

RATIONAL p -ADIC HODGE THEORY FOR RIGID-ANALYTIC VARIETIES

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ABSTRACT. We study a cohomology theory for rigid-analytic varieties over \mathbb{C}_p , without properness or smoothness assumptions, taking values in filtered quasi-coherent complexes over the Fargues–Fontaine curve, which compares to other rational p -adic cohomology theories for rigid-analytic varieties — namely, the rational p -adic pro-étale cohomology, the Hyodo–Kato cohomology, and the infinitesimal cohomology over the positive de Rham period ring. In particular, this proves a conjecture of Le Bras. Such comparison results are made possible thanks to the systematic use of the condensed and solid formalisms developed by Clausen–Scholze. As applications, we deduce some general comparison theorems that describe the rational p -adic pro-étale cohomology in terms of de Rham data, thereby recovering and extending results of Colmez–Nizioł.

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Date: August 14, 2023.

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1. INTRODUCTION

In this introduction, we fix a prime number p . We denote by K a complete discretely valued non-archimedean extension of \mathbb{Q}_p , with perfect residue field k , and ring of integers \mathcal{O}_K . We fix an algebraic closure \overline{K} of K . We denote by $C := \widehat{\overline{K}}$ the completion of \overline{K} , and by \mathcal{O}_C its ring of integers. We let $\mathcal{G}_K := \text{Gal}(\overline{K}/K)$ denote the absolute Galois group of K . We fix a compatible system $(1, \varepsilon_p, \varepsilon_{p^2}, \dots)$ of p -th power roots of unity in \mathcal{O}_C , which defines an element $\varepsilon \in \mathcal{O}_C^\flat$ with Teichmüller lift $[\varepsilon] \in A_{\text{inf}} = W(\mathcal{O}_C^\flat)$.

1.1. Background and motivation. In the last decade, the field of p -adic Hodge theory has witnessed dramatic advances, starting with Scholze’s development of perfectoid geometry, and Fargues–Fontaine’s discovery of *the fundamental curve*. In particular, Scholze initiated the study of the p -adic Hodge theory for rigid-analytic varieties in [Sch13a], proving the finiteness of the geometric p -adic étale cohomology of proper smooth rigid-analytic varieties, as well as the de Rham comparison theorem for such varieties. The latter was known before only for algebraic varieties, and we refer the reader to [Niz21] for a historical account on the de Rham, crystalline, and semistable conjectures for algebraic varieties. After Scholze’s work, there were efforts by a number of people to prove a version of the crystalline/semistable conjecture for proper rigid-analytic varieties having good/semistable reduction, culminating in the following theorem.

Theorem 1.1 ([CN17], [BMS18], [CK19]). *Let \mathfrak{X} be a proper p -adic formal scheme over \mathcal{O}_K with semistable reduction. We write \mathfrak{X}_C for the geometric rigid-analytic generic fiber of \mathfrak{X} . Let $i \geq 0$. There is a natural isomorphism*

$$H_{\text{ét}}^i(\mathfrak{X}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{st}} \cong H_{\text{cris}}^i(\mathfrak{X}_k/W(k)^0) \otimes_{W(k)} B_{\text{st}} \quad (1.1)$$

compatible with the Galois \mathcal{G}_K , Frobenius φ and monodromy N actions, and filtrations.¹

In particular, there is natural \mathcal{G}_K -equivariant isomorphism

$$H_{\text{ét}}^i(\mathfrak{X}_C, \mathbb{Q}_p) \cong (H_{\text{cris}}^i(\mathfrak{X}_k/W(k)^0) \otimes_{W(k)} B_{\text{st}})^{N=0, \varphi=1} \cap \text{Fil}^0(H_{\text{dR}}^i(\mathfrak{X}_K) \otimes_K B_{\text{dR}}). \quad (1.2)$$

We remark that Colmez–Nizioł’s strategy in [CN17] relies on a generalization of the syntomic method initiated by Fontaine–Messing, and later refined by Hyodo, Kato and Tsuji. Instead, Bhatt–Morrow–Scholze’s strategy, in their epoch-making work [BMS18], is based on the construction of a cohomology theory for smooth p -adic formal schemes \mathfrak{Z} over \mathcal{O}_C (generalized to the semistable case by Česnavičius–Koshikawa in [CK19]), called *A_{inf} -cohomology* $R\Gamma_{A_{\text{inf}}}(\mathfrak{Z})$, which in the proper case specializes to the (log-)crystalline cohomology of the special fiber and the étale cohomology of the generic fiber, thus allowing to compare the latter two as in Theorem 1.1.

More recently, in a series of papers, Colmez–Nizioł, partially in joint work with Dospinescu, further generalized the syntomic method to study the rational p -adic Hodge theory of smooth rigid-analytic varieties, which are neither assumed to be proper, nor having semistable reduction, [CDN20b], [CN20], [CN21a], [CN21b]. These works are motivated in part by the desire of finding a geometric incarnation of the p -adic Langlands correspondence in the p -adic cohomology of local Shimura varieties, as partially indicated by [CDN20a].

In order to state the goals of this paper, we will denote by

$$Y_{\text{FF}} := \text{Spa}(A_{\text{inf}}, A_{\text{inf}}) \setminus V(p[p^b])$$

the mixed characteristic punctured open unit disk,

$$\text{FF} := Y_{\text{FF}}/\varphi^{\mathbb{Z}}$$

the adic Fargues–Fontaine curve (relative to C^b and \mathbb{Q}_p), and we fix ∞ the (C, \mathcal{O}_C) -point of FF corresponding to Fontaine’s map $\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C$.

As observed by Fargues, Theorem 1.1 can be reformulated as a natural isomorphism of \mathcal{G}_K -equivariant vector bundles on FF

$$H_{\text{ét}}^i(\mathfrak{X}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{FF}} \cong \mathcal{E}(H_{\text{cris}}^i(\mathfrak{X}_k/W(k)^0)_{\mathbb{Q}_p}, \varphi, N, \text{Fil}) \quad (1.3)$$

where the right-hand side of (1.3) denotes the vector bundle on FF associated to the filtered (φ, N) -module $H_{\text{cris}}^i(\mathfrak{X}_k/W(k)^0)_{\mathbb{Q}_p}$. Since the left-hand side of (1.3) depends only on the geometric generic fiber of \mathfrak{X} , it is natural to ask whether one can give a more direct cohomological construction of the right-hand side that also depends only on the generic fiber, that interpolates between $H_{\text{ét}}^i(\mathfrak{X}_C, \mathbb{Q}_p)$ and the filtered (φ, N) -module $H_{\text{cris}}^i(\mathfrak{X}_k/W(k)^0)_{\mathbb{Q}_p}$, and that allows to prove extensions of the comparison (1.3) to any rigid-analytic variety over C .

Our first goal in this article will be to give a positive answer to the latter question by extending Bhatt–Morrow–Scholze’s strategy and building crucially upon Le Bras’ work [LB18b]. We will study a cohomology theory for rigid-analytic varieties X over C , taking values in filtered quasi-coherent complexes over the Fargues–Fontaine curve FF, which compares to other rational p -adic cohomology theories for rigid-analytic varieties over C , without properness or smoothness assumptions —

¹Here, we write H_{cris}^i for the log-crystalline cohomology, $W(k)^0$ denotes the log structure on $W(k)$ associated to $(\mathbb{N} \rightarrow W(k), 1 \mapsto 0)$, and \mathfrak{X}_k is endowed with the pullback of the canonical log structure on \mathfrak{X} .

namely, the rational p -adic pro-étale cohomology, the Hyodo–Kato cohomology ([CN21a, §4]),² and the infinitesimal cohomology over B_{dR}^+ ([BMS18, §13], [Guo21]). In particular, this will allow us to obtain a general comparison theorem for rigid-analytic varieties defined over a p -adic field, describing the geometric rational p -adic pro-étale cohomology in terms of de Rham data, extending (1.2), and recovering and generalizing the above-mentioned results of Colmez–Nizioł.

1.2. B -cohomology. In the following, we denote by B the ring of analytic functions on Y_{FF} .

To pursue the goals stated in the previous section, we begin by defining the B -cohomology theory for rigid-analytic varieties over C . Then, we shall explain how this cohomology theory interpolates several other rational p -adic cohomology theories, and how to interpret our main comparison theorems in terms of the Fargues–Fontaine curve FF .

For the reader willing to assume that X is smooth in Definition 1.2 below, we note that, in this case, the éh-site of X ([Guo19, §2], §2.2) can be replaced by the étale site of X (Proposition 2.42). Our main results on the B -cohomology theory are already new in the smooth case.

Definition 1.2 (cf. Definition 2.38). Let X be a rigid-analytic variety over C . We denote by $\alpha : X_v \rightarrow X_{\mathrm{éh}}$ the natural morphism from the v -site to the éh-site of X .

(i) We define the B -cohomology of X as

$$R\Gamma_B(X) := R\Gamma_{\mathrm{éh}}(X, L\eta_t R\alpha_* \mathbb{B})$$

where \mathbb{B} denotes the v -site sheaf theoretic version of the ring B , and we write $L\eta_t(-)$ for the décalage functor with respect to $t = \log([\varepsilon]) \in B$, i.e. Fontaine’s $2\pi i$.

(ii) We define the B_{dR}^+ -cohomology of X as

$$R\Gamma_{B_{\mathrm{dR}}^+}(X) := R\Gamma_{\mathrm{éh}}(X, L\eta_t R\alpha_* \mathbb{B}_{\mathrm{dR}}^+)$$

where $\mathbb{B}_{\mathrm{dR}}^+$ is the v -site sheaf theoretic version of the ring B_{dR}^+ .

We endow both $R\Gamma_B(X)$ and $R\Gamma_{B_{\mathrm{dR}}^+}(X)$ with the *filtration décalée*, coming from Bhatt–Morrow–Scholze’s interpretation of the décalage functor in terms of the connective cover functor for the Beilinson t -structure (Definition 2.9). The Frobenius automorphism of \mathbb{B} induces a φ_B -semilinear automorphism

$$\varphi : R\Gamma_B(X) \rightarrow R\Gamma_B(X)$$

which preserves the filtration décalée.

Remark 1.3 (Le Bras’ work). We recall that, in the paper [LB18b], Le Bras introduced and studied (an overconvergent version of) the B -cohomology theory for smooth rigid-analytic varieties over C . In particular, building upon results of [BMS18], for \mathfrak{Z} a smooth proper p -adic formal scheme over \mathcal{O}_C , he compared the B -cohomology of the rigid-analytic generic fiber of \mathfrak{Z} with the crystalline cohomology of the special fiber of \mathfrak{Z} , [LB18b, Proposition 6.5].

In the following, we denote by F the fraction field of the ring of Witt vectors $W(k)$, we write \check{F} for the completion of the maximal unramified extension of F in \overline{K} and we denote by $\mathcal{O}_{\check{F}}$ its ring of integers.

²As we will see, the *Hyodo–Kato cohomology* is a cohomology theory for rigid-analytic varieties of C which refines the de Rham cohomology and, in the case of Theorem 1.1, compares to the rational log-crystalline cohomology of the special fiber.

To motivate our first main result on the B -cohomology theory, we recall that in [CN21a, §4], for smooth rigid-analytic varieties X over C , Colmez–Nizioł (adapting a construction of Beilinson in the case of algebraic varieties [Bei13]), via the alterations of Hartl and Temkin, defined a *Hyodo–Kato cohomology theory*

$$R\Gamma_{\mathrm{HK}}(X)$$

taking values in the derived category of (φ, N) -modules over \check{F} , which refines the de Rham cohomology $R\Gamma_{\mathrm{dR}}(X)$, and in the case X has a semistable formal model \mathfrak{X} over \mathcal{O}_C it is given by the rational log-crystalline cohomology $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p}$ (see also §3, and in particular Theorem 3.15).

At this point, based on Le Bras’ work (Remark 1.3), it was natural to ask how the B -cohomology compares to the Hyodo–Kato cohomology, and whether, at least in the proper case, the latter cohomology theory (which is defined using log-geometry) can be recovered from the former (which is defined directly in terms of the generic fiber). To answer this question, the difficulty is twofold: the first issue comes from the very definition of the Hyodo–Kato cohomology, which forces us to construct a comparison morphism with the B -cohomology locally, and in a functorial way, using log-geometry;³ the second issue is of topological nature, since, locally, $R\Gamma_{\mathrm{HK}}(X)$ is in general not a perfect complex over \check{F} . To avoid the topological issues, one could instead study an overconvergent version of the desired comparison (cf. [LB18b, Conjecture 6.3]), however this makes the first mentioned difficulty even more challenging.

As in our previous work [Bos21], we overcome the topological issues via the condensed mathematics recently developed by Clausen–Scholze, and we refer the reader to the introduction of *loc. cit.* for a more exhaustive explanation of the relevance of the condensed and solid formalism in the study of the p -adic Hodge theory for rigid-analytic varieties.

Thus, given a condensed ring A ,⁴ we denote by $\mathrm{Mod}_A^{\mathrm{cond}}$ the category of A -modules in condensed abelian groups, and, for A a solid ring, we denote by $\mathrm{Mod}_A^{\mathrm{solid}}$ the symmetric monoidal subcategory of A -modules in solid abelian groups, endowed with the solid tensor product \otimes_A^{\blacksquare} . We denote by $D(\mathrm{Mod}_A^{\mathrm{cond}})$ and $D(\mathrm{Mod}_A^{\mathrm{solid}})$ the respective derived ∞ -categories.

Our first main result is the following.

Theorem 1.4 (cf. Theorem 4.1, Theorem 5.1, Theorem 5.9, Theorem 5.20). *Let X be a connected, paracompact, rigid-analytic variety defined over C .*

(i) *We have a natural isomorphism in $D(\mathrm{Mod}_B^{\mathrm{solid}})$*

$$R\Gamma_B(X) \simeq (R\Gamma_{\mathrm{HK}}(X) \otimes_{\check{F}}^{\blacksquare} B_{\mathrm{log}})^{N=0} \quad (1.4)$$

compatible with the action of Frobenius φ , and the action of Galois \mathcal{G}_K in the case when X is the base change to C of a rigid-analytic variety defined over K .

³Likewise, as explained to us by Česnavičius and Le Bras, it is a priori not clear whether the absolute crystalline comparison isomorphism for the A_{inf} -cohomology in the semistable case, constructed in [CK19, Theorem 5.4], is functorial.

⁴All condensed rings will be assumed to be commutative and unital. Moreover, we refer the reader to 1.7 for the set-theoretic conventions we adopt.

Here, B_{\log} denotes the log-crystalline condensed period ring (see §2.3), and $R\Gamma_{\text{HK}}(X)$ denotes the Hyodo–Kato cohomology of X (Definition 3.14).⁵

(ii) We have natural isomorphisms in $D(\text{Mod}_{B_{\text{dR}}^+}^{\text{solid}})$

$$R\Gamma_B(X) \otimes_B^{\text{L}\blacksquare} B_{\text{dR}}^+ \simeq R\Gamma_{B_{\text{dR}}^+}(X) \simeq R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+)$$

compatible with the isomorphism (1.4). Here, $R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+)$ denotes the infinitesimal cohomology over B_{dR}^+ ([Guo21], §5.2.1).

If X is the base change to C of a rigid-analytic variety X_0 defined over K , then we have a natural isomorphism in $D(\text{Mod}_{B_{\text{dR}}^+}^{\text{solid}})$

$$R\Gamma_{B_{\text{dR}}^+}(X) \simeq R\Gamma_{\text{dR}}(X_0) \otimes_K^{\text{L}\blacksquare} B_{\text{dR}}^+$$

compatible with the action of \mathcal{G}_K , and with filtrations. Here, $R\Gamma_{\text{dR}}(X_0)$ denotes the de Rham cohomology of X_0 (Definition 2.16).

The proof of Theorem 1.4 proceeds by constructing functorial local isomorphisms, which are then globalized using some magical properties of the solid tensor product proved by Clausen–Scholze, which rely on the theory of nuclear modules.

Remark 1.5 (Overconvergent Fargues–Fontaine cohomology). Thanks to the properties of the solid tensor product, one can also easily deduce a version of Theorem 1.4 for X a dagger variety over C . In particular, reinterpreting the latter result in terms of the Fargues–Fontaine curve (see §6.2), one can deduce a generalization of [LB18b, Conjecture 6.3], as shown in Theorem 6.17: for $i \geq 0$, given X a qcqs dagger variety over C , the cohomology group $H_B^i(X)$ is a finite projective φ -module over B ; then, denoting by $\mathcal{H}_{\text{FF}}^i(X)$ the associated vector bundle on FF, we have a natural isomorphism

$$\mathcal{H}_{\text{FF}}^i(X) \cong \mathcal{E}(H_{\text{HK}}^i(X)) \tag{1.5}$$

where $H_{\text{HK}}^i(X)$ is a finite (φ, N) -module over \check{F} , and the right-hand side denotes the associated vector bundle on FF; moreover, the completion at ∞ of (1.5) gives a natural isomorphism

$$\mathcal{H}_{\text{FF}}^i(X)_{\infty}^{\wedge} \cong H_{\text{inf}}^i(X/B_{\text{dR}}^+).$$

We note that (1.5) implies in particular that the vector bundle $\mathcal{H}_{\text{FF}}^i(X)$ determines, up to isomorphisms, the φ -module structure on $H_{\text{HK}}^i(X)$, and, while the latter is defined via log-geometry, the former is defined directly on the generic fiber. In addition, one can also recover from $\mathcal{H}_{\text{FF}}^i(X)$ the (φ, N) -module structure on $H_{\text{HK}}^i(X)$ (see Remark 6.18).⁶

As applications, using Theorem 1.4, via the relative fundamental exact sequence of p -adic Hodge theory, we show the following result.

Theorem 1.6 (Theorem 7.1). *Let X be a qcqs rigid-analytic variety defined over K . We have a \mathcal{G}_K -equivariant pullback square in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

⁵Forgetting the condensed structure, the Hyodo–Kato cohomology of X agrees with the one defined by Colmez–Nizioł ([CN21a, §4]) in the case when X is smooth.

⁶We also remark that, recently, Binda–Kato–Vezzani, via a motivic approach, proposed a definition of the overconvergent Hyodo–Kato cohomology theory without using log-geometry, [BKV22, Appendix A].

$$\begin{array}{ccc}
R\Gamma_{\text{proét}}(X_C, \mathbb{Q}_p) & \longrightarrow & (R\Gamma_{\text{HK}}(X_C) \otimes_{\mathbb{F}}^{\mathbb{L}} B_{\log}[1/t])^{N=0, \varphi=1} \\
\downarrow & & \downarrow \\
\text{Fil}^0(R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbb{L}} B_{\text{dR}}) & \longrightarrow & R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbb{L}} B_{\text{dR}}.
\end{array}$$

We note that Theorem 1.6 can be regarded as a derived generalization of (1.2): it tells us that the rational p -adic (pro-)étale cohomology of X_C can be recovered from the Hyodo–Kato cohomology of X_C and the de Rham cohomology of X together with its Hodge filtration.

1.3. Syntomic Fargues–Fontaine cohomology. The search for a theorem comparing the rational p -adic pro-étale cohomology of any rigid-analytic variety over C in terms of the B -cohomology and its filtration led us to define the following cohomology theory.

Definition 1.7. Let X be a rigid-analytic variety over C . Let $i \geq 0$ be an integer. We define the *syntomic Fargues–Fontaine cohomology of X with coefficients in $\mathbb{Q}_p(i)$* as the complex of $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$

$$R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) := \text{Fil}^i R\Gamma_B(X)^{\varphi=p^i}$$

where $R\Gamma_B(X)$ is endowed with the filtration décalée.

The first main result on the syntomic Fargues–Fontaine cohomology is the following.

