Representations up to homotopy via Kan extension to differentiable stacks

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IMPRS Moduli Spaces

Representations of Lie groupoids

A Lie groupoid is a groupoid object $\mathcal{G}_1 \overset{t}{\underset{s}{\rightrightarrows}} \mathcal{G}_0$ in manifolds such that the source and target maps s, t are submersions. This contains the following fundamental examples:

- Lie group $G \rightrightarrows *$.
- manifold $M \rightrightarrows M$ where all arrows are units.
- action groupoid $G \ltimes M \rightrightarrows M$ associated to an action $G \curvearrowright M$.

We write $\mathcal{G}_n$ for the manifold of n-composable arrows $(g_1, \dots, g_n)$ and we write $g_1 g_2 \dots g_n$ for the multiplication. This is the n^{th} level of the simplicial nerve.

Definition 1. A *representation* of $\mathcal{G}$ is a vector bundle over the base $E \to \mathcal{G}_0$ and an action $a_g : E_{s(g)} \to E_{t(g)}$ that satisfies $a_g a_h = a_{gh}$ for composable arrows $g, h \in \mathcal{G}_1$ and $a_{\mathrm{id}_x} = \mathrm{id}$.

This entails:

- representation of G on a vector space.
- vector bundle over M.
- G-equivariant vector bundle over $G \curvearrowright M$.

For a Lie group G, an important representation is the adjoint representation $G \curvearrowright \mathfrak{g}$ on its Lie algebra. The analogue of the Lie algebra for a Lie groupoid $\mathcal{G}$ is its Lie algebroid A or more precisely, the following 2-term complex of vector bundles over $\mathcal{G}_0$:

$$A \longrightarrow T\mathcal{G}_0 \ .$$

This cannot be made into a representation. Rather, there will be a *quasi-action* on $T\mathcal{G}_0$ that is associative up to the differential coming from A. This motivates the following definition.

Definition 2 (Abad, Crainic [AC09])**.** A *representation up to homotopy* consists of a dg vector bundle (E^*, ∂) on $\mathcal{G}_0$ together with cochain maps $R_n(g_1, \dots, g_n) : E_{s(g_n)} \to E_{t(g_1)}$ of degree $1-n$, $n \geq 1$,i.e. $R_n \in \mathrm{dg}\mathcal{V}\mathrm{ec}^{1-n}_{\mathcal{G}_n}(s^*E, t^*E)$, such that:

$$\sum_{i=1}^{n-1} (-1)^i R_{n-1}(g_1, \dots, g_i g_{i+1}, \dots, g_n) = \sum_{i=0}^{n} (-1)^i R_i(g_0, \dots, g_i) R_{n-i}(g_{i+1}, \dots, g_n)$$

Additionally, $R_1(\mathrm{id}) = \mathrm{id}$ and $R_n, n > 1$ vanishes when inserting units.

There is a dg-category $\mathrm{Rep}_\infty(\mathcal{G})$ of representations up to homotopy.

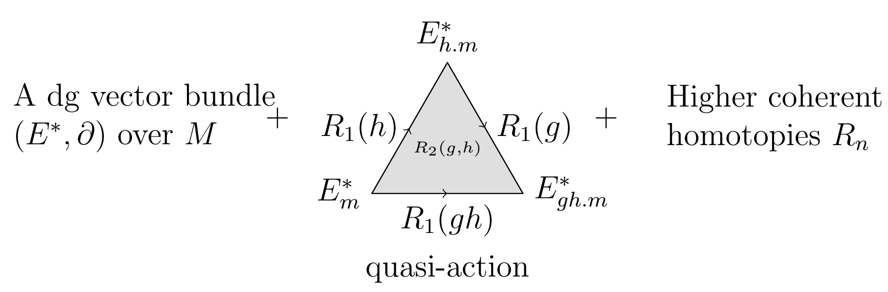


Figure 1: A representation up to homotopy of an action groupoid $G \ltimes M \rightrightarrows M$.

Conjecture: [AC09]

The functor $\mathrm{Rep}_\infty : \mathsf{LieGrpd}^{\mathrm{op}} \to \mathrm{dg}\mathcal{C}\mathrm{at}$ maps Morita equivalences to Dwyer-Kan equivalences. Equivalently, Rep_∞ factors through $\mathsf{LieGrpd} \to \mathsf{Stacks}$.

To explain the conjecture, we need to explain some of the terms.

