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Twisted Cohomology via twisted $0|1$-dimensional Field Theories

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IMPRS Moduli Spaces

Introduction: What are TFTs?

Topological Field Theories are important in various areas of mathematics and physics. This is due to the great flexibility of Atiyah's original definition.
An n-dimensional *topological field theory* is a symmetric monoidal functor Z from the bordism category Bord_n to the category of vector spaces Vect.

$$Z : \mathsf{Bord}_n \to \mathsf{Vect} \tag{1}$$

Explicitly, this means that Z assigns to every closed $(n-1)$-manifold Y a vector space $Z(Y)$ and to every n-manifold X with boundary $\partial X = Y_0 \sqcup Y_1$ a linear map $Z(X) : Z(Y_0) \to Z(Y_1)$. Furthermore being monoidal requires that $Z(X \sqcup X') = Z(X) \otimes Z(X')$.
This definition is similar to the definition of a representation. The zoo of examples is very large.

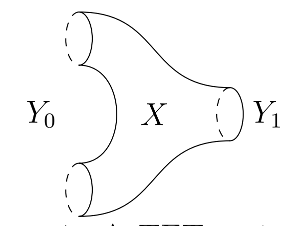

Figure 1: A TFT assigns to this bordism a linear map between vector spaces $Z(X) : Z(Y_0) \to Z(Y_1)$. Here, this could be a multiplication $Z(S^1) \otimes Z(S^1) \to Z(S^1)$.

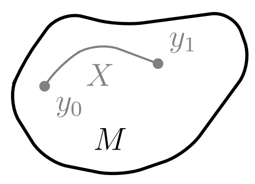

Figure 2: A 1-dimensional TFT over M assigns vector spaces to points and linear maps to paths in M.

There are a number of modifications that can be made in the definition leading to various types of TFTs. For example, we can replace the bordism category Bord_n by $\mathsf{Bord}_n(M)$ and require bordisms to be submanifolds of M.
Below we describe a variation where the bordisms are supermanifolds of superdimension $0|1$. In this case the bordisms will not have boundary since they are points with an additional "superdirection". It turns out that already this degenerate case is interesting.
This poster is mainly a review of [HKST11] and most of the results can be found there. However, the classification of twists and the relation to twisted cohomology is unpublished.
Below, we explain how twisted supersymmetric field theories recovers twisted de Rham cohomology.
This fits into the Stolz-Teichner Program which aims to interpret field theories over a manifold X as cocycles for cohomology theories. (c.f. [ST11])

Basic Notions from Supergeometry

Supergeometry originated in physics to treat elementary particles - bosons and fermions - simultaneously. It is an extension of ordinary geometry that incorporates so-called *odd coordinates*. This means that the commutative algebra of functions is replaced by a *supercommutative* algebra, i.e. a $\mathbb{Z}_2$-graded algebra whose elements commute or anticommute according to their parity.

The standard model is $\mathbb{R}^{p|q}$. This is defined to be the topological manifold $\mathbb{R}^p$ together with the sheaf of functions $\mathcal{O}_{p|q}(U) = C^\infty(U) \otimes \bigwedge^*(\theta_1, \ldots, \theta_q)$. The generators θ_i of the exterior algebra live in degree 1 and are called odd coordinates.

A *supermanifold* $M = (|M|, \mathcal{O}_M)$ is a topological manifold $|M|$ with a structure sheaf $\mathcal{O}_M$ that is locally isomorphic to $\mathbb{R}^{p|q}$.

Example 1. Any smooth manifold can be regarded as a supermanifold whose structure sheaf lives only in degree 0.

Example 2. Let M be a smooth manifold. The *odd tangent bundle* ΠTM is the supermanifold (M, Ω^*_M) whose algebra of functions are the differential forms. In a coordinate patch, the odd coordinates are given by dx^i.

The following is a miraculous way in which supergeometry gives rise to differential forms.

Theorem 3. *The odd tangent bundle ΠTM can naturally be identified with the internal hom $\hom(\mathbb{R}^{0|1}, M)$ in the category of supermanifolds.*

The action of the super Lie group $\mathrm{Diff}(\mathbb{R}^{0|1})$ on ΠTM has infinitesimal generators the de Rham differential and the Euler vector field which multiplies a form by its degree.

Sketch of Proof. A map of supermanifolds $\mathbb{R}^{0|1} \to M$ is an algebra homomorphism $C^\infty(M) \to \bigwedge^*(\theta) = \mathbb{R}[\theta]/(\theta^2)$. The θ-component must be a derivation of $C^\infty(M)$. But $\Omega^*(M)$ carries the universal derivation d and hence this is equivalently a map $\Omega^*(M) \to \mathbb{R}$, i.e. a point in ΠTM. $\square$

The super Lie group of diffeomorphisms of $\mathbb{R}^{0|1}$ is

$$\mathrm{Diff}(\mathbb{R}^{0|1}) = \mathbb{R}^\times \ltimes \mathbb{R}^{0|1}, \tag{2}$$

where the even part $\mathbb{R}^\times$ acts by dilation and the odd part $\mathbb{R}^{0|1}$ acts by translation on $\mathbb{R}^{0|1}$.