Theorem 1.8 (Theorem 6.3). *Let X be a rigid-analytic variety over C . Let $i \geq 0$.*

(i) *We have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$*

$$\tau^{\leq i} R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_{\text{proét}}(X, \mathbb{Q}_p(i)).$$

(ii) *We have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$*

$$R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \simeq \text{fib}(R\Gamma_B(X)^{\varphi=p^i} \rightarrow R\Gamma_{B_{\text{dR}}^+}(X)/\text{Fil}^i).$$

We remark that the construction of the comparison morphisms in Theorem 1.8 is global in nature and it can be extended to coefficients (see §7.5).

Combining Theorem 1.4 with Theorem 1.8, we obtain the following result. Cf. [LB18b, §6] and [AMMN20, Theorem 7.13] for some related results in the proper good reduction case, and [CN21a, Theorem 1.1] for smooth rigid-analytic varieties.

Theorem 1.9 (Theorem 7.2). *Let X be a connected, paracompact, rigid-analytic variety defined over K . For any $i \geq 0$, we have a \mathcal{G}_K -equivariant isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

$$\tau^{\leq i} R\Gamma_{\text{proét}}(X_C, \mathbb{Q}_p(i)) \simeq \tau^{\leq i} \text{fib}((R\Gamma_{\text{HK}}(X_C) \otimes_{\mathbb{F}}^{\mathbb{L}} B_{\log})^{N=0, \varphi=p^i} \rightarrow (R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbb{L}} B_{\text{dR}}^+)/\text{Fil}^i).$$

Another interesting fact about the syntomic Fargues–Fontaine cohomology concerns its close relationship with the curve FF, as it can be guessed from its very definition. As explained in §6.2, for any X qcqs rigid-analytic variety over C , the φ -equivariant filtered complex $\text{Fil}^* R\Gamma_B(X)$ lifts to a filtered object

$$\text{Fil}^* \mathcal{H}_{\text{FF}}(X)$$

of the ∞ -category of quasi-coherent complexes $\text{QCoh}(\text{FF})$, in the sense of Clausen–Scholze (see §A.3). Then, relying in particular on results of Andreychev on the analytic descent for nuclear complexes on analytic adic spaces, [And21], we show the following theorem.

Theorem 1.10 (Theorem 6.19). *Let X be a qcqs rigid-analytic variety over C . Let $i \geq 0$. Consider the quasi-coherent complex on FF defined by*

$$\mathcal{H}_{\mathrm{syn}}(X)(i) := \mathrm{Fil}^i \mathcal{H}_{\mathrm{FF}}(X) \otimes \mathcal{O}(i).$$

We have

$$R\Gamma(\mathrm{FF}, \mathcal{H}_{\mathrm{syn}}(X)(i)) = R\Gamma_{\mathrm{syn}, \mathrm{FF}}(X, \mathbb{Q}_p(i)).$$

If X is proper, the complex $\mathcal{H}_{\mathrm{syn}}(X)(i)$ is perfect, in particular the complex $R\Gamma_{\mathrm{syn}, \mathrm{FF}}(X, \mathbb{Q}_p(i))$ identifies with the C -points of a bounded complex of Banach–Colmez spaces.

Remark 1.11 (Comparison with Fontaine–Messing/Colmez–Nizioł’s syntomic cohomology). Fix $i \geq 0$. In [CN20], via the alterations of Hartl and Temkin, Colmez–Nizioł, starting from the syntomic cohomology of Fontaine–Messing, defined a syntomic cohomology theory for any smooth rigid-analytic variety X over C , that here we will denote by $R\Gamma_{\mathrm{syn}, \mathrm{CN}}(X, \mathbb{Q}_p(i))$. We observe that $H_{\mathrm{syn}, \mathrm{FF}}^j(X, \mathbb{Q}_p(i))$ is isomorphic to $H_{\mathrm{syn}, \mathrm{CN}}^j(X, \mathbb{Q}_p(i))$ for any integer $j \leq i$, as in this case the two cohomology groups are both isomorphic to $H_{\mathrm{pro\acute{e}t}}^j(X, \mathbb{Q}_p(i))$; however, in general, for $j > i$, the two cohomology groups are not isomorphic (see Example 6.36). This difference is reflected in the fact that, for X proper, the complex $R\Gamma_{\mathrm{syn}, \mathrm{FF}}(X, \mathbb{Q}_p(i))$ canonically lifts to a complex of vector bundles on FF , as shown by Theorem 1.10, while the complex $R\Gamma_{\mathrm{syn}, \mathrm{CN}}(X, \mathbb{Q}_p(i))$ canonically lifts to a complex of φ -modules *jaugés* over B^+ , in the sense of Fargues ([Far15, Définition 4.15]), cf. [Niz19, Theorem 1.1].⁷

1.4. Semistable conjectures. From the general derived comparison results we have stated above, in particular from Theorem 1.6 and Theorem 1.9, one can deduce in some special cases a refined description of the single rational p -adic (pro-)étale cohomology groups in terms of de Rham data.

For X a proper (possibly singular) rigid-analytic variety over C , we prove in Theorem 7.4 a version of the semistable conjecture for X , generalizing Theorem 1.1. In the case when X is the base change to C of a rigid-analytic variety X_0 defined over K , this result relies on the degeneration at the first page of the Hodge-de Rham spectral sequence associated to X_0 ([Sch13a, Corollary 1.8], [Guo19, Proposition 8.0.8]). In general, we reduce to the previous case via a combination of Conrad–Gabber’s spreading out for proper rigid-analytic varieties and a generic smoothness result recently proved by Bhatt–Hansen, [BH22].

Another case in which the Hodge-de Rham spectral sequence simplifies is for smooth Stein spaces, thanks to Kiehl’s acyclicity theorem. In this case, we show the following theorem which reproves results of Colmez–Dospinescu–Nizioł [CDN20b] (in the semistable reduction case) and Colmez–Nizioł [CN21b].

Theorem 1.12 (cf. Theorem 7.7). *Let X be a smooth Stein space over C . For any $i \geq 0$, we have a short exact sequence in $\mathrm{Mod}_{\mathbb{Q}_p}^{\mathrm{solid}}$*

$$0 \rightarrow \Omega^{i-1}(X)/\ker d \rightarrow H_{\mathrm{pro\acute{e}t}}^i(X, \mathbb{Q}_p(i)) \rightarrow (H_{\mathrm{HK}}^i(X) \otimes_{\mathbb{F}}^{\blacksquare} B_{\log})^{N=0, \varphi=p^i} \rightarrow 0.$$

Remark 1.13. A recent conjecture of Hansen, [Han21, Conjecture 1.10], suggests that any local Shimura variety is a Stein space, therefore Theorem 1.12 potentially applies to any such variety.

⁷We recall that the category of φ -modules *jaugés* over B^+ is equivalent to the category of modifications of vector bundles on FF , [Far15, §4.2]. However, this is not an equivalence of exact categories, in the sense of Quillen.

Remark 1.14 (A p -adic Artin vanishing for smooth Stein spaces). Let X be a smooth Stein space over C . As a corollary of Theorem 1.12, we have that

$$H_{\text{proét}}^i(X, \mathbb{Q}_p) = 0$$

for all $i > \dim X$ (Corollary 7.10).

For smooth affinoid rigid spaces, the Hodge-de Rham spectral sequence simplifies similarly to smooth Stein spaces, thanks to Tate's acyclicity theorem (see also Proposition 7.3). Therefore, in view of Theorem 1.12, we are led to state the following conjecture.

Conjecture 1.15 (cf. Conjecture 7.11). *Let X be a smooth affinoid rigid space over C . For any $i \geq 0$, we have a short exact sequence in $\text{Mod}_{\mathbb{Q}_p}^{\text{solid}}$*

$$0 \rightarrow \Omega^{i-1}(X)/\ker d \rightarrow H_{\text{proét}}^i(X, \mathbb{Q}_p(i)) \rightarrow (H_{\text{HK}}^i(X) \otimes_{\mathbb{F}}^{\square} B_{\log})^{N=0, \varphi=p^i} \rightarrow 0.$$

We remark that before the advent of condensed mathematics one couldn't even dare to formulate a conjecture in the spirit of the one above; in fact, for a smooth affinoid rigid space X over C , the de Rham $H_{\text{dR}}^i(X)$ and Hyodo–Kato $H_{\text{HK}}^i(X)$ cohomology groups are in general pathological if regarded as topological vector spaces (see [Bos21, Remark 5.14]).⁸ Instead, regarded as condensed vector spaces, these objects are perfectly well-behaved, even though they are non-quasi-separated. Then, exploiting the possibility (provided by the solid formalism) of doing functional analysis with such new objects, we prove the following result.

Theorem 1.16 (cf. Theorem 7.12). *Conjecture 7.11 holds true for X a smooth affinoid rigid space over C of dimension 1.*

We discuss in §7.4 the obstruction to proving Conjecture 7.11 in dimension higher than 1. We note however that an analogue of Remark 1.14 for smooth affinoid spaces over C is known in any dimension, thanks to a result of Bhatt–Mathew (see Lemma 7.13).

1.5. Link to prismatic cohomology: toward an integral theory. We conclude this introduction by conjecturing the existence of an integral variant of the B -cohomology theory for rigid-analytic varieties, which in particular better explains the relation between the results of this paper and the work of Bhatt–Morrow–Scholze and Česnavičius–Koshikawa.

In the following, we write y_C for the (C, \mathcal{O}_C) -point of Y_{FF} corresponding to Fontaine's map $\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C$ (and projecting to the point ∞ of FF). Let us recall the following result of Fargues.

Proposition 1.17 (Fargues, cf. [Far15, Proposition 4.45], [SW20, Theorem 14.1.1]). *The following two categories are equivalent:*

- (1) *Shtukas (of vector bundles) over $\text{Spa } C^{\flat}$ relative to $\text{Spa } \mathbb{Q}_p$ with one leg at $\varphi^{-1}(y_C)$, i.e. vector bundles \mathcal{E} on $\text{Spa } C^{\flat} \dot{\times} \text{Spa } \mathbb{Q}_p := Y_{\text{FF}}$ ⁹ together with an isomorphism*

$$\varphi\mathcal{E} : (\varphi^*\mathcal{E})|_{Y_{\text{FF}} \setminus \varphi^{-1}(y_C)} \cong \mathcal{E}|_{Y_{\text{FF}} \setminus \varphi^{-1}(y_C)}$$

which is meromorphic at $\varphi^{-1}(y_C)$.

⁸There were previously some partial and ad hoc solutions to this issue. For example, in [CDN21], for X a smooth affinoid over C of dimension 1, Colmez–Dospinescu–Nizioł considered the maximal Hausdorff quotient of the topological \mathbb{F} -vector space $H_{\text{HK}}^1(X)$.

⁹To motivate the notation and the terminology, we recall that $Y_{\text{FF}}^{\diamond} \cong (\text{Spa } C)^{\diamond} \times (\text{Spa } \mathbb{Q}_p)^{\diamond}$ as diamonds.

- (2) *Admissible modifications of vector bundles on FF at ∞ , i.e. triples $(\mathcal{F}_1, \mathcal{F}_2, \alpha)$ where \mathcal{F}_1 and \mathcal{F}_2 are vector bundles on FF with \mathcal{F}_1 semistable of slope 0, and $\alpha : \mathcal{F}_1|_{\text{FF} \setminus \{\infty\}} \cong \mathcal{F}_2|_{\text{FF} \setminus \{\infty\}}$ is an isomorphism.*

Now, given X a proper rigid-analytic variety over C , for $i \geq 0$ we consider the modification of the vector bundle $\mathcal{H}_{\text{FF}}^i(X)$ at ∞ given by the B_{dR}^+ -lattice

$$\text{Fil}^0(H_{B_{\text{dR}}^+}^i(X) \otimes_{B_{\text{dR}}^+} B_{\text{dR}}) \subset \mathcal{H}_{\text{FF}}^i(X)_{\infty}^{\wedge} \otimes_{B_{\text{dR}}^+} B_{\text{dR}} \cong H_{\text{ét}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}}$$

which gives an admissible modification of vector bundles on FF at ∞

$$(H_{\text{ét}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{FF}}, \mathcal{H}_{\text{FF}}^i(X), \alpha) \quad (1.6)$$

(see Theorem 7.4). Then, inspired by Bhatt–Morrow–Scholze’s work, it is natural to wonder whether one can give a direct geometric cohomological construction of the shtuka corresponding to (1.6) via the above recalled Fargues’ equivalence, and how the latter compares to the A_{inf} -cohomology theory in the semistable reduction case. More precisely, we formulate the following conjecture (for simplicity, we restrict ourselves to proper rigid-analytic varieties).

We will consider the analytic adic space

$$\mathcal{Y}_{\text{FF}} := \text{Spa}(A_{\text{inf}}, A_{\text{inf}}) \setminus V([p^b])$$

and denote by A the ring of analytic functions on \mathcal{Y}_{FF} . We note that $Y_{\text{FF}} \subset \mathcal{Y}_{\text{FF}}$ is the open subset defined by the locus where $p \neq 0$.

Conjecture 1.18. *There exists a cohomology theory*

$$R\Gamma_{\text{sht}}(X/\mathcal{Y}_{\text{FF}})$$

for proper rigid-analytic varieties X over C , taking values in shtukas of perfect complexes over $\text{Spa } C^b$ relative to $\text{Spa } \mathbb{Z}_p$ with one leg at $\varphi^{-1}(y_C)$ (i.e. perfect complexes \mathcal{E} on $\text{Spa } C^b \times \text{Spa } \mathbb{Z}_p := \mathcal{Y}_{\text{FF}}$ together with an isomorphism $\varphi_{\mathcal{E}} : (\varphi^* \mathcal{E})|_{\mathcal{Y}_{\text{FF}} \setminus \varphi^{-1}(y_C)} \cong \mathcal{E}|_{\mathcal{Y}_{\text{FF}} \setminus \varphi^{-1}(y_C)}$ which is meromorphic at $\varphi^{-1}(y_C)$), and satisfying the following properties.

- (i) *If X is the generic fiber of a proper p -adic formal scheme \mathfrak{X} over \mathcal{O}_C with semistable reduction, there is a natural isomorphism between $R\Gamma_{\text{sht}}(X/\mathcal{Y}_{\text{FF}})$ and the shtuka of perfect complexes over $\text{Spa } C^b$ relative to $\text{Spa } \mathbb{Z}_p$ with one leg at $\varphi^{-1}(y_C)$ associated to $R\Gamma_{A_{\text{inf}}}(X) \otimes_{A_{\text{inf}}} A$.*
- (ii) *Denoting by $R\Gamma_{\text{sht}}(X/Y_{\text{FF}})$ the restriction of $R\Gamma_{\text{sht}}(X/\mathcal{Y}_{\text{FF}})$ to Y_{FF} , the cohomology groups $H_{\text{sht}}^i(X/Y_{\text{FF}})$ are shtukas of vector bundles over $\text{Spa } C^b$ relative to $\text{Spa } \mathbb{Q}_p$ with one leg at $\varphi^{-1}(y_C)$, and the admissible modification of vector bundles on FF at ∞ defined in (1.6) corresponds to $H_{\text{sht}}^i(X/Y_{\text{FF}})$ via Fargues’ equivalence (Proposition 1.17).*

In the notation of Definition 1.2, a natural candidate for the cohomology theory conjectured above is given by a lift of the complex $R\Gamma_{\text{éh}}(X, L\eta_{\mu} R\alpha_* \mathbb{A})$ to \mathcal{Y}_{FF} , where \mathbb{A} is the v -site sheaf theoretic version of the ring A , and $\mu = [\varepsilon] - 1 \in A_{\text{inf}}$. However, it would be more interesting (especially for questions related to cohomological coefficients) to give a definition of $R\Gamma_{\text{sht}}(X/\mathcal{Y}_{\text{FF}})$ in the spirit of prismatic cohomology ([BS22], [Dri22], [BL22]).¹⁰ We hope to come back on these questions in a future work.

¹⁰More precisely, such definition would give a Frobenius descent of $R\Gamma_{\text{sht}}(X/\mathcal{Y}_{\text{FF}})$.

1.6. Leitfaden of the paper. We have organized the paper as follows. We begin by defining the B -cohomology and the B_{dR}^+ -cohomology, together with their filtration décalée, in §2. In §3, we revisit, in the condensed setting, the Hyodo–Kato cohomology of Colmez–Nizioł, and we extend it to singular rigid-analytic varieties over C . In §4 and §5, we prove the first main result of the paper: Theorem 1.4. We then proceed in §6 by introducing the syntomic Fargues–Fontaine cohomology and proving Theorem 1.8 and Theorem 1.10; on the way, we also study nuclear complexes on the Fargues–Fontaine curve. In §7, we give applications of the main results proven in the previous sections, showing in particular Theorem 1.6 and Theorem 1.9. We end with Appendix A in which we collect some complements on condensed mathematics used in the main body of the paper.

1.7. Notation and conventions.

Ground fields. Fix a prime number p . We denote by K a complete discretely valued non-archimedean extension of \mathbb{Q}_p , with perfect residue field k , and ring of integers \mathcal{O}_K . We choose a uniformizer ϖ of \mathcal{O}_K .

We fix an algebraic closure \overline{K} of K . We denote by $C := \widehat{\overline{K}}$ the completion of \overline{K} , and by \mathcal{O}_C its ring of integers. We denote by F the fraction field of the ring of Witt vectors $W(k)$, we write \check{F} for the completion of the maximal unramified extension of F in \overline{K} , and we denote by $\mathcal{O}_{\check{F}}$ its ring of integers.

Moreover, we let $\mathcal{G}_K := \text{Gal}(\overline{K}/K)$ denote the absolute Galois group of K .

∞ -categories. We will adopt the term ∞ -category to indicate a $(\infty, 1)$ -category, i.e. a higher category in which all n -morphisms for $n > 1$ are invertible. We will use the language of ∞ -categories, [Lur09], and higher algebra, [Lur17].

We denote by Δ the simplicial category and, for every integer $m \geq 0$, we write $\Delta_{\leq m}$ for the full subcategory of Δ having as objects $[n]$ for $0 \leq n \leq m$.

We denote by $\text{Ani} := \text{Ani}(\text{Set})$ the ∞ -category of *anima*, i.e. the ∞ -category of animated sets, [CS21, §5.1.4].

Condensed mathematics. We fix an uncountable cardinal κ as in [Sch21, Lemma 4.1]. Unless explicitly stated otherwise, all condensed sets will be κ -condensed sets (and often the prefix “ κ ” is tacit). We will denote by CondAb the category of κ -condensed abelian groups, and by $\text{Solid} \subset \text{CondAb}$ the full subcategory of κ -solid abelian groups.

Unless stated otherwise, all condensed rings will be κ -condensed commutative unital rings. Given a (κ) -condensed ring A , we denote by $\text{Mod}_A^{\text{cond}}$ the category of A -modules in CondAb , and, for A a solid ring, we denote by $\text{Mod}_A^{\text{solid}}$ the symmetric monoidal subcategory of A -modules in Solid , endowed with the solid tensor product \otimes_A^\bullet . We denote by $D(\text{Mod}_A^{\text{cond}})$ and $D(\text{Mod}_A^{\text{solid}})$ the respective derived ∞ -categories, and sometimes we abbreviate $D(A) = D(\text{Mod}_A^{\text{cond}})$. Moreover, we write $\underline{\text{Hom}}_A(-, -)$ for the internal Hom in the category $\text{Mod}_A^{\text{cond}}$ (and in the case $A = \mathbb{Z}$, we often omit the subscript \mathbb{Z}).

Throughout the paper, we use Clausen–Scholze’s non-archimedean condensed function analysis, for which we refer the reader to [Bos21, Appendix A].

Condensed group cohomology. Given a condensed group G , and a G -module M in CondAb , the *condensed group cohomology of G with coefficients in M* will be denoted by

$$R\Gamma_{\text{cond}}(G, M) := R\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) \in D(\text{CondAb})$$

where \mathbb{Z} is endowed with the trivial G -action (see e.g. [Bos21, Appendix B]).

