Homotopy reduction of multisymplectic structures in Lagrangian field theory

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In geometric mechanics, symplectic reduction allows to reduce the degrees of freedom of a mechanical system with hamiltonian symmetries by means of Noether's first theorem. Symplectic structures as they appear in mechanics are replaced by multisymplectic structures in the case of field theories. However, in general relativity (and other field theories with diffeomorphism symmetry) Noether's first theorem does not give rise to a Hamiltonian momentum map. A possible approach to this problem is to move on to higher algebraic structures like the L_∞-algebra of hamiltonian forms and homotopy momentum maps. We investigate the use of these higher structures in a possible multisymplectic reduction method.

Hamiltonian symmetries in symplectic geometry

An action $\rho : \mathfrak{g} \to \mathfrak{X}(X)$ on a symplectic manifold (X, ω) is **Hamiltonian** if

(i) $\forall a \in \mathfrak{g} \quad \mathcal{L}_{\rho(a)}\omega = 0$, and

(ii) $\exists\, \mu : \mathfrak{g} \to C^\infty_{\text{ham}}(X)$ momentum map.

Example: Angular momentum

$\mathbb{R}^{2n}$ with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ and symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ is the local model of symplectic geometry. In geometric mechanics, $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ is the classical phase space for the movement of a particle in n-dimensional space with position coordinate $\vec{q}$ and momentum coordinate $\vec{p}$.

Consider the SO(3)-action on $\mathbb{R}^6$ by rotations:

$$\rho : \text{so}(3) \cong \mathbb{R}^3 \longrightarrow \mathfrak{X}(\mathbb{R}^6), \rho(\vec{a})(\vec{q}, \vec{p}) = (\vec{a} \times \vec{q}, \vec{a} \times \vec{p})$$

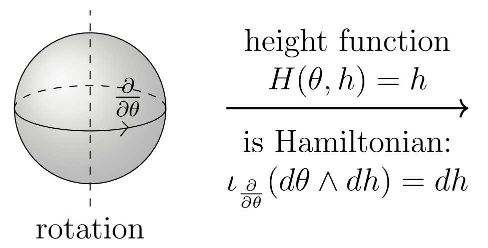
This action is Hamiltonian with momentum map

$$\mu : \text{so}(3) \cong \mathbb{R}^3 \longrightarrow C^\infty(\mathbb{R}^6), \mu(\vec{a})(\vec{q}, \vec{p}) = (\vec{q} \times \vec{p}) \cdot \vec{a}$$

given by the components of angular momentum.

Example: 2-Sphere

S^2 with cylindrical polar coordinates (θ, h) is a symplectic manifold with $\omega = d\theta \wedge dh$.



Symplectic reduction [MW74]

$$\begin{array}{ccccc} \mu^{-1}(\{0\}) & \xrightarrow{\;i\;} & X & & \omega \in \Omega^2(X) \\ \pi \big\downarrow & & & & \text{s.t. } \pi^*\omega_0 = i^*\omega \\ \mu^{-1}(\{0\})/G := X_0 & & & & \omega_0 \in \Omega^2(X_0) \end{array}$$

Note: $I_{\mu^{-1}(\{0\})} = \{f \in C^\infty(X) \,|\, f|_{\mu^{-1}(\{0\})} = 0\}$ is **coisotropic**, i.e. closed under the Poisson bracket.

From symplectic to multisymplectic geometry

(pre-)symplectic $(n = 1)$

$\omega \in \Omega^2(X)$ s.t. $d\omega = 0, \omega$ non-degenerate

$f \in C^\infty_{\text{ham}}(X) \Leftrightarrow \exists v_f \in \mathfrak{X}(X)$ with $\iota_{v_f}\omega = -df$

$C^\infty_{\text{ham}}(X)$ is a Poisson algebra with $\{f, g\} = \iota_{v_g}\iota_{v_f}\omega$

$\rho : \mathfrak{g} \to \mathfrak{X}(X)$ hamiltonian $\Leftrightarrow \exists \mu : \mathfrak{g} \to C^\infty_{\text{ham}}(X)$ s.t.
(1) $\forall a \in \mathfrak{g} \quad \iota_{\rho(a)}\omega = -d\mu(a)$
(2) μ is a morphism of Lie algebras

(pre-)multisymplectic $(n \geq 1)$

$\omega \in \Omega^{n+1}(X)$ s.t. $d\omega = 0, \omega$ non-degenerate

$\alpha \in \Omega^{n-1}_{\text{ham}}(X) \Leftrightarrow \exists v_\alpha \in \mathfrak{X}(X)$ with $\iota_{v_\alpha}\omega = -d\alpha$