Twisted Cohomology

There are a few equivalent ways of defining what twisted cohomology is. We stick to a straight forward geometric definition. A *twist* for the de Rham cohomology of a manifold M is a real rank 1 vector bundle $\mathbb{L}$ over M together with a connection ∇. The *twisted cohomology* $H^*(M;\mathbb{L})$ is the sheaf cohomology of the sheaf of sections s satisfying $\nabla s = 0$.

If $\mathbb{L} = M \times \mathbb{R}$ is a trivial bundle equipped with the trivial connection, then this computes the cohomology of the constant sheaf $\underline{\mathbb{R}}$, i.e. ordinary de Rham cohomology of M.

Suppose that ∇ is a flat connection. This can always be achieved for real line bundles. Then, there is a canonical way to compute $H^*(M;\mathbb{L})$. The connection extends to a unique differential d_∇ on the $\mathbb{L}$-valued differential forms $\Omega^*(M;\mathbb{L})$ that satisfies the Leibniz rule. The twisted cohomology is then the cohomology of the following *twisted de Rham complex*:

$$0 \longrightarrow \Gamma(M;\mathbb{L}) \xrightarrow{\nabla = d_\nabla} \Omega^1(M;\mathbb{L}) \xrightarrow{d_\nabla} \Omega^2(M;\mathbb{L}) \xrightarrow{d_\nabla} \ldots \tag{3}$$

0|1-TFTs and de Rham cohomology

We now want to unpack what a topological field theory of dimension $0|1$ on a manifold X is supposed to be. By definition, it is a monoidal functor from a certain bordism category $0|1\text{-}\mathsf{B}(X)$ to vector spaces.

- objects: only $\emptyset$, since this is the only $-1|1$-dimensional manifold.
- morphisms: diffeomorphism classes of finite collections of superpoints in X. A superpoint in X is a map $\mathbb{R}^{0|1} \to X$.
- The composition and monoidal structure agree. They are given by disjoint union.

Any monoidal functor must map the only object - the monoidal unit - to $\mathbb{R}$. The morphisms are mapped to endomorphisms of $\mathbb{R}$, i.e. multiplication by real numbers. Also, the functor is determined by its value on single superpoints. Any collection of such must act by the product of the individual values since the functor is monoidal. This suggests to only look at the category $0|1\text{-}\mathsf{B}_{\mathrm{conn}}(X)$ whose objects are superpoints in X and whose morphisms are diffeomorphisms of such.

$$\mathrm{Fun}^{\otimes}(0|1\text{-}\mathsf{B}(X), \mathsf{Vect}) \cong \mathrm{Fun}(0|1\text{-}\mathsf{B}_{\mathrm{conn}}(X), \mathbb{R}) \tag{4}$$

By what we said above and crucially by 3, we arrive at the following result.

Theorem 4. *We can identify the connected $0|1$-bordism category with the stacky quotient of the odd tangent bundle.* [1]

$$0|1\text{-}\mathsf{B}_{conn}(X) \cong \Pi TX // \mathrm{Diff}(\mathbb{R}^{0|1}). \tag{5}$$

Putting everything together, we arrive at:

$$0|1-\mathrm{TFT}(X) \cong \mathrm{Fun}(\Pi TX//\mathrm{Diff}(\mathbb{R}^{0|1}), \mathbb{R}) \tag{6}$$
$$\cong \Omega^*(X)^{\mathrm{Diff}(\mathbb{R}^{0|1})} \tag{7}$$

It turns out that the invariant differential forms are precisely those of degree zero with vanishing differential.

Theorem 5. *There is a bijection between $0|1$-dimensional TFTs over X and closed 0-forms:*

$$0|1-\mathrm{TFT}(X) \cong \Omega^0_{closed}(X) \tag{8}$$

To get higher degree forms we need to introduce twists.

Twists for $0|1$-TFTs

Definition 6. A *twist* for a $0|1$-dimensional TFT over X is a line bundle $\mathcal{T}$ over the quotient stack $\Pi TX//\mathrm{Diff}(\mathbb{R}^{0|1})$. A *$\mathcal{T}$-twisted $0|1$-dimensional TFT* is a section of the line bundle $\mathcal{T}$.

Bundles over the quotient stack can be thought of as bundles on ΠTX with an $\mathrm{Diff}(\mathbb{R}^{0|1})$-action. The fibers are super vector spaces.

Example 7. The trivial bundle is a line bundle whose sections are just smooth functions. This recovers non-twisted TFTs.