Adic spaces. We say that an analytic adic space X is κ -small if the cardinality of the underlying topological space $|X|$ is less than κ , and for all open affinoid subspaces $\text{Spa}(R, R^+) \subset X$, the ring R has cardinality less than κ . In this paper, all the analytic adic spaces will be assumed to be κ -small.

Throughout the article, all Huber rings will be assumed to be complete, and will be regarded as condensed rings.

Pro-étale topology. We recall that there is a natural functor $X \mapsto X^\diamond$ from the category of analytic adic spaces defined over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ to the category of locally spatial diamonds, satisfying $|X| = |X^\diamond|$ and $X_{\text{ét}} \cong X_{\text{ét}}^\diamond$, [Sch21, Definition 15.5, Lemma 15.6].

For X an analytic adic space defined over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, we denote by

$$X_{\text{proét}} := X_{\text{qproét}}^\diamond$$

its (κ -bounded) *pro-étale site*, [Sch21, Definition 14.1].

Given $f : X \rightarrow \text{Spa}(C, \mathcal{O}_C)$ an analytic adic space over C , and \mathcal{F} a sheaf of abelian groups on $X_{\text{proét}}$, we define the complex of $D(\text{CondAb})$

$$R\Gamma_{\text{proét}}(X, \mathcal{F}) := Rf_{\text{proét}*}\mathcal{F}$$

(see also [Bos21, Definition 2.7, Remark 2.9]).

Fargues–Fontaine curves. For $S = \text{Spa}(R, R^+)$ an affinoid perfectoid space over \mathbb{F}_p , we let

$$Y_{\text{FF}, S} := \text{Spa}(W(R^+), W(R^+)) \setminus V(p[p^b]).$$

We recall that $Y_{\text{FF}, S}$ defines an analytic adic space over \mathbb{Q}_p , [FS21, Proposition II.1.1]. The p -th power Frobenius on R^+ induces an automorphism φ of $Y_{\text{FF}, S}$ whose action is free and totally discontinuous, [FS21, Proposition II.1.16]. The *Fargues–Fontaine curve relative to S* (and \mathbb{Q}_p) will be denoted by

$$\text{FF}_S := Y_{\text{FF}, S}/\varphi^{\mathbb{Z}}. \tag{1.7}$$

For $I = [s, r] \subset (0, \infty)$ an interval with rational endpoints, we define the open subset

$$Y_{\text{FF}, S, I} := \{|p|^r \leq |[p^b]| \leq |p|^s\} \subset Y_{\text{FF}, S}. \tag{1.8}$$

We note that $Y_{\text{FF}, S, I}$ is an affinoid space, as it is a rational open subset of $\text{Spa}(W(R^+), W(R^+))$.

We denote by $\text{Bun}(\text{FF}_S)$ the category of vector bundles on FF_S , and by $\text{Isoc}_{\overline{\mathbb{F}}_p}$ the category of isocrystals over $\overline{\mathbb{F}}_p$ (also called finite φ -modules over $\check{\mathbb{Q}}_p$), i.e. the category of pairs (V, φ) with V a finite-dimensional $\check{\mathbb{Q}}_p$ -vector space and φ a σ -semilinear automorphism of V , where σ is the automorphism of $\check{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p]$ induced by the p -th power Frobenius on $\overline{\mathbb{F}}_p$.

Recall that we have a natural exact \otimes -functor

$$\text{Isoc}_{\overline{\mathbb{F}}_p} \rightarrow \text{Bun}(\text{FF}_S), \quad (V, \varphi) \mapsto \mathcal{E}(V, \varphi).$$

For $\lambda \in \mathbb{Q}$, we denote by $(D_\lambda, \varphi_\lambda)$ the simple isocrystal over $\overline{\mathbb{F}}_p$ of slope λ in the Dieudonné–Manin classification, and we let

$$\mathcal{O}_{\mathrm{FF}_S}(-\lambda) := \mathcal{E}(D_\lambda, \varphi_\lambda).$$

In particular, for $n \in \mathbb{Z}$, we have

$$\mathcal{O}_{\mathrm{FF}_S}(n) = \mathcal{E}(\check{\mathbb{Q}}_p, p^{-n}\sigma).$$

In the case $S = \mathrm{Spa}(C^b, \mathcal{O}_{C^b})$, we omit the subscript S from (1.7) and (1.8).

We will often use the classification of vector bundles on FF (see [FF18, §8], [FS21, Theorem II.0.3]): the functor $\mathrm{Isoc}_{\overline{\mathbb{F}}_p} \rightarrow \mathrm{Bun}(\mathrm{FF})$ induces a bijection on isomorphism classes; in particular, any vector bundle on FF is isomorphic to a direct sum of vector bundles of the form $\mathcal{O}_{\mathrm{FF}}(\lambda)$ with $\lambda \in \mathbb{Q}$.

We will denote by ∞ the (C, \mathcal{O}_C) -point of the curve FF corresponding to Fontaine’s map $\theta : W(\mathcal{O}_{C^b}) \rightarrow \mathcal{O}_C$, and

$$\iota_\infty : \mathrm{Spa}(C, \mathcal{O}_C) \rightarrow \mathrm{FF}$$

the inclusion map.

Rigid-analytic varieties. All rigid-analytic varieties, and all dagger varieties ([GK00]), occurring in this work will be assumed to be quasi-separated, and of finite dimension.

We say that a rigid-analytic/dagger variety X is *paracompact* if it admits an admissible locally finite affinoid covering, i.e. there exists an admissible covering $\{U_i\}_{i \in I}$ of X by affinoid subspaces such that for each index $i \in I$ the intersection $U_i \cap U_j$ is non-empty for at most finitely many indices $j \in I$.

We recall that a paracompact rigid-analytic variety is taut, [Hub96, Definition 5.1.2, Lemma 5.1.3], and it is the admissible disjoint union of connected paracompact rigid-analytic varieties of countable type, i.e. having a countable admissible affinoid covering, [dJvdP96, Lemma 2.5.7]. We refer the reader to [Bos21, §5.2] for further recollections on paracompact rigid-analytic varieties.

Formal schemes. Unless explicitly stated otherwise, all formal schemes will be assumed to be p -adic and locally of finite type.

Acknowledgments. I am very grateful to Grigory Andreychev, Ko Aoki, Kęstutis Česnavičius, Dustin Clausen, Pierre Colmez, Gabriel Dospinescu, Haoyang Guo, David Hansen, Hiroki Kato, Teruhisa Koshikawa, Arthur-César Le Bras, Lucas Mann, Akhil Mathew, Matthew Morrow, Wiesława Nizioł, Peter Scholze, and Alberto Vezzani for helpful conversations, or for their comments on a draft version of this paper. Special thanks go to Teruhisa Koshikawa for a decisive suggestion, to Arthur-César Le Bras for what he thought me about the Fargues–Fontaine curve and for introducing me to [LB18b, Conjecture 6.3], to Wiesława Nizioł for reading several drafts of this manuscript, and to Peter Scholze for his crucial remarks and corrections as well as for discussions related to Conjecture 1.18.

I would like to thank the organizers of the RAMpAGE seminar for their invitation in June 2021, on which occasion the main results of this paper were first announced.

This project was carried out while the author was a Ph.D. student at Sorbonne Université, within the Institut de Mathématiques de Jussieu–Paris Rive Gauche. Moreover, parts of this manuscript

were written while visiting the Banach Center, at the invitation of Piotr Achinger, and the Max-Planck-Institut für Mathematik. I thank all these institutions for their hospitality and support.

2. PRELIMINARIES

In this first section, our goal is to define the B -cohomology together with its *filtration décalée*. Along the way, we will establish several preliminary technical results, which will be used in the rest of the paper.

2.1. Décalage functors and Beilinson t -structure. In this subsection, we will recall an interpretation of the décalage functor in terms of the connective cover functor for the Beilinson t -structure.

2.1.1. *Décalage functors.* We shall use the following notation.

Notation 2.1. Let (T, \mathcal{O}_T) be a ringed topos and let $(f) \subset \mathcal{O}_T$ be an invertible ideal sheaf. We will write $D(\mathcal{O}_T)$ for the derived category of \mathcal{O}_T -modules.

The following slight generalization of [BMS18, Definition 6.2], which goes back to Berthelot–Ogus, will be used in particular in §6.3.

Definition 2.2 (cf. [BO78, Definition 8.6]). Let $\delta : \mathbb{Z} \rightarrow \mathbb{Z}$ be a function. Let M^\bullet be an f -torsion-free complex of \mathcal{O}_T -modules. We denote by $\eta_{\delta,f}(M^\bullet)$ the subcomplex of $M^\bullet[1/f]$ defined by

$$\eta_{\delta,f}(M^\bullet)^i := \{x \in f^{\delta(i)}M^i : dx \in f^{\delta(i+1)}M^{i+1}\}.$$

In the case $\delta = \text{id}$, we put $\eta_f(-) := \eta_{\text{id},f}(-)$.

Remark 2.3. We note that the definition of $\eta_{\delta,f}(-)$ depends on the ideal sheaf $(f) \subset \mathcal{O}_T$ and it is independent on the chosen generator of the latter.

Proposition 2.4 (cf. [BO78, Proposition 8.19]). *Let $\delta : \mathbb{Z} \rightarrow \mathbb{Z}$ be a non-decreasing function. The functor $\eta_{\delta,f}$ from f -torsion-free complexes of \mathcal{O}_T -modules to $D(\mathcal{O}_T)$ factors canonically over the décalage functor (relative to (f) and δ)*

$$L\eta_{\delta,f} : D(\mathcal{O}_T) \rightarrow D(\mathcal{O}_T).$$

Proof. First, recall that every complex of \mathcal{O}_T -modules is quasi-isomorphic to an f -torsion-free complex of \mathcal{O}_T -modules, [BMS18, Lemma 6.1]. We want to show that the endo-functor $\eta_{\delta,f}$ on the category of f -torsion-free complexes of \mathcal{O}_T -modules preserves quasi-isomorphisms. The latter assertion is implied by the following claim (cf. [BMS18, Lemma 6.4]): given M^\bullet an f -torsion-free complex of \mathcal{O}_T -modules, for all $i \in \mathbb{Z}$, the multiplication by $f^{\delta(i)}$ map, i.e. tensoring by $- \otimes_{\mathcal{O}_T} (f^{\delta(i)})$, induces an isomorphism

$$H^i(M^\bullet)/H^i(M^\bullet)[f^{\delta(i)-\delta(i-1)}] \xrightarrow{\simeq} H^i(\eta_{\delta,f}M^\bullet)$$

(note that $\delta(i) - \delta(i-1) \geq 0$ by assumption on the function $\delta : \mathbb{Z} \rightarrow \mathbb{Z}$). For this, let $Z^i(M^\bullet) \subset M^i$ and $Z^i(\eta_{\delta,f}M^\bullet) \subset (\eta_{\delta,f}M^\bullet)^i$ denote the cocycles. By f -torsion-freeness of the terms of the complex M^\bullet , the multiplication by $f^{\delta(i)}$ map induces an isomorphism

$$Z^i(M^\bullet) \cong Z^i(\eta_{\delta,f}M^\bullet)$$

which in turn induces a surjection

$$H^i(M^\bullet) \twoheadrightarrow H^i(\eta_{\delta,f}M^\bullet).$$

Moreover, given a cocycle $z \in Z^i(M^\bullet)$ mapping to the zero class of $H^i(\eta_{\delta,f}M^\bullet)$ via multiplication by $f^{\delta(i)}$, we have that $f^{\delta(i)}z = d(f^{\delta(i-1)}y)$ for some $y \in M^{i-1}$, i.e. $f^{\delta(i)-\delta(i-1)}z = dy$, which means that the image of z in $H^i(M^\bullet)$ is $f^{\delta(i)-\delta(i-1)}$ -torsion, as claimed. \square

2.1.2. *Beilinson t -structure.* Next, as promised, we want to recall the ∞ -categorical interpretation of the décalage functor $L\eta_f(-)$ as the connective cover functor for the Beilinson t -structure.

Notation 2.5. Let A be a commutative unital ring. Let $D(A)$ denote the derived ∞ -category of A -modules. We write

$$DF(A) := \text{Fun}(\mathbb{Z}^{\text{op}}, D(A))$$

for the *filtered derived ∞ -category of A -modules*. Given $F \in DF(A)$, for $i \in \mathbb{Z}$, we define the i -th *graded piece of F* as the cofiber

$$\text{gr}^i(F) := F(i)/F(i+1).$$

We refer the reader to [BMS19, §5.1] for recollections on filtered derived ∞ -categories.

Definition 2.6 ([BMS19, Definition 5.3]). Let $DF^{\leq 0}(A) \subset DF(A)$ be the full ∞ -subcategory spanned by those F such that $\text{gr}^i(F) \in D^{\leq i}(A)$ for all $i \in \mathbb{Z}$, and let $DF^{\geq 0}(A) \subset DF(A)$ be the full ∞ -subcategory spanned by those F such that $F(i) \in D^{\geq i}(A)$ for all $i \in \mathbb{Z}$. The pair

$$(DF^{\leq 0}(A), DF^{\geq 0}(A))$$

is called the *Beilinson t -structure* on $DF(A)$.

Note that the t -structure depends only on the triangulated category underlying the derived ∞ -category $D(A)$. The definition above is justified by the following result.

Proposition 2.7 ([BMS19, Theorem 5.4]). *Fix notation as in Definition 2.6.*

- (i) *The Beilinson t -structure $(DF^{\leq 0}(A), DF^{\geq 0}(A))$ is a t -structure on $DF(A)$.*
- (ii) *Denoting by*

$$\tau_{\text{Beil}}^{\leq 0} : DF(A) \rightarrow DF^{\leq 0}(A)$$

the connective cover functor for the Beilinson t -structure on $DF(A)$, there is a natural isomorphism

$$\text{gr}^i \circ \tau_{\text{Beil}}^{\leq 0}(-) \simeq \tau^{\leq i} \circ \text{gr}^i(-).$$

- (iii) *Denote by*

$$H_{\text{Beil}}^0 : DF(A) \rightarrow DF(A)^{\heartsuit} := DF^{\leq 0}(A) \cap DF^{\geq 0}(A)$$

the 0-th cohomology functor for the Beilinson t -structure. The heart $DF(A)^{\heartsuit}$ is equivalent to the abelian category $\text{Ch}(A)$ of chain complexes of A -modules in abelian groups, via sending, for varying $F \in DF(A)$, the 0-th cohomology $H_{\text{Beil}}^0(F) \in DF(A)^{\heartsuit}$ to the chain complex $(H^{\bullet}(\text{gr}^{\bullet}(F)), d)$ with differential d induced by the boundary map for the exact triangle

$$\text{gr}^{i+1}(F) \rightarrow F(i)/F(i+2) \rightarrow \text{gr}^i(F).$$

In the following, let f be a non-zero-divisor in A .

Proposition 2.8. *Let $M \in D(A)$. Define $\text{Fil}^* L\eta_f M \in DF(A)$ the filtration on $L\eta_f M$ whose i -th level is given by $L\eta_{\varepsilon_i, f} M$, where $\varepsilon_i : \mathbb{Z} \rightarrow \mathbb{Z}, j \mapsto \max(i, j)$. Denote by $f^* \otimes M \in DF(A)$ the filtration on M whose i -th level is given by $f^i \otimes_A M$. Then, $\text{Fil}^* L\eta_f M$ identifies with $\tau_{\text{Beil}}^{\leq 0}(f^* \otimes M)$ in $DF(A)$.*

Proof. First, note that the function ε_i is non-decreasing, hence it satisfies the assumptions of Proposition 2.4. Then, the statement is contained in the proof of [BMS19, Proposition 5.8]. \square

Definition 2.9. Given $M \in D(A)$, we call the filtration $\text{Fil}^* L\eta_f M$ defined in Proposition 2.8 the *filtration décalée* on $L\eta_f M$.

The definitions and the results above extend to any ringed topoi (or site). In particular, they extend to the case of the ringed site $(*\kappa\text{-proét}, A)$, for κ cut-off cardinal as in §1.7, where $*\kappa\text{-proét}$ is the site of κ -small profinite sets, with coverings given by finite families of jointly surjective maps, and A is a κ -condensed ring.

2.2. The éh-topology. In this subsection, we recall the definition of the éh-site for rigid-analytic varieties, introduced by Guo, and we consider its variant for dagger varieties. This site will be used crucially in the definition of the B -cohomology theory for arbitrary (possibly singular) rigid-analytic/dagger varieties.

2.2.1. Definition of the éh-site. We will use the following notation and conventions.

Notation and conventions 2.10. We denote by L a characteristic 0 complete valued field with a non-archimedean valuation of rank 1 and residue characteristic p . We write Rig_L (resp. Rig_L^\dagger) for the category of rigid-analytic (resp. dagger) varieties over L , and we denote by RigSm_L (resp. RigSm_L^\dagger) the category of smooth rigid-analytic (resp. dagger) varieties over L .

We refer the reader to [GK00] for the foundations of *dagger varieties* (also called *overconvergent rigid varieties*), and to [Vez18, §2] for a quick recollection of the definitions and the main results on the subject, which we will freely use in the following.

Given a dagger variety $X = (\widehat{X}, \mathcal{O}^\dagger)$ over L with underlying rigid-analytic variety \widehat{X} and *overconvergent structure sheaf* \mathcal{O}^\dagger , we say that \widehat{X} is the *limit* of X , and vice versa that X is a *dagger structure* on \widehat{X} , [Vez18, Definition 2.22]; moreover, we regard \mathcal{O}^\dagger as a sheaf with values in $\text{Mod}_L^{\text{cond}}$.

Before defining the éh-site for rigid-analytic and dagger varieties, we need to introduce the notion of *blowing-up*. The construction of the blow-up of a rigid-analytic variety along a closed analytic subset, as well as the verification of its universal property, is due to Conrad, [Con06, §4.1]. In turn, such construction relies on the definition of the *relative analytified Proj*, [Con06, §2.3], denoted Proj^{an} . We note that the latter definition translates *verbatim* to dagger varieties (replacing the structure sheaf with the overconvergent structure sheaf). We can then give the following definition (see [Con06, Definition 4.1.1]).

Definition 2.11. Let X be a rigid-analytic (resp. dagger) variety over L , and let $Z = V(\mathcal{I})$ be the Zariski closed subset defined by a coherent ideal sheaf \mathcal{I} over X . The *blow-up of X along Z* is the rigid-analytic (resp. dagger) variety over X defined by

$$\text{Bl}_Z(X) := \text{Proj}^{\text{an}}(\oplus_{n \geq 0} \mathcal{I}^n).$$

Remark 2.12. Keeping the notation above, the blow-up of X along Z has the following universal property (see the discussion after [Con06, Definition 4.1.1]): $\text{Bl}_Z(X) \rightarrow X$ is the final object in the category of morphisms $f : Y \rightarrow X$ in Rig_L (resp. Rig_L^\dagger) such that the coherent pullback $f^*\mathcal{I}$ is invertible.

Definition 2.13 ([Guo19, Definition 2.4.1]). The *big éh-site* $\text{Rig}_{L,\text{éh}}$ (resp. $\text{Rig}_{L,\text{éh}}^\dagger$) is the Grothendieck topology on the category Rig_L (resp. Rig_L^\dagger), such that the covering families are generated by étale

coverings, universal homeomorphisms, and morphisms

$$\mathrm{Bl}_Z(Y) \sqcup Z \rightarrow Y$$

with Z a closed analytic subset of Y .

Given X a rigid-analytic (resp. dagger) variety over L , we define the *small éh-site* $X_{\acute{e}h}$ as the localization of the site $\mathrm{Rig}_{L,\acute{e}h}$ (resp. $\mathrm{Rig}_{L,\acute{e}h}^\dagger$) at the object X .

