^\gamma_\alpha\right)$ *and* $\tilde{X} \in \Gamma\left(M, \bigwedge^{k-\operatorname{codim}(M^\gamma_\alpha)} TM\right)$ *be any extension. Then, the isomorphism maps* X *to the multivector field* $\tilde{X} \wedge Y_\alpha$ *in* $\mathfrak{X}^k(M)$*. This multivector field naturally acts on* $A^k = C^\infty(M)^k$*, by which it is included as a cocycle* $A^{\otimes k} \to A_\gamma$ *into the Hochschild complex* $H^k(A, A_\gamma)$*. Explicitly, this is the cocycle mapping* $f_1, \ldots, f_k \in C^\infty(M)$ *to*

$$\langle \tilde{X} \wedge Y_\alpha, df_1 \wedge \cdots \wedge df_k \rangle \,. \tag{7.60}$$

Corollary 7.15. *The Hochschild cohomology of the convolution algebra as a* $C^\infty(M)$*-bimodule is given by*

$$H^k(A, A \rtimes \Gamma) \cong \bigoplus_{\gamma \in \Gamma} \bigoplus_{M^\gamma_\alpha \in \operatorname{Comp}(M^\gamma)} \Gamma\left(M^\gamma_\alpha, \bigwedge^{k-\dim(M)+\dim(M^\gamma_\alpha)} TM^\gamma_\alpha\right). \tag{7.61}$$

We now investigate what the Γ-action on the left corresponds to on the right. Suppose that $\Phi : A^{\otimes n} \to A \rtimes \Gamma$ is a Hochschild cocycle. As before, this decomposes as $\Phi = \sum_{\gamma \in \Gamma} \Phi^\gamma \delta_\gamma$. We have seen that in cohomology each Φ^γ may be written as a pairing with a special multivector field X^γ. Now the action of $\beta \in \Gamma$ on Φ becomes:

$$\begin{aligned}
(\beta.\Phi)(f_1, \ldots, f_n) &= \delta_\beta * \sum_{\gamma \in \Gamma} \Phi^\gamma(\beta^{-1}.f_1, \ldots, \beta^{-1}.f_n)\delta_\gamma * \delta_{\beta^{-1}} \\
&= \sum_{\gamma \in \Gamma} \beta.\langle X^\gamma, d(\beta^* f_1) \wedge \cdots \wedge d(\beta^* f_n)\rangle \delta_{\beta\gamma\beta^{-1}} \\
&= \sum_{\gamma \in \Gamma} \langle \beta_* X^\gamma, df_1 \wedge \cdots \wedge df_n \rangle \delta_{\beta\gamma\beta^{-1}}
\end{aligned} \tag{7.62}$$

Hence, Φ is invariant if and only if $\beta_* X^\gamma = X^{\beta\gamma\beta^{-1}}$. Denote by $\operatorname{Conj}(\Gamma)$ the set of conjugacy classes in Γ and by $Z(\gamma)$ the centralizer of γ. It is important to note that $\gamma.x = x$ if and only if $\beta\gamma\beta^{-1}.(\beta.x) = \beta.x$. So, $\beta : M^\gamma \to M^{\beta\gamma\beta^{-1}}$ is a diffeomorphism. By the above computation, an invariant cocycle is uniquely determined by its values along M^γ for γ running through any set of representatives for the conjugacy classes $\operatorname{Conj}(\Gamma)$. However, the condition $\beta_* X^\gamma = X^{\beta\gamma\beta^{-1}}$ is still interesting for $\beta \in Z(\gamma)$: The multivector field X^γ needs to be invariant with respect to $Z(\gamma)$. We have proven the following theorem (Proposition 3.10 in [NPPT06]):

Theorem 7.16. *The Hochschild cohomology of the convolution algebra* $A \rtimes \Gamma$ *associated to the action groupoid* $\Gamma \ltimes M \rightrightarrows M$ *is given by*

$$\begin{aligned}
H^k(A \rtimes \Gamma, A \rtimes \Gamma) &\cong H^k(A, A \times \Gamma)^\Gamma \\
&\cong \bigoplus_{\gamma \in \operatorname{Conj}(\Gamma)} \bigoplus_{M^\gamma_\alpha \in \operatorname{Comp}(M^\gamma)} \Gamma\left(M^\gamma_\alpha, \bigwedge^{k-\dim(M)+\dim(M^\gamma_\alpha)} TM^\gamma_\alpha\right)^{Z(\gamma)}.
\end{aligned} \tag{7.63}$$

The following shows how to link this to all of the preceding theory.