Example 8. Consider the map $\rho_n : \mathrm{Diff}(\mathbb{R}^{0|1}) = \mathbb{R}^\times \ltimes \mathbb{R}^{0|1} \to \mathbb{R}^\times$ mapping $(x, y) \mapsto (x^n, y)$. This induces a map $\Pi TX//\mathrm{Diff}(\mathbb{R}^{0|1}) \to \mathrm{pt}//\mathbb{R}^\times$ by which we can pull back the universal bundle over the point to obtain the twist $\mathcal{T}_n$. [2]

Proposition 9. *The $\mathcal{T}_n$-twisted field theories over X are precisely given by closed n-forms:*

$$\Gamma(\Pi TX//\mathrm{Diff}(\mathbb{R}^{0|1}; \mathcal{T}_n) = \Omega^n_{closed}(X). \tag{9}$$

Example 10. Let $\mathbb{L}$ be an ordinary line bundle over X. Then, $\Omega^*(M;\mathbb{L})$ is a locally free $\Omega^*(M)$-module of rank 1 and hence an even line bundle over ΠTX. If we denote the projection by $\pi : \Pi TX \to X$, this is the bundle $\pi^*\mathbb{L}$.

These examples cover almost all possible twists. Furthermore, we can recover twisted de Rham cohomology.

Theorem 11 (Classification of Twists)**.** *There is a bijection between even line bundles over $\Pi TX//\mathrm{Diff}(\mathbb{R}^{0|1})$ and pairs of a line bundle over X together with a group homomorphism $f : \mathbb{R}^\times \to \mathbb{R}^\times$. Denote the corresponding bundle by $(\pi^*\mathbb{L})^f$.*

If $f(x) = x^n$ and n is even, then sections of $(\pi^*\mathbb{L})^f =: \mathcal{T}_n^{\mathbb{L}}$ are given by closed twisted forms of degree n:

$$\Gamma(\Pi TX//\mathrm{Diff}(\mathbb{R}^{0|1}); (\pi^*\mathbb{L})^f) \cong \Omega^n_{closed}(X;\mathbb{L}). \tag{10}$$

If n is odd, the is true if we replace $(\pi^*\mathbb{L})^f$ by its odd partner. Hence, the TFTs over X twisted by $\mathbb{L}$ and f are in bijection with closed degree n differential forms with values in $\mathbb{L}$.

For all other group homomorphisms f there are no global sections.

So far, we only have cocycles of cohomology theories and no coboundaries. For this we need to pass to concordance classes.

Definition 12. Let X be a manifold and a_+, a_- be objects over X that we can pull back. Then, a_+ and a_- are called *concordant* if over $X \times \mathbb{R}$ there is an object b with $i_\pm^* b = \mathrm{pr}_\pm^* a_\pm$.

Here, $i_\pm : X \times (\pm 1, \pm\infty) \to X \times \mathbb{R}$ are the inclusions and $\mathrm{pr}_\pm$ the projections to X.

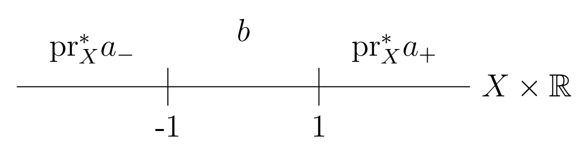

Figure 3: Picture of a concordance b between a_- and a_+.

Concordance makes sense for functions, differential forms, fiber bundles and for TFTs. Two closed forms are concordant if and only if they are cohomologous. We denote concordance classes of TFTs over X by $\mathrm{TFT}[X]$.

Corollary 13. *The concordance classes of twisted $0|1$-dimensional TFTs over X for the twist $\mathcal{T}_n^{\mathbb{L}}$ associated to a line bundle $\mathbb{L} \to X$ are isomorphic to twisted de Rham cohomology.*

$$0|1 - \mathrm{TFT}^{\mathcal{T}_n^{\mathbb{L}}}[X] \cong H^n(X;\mathbb{L}). \tag{11}$$

1 References

[HKST11] Henning Hohnhold, Matthias Kreck, Stephan Stolz, and Peter Teichner. Differential forms and 0-dimensional supersymmetric field theories. *Quantum Topol.*, 2(1):1–41, 2011.

[ST11] Stephan Stolz and Peter Teichner. Supersymmetric field theories and generalized cohomology. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 279–340. Amer. Math. Soc., Providence, RI, 2011.

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[1] To make this precise we would need to use the language of fibered categories and fibered functors. The bordism category $0|1\text{-}\mathsf{B}(X)$ is enhanced to the *family bordism category* fibered over the site of supermanifolds. This contains all families of bordisms parametrized by supermanifolds.

[2] Technically, if n is odd we pull back the odd partner of the universal bundle.