The definition above is designed to make the following result hold true.

Proposition 2.14 (cf. [Guo19, Corollary 2.4.8]). *Let X be a quasi-compact, reduced, rigid-analytic (resp. dagger) variety over L . Then, there exists a proper éh-covering $f : Y \rightarrow X$ with Y a smooth rigid-analytic (resp. dagger) variety over L .*

Proof. We will check that the proof of [Guo19, Corollary 2.4.8] also holds for dagger varieties.

Since X is quasi-compact and reduced, by Temkin's non-embedded desingularization theorem, [Tem12, Theorem 1.2.1, Theorem 5.2.2], there exists a finite sequence of blowups

$$X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 = X$$

such that X_n smooth, with $X_j = \mathrm{Bl}_{Z_{j-1}}(X_{j-1})$ the blowup of X_{j-1} along a smooth Zariski closed subset Z_{j-1} of X_{j-1} .¹¹ In fact, we note that *loc. cit.* also applies in the case when X is a dagger variety, as any dagger L -algebra is an excellent ring: this follows from a criterion of Matsumura [Mat80, Theorem 102], using that Washnitzer algebras are regular, [GK00, Proposition 1.5], and L has characteristic 0.

In conclusion, the morphism

$$Y := X_n \bigsqcup (\bigsqcup_{i=0}^{n-1} Z_i) \rightarrow X$$

is a proper éh-covering with Y smooth. □

Remark 2.15. Let X be a rigid-analytic (resp. dagger) variety over L . By Proposition 2.14, the $Y \in X_{\acute{e}h}$, with Y a smooth rigid-analytic (resp. dagger) variety over L , form a basis of $X_{\acute{e}h}$. In fact, for any rigid-analytic (resp. dagger) variety Z over L , denoting by Z_{red} the reduced subspace of Z , the natural map $Z_{\mathrm{red}} \rightarrow Z$ is a universal homeomorphism, hence it is an éh-covering.

2.2.2. Differential forms and de Rham cohomology of singular varieties. Next, we want to state a condensed version of Guo's descent result for the éh-differentials (Proposition 2.17), which will be useful in the following sections. For this, we refer the reader to [Bos21, §5.1] for a discussion on how to translate classical results on coherent cohomology of rigid-analytic varieties into the condensed setting.

The following definition is based on Proposition 2.14 (and Remark 2.15).

Definition 2.16. Let X be a rigid-analytic variety over L . Denote by $\mathcal{B}_{\acute{e}h}^{\mathrm{sm}}$ the basis of the site $X_{\acute{e}h}$ consisting of all smooth $Y \in X_{\acute{e}h}$. For $i \geq 0$, we define $\Omega_{X_{\acute{e}h}}^i$ as the sheaf on $X_{\acute{e}h}$, with values in $\mathrm{Mod}_L^{\mathrm{cond}}$, associated to the presheaf

$$(\mathcal{B}_{\acute{e}h}^{\mathrm{sm}})^{\mathrm{op}} \rightarrow \mathrm{Mod}_L^{\mathrm{cond}} : Y \mapsto \Omega_Y^i(Y).$$

¹¹We observe that in *loc. cit.* the blow-ups considered are analytification of scheme-theoretic blow-ups. However, by the universal property in Remark 2.12, we have natural comparison morphisms between the blow-up in the sense of Definition 2.11 and the analytification of the scheme-theoretic blow-up, which are isomorphisms.

Denote by $\Omega_{X_{\text{éh}}}^\bullet$ the de Rham complex of X , given by

$$\Omega_{X_{\text{éh}}}^\bullet := [\mathcal{O}_{X_{\text{éh}}} \xrightarrow{d} \Omega_{X_{\text{éh}}}^1 \xrightarrow{d} \Omega_{X_{\text{éh}}}^2 \xrightarrow{d} \cdots].$$

We define the *de Rham cohomology* of X (over L) as

$$R\Gamma_{\text{dR}}(X) := R\Gamma(X, \Omega_{X_{\text{éh}}}^\bullet) \in D(\text{Mod}_L^{\text{cond}})$$

and endow it with the \mathbb{N}^{op} -indexed filtration $\text{Fil}^* R\Gamma_{\text{dR}}(X) := R\Gamma(X, \Omega_{X_{\text{éh}}}^{\geq *})$, called *Hodge filtration*.

The next result shows in particular that in the smooth case the de Rham cohomology defined above agrees with the usual de Rham cohomology.

Proposition 2.17 ([Guo19, Theorem 4.0.2]). *Let X be a smooth rigid-analytic variety over L . Let $\pi : X_{\text{éh}} \rightarrow X_{\text{ét}}$ be the natural morphism of sites. Then, for each $i \geq 0$, we have*

$$R\pi_* \Omega_{X_{\text{éh}}}^i = \Omega_{X_{\text{ét}}}^i[0]$$

as complexes of sheaves with values in $\text{Mod}_L^{\text{cond}}$.

The following boundedness result will also be useful in the sequel.

Proposition 2.18 ([Guo19, Theorem 6.0.2]). *Let X be a qcqs rigid-analytic variety over L of dimension d . Then, $H^i(X, \Omega_{X_{\text{éh}}}^j)$ vanishes if $i > d$ or $j > d$.*

From the proposition above we deduce the following corollary.

Corollary 2.19. *Let X be a qcqs rigid-analytic variety over L of dimension d . Then, the de Rham cohomology complex $R\Gamma_{\text{dR}}(X)$ lies in $D^{\leq 2d}(\text{Mod}_L^{\text{solid}})$.*

2.3. Period sheaves. In this subsection, we first recall the definitions of the pro-étale sheaf-theoretic version of the classical period rings of Fontaine, and we introduce a log-variant of the pro-étale sheaf-theoretic version \mathbb{B} of the ring B of analytic functions on Y_{FF} , i.e. the *log-crystalline pro-étale period sheaf* \mathbb{B}_{log} . Then, after some preliminary complementary results on the pro-étale period sheaves (and condensed period rings), we recall that, thanks to results of Scholze [Sch21], the pro-étale period sheaves satisfy v -descent.

2.3.1. Pro-étale period sheaves.

Definition 2.20. Let X be an analytic adic space over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. We define the *integral pro-ét-structure sheaf* $\widehat{\mathcal{O}}_X^+$ and the *pro-ét-structure sheaf* $\widehat{\mathcal{O}}_X$ as the sheaves on $X_{\text{proét}}$ satisfying respectively

$$\widehat{\mathcal{O}}_X^+(Y) := \mathcal{O}_{Y^\#}^+(Y^\#), \quad \widehat{\mathcal{O}}_X(Y) := \mathcal{O}_{Y^\#}(Y^\#)$$

for all perfectoid spaces $Y \in X_{\text{proét}}$.

We recall that, thanks to [Sch21, Theorem 8.7], $\widehat{\mathcal{O}}_X^+$ and $\widehat{\mathcal{O}}_X$ are indeed sheaves.

Definition 2.21. Let X be an analytic adic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. The following are defined to be sheaves on $X_{\text{proét}}$.

- (i) The *tilted integral pro-ét-structure sheaf* $\widehat{\mathcal{O}}_X^{b+} = \varprojlim_{\varphi} \widehat{\mathcal{O}}_X^+/p$, where the inverse limit is taken along the Frobenius map φ .
- (ii) The sheaves $\mathbb{A}_{\text{inf}} = W(\widehat{\mathcal{O}}_X^{b+})$ and $\mathbb{B}_{\text{inf}} = \mathbb{A}_{\text{inf}}[1/p]$. We have a morphism of pro-étale sheaves $\theta : \mathbb{A}_{\text{inf}} \rightarrow \widehat{\mathcal{O}}_X^+$ that extends to $\theta : \mathbb{B}_{\text{inf}} \rightarrow \widehat{\mathcal{O}}_X$.

- (iii) We define the *positive de Rham sheaf* $\mathbb{B}_{\text{dR}}^+ = \varprojlim_{n \in \mathbb{N}} \mathbb{B}_{\text{inf}} / (\ker \theta)^n$, with filtration given by $\text{Fil}^r \mathbb{B}_{\text{dR}}^+ = (\ker \theta)^r \mathbb{B}_{\text{dR}}^+$.
- (iv) Let t be a generator of $\text{Fil}^1 \mathbb{B}_{\text{dR}}^+$.¹² We define the *de Rham sheaf* $\mathbb{B}_{\text{dR}} = \mathbb{B}_{\text{dR}}^+[1/t]$, with filtration $\text{Fil}^r \mathbb{B}_{\text{dR}} = \sum_{j \in \mathbb{Z}} t^{-j} \text{Fil}^{r+j} \mathbb{B}_{\text{dR}}^+$.

Notation 2.22. In the following, we denote by $v(-)$ the valuation on \mathcal{O}_{C^\flat} defined as follows: for $x \in \mathcal{O}_{C^\flat}$, we define $v(x)$ as the p -adic valuation of $x^\sharp \in \mathcal{O}_C$.

Definition 2.23. Let X an analytic adic space over $\text{Spa}(C, \mathcal{O}_C)$. Let $I = [s, r]$ be an interval of $(0, \infty)$ with rational endpoints, and let $\alpha, \beta \in \mathcal{O}_{C^\flat}$ with valuation $v(\alpha) = 1/r$ and $v(\beta) = 1/s$. We define the following sheaves on $X_{\text{proét}}$

$$\mathbb{A}_{\text{inf}, I} = \mathbb{A}_{\text{inf}} \left[\frac{p}{[\alpha]}, \frac{[\beta]}{p} \right], \quad \mathbb{A}_I = \varprojlim_n \mathbb{A}_{\text{inf}, I} / p^n, \quad \mathbb{B}_I = \mathbb{A}_I[1/p].$$

Moreover, we define the sheaf on $X_{\text{proét}}$

$$\mathbb{B} = \varprojlim_{I \subset (0, \infty)} \mathbb{B}_I$$

where I runs over all the compact intervals of $(0, \infty)$ with rational endpoints.

We recall the following interpretation of the latter period sheaves defined above in terms of the curves $Y_{\text{FF}, S}$ (see §1.7 for the notation).

Lemma 2.24 ([Bos21, Lemma 4.14]). *Let S^\sharp be an affinoid perfectoid space over $\text{Spa}(C, \mathcal{O}_C)$, and let $S = (S^\sharp)^\diamond$. Let $I = [s, r] \subset (0, \infty)$ be an interval with rational endpoints. Then, we have*

$$\mathbb{B}_I(S^\sharp) = \mathcal{O}(Y_{\text{FF}, S, I}), \quad \mathbb{B}(S^\sharp) = \mathcal{O}(Y_{\text{FF}, S}).$$

The following fundamental exact sequences of p -adic Hodge theory summarize the relevant relations between the various rational period sheaves.

Proposition 2.25. *Let X an analytic adic space over $\text{Spa}(C, \mathcal{O}_C)$. Let $i \geq 0$ be an integer. We have the following exact sequences of sheaves on $X_{\text{proét}}$*

$$0 \rightarrow \mathbb{B}^{\varphi=p^i} \rightarrow \mathbb{B} \xrightarrow{\varphi p^{-i}-1} \mathbb{B} \rightarrow 0 \quad (2.1)$$

$$0 \rightarrow \mathbb{Q}_p(i) \rightarrow \mathbb{B}^{\varphi=p^i} \rightarrow \mathbb{B}_{\text{dR}}^+ / \text{Fil}^i \mathbb{B}_{\text{dR}}^+ \rightarrow 0. \quad (2.2)$$

Proof. See e.g. [Bos21, Proposition 4.16]. □

Corollary 2.26. *Let X be an analytic adic space over $\text{Spa}(C, \mathcal{O}_C)$. We have the following exact sequences of sheaves on $X_{\text{proét}}$*

$$0 \rightarrow \mathbb{B}_e \rightarrow \mathbb{B}[1/t] \xrightarrow{\varphi-1} \mathbb{B}[1/t] \rightarrow 0 \quad (2.3)$$

$$0 \rightarrow \mathbb{Q}_p \rightarrow \mathbb{B}_e \rightarrow \mathbb{B}_{\text{dR}} / \mathbb{B}_{\text{dR}}^+ \rightarrow 0 \quad (2.4)$$

where $\mathbb{B}_e := \mathbb{B}[1/t]^{\varphi=1}$.

Proof. See [LB18a, Proposition 8.5] and [Bos21, Corollary 4.18]. □

¹²Such a generator exists locally on $X_{\text{proét}}$, it is a non-zero-divisor and unique up to unit, by [Sch13a, Lemma 6.3].

2.3.2. *Log-crystalline period sheaves.* We recall that Fargues–Fontaine defined in [FF18, §10.3.1] the *log-crystalline period ring*

$$B_{\log} := B \otimes_{\mathrm{Sym}_{\mathbb{Z}} \mathcal{O}_{C^b}^\times} \mathrm{Sym}_{\mathbb{Z}}(C^b)^\times$$

where $\mathrm{Sym}_{\mathbb{Z}}(-)$ denotes the symmetric algebra over \mathbb{Z} . The ring B_{\log} is endowed with an action of the Galois group \mathcal{G}_K , a Frobenius φ , and a monodromy operator N for which $B_{\log}^{N=0} = B$. Moreover, there is a (non-canonical) isomorphism of rings

$$B[U] \xrightarrow{\sim} B_{\log}, \quad U \mapsto \log[p^b]$$

where $B[U]$ denotes the ring of polynomials over B in the variable U .

Now, keeping the notation of Definition 2.23, we introduce a pro-étale sheaf-theoretic version of the ring B_{\log} .

Definition 2.27. Let X be an analytic adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$. Let $I = [s, r]$ be an interval of $(0, \infty)$ with rational endpoints, and let $\alpha, \beta \in \mathcal{O}_{C^b}$ with valuation $v(\alpha) = 1/r$ and $v(\beta) = 1/s$. We define the following sheaves on $X_{\mathrm{pro\acute{e}t}}$

$$\mathbb{B}_{\log} := \mathbb{B}[U], \quad \mathbb{B}_{\log, I} := \mathbb{B}_I[U].$$

We endow \mathbb{B}_{\log} (resp. $\mathbb{B}_{\log, I}$) with a Frobenius φ and a Galois action extending the ones on \mathbb{B} (resp. \mathbb{B}_I) by setting $\varphi(U) := pU$, and, for $g \in \mathcal{G}_K$,

$$g(U) := U + \log[g(p^b)/p^b]. \quad (2.5)$$

Moreover, we equip \mathbb{B}_{\log} and $\mathbb{B}_{\log, I}$ with a monodromy operator $N := -\frac{d}{dU}$.

Let us list some useful basic properties of \mathbb{B}_{\log} .

Remark 2.28. The action of \mathcal{G}_K on \mathbb{B}_{\log} defined above commutes with φ and N , and we have $N\varphi = p\varphi N$.

Remark 2.29. We have the following exact sequence of sheaves on $X_{\mathrm{pro\acute{e}t}}$

$$0 \rightarrow \mathbb{B} \rightarrow \mathbb{B}_{\log} \xrightarrow{N} \mathbb{B}_{\log} \rightarrow 0.$$

Remark 2.30. For $I = [s, r]$ an interval of $(0, \infty)$ with rational endpoints such that $s \leq 1 \leq r$, we have a natural inclusion

$$\mathbb{B}_I \hookrightarrow \mathbb{B}_{\mathrm{dR}}^+.$$

The induced inclusion $\mathbb{B} \hookrightarrow \mathbb{B}_{\mathrm{dR}}^+$ extends to a \mathcal{G}_K -equivariant injection $\mathbb{B}_{\log} \hookrightarrow \mathbb{B}_{\mathrm{dR}}^+$, via sending U to $\log([p^b]/p)$ (see the proof of [FF18, Proposition 10.3.15]).

In this article, we will adopt the following notation and conventions.

Notation and conventions 2.31.

Condensed period rings. We denote by $A_{\mathrm{inf}}, B_{\mathrm{dR}}^+, B_{\mathrm{dR}}, B, B_{\log}$, the condensed rings given respectively by the sheaves $\mathbb{A}_{\mathrm{inf}}, \mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}, \mathbb{B}, \mathbb{B}_{\log}$ on the site $\mathrm{Spa}(C, \mathcal{O}_C)_{\mathrm{pro\acute{e}t}}$ and, for any compact interval $I \subset (0, \infty)$ with rational endpoints, we similarly define $A_{\mathrm{inf}, I}, A_I, B_I, B_{\log, I}$.¹³ In addition, we denote by $A_{\mathrm{cris}}, B_{\mathrm{cris}}^+, B_{\mathrm{st}}^+$ the condensed version¹⁴ of the crystalline and semistable period rings of Fontaine (relative to \mathcal{O}_C), [Fon94].

¹³See [Bos21, Corollary 4.9, Example 4.10] for the relation to the classical topological period rings.

¹⁴We take Fontaine's definitions in condensed sets.

Orientation. We fix a compatible system $(1, \varepsilon_p, \varepsilon_{p^2}, \dots)$ of p -th power roots of unity in \mathcal{O}_C , which defines an element $\varepsilon \in \mathcal{O}_C^\times$. We denote by $[\varepsilon] \in A_{\text{inf}}$ its Teichmüller lift and $\mu = [\varepsilon] - 1 \in A_{\text{inf}}$. Furthermore, we let $\xi = \mu/\varphi^{-1}(\mu) \in A_{\text{inf}}$ and $t = \log[\varepsilon] \in B$.

Let us collect some useful facts on the above-defined (condensed) period rings, that we will repeatedly use in the following.

Remark 2.32. Let us recall that, for a compact interval $I \subset [1/(p-1), \infty)$ with rational endpoints, we have that $A_{\text{cris}} \subset A_I$ (see e.g. [CN17, §2.4.2]). In particular, for any such interval I , we also have $B_{\text{st}}^+ \subset B_{\log, I}$, via the (non-canonical) identification

$$B_{\text{cris}}^+[U] \xrightarrow{\sim} B_{\text{st}}^+, \quad U \mapsto \log[p^\flat] \quad (2.6)$$

where we endow $B_{\text{cris}}^+[U]$ with a Frobenius φ extending the one on B_{cris}^+ by setting $\varphi(U) := pU$, a Galois action extending the one on B_{cris}^+ as in (2.5), and we equip it with a monodromy operator $N := -\frac{d}{dU}$.

We will also need the following result.

Lemma 2.33. *Let $I \subset (0, \infty)$ be a compact interval with rational endpoints. The system of ideals of the ring A_I defined by $(p^n A_I)_{n \geq 1}$ and $(\{x \in A_I : \mu x \in p^n A_I\})_{n \geq 1}$ are intertwined.*

Proof. We will proceed by noetherian approximation, adapting the proof of [BMS18, Lemma 12.8 (ii)]. We define $\Lambda := \mathbb{Z}_p[[T_1, T_2]]$, and we regard A_{inf} as a Λ -module via the \mathbb{Z}_p -linear map

$$\Lambda \rightarrow A_{\text{inf}}, \quad T_1 \mapsto [\varepsilon], \quad T_2 \mapsto [p^\flat]. \quad (2.7)$$

First, note that μ is the image of $T_1 - 1$ under the map (2.7). By setting

$$\Lambda_{\text{inf}, I} := \Lambda \left[\frac{p}{T_2^{1/r}}, \frac{T_2^{1/s}}{p} \right]$$

we have that $A_{\text{inf}, I} = A_{\text{inf}} \otimes_{\Lambda} \Lambda_{\text{inf}, I}$. In particular, denoting by Λ_I the p -adic completion of $\Lambda_{\text{inf}, I}$, we have $A_I = A_{\text{inf}} \widehat{\otimes}_{\Lambda} \Lambda_I$, where the latter completion is p -adic. Then, it suffices to prove the statement with the ring Λ_I in place of A_I . For this, observing that Λ_I is noetherian, we conclude by the Artin-Rees lemma ([Sta20, Tag 00IN]) for the system of ideals $(p^n \Lambda_I)_{n \geq 1}$ of Λ_I , and $(T_1 - 1)\Lambda_I \subset \Lambda_I$. \square

2.3.3. v -descent. As announced, our next goal is to state a consequence of Scholze's v -descent results in [Sch21], which will serve as a tool to prove the main comparison results of this paper for singular rigid-analytic varieties. To state the desired result we need some preliminary definitions.