$\Omega^{n-1}_{\text{ham}}(X)$ is part of an L_∞-algebra $L_\infty(X, \omega)$

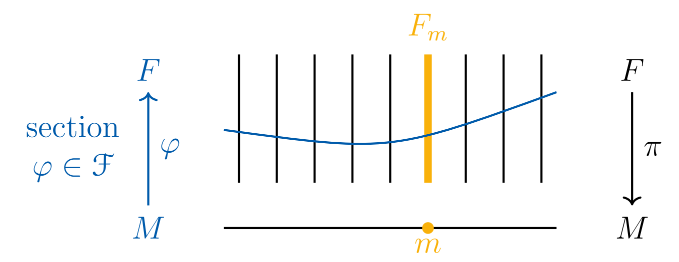
$\rho : \mathfrak{g} \to \mathfrak{X}(X)$ hamiltonian $\Leftrightarrow \exists \mu : \mathfrak{g} \to L_\infty(X, \omega)$ s.t.
(1) $\forall a \in \mathfrak{g} \quad \iota_{\rho(a)}\omega = -d\mu_1(a)$
(2) μ is a morphism of L_∞-algebras

Lagrangian field theory

Definition

A **local Lagrangian field theory** (M, F, L) consists of the following data:

- a smooth manifold M of $\dim(M) = n$ called **spacetime**
- a smooth fiber bundle $F \xrightarrow{\pi} M$ called **configuration bundle**
- a local map $L : \mathcal{F} = \Gamma(M, F) \to \Omega^n(M)$ called **Lagrangian**



Assume

- M compact and oriented,
- $\Sigma \subset M$ closed, cooriented submanifold of codimension 1.

Since the Lagrangian is local, it can be viewed as a form $L \in \Omega^{0,n}(J^\infty F)$ in the variational bicomplex $(\Omega^{\bullet,\bullet}(J^\infty F), d, \delta)$. [And89]

$$\begin{aligned} j^\infty : \quad & \mathcal{F} \times M \longrightarrow J^\infty F, \quad (\varphi, m) \longmapsto j^\infty_m \varphi \\ (j^\infty)^* : \quad & \Omega(J^\infty F) \longrightarrow \Omega(\mathcal{F} \times M) \end{aligned}$$

In this framework, the action principle of variational calculus can be phrased in cohomological terms since the acyclicity theorem [Tak79] guarantees that $\delta L = EL - d\gamma$ for some **boundary form** $\gamma \in \Omega^{1,n-1}(J^\infty F)$.

Action principle

critical point of action $S = \int_M L$	$=$	solution of Euler-Lagrange equation
δL is d-exact at $\varphi \in \mathcal{F}$	$\Leftrightarrow$	$EL\|_\varphi = 0$

The symmetries of Noether's first theorem are precisely those that leave the d-cohomology class of the Lagrangian invariant.

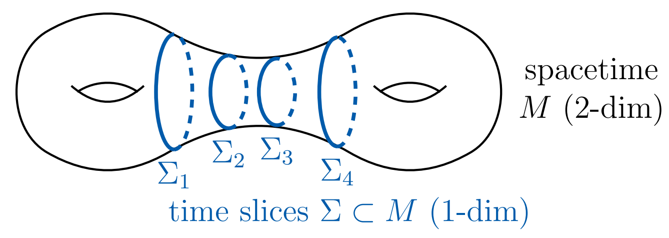
Theorem [Noe18]

If $\xi \in \mathfrak{X}_{\text{loc}}(\mathcal{F}) \subset \mathfrak{X}(J^\infty F)$ is a **Noether symmetry** with $\mathcal{L}_\xi L = d\alpha$, then $j_\xi = \alpha - \iota_\xi\gamma$ is a conserved current, i.e. d-closed on shell.

Proof: $dj_\xi = d(\alpha - \iota_\xi\gamma) = \mathcal{L}_\xi L - d\iota_\xi\gamma = \iota_\xi EL$.