Differentiable Stacks

From now on we will work with the ∞-category Stacks of stacks on manifolds with the Grothendieck topology of open covers. Its key features are: There is a fully faithful Yoneda embedding:

$$\mathcal{M}\mathrm{fld} \hookrightarrow \mathsf{Stacks}$$

Stacks is a reflective subcategory of $\mathsf{PreStacks} = \mathrm{Fun}(\mathcal{M}\mathrm{fld}^{\mathrm{op}}, \infty\mathsf{Grpd})$ consisting of those prestacks $\mathcal{F}$ that satisfy descent for any open cover (U_i) of U:

$$\mathcal{F}(U) = \mathrm{holim}\Big(\prod_i \mathcal{F}(U_i) \Longleftarrow \prod_{i,j} \mathcal{F}(U_{ij}) \Lleftarrow \prod \mathcal{F}(U_{ijk}) \cdots\Big) \qquad (1)$$

Stacks is bicomplete, so we can take weak quotients and this gives a functor $\mathsf{LieGrpd} \to \mathsf{Stacks}$:

$$[\mathcal{G}_0 // \mathcal{G}_1] = \mathrm{hocolim}_{\Delta^{\mathrm{op}}}\big(\mathcal{G}_0 \Longleftarrow \mathcal{G}_1 \Lleftarrow \mathcal{G}_2 \cdots\big)$$

The class W of morphisms in Grpd that are sent to equivalences in Stacks are called *Morita morphisms*. A Morita morphism $\mathcal{G} \to \mathcal{H}$ of Lie groupoids is a map such that $\mathcal{G}_1 \cong \mathcal{H}_1 \times_{\mathcal{H}_0 \times \mathcal{H}_0} (\mathcal{G}_0 \times \mathcal{G}_0)$. The full subcategory $\mathsf{Grpd}[W^{-1}] \subset \mathsf{Stacks}$ is called the category of differentiable stacks. Many functors/constructions on groupoids are known to be Morita invariant, i.e. constructions on the level of differentiable stacks.

Example 3. Consider the stack $\mathcal{V}\mathrm{ec} = BO$. A calculation shows that $\mathsf{Stacks}([\mathcal{G}_0//\mathcal{G}_1], \mathcal{V}\mathrm{ec}) \simeq \mathrm{Rep}(\mathcal{G})$. Hence, the category of representations of a Lie groupoid is a Morita invariant.

Example 4. This margin is too narrow to contain a long geometric motivation for stacks. Geometrically, most examples arise from Lie groupoids.

dg-categories

A dg-category is a category $\mathcal{C}$ enriched in cochain complexes Ch_k. By taking cohomology $H^0(-)$, we get an ordinary category $Ho(\mathcal{C})$ The category $\mathrm{dg}\mathcal{C}\mathrm{at}_k$ carries the Dwyer-Kan model structure.

- Weak equivalences are those enriched functors $F : \mathcal{C} \to \mathcal{D}$ that (a) induce quasi-isomorphisms $\mathcal{C}(c, c') \simeq \mathcal{D}(Fc, Fc')$ and (b) induce an equivalence $Ho(\mathcal{C}) \simeq Ho(\mathcal{D})$
- The fibrations are those functors $F : \mathcal{C} \to \mathcal{D}$ that induce (a) degreewise surjections and (b) an isofibration $Ho(\mathcal{C}) \to Ho(\mathcal{D})$. All objects are fibrant.

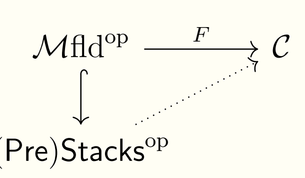
The dg-nerve $N^{dg} : \mathrm{dg}\mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ is a right Quillen functor. This means we can regard dg-categories as ∞-categories. The only advantage of dg-categories is that the algebra is very explicit.

Homotopy Kan extension to stacks

Let F be some sort of "geometric" structure, e.g. a function, a metric, a bundle, etc. An interesting question is: What is the correct version of F for Lie groupoids? "Correct" often means Morita invariant. Here is an abstract nonsense answer to this:

The basic recipe

Start with a functor $F : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathcal{C}$ where $\mathcal{C}$ is an ∞-category. Then, there is a unique way to extend F to a functor $F : \mathsf{PreStacks}^{\mathrm{op}} \to \mathcal{C}$ that maps homotopy colimits to homotopy limits. If F satisfies descent (1), it factors through the stackification functor and has the same continuity property.



We can calculate the value on a differentiable stack $[\mathcal{G}_0//\mathcal{G}_1]$ presented by a Lie groupoid $\mathcal{G}$ as the cosimplicial homotopy limit

$$F([\mathcal{G}_0//\mathcal{G}_1]) \simeq \mathrm{holim}_{[n]\in\Delta} F(\mathcal{G}_n) \, .$$

If F satisfies descent, this is by definition Morita invariant.