Definition 2.34. Let X be an analytic adic space defined over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. We denote by

$$X_v := X_v^\diamond$$

its v -site.

We recall from [Sch21, Theorem 8.7] that the presheaves $\mathcal{O}^+ : Y \mapsto \mathcal{O}_Y^+(Y)$ and $\mathcal{O} : Y \mapsto \mathcal{O}_Y(Y)$ on the v -site of all (κ -small) perfectoid spaces are sheaves. Then, similarly to Definition 2.20, we can give the following definition.

Definition 2.35. Let X be an analytic adic space defined over $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. We define the *integral v -structure sheaf* $\widehat{\mathcal{O}}_X^+$ and the *v -structure sheaf* $\widehat{\mathcal{O}}_X$ on X_v by setting respectively

$$\widehat{\mathcal{O}}_X^+(Y) := \mathcal{O}_{Y^\sharp}^+(Y^\sharp), \quad \widehat{\mathcal{O}}_X(Y) := \mathcal{O}_{Y^\sharp}(Y^\sharp)$$

for all perfectoid spaces $Y \in X_v$.

Then, we can introduce the following notation.

Notation 2.36. For X an analytic adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$, starting from the integral v -structure sheaf $\widehat{\mathcal{O}}_X^+$, we define an analogue of the pro-étale period sheaves in §2.3.1 on the v -site X_v . By a slight abuse of notation, we denote such v -sheaves with the same symbol as the respective pro-étale period sheaves, adding a subscript $(-)_v$, resp. $(-)_{\mathrm{pro\acute{e}t}}$, in case of potential confusion.

Proposition 2.37. *Let $I \subset (0, \infty)$ be a compact interval with rational endpoints, and let $m \geq 1$ be an integer. Let*

$$\mathbf{B} \in \{\mathbb{B}_I, \mathbb{B}, \mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}^+ / \mathrm{Fil}^m\}. \quad (2.8)$$

- (i) *For any Z affinoid perfectoid space over $\mathrm{Spa}(C, \mathcal{O}_C)$, we have $H_v^i(Z, \mathbf{B}) = 0$ for all $i > 0$.*
- (ii) *Let X an analytic adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$. Let $\lambda : X_v \rightarrow X_{\mathrm{pro\acute{e}t}}$ denote the natural morphism of sites. Then, we have*

$$R\lambda_* \mathbf{B}_v = \mathbf{B}_{\mathrm{pro\acute{e}t}}.$$

In particular, the pro-étale cohomology of \mathbf{B} satisfies v -hyperdescent.

Proof. By standard reduction steps (see e.g. [Bos21, Proposition 4.7] and the references therein), part (i) follows from the almost vanishing of $H_v^i(Z, \widehat{\mathcal{O}}_X^+)$ for $i > 0$, which is proven in greater generality in [Sch21, Proposition 8.8].

For part (ii), by definition we have $\lambda_* \mathbf{B}_v = \mathbf{B}_{\mathrm{pro\acute{e}t}}$. Then, we want to show that $R^i \lambda_* \mathbf{B}_v = 0$ for $i > 0$. This follows from part (i) recalling that $R^i \lambda_* \mathbf{B}_v$ is the sheafification of the presheaf $U \mapsto H_v^i(U, \mathbf{B})$ on $X_{\mathrm{pro\acute{e}t}}$, and affinoid perfectoid spaces in $X_{\mathrm{pro\acute{e}t}}$ form a basis of the site. \square

2.4. B -cohomology and B_{dR}^+ -cohomology. Now, we can finally define the B -cohomology and the B_{dR}^+ -cohomology theories for rigid-analytic varieties over C .

In the following, for X a rigid-analytic variety over C , we denote by $X_{\acute{e}t, \mathrm{cond}}$ the site introduced in [Bos21, Definition 2.13], and similarly we define the site $X_{\acute{e}h, \mathrm{cond}}$. Note that we have a natural morphism of sites

$$\alpha : X_v \rightarrow X_{\acute{e}h, \mathrm{cond}}.$$

A feature of the site $X_{\acute{e}h, \mathrm{cond}}$ (as opposed to the site $X_{\acute{e}h}$) is that the pushforward along α retains the information captured by profinite sets.¹⁵

Definition 2.38. Let X be a rigid-analytic variety over C . We denote by $\alpha : X_v \rightarrow X_{\acute{e}h, \mathrm{cond}}$ the natural morphism of sites. Let $I \subset (0, \infty)$ be a compact interval with rational endpoints, and let $m \geq 1$ be an integer. Given

$$\mathbf{B} \in \{\mathbb{B}_I, \mathbb{B}, \mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}^+ / \mathrm{Fil}^m\}$$

we write $\mathcal{B} = \mathbf{B}_{\mathrm{Spa}(C)_{\mathrm{pro\acute{e}t}}}$ for the corresponding condensed period ring.

¹⁵We refer the reader to [Bos21, §2.3] for a more detailed discussion.

We define the \mathcal{B} -cohomology of X as the complex of $D(\text{Mod}_{\mathcal{B}}^{\text{cond}})$

$$R\Gamma_{\mathcal{B}}(X) := R\Gamma_{\text{éht,cond}}(X, L\eta_t R\alpha_* \mathbf{B}).$$

We endow $R\Gamma_{\mathcal{B}}(X)$ with the filtration induced by the filtration décalée of Definition 2.9.

Remark 2.39 (Frobenius on the B -cohomology). Since $\varphi(t) = pt$, the Frobenius automorphism of \mathbb{B} induces a φ_B -semilinear automorphism

$$\varphi : R\Gamma_B(X) \rightarrow R\Gamma_B(X)$$

which preserves the filtration décalée.

Next, we begin to study some basic properties of the B -cohomology theory. As a preparation, we state the following boundedness result which relies on an improved version of the almost purity theorem recently proved by Bhatt–Scholze, [BS22, Theorem 10.9].

Proposition 2.40 (cf. [Guo19, Proposition 7.5.2], [Zav22, Theorem 6.10.3]). *Let X be a rigid-analytic variety over C of dimension d . Let $\nu : X_{\text{proét}} \rightarrow X_{\text{ét,cond}}$ denote the natural morphisms of sites. Let \mathbf{B} any of the period sheaves of (2.8). Then, $R^i \nu_* \mathbf{B}$ vanishes for all $i > d$.*

Proof. We will show that for any affinoid rigid space X over C of dimension d , we have $H_{\text{proét}}^i(X, \mathbf{B}) = 0$ for all $i > d$. By Noether normalization lemma, [Bos14, §3.1, Proposition 3], there exists a finite morphism $f : X \rightarrow \mathbb{D}_C^d$, where the target denotes the d -dimensional unit closed disk over C . Then, by [Zav22, Lemma 6.10.2],¹⁶ the diamond X^\diamond admits a $\mathbb{Z}_p(1)^d$ -torsor for the ν -topology

$$\tilde{X}^\diamond \rightarrow X^\diamond \tag{2.9}$$

where \tilde{X}^\diamond is a diamond representable by an affinoid perfectoid space. Considering the Cartan–Leray spectral sequence associated to (2.9), by [Bos21, Proposition 4.12] and Proposition 2.37, we have an isomorphism

$$R\Gamma_{\text{cond}}(\mathbb{Z}_p(1)^d, H^0(\tilde{X}^\diamond, \mathbf{B})) \xrightarrow{\sim} R\Gamma_{\text{proét}}(X, \mathbf{B}).$$

Then, the statement follows from [Bos21, Proposition B.3], which implies that $\mathbb{Z}_p(1)^d \cong \mathbb{Z}_p^d$ has cohomological dimension d . \square

The following lemma will be useful to reduce the study of the B -cohomology theory to the study of the B_I -cohomology theories for suitable intervals $I \subset (0, \infty)$.

Lemma 2.41 (cf. [LB18b, Lemma 4.3]). *With notation as in Definition 2.38, the natural maps*

$$L\eta_t R\alpha_* \mathbb{B} \rightarrow R\varprojlim_I L\eta_t R\alpha_* \mathbb{B}_I \quad L\eta_t R\alpha_* \mathbb{B}_{\text{dR}}^+ \rightarrow R\varprojlim_m L\eta_t R\alpha_* (\mathbb{B}_{\text{dR}}^+ / \text{Fil}^m). \tag{2.10}$$

are isomorphisms compatible with the filtration décalée.

Proof. We prove that the left map in (2.10) is an isomorphism compatible with filtrations (for the right map in (2.10) the proof is similar and easier). By [BMS19, Lemma 5.2, (1)], it suffices to show that the limit of the filtrations of the source and the target agree, and that such map is an isomorphism on graded pieces.

¹⁶This lemma relies on [BS22, Theorem 10.11], and hence on [BS22, Theorem 10.9].

For the first assertion, by the uniform boundedness of the complexes $R\alpha_*\mathbb{B}$ and $R\alpha_*\mathbb{B}_I$ for varying compact intervals $I \subset (0, \infty)$ with rational endpoints, which follows from Proposition 2.40, we can reduce to showing that the natural map

$$R\alpha_*\mathbb{B} \rightarrow R\varprojlim_I R\alpha_*\mathbb{B}_I$$

is an isomorphism. This follows recalling that the natural map $\mathbb{B} \rightarrow R\varprojlim_I \mathbb{B}_I$ is an isomorphism: in fact, using [Sch13a, Lemma 3.18], we can reduce to checking this on each affinoid perfectoid space Z over $\mathrm{Spa}(C, \mathcal{O}_C)$, where it follows from the topological Mittag-Leffler property of the countable inverse system $\{\mathbb{B}_I(Z)\}_I$ which implies that, for all $j > 0$, we have $R^j\varprojlim_I \mathbb{B}_I(Z) = 0$ ([Bos21, Lemma 4.8]).

Then, by Proposition 2.7(ii) (and by twisting) it remains to prove that, for each $i \geq 0$, the natural map

$$\tau^{\leq i} R\alpha_*(\mathbb{B}/t) \rightarrow R\varprojlim_I \tau^{\leq i} R\alpha_*(\mathbb{B}_I/t) \quad (2.11)$$

is an isomorphism. For this, we observe that, by [FS21, Theorem II.0.1], for any compact interval $I \subset (0, \infty)$ with rational endpoints, we have

$$\mathbb{B}_I/t = \prod_{y \in |Y_{\mathrm{FF}, I}|^{\mathrm{cl}}} \mathbb{B}_{\mathrm{dR}}^+ / t^{\mathrm{ord}_y(t)} \mathbb{B}_{\mathrm{dR}}^+ \quad (2.12)$$

where $|Y_{\mathrm{FF}, I}|^{\mathrm{cl}} \subset |Y_{\mathrm{FF}, I}|$ denotes the subset of classical points (we note that, by compactness of the interval I , the latter product is a finite direct product of copies of $\widehat{\mathcal{O}}$).¹⁷ Then, using again the topological Mittag-Leffler property of the countable inverse system $\{\mathbb{B}_I(Z)\}_I$ for each Z affinoid perfectoid spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$, we have that

$$\mathbb{B}/t = \prod_{y \in |Y_{\mathrm{FF}}|^{\mathrm{cl}}} \mathbb{B}_{\mathrm{dR}}^+ / t^{\mathrm{ord}_y(t)} \mathbb{B}_{\mathrm{dR}}^+ \quad (2.13)$$

where $|Y_{\mathrm{FF}}|^{\mathrm{cl}} \subset |Y_{\mathrm{FF}}|$ denotes the subset of classical points (cf. with [FF18, §2.6]). Moreover, by (2.12) we have that

$$R\varprojlim_I \tau^{\leq i} R\alpha_*(\mathbb{B}_I/t) = \prod_I \tau^{\leq i} R\alpha_*(\mathbb{B}_I/t). \quad (2.14)$$

We conclude that the natural map (2.11) is an isomorphism, combining (2.12), (2.13), (2.14), and fact that cohomology commutes with direct products. \square

The next proposition gives in particular a convenient local description of the B -cohomology theory on a smooth affinoid rigid space over C .

Proposition 2.42. *With notation as in Definition 2.38, let $\nu : X_{\mathrm{pro\acute{e}t}} \rightarrow X_{\acute{e}t, \mathrm{cond}}$ denote the natural morphism of sites.*

(i) *If X is smooth, we have a natural identification in $D(\mathrm{Mod}_{\mathcal{B}}^{\mathrm{cond}})$*

$$R\Gamma_{\mathcal{B}}(X) = R\Gamma(X, L\eta_t R\nu_* \mathbf{B}).$$

¹⁷For $y \in |Y_{\mathrm{FF}}|^{\mathrm{cl}}$, we have $\mathrm{ord}_y(t) \in \{0, 1\}$: in fact, t has a simple zero at ∞ on $\mathrm{FF} = Y_{\mathrm{FF}}/\varphi^{\mathbb{Z}}$.

(ii) If X is a smooth affinoid over C , the natural map of complexes of condensed \mathcal{B} -modules

$$L\eta_t R\Gamma_v(X, \mathbf{B}) \rightarrow R\Gamma(X, L\eta_t R\alpha_* \mathbf{B}) = R\Gamma_{\mathcal{B}}(X) \quad (2.15)$$

is a filtered quasi-isomorphism. Here, on both sides, the filtration on $L\eta_t(-)$ is the filtration décalée of Definition 2.9.

Proof. We first prove part (ii), adapting the proof of [LB18b, Proposition 3.11], and using Proposition 2.8 for the compatibility with the filtrations statement.

Thus, let X be a smooth affinoid over C . To show that (2.15) is a filtered quasi-isomorphism, similarly to the proof of Lemma 2.41, by [BMS19, Lemma 5.2, (1)], it suffices to show that the limit of the filtrations of the source and the target of (2.15) agree, and that such map is an isomorphism on graded pieces. The former statement follows from Proposition 2.40. Then, by Proposition 2.7(ii) (and by twisting) it suffices to prove that, for each $i \geq 0$, the natural map

$$\tau^{\leq i} R\Gamma(X, \mathbf{B}/t) \rightarrow R\Gamma(X, \tau^{\leq i} R\alpha_*(\mathbf{B}/t))$$

is a quasi-isomorphism. By (the proof of) Lemma 2.41, we can reduce to the case $\mathbf{B} \in \{\mathbb{B}_I, \mathbb{B}_{\mathbb{R}}^+\}$. Then, recalling that \mathbb{B}_I/t is isomorphic to a finite direct product of copies of $\widehat{\mathcal{O}}$ (by (2.12) and the compactness of the interval I), we can further reduce to showing that, for each $i \geq 0$, the natural map

$$\tau^{\leq i} R\Gamma(X, \widehat{\mathcal{O}}) \rightarrow R\Gamma(X, \tau^{\leq i} R\alpha_* \widehat{\mathcal{O}})$$

is a quasi-isomorphism. For this, considering the spectral sequences

$$\begin{aligned} H^{j-k}(X, H^k(R\alpha_* \widehat{\mathcal{O}})) &\implies H^j(X, \widehat{\mathcal{O}}) \\ H^{j-k}(X, H^k(\tau^{\leq i} R\alpha_* \widehat{\mathcal{O}})) &\implies H^j(X, \tau^{\leq i} R\alpha_* \widehat{\mathcal{O}}) \end{aligned}$$

it suffices to show that, for $j > i$ and $k \leq i$, we have $H^{j-k}(X, R^k \alpha_* \widehat{\mathcal{O}}) = 0$, or more generally that

$$H^r(X, R^k \alpha_* \widehat{\mathcal{O}}) = 0, \text{ for all } r > 0. \quad (2.16)$$

By (the proof of) [Sch13b, Proposition 3.23], for any Y smooth rigid-analytic variety over $\mathrm{Spa}(C, \mathcal{O}_C)$, denoting by $\nu : Y_{\mathrm{pro\acute{e}t}} \rightarrow Y_{\acute{e}t, \mathrm{cond}}$ the natural morphism of sites, we have a natural isomorphism $\Omega_{Y_{\acute{e}t}}^k(-k) \xrightarrow{\sim} R^k \nu_* \widehat{\mathcal{O}}$ of sheaves with values in $\mathrm{Mod}_C^{\mathrm{cond}}$.¹⁸ Then, by éh-sheafification, we have a natural isomorphism of sheaves with values in $\mathrm{Mod}_C^{\mathrm{cond}}$

$$\Omega_{X_{\acute{e}h}}^k(-k) \xrightarrow{\sim} R^k \alpha_* \widehat{\mathcal{O}}. \quad (2.17)$$

Denoting by $\pi : X_{\acute{e}h} \rightarrow X_{\acute{e}t}$ the natural morphism of sites, by Proposition 2.17 (using that X is smooth), we have that

$$R\pi_* \Omega_{X_{\acute{e}h}}^k = \Omega_{X_{\acute{e}t}}^k[0] \quad (2.18)$$

as complexes of sheaves with values in $\mathrm{Mod}_C^{\mathrm{cond}}$. Then, combining (2.17) and (2.18), we deduce (2.16) from the condensed version of Tate's acyclicity theorem (see [Bos21, Lemma 5.6(i)]). This concludes the proof of part (ii).

For part (i), as the statement is étale local, we can reduce to the case when X is a smooth affinoid rigid space over C . In this case, similarly to part (ii), the natural map

$$L\eta_t R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}) \rightarrow R\Gamma(X, L\eta_t R\nu_* \mathbf{B}) \quad (2.19)$$

¹⁸Working with condensed group cohomology instead of continuous group cohomology in the proof of *loc. cit.*

is a quasi-isomorphism. Hence, combining (2.19) and (2.15), the statement follows from Proposition 2.37. \square

3. HYODO–KATO COHOMOLOGY

In this section, following Colmez–Nizioł, [CN21a], we define the *Hyodo–Kato cohomology theory* for rigid-analytic varieties over C , simplifying the topological treatment given in *op. cit.* and extending it to the singular case.

3.1. Local Hyodo–Kato morphism. We begin by revisiting in the condensed setup the Hyodo–Kato morphism constructed by Beilinson and Colmez–Nizioł.

Notation and conventions 3.1. Let $n \geq 1$ be an integer. For a condensed ring R , we denote by R_n the reduction of R modulo p^n .

Log structures. We define a (pre-)log structure on a given condensed ring as a (pre-)log structure on the underlying ring.

For \mathcal{O} a discrete valuation ring, we denote by \mathcal{O}^\times (resp. \mathcal{O}_n^\times) the canonical log structure on \mathcal{O} (resp. its pullback on \mathcal{O}_n), and we denote by \mathcal{O}^0 (resp. \mathcal{O}_n^0) the log structure on \mathcal{O} associated to $(\mathbb{N} \rightarrow \mathcal{O}, 1 \mapsto 0)$ (resp. its pullback on \mathcal{O}_n).

We denote by \mathcal{O}_C^\times (resp. $\mathcal{O}_{C,n}^\times$) the canonical log structure on \mathcal{O}_C (resp. its pullback on $\mathcal{O}_{C,n}$). We write $A_{\text{cris},n}^\times$ for the unique quasi-coherent, integral, log structure on $A_{\text{cris},n}$ lifting $\mathcal{O}_{C,n}^\times$ (see e.g. [CK19, §5.2]). We denote by A_{cris}^\times the log structure on A_{cris} associated to the pre-log structure

$$\mathcal{O}_{C^b} \setminus \{0\} \rightarrow A_{\text{cris}}, x \mapsto [x].$$

Note that the log structure $A_{\text{cris},n}^\times$ is the pullback of the log structure A_{cris}^\times .