Then the **Noether current** j_ξ can be integrated to the corresponding **Noether charge**

$$q_{\Sigma,\xi} = \int_\Sigma (j^\infty)^* j_\xi : \mathcal{F} \longrightarrow \mathbb{R}\,.$$



We consider the following premultisymplectic form:

$$\omega = (d + \delta)(L + \gamma) = EL + \delta\gamma \in \Omega^{n+1}(J^\infty F)\,.$$

A similar integration procedure yields a 2-form on the space of fields which is closed on shell and for which the charges are hamiltonian functions:

$$\Omega_\Sigma = \int_\Sigma (j^\infty)^*(EL + \delta\gamma) \overset{\text{on shell}}{=} \int_\Sigma (j^\infty)^*\delta\gamma \in \Omega^2(\mathcal{F})\,.$$

Theorem [Blo23]

Let X be a vector field on $J^\infty F$ with vertical component $X^\perp$. Then X is hamiltonian with respect to the premultisymplectic form $\omega = EL + \delta\gamma$ if and only if

(i) $\mathcal{L}_X\omega = 0$, and

(ii) $\iota_{X^\perp}EL = dj$ for some $j \in \Omega^{0,n-1}(J^\infty F)$.

Homotopy momentum maps

The **L_∞-algebra of hamiltonian forms** $L_\infty(J^\infty F, \omega)$ (Thm. 5.2 in [Rog12]) is the L_∞-algebra with

$$L_\infty(J^\infty F, \omega)_i = \begin{cases} \Omega^{n-1}_{\text{ham}}(J^\infty F) & \text{for } i = 0 \\ \Omega^{n-1+i}(J^\infty F) & \text{for } 1 - n \leq i < 0 \\ 0 & \text{otherwise} \end{cases}$$

and brackets $l_k : \wedge^k L_\infty(J^\infty F, \omega) \to L_\infty(J^\infty F, \omega)$, $1 \leq k \leq n + 1$, where

$$\begin{aligned} l_1(\alpha) &= (d + \delta)\alpha, \quad \alpha \in \Omega^i(J^\infty F) \text{ for } i < n - 1\,, \\ l_k(\alpha_1 \wedge \dots \wedge \alpha_k) &= \{\alpha_1, \dots, \alpha_k\} = \pm\iota_{\chi_1} \cdots \iota_{\chi_k}\omega\,, \\ & \qquad \alpha_i \in \Omega^{n-1}_{\text{ham}}(J^\infty F) \text{ with} \\ & \qquad \chi_i \in \mathfrak{X}_{\text{ham}}(J^\infty F)\,. \end{aligned}$$

This is a higher current algebra since

$$\alpha \in \Omega^{n-1}_{\text{ham}}(J^\infty F) \Rightarrow \alpha = \alpha_0 + \alpha_1 + \dots + \alpha_{n-1} \quad \text{with } \alpha_j \in \Omega^{j,n-1-j}(J^\infty F)$$

so that α_0 is a current.

A **homotopy momentum map** (Def./Prop. 5.1 in [CFRZ16]) for an action $\rho : \mathfrak{g} \to \mathfrak{X}(J^\infty F)$ is a morphism of L_∞-algebras $\mu : \mathfrak{g} \to L_\infty(J^\infty F, \omega)$ with components

$$\mu_i : \quad \wedge^i\mathfrak{g} \longrightarrow \Omega^{n-i}(J^\infty F), \quad 1 \leq i \leq n\,,$$

such that

$$\begin{aligned} \iota_{\rho(a)}\omega &= -(d + \delta)\mu_1(a) \\ \mu_1([a, b]) &= \{\mu_1(a), \mu_1(b)\} \pm (d + \delta)\mu_2(a, b) \end{aligned}$$

plus many other compatibility relations with the multibrackets.

Definition [DF99]

A **manifest symmetry** is a vector field $\chi \in \mathfrak{X}(J^\infty F)$ such that

(i) $\chi^\perp \in \mathfrak{X}_{\text{loc}}(\mathcal{F})$ and $\chi^\parallel \in \mathfrak{X}(M)$

(ii) $\mathcal{L}_\chi(L + \gamma) = 0\,. \quad \Rightarrow \mathcal{L}_\chi\omega = 0$

Lemma (Lem. 8.1 in [CFRZ16])

Any action $\rho : \mathfrak{g} \to \mathfrak{X}(J^\infty F)$ by manifest symmetries admits a homotopy momentum map $\mu : \mathfrak{g} \to L_\infty(J^\infty F, \omega = EL + \delta\gamma)$ given by

$$\mu_i(a_1, \dots, a_i) = \pm\iota_{\rho(a_1)} \cdots \iota_{\rho(a_i)}(L + \gamma)\,.$$