Example 7.17. Suppose that the Γ action is free. Then, M/Γ exists. As we have seen before, $\mathcal{A}_{\Gamma \ltimes M} = A \rtimes \Gamma$ is Morita equivalent to $C^\infty(M/\Gamma)$ as unital bornological algebras. In particular, their Hochschild cohomology agrees. The HKR theorem identifies the Hochschild cohomology of $C^\infty(M/\Gamma)$ with multivector fields on M/Γ. These in turn can be identified with invariant multivector fields on M. By the above theorem, we have a different method to compute $H^*(A \rtimes \Gamma, A \rtimes \Gamma)$. Note that in this special case, only the identity e has a fixed point set $M^e = M$ and we precisely recover the HKR theorem:

$$H^*(\mathcal{A}_{\Gamma \ltimes M}, \mathcal{A}_{\Gamma \ltimes M}) \cong \mathfrak{X}^*(M/\Gamma)\,. \tag{7.64}$$

Let us point out that the version of the HKR theorem that we proved earlier together with the 2-functoriality even included a formality result, i.e. that the Hochschild cochain complex is homotopy equivalent to its cohomology.

Remark 7.18. It would be interesting to know if there is a general formality for the Hochschild cohomology of proper étale action groupoids or even for orbifold groupoids (=proper étale). The methods applied above are particularly suited for abelian categories and working out the cohomology in Vect.

Let us conclude by looking at the image of the cochain map $\Phi : C^*_{\mathrm{def}}(\Gamma \ltimes M) \to C^*(A \ltimes \Gamma, A \ltimes \Gamma)$. The action groupoid is étale and hence the cochain map takes the form

$$\Phi(c)(f_1, \dots, f_k)(\gamma, m) = \sum_{\gamma = \gamma_1 \cdots \gamma_k} c(\gamma_1, \dots, \gamma_k, m) f_1 \cdot f_2(\gamma_2, \gamma_3 \dots \gamma_k . m) \dots f_k(\gamma_k, m)\,. \tag{7.65}$$

Since $\Gamma \ltimes M$ is proper, its deformation cohomology vanishes in degrees bigger than 1. Since it is also étale, we have $H^1_{\mathrm{def}}(\Gamma \ltimes M) = \mathfrak{X}(\Gamma \ltimes M)^{\mathrm{inv}}$ and $H^0_{\mathrm{def}}(\Gamma \ltimes M) = 0$. Via Φ, an invariant vector field $X \in \mathfrak{X}(\Gamma \ltimes M)$ acts on $A \rtimes \Gamma$ via the usual action:

$$\Phi(X)(f)(\gamma, m) = X_{(\gamma, m)} f = X f(\gamma, m)\,. \tag{7.66}$$

This means, that in cohomology Φ only includes the invariant vector field into the degree 1 part of $H^k(A \rtimes \Gamma, A \rtimes \Gamma)$ as the summand of the conjugacy class of the identity e.

8 Appendix

8.1 Constructions for proper Groupoids

If X is a Hausdorff space and $R \subset X \times X$ is the graph of an equivalence relation $\sim$, then $X/\sim$ is Hausdorff if and only if R is closed.

If $f : X \to Y$ is a proper continuous map between topological spaces and Y is locally compact, then f is closed.

If $f : X \to X/\sim$ is an open map and X is locally compact, then $X/\sim$ is locally compact.

Hence, if $G \rightrightarrows M$ is a proper groupoid, then M/G is Hausdorff and locally compact. Here, $\pi : M \to M/G$ is open since the saturated sets $\pi^{-1}(\pi(U)) = s(t^{-1}(U))$ are open.

The following is Proposition 6.11 in [Tu99].

Proposition 8.1. *Let $G \rightrightarrows M$ be a proper Lie groupoid equipped with a left Haar system. Then, there exists a smooth function $c : M \to \mathbb{R}$ such that*

1. *$\forall x \in M$ $\int_{t^{-1}(x)} c(s(g))\mathrm{d}g = 1$;*
2. *t: supp$(c \circ s) \to M$ is proper.*

We sometimes denote $c(s(g)) = \lambda(g)$ throughout the text.