Log-crystalline cohomology. We refer the reader to [Bei13, §1] for a review of log-crystalline cohomology, and the terminology used in the following. We write PD as a shortening of *divided power*.

Let $(\mathcal{Y}, M_{\mathcal{Y}}, \mathcal{I}, \gamma)$ be a (p -adic formal) log PD scheme such that $(\mathcal{Y}, M_{\mathcal{Y}})$ is quasi-coherent. Let $(\mathcal{X}, M_{\mathcal{X}})$ be an integral quasi-coherent (p -adic formal) log scheme over $(\mathcal{Y}, M_{\mathcal{Y}}, \mathcal{I}, \gamma)$. We write

$$((\mathcal{X}, M_{\mathcal{X}})/(\mathcal{Y}, M_{\mathcal{Y}}))_{\text{cris}}$$

for the log-crystalline site of $(\mathcal{X}, M_{\mathcal{X}})$ over $(\mathcal{Y}, M_{\mathcal{Y}}, \mathcal{I}, \gamma)$, [Bei13, §1.12], we denote by $\mathcal{O}_{\text{cris}}$ its structure sheaf, regarded as a sheaf with values in condensed abelian groups, and we define the log-crystalline cohomology

$$R\Gamma_{\text{cris}}((\mathcal{X}, M_{\mathcal{X}})/(\mathcal{Y}, M_{\mathcal{Y}})) := R\Gamma(((\mathcal{X}, M_{\mathcal{X}})/(\mathcal{Y}, M_{\mathcal{Y}}))_{\text{cris}}, \mathcal{O}_{\text{cris}}) \in D(\text{CondAb}).$$

In the case the relevant log structures are fixed, they are omitted from the notation.

Condensed period rings. Recall from §1.7 that $\mathcal{O}_F = W(k)$ and $\mathcal{O}_{\bar{F}} = W(\bar{k})$, where \bar{k} is a fixed algebraic closure of k . We fix the unique Frobenius equivariant section $k \rightarrow \mathcal{O}_K/p$ of $\mathcal{O}_K/p \rightarrow k$, in order to regard \mathcal{O}_K as a \mathcal{O}_F -algebra, and \mathcal{O}_C , as well as the condensed period rings of 2.31, as a $\mathcal{O}_{\bar{F}}$ -algebra.

We denote $r_{\varpi}^{\pm} = \mathcal{O}_F[[T]]$ and we equip it with the log structure associated to T . We write r_{ϖ}^{PD} for the p -adic log PD envelope of r_{ϖ}^{\pm} with respect to the kernel of the morphism

$$r_{\varpi}^{\pm} \rightarrow \mathcal{O}_K^{\times}, T \mapsto \varpi$$

and we endow it with a Frobenius induced by $T \mapsto T^p$, and a monodromy defined by $T \mapsto T$. Then, we define the condensed period rings

$$\widehat{A}_{\text{st},n} := H_{\text{cris}}^0(\mathcal{O}_{C,n}^{\times}/r_{\varpi,n}^{\text{PD}}) \simeq R\Gamma_{\text{cris}}(\mathcal{O}_{C,n}^{\times}/r_{\varpi,n}^{\text{PD}}), \quad \widehat{A}_{\text{st}} := \varprojlim_n \widehat{A}_{\text{st},n}, \quad \widehat{B}_{\text{st}}^+ := \widehat{A}_{\text{st}}[1/p]$$

and we equip them with their natural action of \mathcal{G}_K , Frobenius φ , and monodromy N , [Tsu99, §4.6], [CDN20b, §3.2.1].

Theorem 3.2 (Beilinson, Colmez–Nizioł, [CN21a, Theorem 2.12, Corollary 2.20]). *Let \mathcal{X} be an integral quasi-coherent log scheme over $\mathcal{O}_{C,1}^{\times}$. Denote by \mathcal{X}^0 the pullback of \mathcal{X} to $\mathcal{O}_{\check{F},1}^0$. Assume that \mathcal{X} has a descent to \mathcal{Z} a qcqs, fine, log-smooth, log scheme over $\mathcal{O}_{L,1}^{\times}$ of Cartier type,¹⁹ for some finite extension L/K .*

(i) *There exists a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

$$R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \otimes_{\mathcal{O}_{\check{F}}}^{\text{L}\blacksquare} B_{\text{st}}^+ \xrightarrow{\sim} R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^{\times}) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} B_{\text{st}}^+ \quad (3.1)$$

*independent of the descent, and compatible with the actions of Galois, Frobenius φ and monodromy N .*²⁰

(ii) *There exists a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

$$R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \otimes_{\mathcal{O}_{\check{F}}}^{\text{L}\blacksquare} C \xrightarrow{\sim} R\Gamma_{\text{cris}}(\mathcal{X}/\mathcal{O}_C^{\times})_{\mathbb{Q}_p} \quad (3.2)$$

independent of the descent, and compatible with the actions of Galois, Frobenius φ and with the quasi-isomorphism (3.1) via the morphism $R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^{\times}) \rightarrow R\Gamma_{\text{cris}}(\mathcal{X}/\mathcal{O}_C^{\times})$ induced by Fontaine's map $\theta : A_{\text{cris}} \rightarrow \mathcal{O}_C$.

Proof. For part (i), the desired morphism (3.1), satisfying the stated properties, is constructed in [CN21a, Theorem 2.12], and we only need to carry *loc. cit.* to solid \mathbb{Q}_p -vector spaces. By the independence of the descent proven in *loc. cit.* we can assume for simplicity that $L = K$. Relying on [CN21a, §2.3.3], we will construct (3.1) as the composite

$$\varepsilon_{\text{st}} := \delta^{-1} \circ (\widehat{\varepsilon}_{\text{st}})^{N\text{-nilp}} \circ \delta$$

where $M^{N\text{-nilp}} := \text{colim}_{r \in \mathbb{N}} M^{Nr=0}$, we denote by $\delta : B_{\text{st}}^+ \xrightarrow{\sim} \widehat{B}_{\text{st}}^{+,N\text{-nilp}}$ the natural B_{cris}^+ -linear isomorphism,²¹ and

$$\widehat{\varepsilon}_{\text{st}} : R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \otimes_{\mathcal{O}_{\check{F}}}^{\text{L}\blacksquare} \widehat{B}_{\text{st}} \rightarrow R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^{\times}) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} \widehat{B}_{\text{st}}$$

is defined as follows. Considering the morphisms of PD thickenings

¹⁹See [Kat89, Definition 4.8] for the definition of *Cartier type*.

²⁰On the right-hand side of (3.1) the operator N is the monodromy of B_{st}^+ , and on the left-hand side of (3.1) it combines the monodromy of both factors of the tensor product.

²¹Which is compatible with Galois, Frobenius, and monodromy actions, [Kat94, Theorem 3.7].

$$\begin{array}{ccc} r_{\varpi,n}^{\text{PD}} & \longrightarrow & \mathcal{O}_{K,1}^\times \\ \downarrow & & \downarrow \\ \widehat{A}_{\text{st},n} & \longrightarrow & \mathcal{O}_{C,1}^\times \end{array}$$

for varying $n \geq 1$, by base change, [Bei13, (1.11.1)], we have quasi-isomorphisms

$$R\Gamma_{\text{cris}}(\mathcal{Z}/(r_{\varpi,n}^{\text{PD}}, \mathcal{O}_{K,1}^\times)) \otimes_{r_{\varpi,n}^{\text{PD}}}^{\text{L}} \widehat{A}_{\text{st},n} \xrightarrow{\sim} R\Gamma_{\text{cris}}(\mathcal{X}/(\widehat{A}_{\text{st},n}, \mathcal{O}_{C,1}^\times)). \quad (3.3)$$

Now, denoting by $\widehat{\otimes}^{\text{L}}$ the derived p -adic completion, again by base change [Bei13, (1.11.1)], after taking the derived inverse limit over $n \geq 1$, and then inverting p in (3.3), the right-hand side identifies with $(R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times) \widehat{\otimes}_{A_{\text{cris}}}^{\text{L}} \widehat{A}_{\text{st}})_{\mathbb{Q}_p}$, and, by [CN21a, Theorem 2.6], the left-hand side is quasi-isomorphic to

$$(R\Gamma_{\text{cris}}(\mathcal{Z}^0/\mathcal{O}_F^0) \widehat{\otimes}_{\mathcal{O}_F}^{\text{L}} \widehat{A}_{\text{st}})_{\mathbb{Q}_p} \simeq (R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \widehat{\otimes}_{\mathcal{O}_{\check{F}}}^{\text{L}} \widehat{A}_{\text{st}})_{\mathbb{Q}_p}.$$

where \mathcal{Z}^0 is the pullback of \mathcal{Z} to $\mathcal{O}_{F,1}^0$. We denote by $\widehat{\varepsilon}_{\text{st}}$ the induced from (3.3) morphism

$$\widehat{\varepsilon}_{\text{st}} : (R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \widehat{\otimes}_{\mathcal{O}_{\check{F}}}^{\text{L}} \widehat{A}_{\text{st}})_{\mathbb{Q}_p} \xrightarrow{\sim} (R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times) \widehat{\otimes}_{A_{\text{cris}}}^{\text{L}} \widehat{A}_{\text{st}})_{\mathbb{Q}_p}. \quad (3.4)$$

To write the target of (3.4) in terms of the derived solid tensor product, we note that, since the A_{cris} -algebra \widehat{A}_{st} is isomorphic to the p -adic completion of a divided power polynomial algebra of the form $A_{\text{cris}}\langle x \rangle$, applying Proposition A.3 with $M = R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times)$ and $N = A_{\text{cris}}\langle x \rangle$ regarded in $D(\text{Mod}_{A_{\text{cris}}}^{\text{solid}})$, we obtain the identification

$$R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times) \widehat{\otimes}_{A_{\text{cris}}}^{\text{L}} \widehat{A}_{\text{st}} = R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} \widehat{A}_{\text{st}}. \quad (3.5)$$

Next, we want to show that

$$(R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \widehat{\otimes}_{\mathcal{O}_{\check{F}}}^{\text{L}} \widehat{A}_{\text{st}})_{\mathbb{Q}_p} = (R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \otimes_{\mathcal{O}_{\check{F}}}^{\text{L}\blacksquare} \widehat{A}_{\text{st}})_{\mathbb{Q}_p}. \quad (3.6)$$

We note that, choosing a basis for the \check{F} -Banach space $\widehat{B}_{\text{st}}^+$, we can identify it with $N_p^\wedge[1/p]$, where $N = \bigoplus_I \mathcal{O}_{\check{F}}$ for some set I .²² Since \widehat{A}_{st} is a lattice in $\widehat{B}_{\text{st}}^+$, there exist $n, m \in \mathbb{Z}$ such that $p^n N_p^\wedge \subset \widehat{A}_{\text{st}} \subset p^m N_p^\wedge$. Then, (3.6) follows applying Proposition A.3 with $M = R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0)$ and $N = \bigoplus_I \mathcal{O}_{\check{F}}$ regarded in $D(\text{Mod}_{\mathcal{O}_{\check{F}}}^{\text{solid}})$.

Therefore, in view of (3.5) and (3.6), using that the derived solid tensor product commutes with filtered colimits, the composite $\varepsilon_{\text{st}} = \delta^{-1} \circ (\widehat{\varepsilon}_{\text{st}})^{N\text{-nilp}} \circ \delta$ is given by

$$R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \otimes_{\mathcal{O}_{\check{F}}}^{\text{L}\blacksquare} B_{\text{st}}^+ \xrightarrow{\sim} (R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\check{F}}^0) \otimes_{\mathcal{O}_{\check{F}}}^{\text{L}\blacksquare} \widehat{B}_{\text{st}}^+)^{N\text{-nilp}} \quad (3.7)$$

$$\xrightarrow{\sim} (R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times) \otimes_{\mathcal{O}_{\check{F}}}^{\text{L}\blacksquare} \widehat{B}_{\text{st}}^+)^{N\text{-nilp}} \quad (3.8)$$

$$\xrightarrow{\sim} R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} B_{\text{st}}^+ \quad (3.9)$$

²²In fact, as \check{F} is discretely valued, combining [Bos21, Lemma A.30, Proposition A.55(i), Lemma A.52], any \check{F} -Banach space is isomorphic to $(\bigoplus_I \mathbb{Z}_p)_p^\wedge \otimes_{\mathbb{Z}_p}^{\blacksquare} \check{F}$, for some set I , and the latter is isomorphic to $(\bigoplus_I \mathcal{O}_{\check{F}})_p^\wedge[1/p]$ by Proposition A.3.

where in (3.7) we used that the monodromy operator N on $R\Gamma_{\text{cris}}(\mathcal{X}^0/\mathcal{O}_{\mathbb{F}}^0)$ is nilpotent by Lemma 3.3 (and base change), and in (3.9) we used the triviality of the action of N on $R\Gamma_{\text{cris}}(\mathcal{X}/A_{\text{cris}}^\times)$. This shows part (i).

Part (ii) follows from (i). In fact, under the (non-canonical) identification $B_{\text{st}}^+ = B_{\text{cris}}^+[U]$, given by (2.6), applying to (3.1) the (non-Galois-equivariant) map $B_{\text{st}}^+ \rightarrow B_{\text{cris}}^+ : U \mapsto 0$, and then Fontaine's map $\theta : B_{\text{cris}}^+ \rightarrow C$, by base change we get (3.2). The compatibility of (3.2) with the Galois action is checked in the proof of [CN21a, Corollary 2.20]. \square

Lemma 3.3. *Let \mathcal{Z} be a quasi-separated, fine, saturated, log-smooth, locally of finite type log scheme over $\mathcal{O}_{K,1}^\times$ of dimension d . Then, the monodromy operator N on $R\Gamma_{\text{cris}}(\mathcal{Z}/W(k)^\times)$ is nilpotent with nilpotency index bounded above by a function depending on d .*

Proof. By [Niz06, Theorem 5.10] there exists a log-blow-up $\mathcal{Y} \rightarrow \mathcal{Z}$ over $\mathcal{O}_{K,1}^\times$ that resolves singularities, and by the proof of [Niz08, Proposition 2.3] we have a natural quasi-isomorphism

$$R\Gamma_{\text{cris}}(\mathcal{Z}/W(k)^\times) \xrightarrow{\sim} R\Gamma_{\text{cris}}(\mathcal{Y}/W(k)^\times)$$

compatible with monodromy N . Then, the statement follows from [Mok93, §3]. \square

3.2. Beilinson bases and ∞ -categories of hypersheaves. In this subsection, we collect some ∞ -categorical tools that we will need to extend to rigid-analytic varieties over C the local Hyodo-Kato morphism of §3.1.

Notation 3.4 (Hypersheaves and hypercompletion). Let \mathcal{C} be a site, and let \mathcal{D} be a presentable ∞ -category. We denote by $\text{Shv}(\mathcal{C}, \mathcal{D})$ the ∞ -category of sheaves on \mathcal{C} with values in \mathcal{D} .

We recall that

$$\text{Shv}(\mathcal{C}, \mathcal{D}) = \text{Shv}(\mathcal{C}, \text{Ani}) \otimes \mathcal{D}$$

[Lur18, Remark 1.3.1.6], where \otimes denotes the tensor product of ∞ -categories [Lur17, §4.8.1].

We denote by $\text{Shv}^{\text{hyp}}(\mathcal{C}, \text{Ani})$ the full ∞ -subcategory of $\text{Shv}(\mathcal{C}, \text{Ani})$ spanned by the hypercomplete objects, [Lur09, §6.5], and we define the ∞ -category of *hypersheaves on \mathcal{C} with values in \mathcal{D}* as

$$\text{Shv}^{\text{hyp}}(\mathcal{C}, \mathcal{D}) := \text{Shv}^{\text{hyp}}(\mathcal{C}, \text{Ani}) \otimes \mathcal{D}.$$

The inclusion $\text{Shv}^{\text{hyp}}(\mathcal{C}, \mathcal{D}) \hookrightarrow \text{Shv}(\mathcal{C}, \mathcal{D})$ admits a left adjoint

$$(-)^{\text{hyp}} : \text{Shv}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Shv}^{\text{hyp}}(\mathcal{C}, \mathcal{D}) \quad (3.10)$$

called *hypercompletion*, [Lur18, Remark 1.3.3.2].

The following generalization of the notion of Grothendieck basis for a site is due to Beilinson, [Bei12, §2.1].

Definition 3.5. Let \mathcal{C} be a small site. A *Beilinson basis* for \mathcal{C} is a pair (\mathcal{B}, \beth) where \mathcal{B} is a small category and $\beth : \mathcal{B} \rightarrow \mathcal{C}$ is a faithful functor satisfying the following property:

for any $V \in \mathcal{C}$ and any finite family of pairs $\{(U_\alpha, f_\alpha)\}$ with $U_\alpha \in \mathcal{B}$ and $f_\alpha : V \rightarrow \beth(U_\alpha)$, there exists a family $\{U'_\beta\}$ with $U'_\beta \in \mathcal{B}$ and a covering family $\{\beth(U'_\beta) \rightarrow V\}$ such that each composition

$$\beth(U'_\beta) \rightarrow V \rightarrow \beth(U_\alpha)$$

lies in the image of $\text{Hom}(U'_\beta, U_\alpha) \hookrightarrow \text{Hom}(\beth(U'_\beta), \beth(U_\alpha))$.

We endow \mathcal{B} with the Grothendieck topology induced from that of \mathcal{C} : a sieve in \mathcal{B} is a covering sieve if its image under $\square : \mathcal{B} \rightarrow \mathcal{C}$ generates a covering sieve in \mathcal{C} .

We will use repeatedly the following result.

Lemma 3.6. *Let \mathcal{C} be a small site, and let (\mathcal{B}, \square) be a Beilinson basis for \mathcal{C} . For any presentable ∞ -category \mathcal{D} , the functor $\square : \mathcal{B} \rightarrow \mathcal{C}$ induces an equivalence of ∞ -categories*

$$\mathrm{Shv}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C}, \mathcal{D}) : \mathcal{F} \mapsto \mathcal{F}^\square$$

where the hypersheaf \mathcal{F}^\square is defined via sending $V \in \mathcal{C}$ to

$$\mathcal{F}^\square(V) = \mathrm{colim}_{U_\bullet} \lim_{[n] \in \Delta} \mathcal{F}(U_n)$$

the colimit running over all simplicial objects U_\bullet of \mathcal{B} such that $\square(U_\bullet) \rightarrow V$ is a hypercover; furthermore, chosen such a U_\bullet , the natural map

$$\mathcal{F}^\square(V) \rightarrow \lim_{[n] \in \Delta} \mathcal{F}(U_n)$$

is an isomorphism in \mathcal{D} .

Proof. It suffices to show the statement in the case \mathcal{D} is the ∞ -category of anima Ani . By [Bei12, §2.1 Proposition], the functor $\square : \mathcal{B} \rightarrow \mathcal{C}$ is continuous and induces an equivalence of topoi $\mathcal{B}^\sim \rightarrow \mathcal{C}^\sim$. Then, the statement follows interpreting the notion of hypercompleteness in terms of the Brown–Joyal–Jardine theory of simplicial presheaves, via [Lur09, Proposition 6.5.2.14]. \square

Remark 3.7. We will often apply Lemma 3.6 in the case $\mathcal{D} = D(\mathrm{Mod}_A^{\mathrm{cond}})$ is the derived ∞ -category of A -modules in CondAb , for a given condensed ring A . Note that such \mathcal{D} is indeed presentable, since it is compactly generated, as it follows from [CS19, Theorem 2.2].²³ Moreover, by [Lur18, Corollary 2.1.2.3], we have an equivalence of ∞ -categories

$$D(\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_A^{\mathrm{cond}})) \xrightarrow{\sim} \mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C}, D(\mathrm{Mod}_A^{\mathrm{cond}}))$$

sending $M \in D(\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_A^{\mathrm{cond}}))$ to the hypersheaf $U \mapsto R\Gamma(U, M)$.