Consider the bicomplex

$$\Omega^{i,j}(\mathfrak{g}, J^\infty F) = \text{Hom}_{\mathbb{R}}(\wedge^i\mathfrak{g}, \Omega^j(J^\infty F))$$

with differential $\overline{d} = d_{\mathfrak{g}} + (d + \delta)$. Then:

$$\begin{aligned} \overline{\mu} &= \mu_1 \pm \mu_2 \pm \dots + \mu_n & &\in \Omega^n(\mathfrak{g}, J^\infty F) \\ \overline{\omega} &= \omega_1 \pm \omega_2 \pm \dots + \omega_n + \omega_{n+1} & &\in \Omega^{n+1}(\mathfrak{g}, J^\infty F) \end{aligned}$$

satisfy $\overline{d}\overline{\mu} = \overline{\omega}$ if and only if μ is a homotopy momentum map. (Prop. 2.5 in [FLGZ15]) Here we set:

$$\omega_i(a_1, \dots, a_i) = \pm\iota_{\rho(a_1)} \cdots \iota_{\rho(a_i)}\omega\,.$$

The homotopy zero locus

Definition (B., Blohmann)

Let $\overline{\mu} \in \Omega^n(\mathcal{A}, J^\infty F)$ be a homotopy momentum map for the local action $\rho : \mathcal{A} \to \mathfrak{X}_{\text{loc}}(\mathcal{F}) \times \mathfrak{X}(M), a \mapsto \rho(a) = (\xi_a, v_a)$ on $(J^\infty F, \omega)$ by manifest symmetries. The **homotopy zero locus** of $\overline{\mu}$ is the set

$$Z := \{\varphi \in \mathcal{F} \,|\, (j^\infty\varphi)^*\overline{\mu} \text{ is exact in } \Omega(\mathcal{A}, M)\} \subseteq \mathcal{F}\,.$$

The following theorem characterizes the homotopy zero locus of a local homotopy momentum map in Lagrangian field theory:

Theorem (B., Blohmann)

Let μ be a local homotopy momentum map for the local action $\rho : \mathcal{A} \to \mathfrak{X}_{\text{loc}}(\mathcal{F}) \times \mathfrak{X}(M), a \mapsto \rho(a) = (\xi_a, v_a)$ by manifest symmetries. A field $\varphi \in \mathcal{F}$ is in the homotopy zero locus of $\overline{\mu}$ if and only if the following two conditions are satisfied:

(i) $\forall a \in \mathcal{A} \quad d\left((j^\infty\varphi)^* j_a\right) = 0$

(ii) $\forall a, b \in \mathcal{A} \quad (j^\infty\varphi)^*(\iota_{\xi_a}\iota_{\xi_b}\delta\gamma) = 0$.

Interpretation: Assume that M is compact and oriented with a closed, cooriented submanifold $\Sigma \subset M$ of codimension 1. Then (i) implies that a field in the homotopy zero locus is in the zero locus of the charges:

$$\begin{aligned} \varphi \in Z \Rightarrow \quad & \forall a \in \mathcal{A} \quad d\left((j^\infty\varphi)^* j_a\right) = 0 & & \text{by (i)} \\ \Rightarrow \quad & \exists \alpha \in \Omega^{1,n-2}_{\text{loc}}(\mathcal{A}, M) \quad (j^\infty\varphi)^* = d\alpha & & \text{by acyclicity of } \Omega^{1,\bullet}(J^\infty A) \\ \Rightarrow \quad & \forall a \in \mathcal{A} \quad q_{\Sigma,a}(\varphi) = \int_\Sigma (j^\infty\varphi)^* j_a = \int_\Sigma d\alpha(a) = \int_{\partial\Sigma} \alpha(a) = 0 & & \text{by Stokes' theorem.} \end{aligned}$$

Condition (ii) yields an isotropy condition on the zero locus of the charges:

$$\begin{aligned} \varphi \in Z \Rightarrow \quad & \forall a, b \in \mathcal{A} \quad (j^\infty\varphi)^*(\iota_{\xi_a}\iota_{\xi_b}\delta\gamma) = 0 & & \text{by (ii)} \\ \Rightarrow \quad & \forall a, b \in \mathcal{A} \quad \{q_{\Sigma,a}, q_{\Sigma,b}\}(\varphi) = \Omega_\Sigma(\xi_a|_\varphi, \xi_b|_\varphi) = \int_\Sigma (j^\infty\varphi)^*(\iota_{\xi_a}\iota_{\xi_b}\delta\gamma) = 0 & & \text{by definition of } \Omega_\Sigma\,. \end{aligned}$$

Hence, the homotopy zero locus can be interpreted as the "universal coisotropic zero locus" of the corresponding charges.

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