The main challenge is now the computation of the homotopy limit.

Example 5.

- Let $F = C^\infty : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathrm{Ch}_{\geq 0}(\mathbb{R})$. Then $\mathrm{holim}_n C^\infty(\mathcal{G}_n)$ is computed via totalization of the resulting double complex. The resulting cochain complex is otherwise known as differentiable groupoid cohomology $C^*(\mathcal{G})$ and is a smooth version of group cohomology.
- Let $F = (\Omega^*, d)$ be the de Rham complex. Then $\mathrm{holim}_n \Omega^*(\mathcal{G}_n)$ is again a double complex known as the Bott-Shulman-Stasheff complex. It can be used to compute the equivariant de Rham cohomology.
- Let $F = \mathcal{V}\mathrm{ec} : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$. Then the homotopy limit can be identified with the category of representations $\mathrm{Rep}(\mathcal{G})$.

Consider the functor $\mathrm{dg}\mathcal{V}\mathrm{ec}$ which assigns to a manifold the dg-category of bounded complexes of vector bundles.

Theorem: (DA)

Rep_∞ is the homotopy Kan extension of $\mathrm{dg}\mathcal{V}\mathrm{ec} : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathrm{dg}\mathcal{C}\mathrm{at}$ along the Yoneda embedding to $\mathsf{PreStacks}$, i.e.

$$\mathrm{Rep}_\infty(\mathcal{G}) \simeq \mathrm{holim}_\Delta \, \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n) \, .$$

$\mathrm{dg}\mathcal{V}\mathrm{ec}$ *does not* satisfy descent and hence Rep_∞, as defined above, is not Morita invariant.

Thus the primary answer to the conjecture is negative. **But it is really close to being true.** The problem is that the Tor-amplitude might be globally unbounded.

Computation of the homotopy limit

The Bousfield-Kan formula tells us that under some fibrancy assumptions the homotopy limit may be computed as the end

$$\mathrm{holim}_\Delta \, \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n) \simeq \int_\Delta \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n)^{\Delta^n} \, .$$

In [AO19] it is shown in general that these assumptions are always met in the case of dg categories. The dg-category $\mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n)^{\Delta^n}$ is a thickened version of $\mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n)$. A generic object for $n = 2$ looks like this:

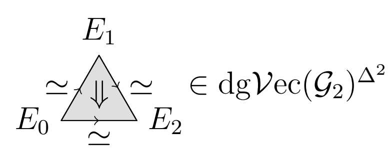


Figure 2: A 2-simplex decorated with weakly equivalent dg vector bundles over $\mathcal{G}_2$.

Computing the end amounts to glueing these simplices together.

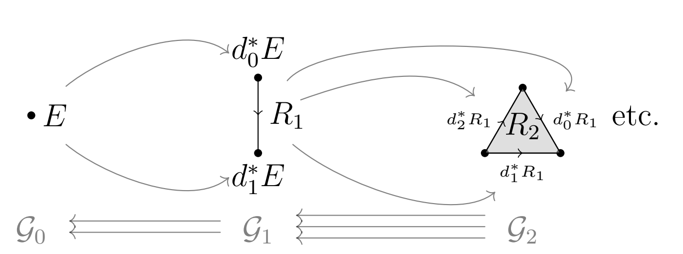


Figure 3: An object in the dg-category computed by the end is a dg vector bundle E over $\mathcal{G}_0$ together with a map $R_1(g) : E_{s(g)} \to E_{t(g)}$ that is associative up to a homotopy $R_2(g, h)$ and higher coherences.

Doing this carefully, we arrive at the identification of $\mathrm{Rep}_\infty(\mathcal{G})$ and $\mathrm{holim}_\Delta \, \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n)$.

Outlook

- The failure of Rep_∞ to be Morita invariant is the failure of $\mathrm{dg}\mathcal{V}\mathrm{ec}$ to satisfy descent. The ∞-category of perfect complexes of sheaves should be the correct replacement. Using such a refined theory, we get a category of perfect complexes on all higher stacks $\widetilde{\mathrm{Rep}}_\infty : \mathsf{Stacks}^{\mathrm{op}} \to \mathrm{dg}\mathcal{C}\mathrm{at}\mathcal{C}\mathrm{at}_\infty$.
- This is a proof-of-concept and there are many constructions for $\mathsf{LieGrpd}$ that could be revisited homotopically.

References

[AC09] Camilo Arias Abad and Marius Crainic. Representations up to homotopy and Bott's spectral sequence for Lie groupoids, 2009.

[AO19] Sergey Arkhipov and Sebastian Oersted. Homotopy limits in the category of dg-categories in terms of A_∞-comodules, 2019.

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