Proof. Since G is proper, M/G is locally compact Hausdorff and $\pi : M \to M/G$ is open. Then, there is a family of functions $f_i \in C_c^\infty(G, [0, 1])$ such that $\pi(\{f_i > 0\})$ is a locally finite cover of M/G. (To construct such a family, take any cover of M by $U_i = \{f_i > 0\}$ and $f_i \in C_c^\infty(M)$. Then, pick a locally finite subcover of $\pi^{-1}(\pi(U_i))$. The remaining index set yields the family f_i).

Let $d = \sum_i f_i$. The sum is finite on every compact subset and hence defines a smooth function in $C^\infty(M)$. Let $K \subset M$ be compact. Denote the saturation of K by $\overline{K} := s(t^{-1}(K))$. Then, by local finiteness, $\pi(\overline{K}) = \pi(K)$ intersects only finitely many $\pi(\{f_i > 0\})$. Hence, $\mathrm{supp}(d) \cap \overline{K} \subset \bigcup_{j=1}^n \mathrm{supp}(f_{i_j})$ is a compact set. So, $g \in t^{-1}(K) \cap \mathrm{supp}(d \circ s)$ if and only if $t(g) \in K$ and $s(g) \in \mathrm{supp}(d)$. Reformulating, we get that $s(g) \in \mathrm{supp}(d) \cap \overline{K}$. Hence $g \in s^{-1}(\mathrm{supp}(d) \cap \overline{K}) \cap t^{-1}(K)$. By properness of G, this set is compact and hence $t : \mathrm{supp}(d \circ s) \to M$ is proper.

Finally, we renormalize d to

$$c(x) := \frac{d(x)}{\int_{t^{-1}(x)} d(s(g))\mathrm{d}g} . \tag{8.1}$$

This is well-defined, since the integral will always be positive by construction. The properness property is not altered by this renormalization. □

8.2 Additional Material

Here we exhibit explicit formulas for an equivalence between the Hochschild cochain complexes based on the datum of a Morita context between algebras. This is inspired by [Lod98, Chapter 1.2].

Definition 8.2. A Morita context between two unital algebras A, B consists of two bimodules ${}_AP_B, {}_BQ_A$ and bimodule pairings

$$\langle \cdot, \cdot \rangle_A : P \otimes_B Q \to A \tag{8.2}$$

$$\langle \cdot, \cdot \rangle_B : Q \otimes_A P \to B \tag{8.3}$$

that are A- (resp. B-) bilinear, surjective and satisfy the interchange property:

$$\langle p,q\rangle_A p' = p\langle q,p'\rangle_B \tag{8.4}$$
$$\langle q,p\rangle_B q' = q\langle p,q'\rangle_A \tag{8.5}$$

Remark 8.3. A bimodule pairing is equivalently given by a bilinear map $\langle\cdot,\cdot\rangle : P\times Q \to A$ satisfying the following equalities:

$$\langle ap, qa'\rangle = a\langle p,q\rangle a' \tag{8.6}$$
$$\langle pb, q\rangle = \langle p, bq\rangle \tag{8.7}$$

We can reduce surjectivity to the basic assertion that there exist $p_i, y_j \in P$ and $q_i, x_j \in Q$ such that

$$1_A = \sum_i \langle p_i, q_i\rangle_A \tag{8.8}$$
$$1_B = \sum_j \langle x_j, y_j\rangle_B \tag{8.9}$$

Proposition 8.4 (Morita Invariance of Hochschild Cohomology)**.** *Let A, B be unital algebras and $(P, Q, \langle\cdot,\cdot\rangle_A, \langle\cdot,\cdot\rangle_B)$ a Morita context.*
Then, there is a chain homotopy equivalence between the Hochschild cochain complexes given by:

$$C^\bullet(A,A) \underset{\Psi}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} C^\bullet(B,B) \tag{8.10}$$