3.3. Globalization. In this subsection, we extend to rigid-analytic varieties over C the local Hyodo–Kato morphism of §3.1, from a suitable Beilinson basis for the site $\mathrm{Rig}_{C, \acute{e}h}$.

Notation 3.8.

Semistable formal schemes. For each prime ℓ , we fix a compatible system $(p, p^{1/\ell}, p^{\ell^2}, \dots)$ of ℓ -th power roots of p in \mathcal{O}_C . We denote by $\mathcal{M}_{\mathrm{ss}}$ the category of *semistable p -adic formal schemes over $\mathrm{Spf}(\mathcal{O}_C)$* , that is the category of p -adic formal schemes over $\mathrm{Spf}(\mathcal{O}_C)$ having in the Zariski topology a covering by open affines \mathfrak{U} with *semistable coordinates*, i.e. admitting an étale $\mathrm{Spf}(\mathcal{O}_C)$ -morphism $\mathfrak{U} \rightarrow \mathrm{Spf}(R^\square)$ with

$$R^\square := \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\} / (t_0 \cdots t_r - p^q)$$

for some $0 \leq r \leq d$, and $q \in \mathbb{Q}_{>0}$ (that may depend on \mathfrak{U}). We denote by $\mathcal{M}_{\mathrm{ss}, \mathrm{qcqs}}$ the subcategory of $\mathcal{M}_{\mathrm{ss}}$ consisting of the qcqs formal schemes. We write

$$(-)_\eta : \mathcal{M}_{\mathrm{ss}} \rightarrow \mathrm{Rig}_C$$

for the generic fiber functor.

²³Recall also our set-theoretic conventions in §1.7.

Log structures. Unless stated otherwise, we equip $\mathfrak{X} \in \mathcal{M}_{\text{ss}}$ (resp. $\mathfrak{X}_{\mathcal{O}_C/p^n}$) with the canonical log structure, [CK19, §1.6], i.e. the log structure given by the subsheaf associated to the subsheaf $\mathcal{O}_{\mathfrak{X},\text{ét}} \cap (\mathcal{O}_{\mathfrak{X},\text{ét}}[1/p])^\times \hookrightarrow \mathcal{O}_{\mathfrak{X},\text{ét}}$ (resp. its pullback). For $\mathfrak{X} \in \mathcal{M}_{\text{ss}}$, we denote by $\mathfrak{X}_{\mathcal{O}_C/p}^0$ the pullback to $\mathcal{O}_{\mathbb{F},1}^0$ of the log scheme $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathcal{O}_{C,1}^\times$.

Log de Rham cohomology. For $\mathfrak{X} \in \mathcal{M}_{\text{ss}}$, we denote by $\Omega_{\mathfrak{X},\log}^\bullet$ the logarithmic de Rham complex of \mathfrak{X} over \mathcal{O}_C , and we define the *log de Rham cohomology* of X (over \mathcal{O}_C) as

$$R\Gamma_{\log \text{dR}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}, \Omega_{\mathfrak{X},\log}^\bullet) \in D(\text{Mod}_{\mathcal{O}_C}^{\text{cond}}).$$

Remark 3.9.

- (i) For any affine $\mathfrak{X} \in \mathcal{M}_{\text{ss}}$ with semistable coordinates there exists a finite extension L/K and a p -adic formal scheme $\mathfrak{X}' \rightarrow \text{Spf}(\mathcal{O}_L)$ admitting an étale $\text{Spf}(\mathcal{O}_L)$ -morphism $\mathfrak{X}' \rightarrow \text{Spf}(R')$ with

$$R' := \mathcal{O}_L\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q)$$

for some $0 \leq r \leq d$, and $q \in \mathbb{Q}_{>0}$, such that $\mathfrak{X} = \mathfrak{X}' \times_{\text{Spf}(\mathcal{O}_L)} \text{Spf}(\mathcal{O}_C)$: this follows from [Sta20, 04D1, 00U9]. By [CK19, Claims 1.6.1 and 1.6.2], the p -adic formal scheme \mathfrak{X}' can be endowed with a fine log structure, whose base change to $\text{Spf}(\mathcal{O}_C)$ gives the log structure on \mathfrak{X} we started with.

- (ii) For any $\mathfrak{X} \in \mathcal{M}_{\text{ss},\text{qcqs}}$ there exist a finite extension L/K and a descent of $\mathfrak{X}_{\mathcal{O}_C/p}$ to a qcqs, fine, log-smooth, log scheme over $\mathcal{O}_{L,1}^\times$ of Cartier type: covering \mathfrak{X} by a finite number of open affines with semistable coordinates, this follows from part (i) and fact that morphisms of Cartier type are stable under base change.

The following Beilinson basis will be used to define the Hyodo–Kato cohomology for rigid-analytic varieties over C starting from the semistable reduction case.

Proposition 3.10. *The pair $(\mathcal{M}_{\text{ss}}, (-)_\eta)$ is a Beilinson basis for the site $\text{Rig}_{C,\text{éh}}$.*

Proof. By Proposition 2.14 (and Remark 2.15), it suffices to show that $(\mathcal{M}_{\text{ss}}, (-)_\eta)$ is a Beilinson basis for $\text{RigSm}_{C,\text{ét}}$, i.e. the big étale site of smooth rigid-analytic varieties over C . This follows from Temkin’s alteration theorem [Tem17, Theorem 3.3.1], as shown in [CN20, Proposition 2.8]. \square

3.3.1. Condensed (φ, N) -modules. Before defining the Hyodo–Kato cohomology for rigid-analytic varieties over C , we need to establish the following terminology.

Definition 3.11. Let $\sigma = \sigma_{\check{F}} : \check{F} \rightarrow \check{F}$ denote the automorphism induced by the p -th power Frobenius on the residue field.

- (i) A *condensed φ -module over \check{F}* is a pair (V, φ) with $V \in \text{Mod}_{\check{F}}^{\text{cond}}$ and $\varphi : V \rightarrow V$ a σ -semilinear automorphism, called *Frobenius*. A morphism of condensed φ -modules over \check{F} is a morphism of condensed modules over \check{F} , which is compatible with the Frobenius.
- (ii) A *condensed (φ, N) -module over \check{F}* is a triple (V, φ, N) with (V, φ) a condensed φ -module over \check{F} and $N : V \rightarrow V$ a \check{F} -linear endomorphism, called *monodromy operator*, such that $N\varphi = p\varphi N$. A morphism of condensed (φ, N) -modules over \check{F} is a morphism of condensed modules over \check{F} , which is compatible with the Frobenius and the monodromy operator.

Note that the category of condensed (φ, N) -modules over \check{F} is an abelian category. We denote by $D_{(\varphi, N)}(\text{Mod}_{\check{F}}^{\text{cond}})$ the corresponding derived ∞ -category; we abbreviate $D_{(\varphi, N)}(\check{F}) = D_{(\varphi, N)}(\text{Mod}_{\check{F}}^{\text{cond}})$.

Remark 3.12. The category of condensed (φ, N) -modules over \check{F} can be identified with the category

$$\mathrm{LMod}_{\check{F}\{\varphi^\pm, N\}}(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}})$$

of left modules in $\mathrm{Mod}_{\check{F}}^{\mathrm{cond}}$ over the condensed non-commutative algebra $\check{F}\{\varphi^\pm, N\}$ over \check{F} defined as follows: for any (κ -small, for κ cut-off cardinal as in §1.7) extremally disconnected set S , we define $\check{F}\{\varphi^\pm, N\}(S)$ as the twisted polynomial ring

$$\mathcal{C}^0(S, \check{F})\{\varphi, \varphi^{-1}, N\}$$

with underlying abelian group $\mathcal{C}^0(S, \check{F})[\varphi, \varphi^{-1}, N]$ and multiplication rules $\varphi\varphi^{-1} = \varphi^{-1}\varphi = 1$, $\varphi a = \sigma(a)\varphi$ and $Na = aN$ for all $a \in \mathcal{C}^0(S, \check{F})$, and $N\varphi = p\varphi N$.

Then, we note that we have an equivalence of ∞ -categories

$$D_{(\varphi, N)}(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}}) \simeq \mathrm{LMod}_{\check{F}\{\varphi^\pm, N\}} D(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}})$$

as it follows from [Lur17, Theorem 7.1.3.1] (applied to the ∞ -category of κ -condensed spectra).²⁴

Thus, we can regard objects in $D_{(\varphi, N)}(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}})$ as complexes $M \in D(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}})$ endowed with a σ -semilinear automorphism $\varphi : M \rightarrow M$, called *Frobenius*, and a \check{F} -linear endomorphism $N : M \rightarrow M$, called *monodromy operator*, such that $N\varphi = p\varphi N$.

Remark 3.13. For $\mathfrak{X} \in \mathcal{M}_{\mathrm{ss}}$, we have that

$$R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \in D_{(\varphi, N)}(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}}).$$

In fact, we recall that the complex $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \in D(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}})$ is naturally equipped with a σ -semilinear endomorphism φ called Frobenius, and, by the arguments in [Bei11, §1.16, §1.19], it comes equipped with a monodromy operator N , satisfying $N\varphi = p\varphi N$; by (the proof of) [HK94, Proposition 2.24] the Frobenius φ is a σ -semilinear automorphism on it.

Then, we are ready to give the following definition, which is based on Lemma 3.6 and Proposition 3.10.

Definition 3.14 (Hyodo–Kato cohomology). We denote by $\mathcal{F}_{\mathrm{HK}}$ the hypersheaf on $\mathrm{Rig}_{C, \mathrm{eh}}$ with values in $D_{(\varphi, N)}(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}})$ associated to the presheaf

$$(\mathcal{M}_{\mathrm{ss}})^{\mathrm{op}} \rightarrow D_{(\varphi, N)}(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}}) : \mathfrak{X} \mapsto R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p}, \quad (3.11)$$

For X a rigid-analytic variety over C , we define the *Hyodo–Kato cohomology* of X as

$$R\Gamma_{\mathrm{HK}}(X) := R\Gamma(X, \mathcal{F}_{\mathrm{HK}}) \in D_{(\varphi, N)}(\mathrm{Mod}_{\check{F}}^{\mathrm{cond}}).$$

The following result shows in particular that the Hyodo–Kato cohomology of X is a refinement of the de Rham cohomology of X (Definition 2.16).

We refer the reader to [Bos21, §A.6] and §A.2 for a review of the notion of *nuclearity*, introduced by Clausen–Scholze, used in the following statement and the rest of the paper.

Theorem 3.15. *Let X be a rigid-analytic variety over C .*

²⁴In fact, applying *loc. cit.* to the ∞ -category of κ -condensed spectra CondSp (for κ cut-off cardinal as in §1.7), we have, for any κ -condensed commutative ring R and any R -algebra A in CondAb , an equivalence of ∞ -categories $\mathrm{LMod}_A(\mathrm{LMod}_R(\mathrm{CondSp})) \simeq \mathrm{LMod}_A(\mathrm{CondSp})$. Then, it remains to observe that we have an equivalence of ∞ -categories $\mathrm{LMod}_A(\mathrm{CondSp}) \simeq D(\mathrm{LMod}_A(\mathrm{CondAb}))$, cf. [Lur18, Theorem 2.1.2.2].

(i) (Local-global compatibility) Assume X is the generic fiber of $\mathfrak{X} \in \mathcal{M}_{\text{ss}}$, then the natural map

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \rightarrow R\Gamma_{\text{HK}}(X)$$

is an isomorphism in $D_{(\varphi, N)}(\text{Mod}_{\check{F}}^{\text{cond}})$.

(ii) (Boundedness and nuclearity) If X is qcqs of dimension d , then $R\Gamma_{\text{HK}}(X)$ is represented by a complex of nuclear (solid) \check{F} -vector spaces, and it lies in $D^{\leq 2d}(\text{Mod}_{\check{F}}^{\text{solid}})$. Moreover, the monodromy operator N on $R\Gamma_{\text{HK}}(X)$ is nilpotent with nilpotency index bounded above by a function depending on d .

(iii) (Hyodo–Kato isomorphism) Assume X is connected and paracompact, then we have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$

$$\iota_{\text{HK}} : R\Gamma_{\text{HK}}(X) \otimes_{\check{F}}^{\text{L}\square} C \xrightarrow{\sim} R\Gamma_{\text{dR}}(X).$$

Proof. We start with some preliminary observations. First, we observe that, for any $\mathfrak{X} \in \mathcal{M}_{\text{ss}, \text{qcqs}}$, by [Bei13, (1.8.1)], we have a natural quasi-isomorphism

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/\mathcal{O}_C^{\times})_{\mathbb{Q}_p} \simeq R\Gamma_{\log \text{dR}}(\mathfrak{X})_{\mathbb{Q}_p} \simeq R\Gamma_{\text{dR}}(\mathfrak{X}_C) \quad (3.12)$$

and then, by Theorem 3.2, which applies thanks to Remark 3.9, we have a natural quasi-isomorphism

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \otimes_{\check{F}}^{\text{L}\square} C \xrightarrow{\sim} R\Gamma_{\text{dR}}(\mathfrak{X}_C). \quad (3.13)$$

Moreover, we claim that $R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p}$ is represented by a complex of \check{F} -Banach spaces. For this, we note that, as \check{F} is discretely valued we can choose a basis of the \check{F} -Banach space C , and then there exists a \check{F} -Banach space V and an isomorphism

$$C \cong \check{F} \oplus V \text{ in } \text{Mod}_{\check{F}}^{\text{solid}}. \quad (3.14)$$

Now, the claim follows using the quasi-isomorphism (3.13) combined with the isomorphism (3.14), observing that, as \mathfrak{X}_C is qcqs, $R\Gamma_{\text{dR}}(\mathfrak{X}_C)$ is represented by a complex C -Banach spaces (and hence \check{F} -Banach spaces), and a direct summand of a Banach space is a Banach space.

For part (i), it suffices to show that given $\mathfrak{X} \in \mathcal{M}_{\text{ss}, \text{qcqs}}$ with generic fiber X , then, for any simplicial object \mathfrak{U}_{\bullet} of $\mathcal{M}_{\text{ss}, \text{qcqs}}$ such that $\mathfrak{U}_{\bullet, \eta} \rightarrow X$ is a $\acute{e}h$ -hypercover, the natural map

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \rightarrow \lim_{[n] \in \Delta} R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \quad (3.15)$$

is a quasi-isomorphism. First, we note that the map (3.15) is compatible with the Frobenius and the monodromy operator. Next, we will use an idea from the proof of [CN21a, Proposition 3.5]. By (3.13) we have the following commutative diagram

$$\begin{array}{ccc} R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \otimes_{\check{F}}^{\text{L}\square} C & \longrightarrow & \lim_{[n] \in \Delta} (R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \otimes_{\check{F}}^{\text{L}\square} C) \\ \downarrow \wr & & \downarrow \wr \\ R\Gamma_{\text{dR}}(\mathfrak{X}_C) & \longrightarrow & \lim_{[n] \in \Delta} R\Gamma_{\text{dR}}(\mathfrak{U}_{n, C}). \end{array}$$

The bottom horizontal arrow is a quasi-isomorphism, as $R\Gamma_{\text{dR}}(-)$ satisfies $\acute{e}h$ -hyperdescent, hence the top horizontal arrow is a quasi-isomorphism too. Moreover, setting $M_n := R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p}$ we have

$$\lim_{[n] \in \Delta} (M_n \otimes_{\check{F}}^{\text{L}\square} C) = (\lim_{[n] \in \Delta} M_n) \otimes_{\check{F}}^{\text{L}\square} C$$

as it follows from [Bos21, Corollary A.67(ii)], recalling that each M_n is represented by a complex of \check{F} -Banach spaces (and hence nuclear \check{F} -vector spaces by [Bos21, Corollary A.61]), and using that²⁵

$$\lim_{[n] \in \Delta} M_n = \lim_{\substack{\longleftarrow \\ m \in \mathbb{N}}} \lim_{[n] \in \Delta_{\leq m}} M_n. \quad (3.16)$$

Then, considering the fibers of the horizontal arrows in the following commutative diagram

$$\begin{array}{ccc} R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} & \longrightarrow & \lim_{[n] \in \Delta} R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \\ \downarrow & & \downarrow \\ R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \otimes_{\check{F}}^{\mathbf{L}} C & \xrightarrow{\sim} & \lim_{[n] \in \Delta} (R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p}) \otimes_{\check{F}}^{\mathbf{L}} C \end{array}$$

in order to show that the top horizontal arrow is a quasi-isomorphism, it suffices to prove that, for any $M \in D(\text{Mod}_{\check{F}}^{\text{solid}})$,

$$M \otimes_{\check{F}}^{\mathbf{L}} C \text{ acyclic} \implies M \text{ acyclic.}$$

This immediately follows using the isomorphism (3.14).

For part (ii), to show that $R\Gamma_{\text{HK}}(X)$ lies in $D^{\leq 2d}(\text{Mod}_{\check{F}}^{\text{solid}})$, using the quasi-isomorphism (3.13) combined with the isomorphism (3.14) and $\acute{e}h$ -hyperdescent, we can reduce to the analogous statement for the de Rham cohomology, which follows from Corollary 2.19. To show that $R\Gamma_{\text{HK}}(X)$ is represented by a complex of nuclear \check{F} -vector spaces, taking a simplicial object \mathfrak{U}_{\bullet} of $\mathcal{M}_{\text{ss, qcqs}}$ such that $\mathfrak{U}_{\bullet, \eta} \rightarrow X$ is a $\acute{e}h$ -hypercover, by $\acute{e}h$ -hyperdescent and part (i), we have

$$R\Gamma_{\text{HK}}(X) = \lim_{[n] \in \Delta} R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \quad (3.17)$$

and then, by [Bos21, Corollary A.61, Theorem A.43(i)], we can reduce to the fact that each complex $R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p}$ is represented by a complex of \check{F} -Banach spaces, which was shown above. The last statement of part (ii) follows from Lemma 3.3.

For part (iii), we first assume X qcqs. In this case, using (3.17), the statement follows (3.13) and [Bos21, Corollary A.67(ii)], which applies thanks to part (ii). For a general X connected and paracompact, choosing a quasi-compact admissible covering $\{U_n\}_{n \in \mathbb{N}}$ of X such that $U_n \subseteq U_{n+1}$, the statement follows from the previous case, using again [Bos21, Corollary A.67(ii)] and part (ii). \square

As a consequence of the Hyodo–Kato isomorphism, we have the following result.

Corollary 3.16. *Let X be a connected, paracompact, rigid-analytic variety over C . Then, the Hyodo–Kato complex $R\Gamma_{\text{HK}}(X)$ and the de Rham complex $R\Gamma_{\text{dR}}(X)$ have the same cohomological dimension.*

Proof. By Theorem 3.15(iii) and the flatness of C for the solid tensor product $\otimes_{\check{F}}^{\mathbf{L}}$ ([Bos21, Corollary A.65]), for any $i \geq 0$, we have an isomorphism

$$H_{\text{HK}}^i(X) \otimes_{\check{F}}^{\mathbf{L}} C \cong H_{\text{dR}}^i(X).$$

Therefore, if $H_{\text{HK}}^i(X)$ vanishes then $H_{\text{dR}}^i(X)$ vanishes as well, and the converse statement follows using the isomorphism (3.14). \square

²⁵Here, all the limits are derived.

3.4. Finiteness in the overconvergent case. In this subsection, we extend the Hyodo–Kato morphism to dagger varieties over C . As we will see, this will follow easily from the results of the previous subsection, using that the solid tensor product commutes with colimits. Moreover, we will prove a finiteness result for the Hyodo–Kato cohomology of qcqs dagger varieties over C , generalizing already known results to the singular case.

