A NOTE ON DE RHAM COHOMOLOGY FOR HIGHER DIFFERENTIABLE STACKS

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Abstract. The goal of this note is to show that the Bott-Shulman-Stasheff complex associated to a Lie groupoid computes de Rham cohomology of the associated quotient stack. As a consequence, any hypercover (also called Morita equivalence) of Lie groupoids induces a quasiisomorphism on the Bott-Shulman-Stasheff complex. We will also prove the de Rham theorem along the way.

Let X be a simplicial manifold. The *Bott-Shulman-Stasheff complex* Ω_{BSS} is defined to be the total complex of the following double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^2(X_0) & \xrightarrow{\delta} & \Omega^2(X_1) & \xrightarrow{\delta} & \Omega^2(X_2) & \xrightarrow{\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^1(X_0) & \xrightarrow{\delta} & \Omega^2(X_1) & \xrightarrow{\delta} & \Omega^2(X_2) & \xrightarrow{\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^0(X_0) & \xrightarrow{\delta} & \Omega^2(X_1) & \xrightarrow{\delta} & \Omega^2(X_2) & \xrightarrow{\delta} & \dots
\end{array}$$

The horizontal differential is the alternating sum of the pullback along the face maps $\delta = \sum_{i=0}^n (-1)^i \delta_i^*$.

Theorem 0.1. *Let X be a simplicial manifold that presents the ∞-stack $\mathcal{X}$. Then the Bott-Shulman-Stasheff complex Ω_{BSS} computes the de Rham cohomology of $\mathcal{X}$.*

Corollary 0.2. *Let $f : X \to Y$ be a finite height hypercover of simplicial manifolds. Then the induced map f^* on the Bott-Shulman-Stasheff complexes is a quasi-isomorphism. In particular, this holds for all hypercovers between Lie n-groupoids.*

We have to give a few definitions. In the following we will not need Lie n-groupoids and we will instead refer to [Nui] for a discussion of hypercovers, Morita equivalence, etc. To talk about higher stacks on the site of manifolds we will use simplicial presheaves $\mathrm{s}\mathcal{P}\mathrm{re}\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fld})$ satisfying a descent condition.

Definition 0.3. A ∞-stack is a simplicial presheaf $F : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}$ satisfying the *descent condition*

$$F(M) \xrightarrow{\simeq} \mathrm{holim}_{\Delta^{\mathrm{op}}}\Big(\prod F(U_i) \rightrightarrows \prod F(U_{ij}) \Rrightarrow \prod F(U_{ijk}) \dots\Big) \qquad \text{(D)}$$

For any manifold M, the Yoneda embedding $y(M)$ defines a presheaf valued in $\mathcal{S}\mathrm{et} \subset \mathrm{s}\mathcal{S}\mathrm{et}$ satisfying descent. For any simplicial presheaf F there is an ∞-stack

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$\hat{F}$ and a map $F \to \hat{F}$ such that any map $F \to G$ into an ∞-stack factors through $\hat{F}$. $\hat{F}$ is called the *stackification* of F. The stackification of the (levelwise) Yoneda embedding defines an ∞-functor $\mathcal{M}\mathrm{fld}^{\Delta^{\mathrm{op}}} \to \mathcal{S}\mathrm{tacks}$.

$$\begin{aligned} \mathcal{M}\mathrm{fld}^{\Delta^{\mathrm{op}}} &\longrightarrow \mathcal{S}\mathrm{tacks} \\ X &\longmapsto \operatorname{hocolim}_{\Delta^{\mathrm{op}}} y(X_n) \end{aligned}$$

The ∞-category $\mathcal{S}\mathrm{tacks}$ has the following universal property. For any contravariant ∞-functor $F : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathcal{D}$ that satisfies descent (D) there is a unique ∞-functor $\mathrm{Ran}_{y^{\mathrm{op}}} F$ mapping homotopy colimits in $\mathcal{S}\mathrm{tacks}$ to homotopy limits in $\mathcal{D}$ and that makes the diagram below commutative:

$$\begin{array}{ccc} \mathcal{M}\mathrm{fld}^{\mathrm{op}} & \xrightarrow{F} & \mathcal{D} \\ \big\downarrow{\scriptstyle y^{\mathrm{op}}} & \nearrow{\scriptstyle \mathrm{Ran}_{y^{\mathrm{op}}} F} & \\ \mathcal{S}\mathrm{tacks}^{\mathrm{op}} & & \end{array}$$

We will be interested in the functor $\Omega : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathrm{Ch}_{\mathbb{R}}[q.i.^{-1}]$ that assigns to a manifold the cochain complex of differential forms. The target is $\mathcal{D}(\mathbb{R}) = \mathrm{Ch}_{\mathbb{R}}[q.i.^{-1}]$, the derived ∞-category of $\mathbb{R}$. The proof of Theorem 0.1 consists of 3 steps.

(1) We need to check that Ω satisfies the descent condition (D).
(2) Ω extends to a functor on stacks mapping colimits to limits. By abuse of notation we will also write Ω for this functor.
(3) We can *calculate* Ω on the stack $\mathcal{X}$ presented by a simplicial manifold X and show that it is quasiisomorphic to $\Omega_{BSS}(X)$.

To prove Corollary 0.2 we also need the following:

(4) Under the map $\mathcal{M}\mathrm{fld}^{\Delta^{\mathrm{op}}} \to \mathcal{S}\mathrm{tacks}$, hypercovers of simplicial manifolds are mapped to equivalences of stacks.