Where a cocycle $f : A^{\otimes n} \to A$ in $C^n(A,A)$ is mapped to the following map $\Phi(f) : B^{\otimes n} \to B$:

$$\begin{aligned}&\Phi(f)(b_1,\dots,b_n)\\ &\quad= \sum_{j_0,j_1,\dots,j_n} \langle x_{j_0}, f\Big(\langle y_{j_0}, b_1 x_{j_1}\rangle_A \otimes \langle y_{j_1}, b_2 x_{j_2}\rangle_A \otimes \cdots \otimes \langle y_{j_{n-1}}, b_n x_{j_n}\rangle_A\Big) y_{j_n}\rangle_B\end{aligned} \tag{8.11}$$

Analogously, for $g \in C^n(B,B)$:

$$\begin{aligned}&\Psi(g)(a_1,\dots,a_n)\\ &\quad= \sum_{i_0,i_1,\dots,i_n} \langle p_{i_0}, f\Big(\langle q_{i_0}, a_1 p_{i_1}\rangle_B \otimes \langle q_{i_1}, a_2 p_{i_2}\rangle_B \otimes \cdots \otimes \langle q_{i_{n-1}}, a_n p_{i_n}\rangle_B\Big) q_{j_n}\rangle_A\end{aligned} \tag{8.12}$$

A (pre-cosimplicial) chain homotopy $\Psi\Phi \simeq id$ is given by

$$h = \sum_{m=0}^{n} (-1)^m h_m : C^{n+1}(A,A) \to C^n(A,A)\,, \tag{8.13}$$

$$\begin{aligned}h_m(f)(a_1,\dots,a_n) = \sum_{j_0,\dots,j_m,k_0\dots,k_m} &\langle p_{k_0}, x_{i_0}\rangle_A f\Big(\langle y_{i_0}, q_{k_0}\rangle a_1 \langle p_{k_1}, x_{j_1}\rangle, \dots,\\ &\langle y_{j_{m-1}}, q_{k_{m-1}}\rangle a_m \langle p_{k_m}, x_{j_m}\rangle, \langle y_{j_m}, q_{j_m}\rangle, a_{m+1}, \dots, a_n\Big)\end{aligned} \tag{8.14}$$

Remark 8.5. By the Dold-Kan correspondence, the category of simplicial abelian groups is equivalent to the category of chain complexes in nonnegative degrees. Analogously, cosimplicial abelian groups are equivalent to cochain complexes where the differential is the alternating sum of cosimplicial face maps. If we have a graded abelian group without degeneracy maps and only face maps d_i we call it precosimplical.

Actually, Φ, Ψ in the proposition above are maps between precosimplical abelian groups, that is $d_i\Phi = \Phi d_i$. Let $f, g : C^* \to D^*$ be maps between precosimplical abelian groups.
By a precosimplicial homotopy h from f to g we mean a set of maps $h^n_m : C^n \to D^{n-1}$, $0 \leq m \leq n-1$ such that

$$h_i d_j = \begin{cases} d_{j-1}h_i & 0 \leq i+1 < j \leq n+1 \\ d_j h_{i-1} & 0 \leq j < i \\ h_{i+1}d_j & j = i+1 \end{cases} \tag{8.15}$$

and, crucially, $h_0 d_0 = f$, $h_n d_{n+1} = g$. Then

$$h^n = \sum_{m=0}^{n-1} (-1)^m h_m : C^n \to D^{n-1} \tag{8.16}$$

is a homotopy of the cochain maps f, g on the respective cochain complexes.

9 References

[AAC13] Camilo Arias Abad and Marius Crainic. Representations up to homotopy and Bott's spectral sequence for Lie groupoids. *Advances in Mathematics*, 248:416–452, 2013.

[Beh02] Kai Behrend. Cohomology of stacks. Lecture Notes, 2002. Lecture Notes, availabe at .

[Blo08] Christian Blohmann. Stacky Lie groups. *International Mathematics Research Notices. IMRN*, 2008. rnn082.

[BX11] Kai Behrend and Ping Xu. Differentiable stacks and gerbes. *The Journal of Symplectic Geometry*, 9(3):285–341, 2011.

[Bü10] Theo Bühler. Exact categories. *Expositiones Mathematicae*, 28(1):1–69, 2010.

[CdSW99] Ana Cannas da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[CFM21] Marius Crainic, Rui Loja Fernandes, and Ioan Mărcuţ. *Lectures on Poisson geometry*, volume 217 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2021.

[CM01] Marius Crainic and Ieke Moerdijk. Foliation groupoids and their cyclic homology. *Advances in Mathematics*, 157(2):177–197, 2001.

[CMS15] Marius Crainic, João Nuno Mestre, and Ivan Struchiner. Deformations of Lie groupoids. *International Mathematics Research Notices, Volume 2020, Issue 21, November 2020, Pages 7662-7746*, 2015.

[Con85] Alain Connes. Noncommutative differential geometry. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, (62):257–360, 1985.

[Cra03] Marius Crainic. Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes. *Commentarii Mathematici Helvetici*, 78(4):681–721, 2003.

[CST04] Joachim Cuntz, Georges Skandalis, and Boris Tsygan. *Cyclic homology in non-commutative geometry*, volume 121 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Operator Algebras and Non-commutative Geometry, II.

[GM04] L. Grunenfelder and M. Mastnak. Cohomology of abelian matched pairs and the Kac sequence. *Journal of Algebra*, 276(2):706–736, 2004.

[Grø96] Niels Grønbæk. Morita equivalence for self-induced Banach algebras. *Houston Journal of Mathematics*, 22(1):109–140, 1996.

[HG07] Andre Henriques and David Gepner. Homotopy theory of orbispaces. Preprint, arXiv:math/0701916, 2007.

[HN77] Henri Hogbe-Nlend. *Bornologies and functional analysis.* North-Holland Mathematics Studies, Vol. 26. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Introductory course on the theory of duality topology-bornology and its use in functional analysis, Translated from the French by V. B. Moscatelli.

[Kha13] Masoud Khalkhali. *Basic noncommutative geometry.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, second edition, 2013.

[KM97] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[Kon03] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Letters in Mathematical Physics*, 66(3):157–216, 2003.

[KP21] Bjarne Kosmeijer and Hessel Posthuma. Lie Groupoid Deformations and Convolution Algebras. Preprint, arXiv:2107.14301, 2021.

[Lan00] N. P. Landsman. The Muhly-Renault-Williams theorem for Lie groupoids and its classical counterpart. *Letters in Mathematical Physics*, 54(1):43–59, 2000.

[Lan06] N. P. Landsman. Lie groupoids and Lie algebroids in physics and noncommutative geometry. *Journal of Geometry and Physics*, 56(1):24–54, 2006.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2013.

[Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[Mey04] Ralf Meyer. Embeddings of derived categories of bornological modules. Preprint, arXiv:math/0410596, 2004.

[Mey07] Ralf Meyer. *Local and analytic cyclic homology*, volume 3 of *EMS Tracts in Mathematics.* European Mathematical Society (EMS), Zürich, 2007.

[Mey11] Ralf Meyer. Smooth and rough modules over self-induced algebras. *Münster Journal of Mathematics*, 4:29–51, 2011.

[Nac49] Leopoldo Nachbin. Sur les algèbres denses de fonctions différentiables sur une variété. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 228:1549–1551, 1949.

[NPPT06] N. Neumaier, M. J. Pflaum, H. B. Posthuma, and X. Tang. Homology of formal deformations of proper étale Lie groupoids. *Journal für die Reine und Angewandte Mathematik*, 593:117–168, 2006.

[PPT20] Markus J. Pflaum, Hessel B. Posthuma, and Xiang Tang. On the Hochschild homology of convolution algebras of proper Lie groupoids. Preprint, arXiv:2009.03216, 2020.

[PS01] Fabienne Prosmans and Jean-Pierre Schneiders. A Homological Study of Bornological Spaces. Preprint, `http://www.analg.ulg.ac.be/jps/rec/hsbs.pdf`, 2001.

[Trè06] François Trèves. *Topological vector spaces, distributions and kernels.* Mineola, NY: Dover Publications, unabridged republication of the 1967 original edition, 2006.

[Tu99] Jean Louis Tu. La conjecture de Novikov pour les feuilletages hyperboliques. *K-Theory. An Interdisciplinary Journal for the Development, Application, and Influence of K-Theory in the Mathematical Sciences*, 16(2):129–184, 1999.

[Wal18] James Wallbridge. Homotopy theory in a quasi-abelian category. Preprint, arXiv:1510.04055, 2018.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.