Remark 0.4. Finally, it is worthwhile to remark that the unqualified term ∞-category refers to any of the equivalent models such as quasicategories. For computations it is often easiest to work in model categories and all of this note could have been phrased in model categories. We will not discuss any comparison results.

1. Proofs

To compute homotopy limits of cochain complexes we will need the following Lemma. We will always consider the localization of $\mathrm{Ch}_{\mathbb{R}}$ at quasi-isomorphisms, i.e. the derived ∞-category $\mathcal{D}(\mathbb{R})$.

Lemma 1.1 (Prop 3.20 in [Ara23])**.** *Let $C : \Delta \to \mathrm{Ch}_{\mathbb{R}}$ be a cosimplicial diagram of cochain complexes. Then $\operatorname{holim}_{\Delta} C \simeq \mathrm{Tot}(DK(C))$ is the total complex of the double complex obtained from C by taking the differential to be alternating sums of face maps.*

We now show Step (1).

Lemma 1.2. *Ω satisfies descent.*

Proof. Let $\mathcal{U}$ be a cover of M. Using Lemma 1.1 for $[n] \mapsto \prod \Omega(U_{i_0 \dots i_n})$ we compute the homotopy limit to be the total complex of the Čech-de Rham double complex. We conclude by invoking [BT82][Theorem 8.8]. Their proof shows that by the generalized Mayer-Vietoris argument the rows of the double complex augmented by

$\Omega(M)$ are exact and hence the total complex also computes de Rham cohomology of M. □

Remark 1.3. The real singular cochains $M \mapsto C^*(M, \mathbb{R})$ also satisfy descent. Both Ω and $C^*(-, \mathbb{R})$ are invariant under smooth homotopies, i.e. $\mathrm{pr}^* : \Omega(M) \to \Omega(M \times \mathbb{R})$ is a quasiisomorphism.

Corollary 1.4 (De Rham Theorem)**.** *The natural map* $\int : \Omega^*(M) \to C^*(M, \mathbb{R})$ *is a quasiisomorphism for all* M*.*

Proof. Both agree on $M = *$ and by homotopy invariance also on contractible open sets. The integration is also an equivalence globally by using a good open cover and the descent property. □

Lemma 1.5. *Let* X *be a simplicial manifold. The Bott-Shulman-Stasheff complex computes the homotopy limit* $\operatorname{holim}_\Delta \Omega(X_n)$ *in* $\mathcal{D}(\mathbb{R})$*.*

Proof. This follows directly from Lemma 1.1 and the definition of $\Omega_{BSS}(X)$. □

We are now ready to prove Step (3).

Corollary 1.6. *Let* X *be a simplicial manifold presenting the stack* $\mathcal{X}$*. Then the de Rham complex of* $\mathcal{X}$ *is computed by the Bott-Shulman-Stasheff complex:* $\Omega(\mathcal{X}) \simeq \Omega_{BSS}(X)$*.*

Proof. This follows directly from the previous Lemma and the definition of $\Omega(\mathcal{X})$ in Step (2):

$$\Omega(\mathcal{X}) \simeq \Omega(\operatorname{hocolim}_{\Delta^{\mathrm{op}}} y(X_n)) \simeq \operatorname{holim}_\Delta \Omega(X_n) \simeq \Omega_{BSS}(X)\,. \qquad \square$$

Consider the functor that maps M to its underlying topological space:

$$\begin{aligned} \mathcal{M}\mathrm{fld} &\longrightarrow \mathcal{T}\mathrm{op} \\ M &\longmapsto |M| \end{aligned}$$

If we consider the relative category $\mathcal{T}\mathrm{op}[W^{-1}]$ where we invert weak homotopy equivalences, then $|-|$ satisfies a version of descent: $|M| \simeq \operatorname{hocolim}_{\Delta^{\mathrm{op}}} \coprod |U_{i_0 \dots i_n}|$. By the universal property of $\mathcal{S}$tacks we can extend this to a functor $|-| : \mathcal{S}\mathrm{tacks} \to \mathcal{T}\mathrm{op}[W^{-1}]$ that preserves homotopy colimits. We refer to this as *geometric realization.* We could now define singular cohomology as the singular cohomology of the geometric realization $\mathcal{X} \mapsto C^*(|\mathcal{X}|, \mathbb{R})$. The various descent properties imply:

Corollary 1.7. *Let* X *be a simplicial manifold presenting a stack* $\mathcal{X}$*. The singular cohomology of* $\mathcal{X}$ *coincides with de Rham cohomology and is computed by the Bott-Shulman-Stasheff complex* $\Omega_{BSS}(X)$*.*

To conclude we provide a reference for Step (4). The preprint [Nui] in fact proves much stronger structure theorems, but we are content with the following:

Theorem 1.8 (Prop. 4.9 in [Nui])**.** *Let* $f : X \to Y$ *be a hypercover of height* $< \infty$ *between simplicial manifolds. Then the induced map* $\mathcal{X} \to \mathcal{Y}$ *is an equivalence of stacks. In particular, this includes hypercovers of Lie* n*-groupoids for* $n < \infty$*.*

References

[Ara23] Kensuke Arakawa. Homotopy limits and homotopy colimits of chain complexes, 2023.

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