3.4.1. Hyodo–Kato cohomology of dagger varieties over C . We begin with a general construction that will allow us to canonically define a cohomology theory on $\mathrm{Rig}_{L,\acute{e}h}^\dagger$ starting from a cohomology theory defined on $\mathrm{Rig}_{L,\acute{e}h}$. Then, we will specialize this construction to the Hyodo–Kato and the de Rham cohomology theories.

Notation 3.17. In the following, we keep notation and conventions from 2.10. In particular, we denote by L a characteristic 0 complete valued field with a non-archimedean valuation of rank 1 and residue characteristic p .

Construction 3.18. Let \mathcal{D} be a presentable ∞ -category. The continuous functor

$$l : \mathrm{Rig}_{L,\acute{e}h}^\dagger \rightarrow \mathrm{Rig}_{L,\acute{e}h} : X \mapsto \widehat{X} \quad (3.18)$$

given by sending a dagger variety X to its limit \widehat{X} , induces an adjunction

$$l_* : \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{L,\acute{e}h}^\dagger, \mathcal{D}) \rightleftarrows \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{L,\acute{e}h}, \mathcal{D}) : l^{*\mathrm{hyp}}$$

where $l^{*\mathrm{hyp}}$ is given by the composite of the pullback functor $l^* : \mathrm{Shv}(\mathrm{Rig}_{L,\acute{e}h}, \mathcal{D}) \rightarrow \mathrm{Shv}(\mathrm{Rig}_{L,\acute{e}h}^\dagger, \mathcal{D})$ and the hypercompletion functor (3.10). For $\mathcal{F} \in \mathrm{Shv}(\mathrm{Rig}_{L,\acute{e}h}, \mathcal{D})$, we denote

$$\mathcal{F}^\dagger := l^{*\mathrm{hyp}} \mathcal{F} \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{L,\acute{e}h}^\dagger, \mathcal{D}).$$

Now, using Construction 3.18 in the case $\mathcal{D} = D(A) = D(\mathrm{Mod}_A^{\mathrm{cond}})$ (see Remark 3.7), we can give the following definition.

Definition 3.19 (de Rham and Hyodo–Kato cohomology of dagger varieties).

(i) Let X be a dagger variety over L . Denote by

$$\mathcal{F}_{\mathrm{dR}} \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{C,\acute{e}h}, D(L))$$

the hypersheaf given by $R\Gamma_{\mathrm{dR}}(-)$. We define the *de Rham cohomology* of X as

$$R\Gamma_{\mathrm{dR}}(X) := R\Gamma(X, \mathcal{F}_{\mathrm{dR}}^\dagger) \in D(L).$$

(ii) Let X be a dagger variety over C . Consider the hypersheaf

$$\mathcal{F}_{\mathrm{HK}} \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{C,\acute{e}h}, D_{(\varphi,N)}(\check{F}))$$

introduced in Definition 3.14. We define the *Hyodo–Kato cohomology* of X as

$$R\Gamma_{\mathrm{HK}}(X) := R\Gamma(X, \mathcal{F}_{\mathrm{HK}}^\dagger) \in D_{(\varphi,N)}(\check{F}).$$

3.4.2. Presentation of a dagger structure. In order to construct the Hyodo–Kato morphism for dagger varieties over C , we will rely on its analogue for rigid-analytic varieties over C , that is Theorem 3.15. For this, we will need to express more explicitly the Hyodo–Kato/de Rham cohomology of a smooth dagger affinoid over C in terms of the respective cohomology of smooth affinoid rigid spaces over C . This is our next goal.

We recall from 2.10 that given a dagger variety $X = (\widehat{X}, \mathcal{O}^\dagger)$ over L with underlying rigid-analytic variety \widehat{X} we say that X is a dagger structure on \widehat{X} .

We have the following important example of dagger structure.

Remark 3.20 (Dagger structure on smooth affinoid rigid spaces). We note that any smooth affinoid rigid space $\widehat{X} = \text{Spa}(R, R^\circ)$ over L has a dagger structure. In fact, by [Elk73, Theorem 7 and Remark 2], there exist f_1, \dots, f_m elements of the Washnitzer algebra $L\langle \underline{T} \rangle^\dagger$ such that $R \cong L\langle \underline{T} \rangle / (f_1, \dots, f_m)$.²⁶ In particular, the dagger variety associated to the dagger algebra $L\langle \underline{T} \rangle^\dagger / (f_1, \dots, f_m)$ defines a dagger structure on \widehat{X} .

Next, we recall the following convenient definition.

Definition 3.21 ([Vez18, Definition A.19]). Let X be an affinoid rigid space over L . A *presentation of a dagger structure* on X is a pro-(affinoid rigid space over L) $\varprojlim_{h \in \mathbb{N}} X_h$ with X and X_h rational subspaces of X_1 , such that $X \subseteq X_{h+1} \subseteq X_h$,²⁷ and this system is coinital among rational subspaces containing X .

A morphism of presentations of a dagger structure on an affinoid rigid space over L is a morphism of pro-objects.

The next lemma relates affinoid dagger spaces to presentations of a dagger structure on an affinoid rigid space.

Lemma 3.22. *Let \widehat{X} be an affinoid rigid space over L , and let $\varprojlim_h X_h$ be a presentation of a dagger structure on \widehat{X} . We denote by X^\dagger the dagger affinoid over L associated to the dagger algebra $R = \varinjlim_h \mathcal{O}(X_h)$.*

(i) *The functor*

$$\varprojlim_h X_h \mapsto X^\dagger$$

from the category of presentations of a dagger structure on an affinoid rigid space over L to the category of affinoid dagger spaces over L is an equivalence.

(ii) *Let $(\varprojlim_h X_h)_{\text{ét}}$ denote the (small) étale site of $\varprojlim_h X_h$, [Vez18, Definition A.24]. We have natural morphisms of sites*

$$\widehat{X}_{\text{ét}} \rightarrow (X^\dagger)_{\text{ét}} \rightarrow (\varprojlim_h X_h)_{\text{ét}}$$

which induce an equivalence on the associated topoi.

Proof. Part (i) is [Vez18, Proposition A.22(2)], and part (ii) is [Vez18, Corollary A.28]. \square

²⁶Here, we write \underline{T} for T_1, \dots, T_n where n is the dimension of \widehat{X} .

²⁷For $Y \subset Z$ an open immersion of rigid-analytic varieties over L , we write $Y \subseteq Z$ if the inclusion map of Y into Z factors over the adic compactification of Y over L .

Remark 3.23. Entering the proof of [Vez18, Proposition A.22(2)], we see that given an affinoid dagger space $(\widehat{X}, \mathcal{O}^\dagger)$ over L , associated to a dagger algebra $L\langle \underline{T} \rangle^\dagger / (f_1, \dots, f_m)$, then the corresponding presentation of a dagger structure $\varprojlim_h X_h$ on \widehat{X} can be defined as follows: since $L\langle \underline{T} \rangle^\dagger$ is noetherian, [GK00, §1.4], there exists an integer H sufficiently big such that $f_1, \dots, f_m \in L\langle \pi^{1/H} \underline{T} \rangle$, where π is a pseudo-uniformizer of \mathcal{O}_L ; then, we set $X_h := \mathrm{Spa}(R_h, R_h^\circ)$ with

$$R_h := L\langle \pi^{1/(h+H)} \underline{T} \rangle / (f_1, \dots, f_m).$$

Then, the next result follows formally from Lemma 3.22 by étale hyperdescent (cf. [CN20, Lemma 3.13]).

Lemma 3.24. *Fix notation as in 2.10 and Construction 3.18. Let $\mathcal{F} \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{L, \acute{e}h}, \mathcal{D})$ that is the pullback of an hypersheaf in $\mathrm{Shv}^{\mathrm{hyp}}(\mathrm{RigSm}_{L, \acute{e}t}, \mathcal{D})$. Let X be a smooth dagger affinoid over L with corresponding presentation $\varprojlim_h X_h$. Then, we have*

$$R\Gamma(X, \mathcal{F}^\dagger) = \mathrm{colim}_{h \in \mathbb{N}} R\Gamma(X_h, \mathcal{F}).$$

The assumptions of the previous lemma are designed to be satisfied by the hypersheaves defining the de Rham and the Hyodo–Kato cohomology:

Remark 3.25. We note that Lemma 3.24 applies to $\mathcal{F} = \mathcal{F}_{\mathrm{dR}}$, thanks to Proposition 2.17, and it applies to $\mathcal{F} = \mathcal{F}_{\mathrm{HK}}$ thanks to Theorem 3.15(i).

Next, we recall that the category of partially proper dagger varieties is equivalent to the category of partially proper rigid-analytic varieties, via the functor (3.18), [GK00, Theorem 2.27]. In the situation of Lemma 3.24, such equivalence preserves cohomology:

Proposition 3.26. *Fix notation as in 2.10 and Construction 3.18. Let $\mathcal{F} \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{L, \acute{e}h}, \mathcal{D})$ that is the pullback of an hypersheaf in $\mathrm{Shv}^{\mathrm{hyp}}(\mathrm{RigSm}_{L, \acute{e}t}, \mathcal{D})$. Let X be a partially proper dagger variety over L . Then, there exists a natural isomorphism*

$$R\Gamma(X, \mathcal{F}^\dagger) \xrightarrow{\sim} R\Gamma(\widehat{X}, \mathcal{F}).$$

Proof. Recalling that any partially proper dagger variety admits an admissible covering by Stein spaces (see the proof of [GK00, Theorem 2.26]), we may assume that X is a Stein space. Then, let $\{U_n\}_{n \in \mathbb{N}}$ be a Stein covering of X . Writing $R\Gamma(X, \mathcal{F}^\dagger) = R\varprojlim_{n \in \mathbb{N}} R\Gamma(U_n, \mathcal{F}^\dagger)$, and similarly $R\Gamma(\widehat{X}, \mathcal{F}) = R\varprojlim_{n \in \mathbb{N}} R\Gamma(\widehat{U}_n, \mathcal{F})$, it suffices to show that, for a fixed $n \in \mathbb{N}$, the natural map $R\Gamma(U_{n+1}, \mathcal{F}^\dagger) \rightarrow R\Gamma(U_n, \mathcal{F}^\dagger)$ factors through $R\Gamma(\widehat{U}_{n+1}, \mathcal{F})$. Renaming $V := U_n$ and $W := U_{n+1}$, by Proposition 2.14 (and Remark 2.15), we can choose an $\acute{e}h$ -hypercover $W_\bullet \rightarrow W$ with each W_m smooth dagger affinoid over L ; via pullback along the open immersion $V \rightarrow W$, we obtain an $\acute{e}h$ -hypercover $V_\bullet \rightarrow V$ with each V_m smooth dagger affinoid over L . Then, we may reduce to the case V and W are smooth over L , which follows from Lemma 3.24. \square

3.4.3. Semistable weak formal schemes. In order to study the Hyodo–Kato cohomology of dagger varieties over C (Definition 3.19), we will define a convenient Beilinson basis for the site $\mathrm{Rig}_{C, \acute{e}h}^\dagger$. In addition to Notation 3.8, we introduce the following notation. We refer the reader to [Mer72] for the basics on the theory of *weak formal schemes*, and to [LM13] for an analogue of Raynaud’s theorem relating the categories of weak formal schemes and dagger varieties.

Notation 3.27. We denote by $\mathcal{M}_{\text{ss}}^\dagger$ the category of weak formal schemes over $\text{Spf}(\mathcal{O}_C)$ having in the Zariski topology a covering by open affines \mathfrak{U} with *semistable coordinates*, i.e. admitting an $\text{Spf}(\mathcal{O}_C)$ -morphism $\mathfrak{U} \rightarrow \text{Spf}(R^{\square\dagger})$ with

$$R^{\square\dagger} := \mathcal{O}_C[t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}]^\dagger / (t_0 \cdots t_r - p^q)$$

for some $0 \leq r \leq d$, and $q \in \mathbb{Q}_{>0}$ (that may depend on \mathfrak{U}). We denote by $\mathcal{M}_{\text{ss}, \text{qcqs}}^\dagger$ the subcategory of $\mathcal{M}_{\text{ss}}^\dagger$ consisting of the qcqs formal schemes. We write

$$(-)_\eta : \mathcal{M}_{\text{ss}}^\dagger \rightarrow \text{Rig}_C^\dagger$$

for the generic fiber functor.

Proposition 3.28. *The pair $(\mathcal{M}_{\text{ss}}^\dagger, (-)_\eta)$ is a Beilinson basis for the site $\text{Rig}_{C, \text{eh}}^\dagger$.*

Proof. As in the proof of Proposition 3.10, the statement follows from Proposition 2.14 (and Remark 2.15) combined with [CN20, Proposition 2.13]. \square

The following result is an overconvergent version of Theorem 3.15.

Theorem 3.29. *Let X be a dagger variety over C .*

(i) (Local description) *Assume X is the generic fiber of $\mathfrak{X} \in \mathcal{M}_{\text{ss}}^\dagger$, then there is a natural quasi-isomorphism*

$$R\Gamma_{\text{HK}}(X) \simeq R\Gamma_{\text{rig}}(\mathfrak{X}_{\bar{k}}/\mathcal{O}_{\check{F}}^0)$$

compatible with Frobenius φ and monodromy N . Here, the right-hand side denotes the (rational) log-rigid cohomology of $\mathfrak{X}_{\bar{k}}$ over $\mathcal{O}_{\check{F}}^0$, [GK05, §1], [CDN20b, §3.1.2].

(ii) (Hyodo–Kato isomorphism) *Assume X is connected and paracompact, then we have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

$$\iota_{\text{HK}} : R\Gamma_{\text{HK}}(X) \otimes_{\check{F}}^{\text{L}\blacksquare} C \xrightarrow{\sim} R\Gamma_{\text{dR}}(X).$$

Proof. Part (i) follows from [CN21a, §4.2.1, (iv)].²⁸ Part (ii) for X smooth affinoid follows from Theorem 3.15(ii) and Lemma 3.24 (together with Remark 3.25), using that the tensor product $\otimes_{\check{F}}^{\text{L}\blacksquare}$ commutes with filtered colimits. From Lemma 3.24 we also deduce that, for X smooth affinoid, $R\Gamma_{\text{HK}}(X)$ is represented by a complex of nuclear \check{F} -vector spaces (recall that the category of nuclear \check{F} -vector spaces is closed under colimits). Therefore, the same argument used in the proof of Theorem 3.15(iii) shows part (ii) in general. \square

3.4.4. Finiteness. Now, we state the promised finiteness result for the Hyodo–Kato cohomology groups of a qcqs dagger variety over C , and we give a bound on the slopes of such cohomology groups regarded as φ -modules.

Theorem 3.30. *Let X be a qcqs dagger variety over C . Let $i \geq 0$.*

(i) *The condensed cohomology group $H_{\text{HK}}^i(X)$ (resp. $H_{\text{dR}}^i(X)$) is a finite-dimensional condensed vector space over \check{F} (resp. over C).*

(ii) *The vector bundle on FF associated to the finite φ -module $H_{\text{HK}}^i(X)$ over \check{F} has Harder–Narasimhan slopes $\geq -i$.*

²⁸Note that Theorem 3.15(i) implies that, for X a smooth rigid-analytic/dagger variety over C , the Hyodo–Kato cohomology $R\Gamma_{\text{HK}}(X)$ agrees with the one defined in [CN21a] considered in $D(\text{CondAb})$.

Proof. In the case when X is the generic fiber of $\mathfrak{X} \in \mathcal{M}_{\text{ss}, \text{qcqs}}^\dagger$ by Theorem 3.29(i) part (i) follows from a result of Grosse-Klönne, [GK05, Theorem 5.3] (and base change), and part (ii) follows from [CLS99, Théorème 3.1.2]. In the general case, we take a simplicial object \mathfrak{U}_\bullet of $\mathcal{M}_{\text{ss}, \text{qcqs}}^\dagger$ such that $\mathfrak{U}_{\bullet, \eta} \rightarrow X$ is a éh-hypercover, and we consider the spectral sequence

$$E_1^{j, i-j} = H_{\text{HK}}^{i-j}(\mathfrak{U}_{j, \eta}) \implies H_{\text{HK}}^i(X). \quad (3.19)$$

Then, part (i) for the Hyodo–Kato cohomology follows immediately from the previous case, the spectral sequence (3.19), and [Bos21, Lemma A.33]. Similarly, part (i) for the de Rham cohomology follows from the previous case and an analogous spectral sequence for the de Rham cohomology.²⁹

For part (ii), applying to (3.19) the exact functor $\mathcal{E}(-)$ sending a finite φ -module over \check{F} to the associated vector bundle on FF, and then twisting by $\mathcal{O}(i)$, we deduce that the vector bundle $\mathcal{E}(H_{\text{HK}}^i(X)) \otimes \mathcal{O}(i)$ has non-negative Harder–Narasimhan slopes: in fact, by the previous case, for all j , the vector bundle $\mathcal{E}(H_{\text{HK}}^{i-j}(\mathfrak{U}_{j, \eta})) \otimes \mathcal{O}(i)$ has non-negative Harder–Narasimhan slopes, and then the claim follows from the classification of vector bundles on FF. \square

4. B -COHOMOLOGY

This section is devoted to the proof of the following main result, which compares the B -cohomology with the Hyodo–Kato cohomology.

Theorem 4.1. *Let X be a connected, paracompact, rigid-analytic variety defined over C . Then, we have a natural isomorphism in $D(\text{Mod}_B^{\text{solid}})$*

$$R\Gamma_B(X) \simeq (R\Gamma_{\text{HK}}(X) \otimes_{\check{F}}^{\text{L}\blacksquare} B_{\log})^{N=0} \quad (4.1)$$

compatible with the action of Frobenius φ . If X is the base change to C of a rigid-analytic variety defined over K , then (4.1) is \mathcal{G}_K -equivariant.

We will first prove Theorem 4.1 in the case when X has semistable reduction. This will be done in two main steps: we first compare, in §4.1, the B -cohomology with the log-crystalline cohomology over A_{cris} , and then, in §4.2, we relate the latter with the Hyodo–Kato cohomology.

Notation and conventions 4.2. In the following, we keep the notation and conventions introduced in 3.1 and 3.8.

4.1. The comparison with the log-crystalline cohomology over A_{cris} . We begin by comparing the B -cohomology with the log-crystalline cohomology over A_{cris} .

Theorem 4.3. *Let \mathfrak{X} be a qcqs semistable p -adic formal scheme over $\text{Spf}(\mathcal{O}_C)$ and let $I \subset [1/(p-1), \infty)$ be a compact interval with rational endpoints. Then, there is a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

$$R\Gamma_{B_I}(\mathfrak{X}_C) \simeq R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^\times) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} B_I \quad (4.2)$$

compatible with the action of Frobenius φ .

In the first instance, we prove a local version of Theorem 4.3, and then we globalize the result. Therefore, we begin by defining the local setting in which we will work.

²⁹Alternatively, part (i) for the de Rham cohomology follows from part (i) for the de Hyodo–Kato cohomology and the Hyodo–Kato isomorphism, Theorem 3.29(ii).

Notation 4.4. Let $\mathfrak{X} = \mathrm{Spf}(R)$ be a connected affine p -adic formal scheme over $\mathrm{Spf}(\mathcal{O}_C)$ admitting an étale $\mathrm{Spf}(\mathcal{O}_C)$ -morphism $\mathfrak{X} \rightarrow \mathrm{Spf}(R^\square)$ with

$$R^\square := \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q)$$

for some $0 \leq r \leq d$, and $q \in \mathbb{Q}_{>0}$.

We denote by R_∞^\square the perfectoid R^\square -algebra defined by $R_\infty^\square := (\varinjlim_m R_m^\square)_p^\wedge$ with

$$R_m^\square := \mathcal{O}_C\{t_0^{1/p^m}, \dots, t_r^{1/p^m}, t_{r+1}^{\pm 1/p^m}, \dots, t_d^{\pm 1/p^m}\}/(t_0 \cdots t_r - p^{q/p^m})$$

and we put $\mathfrak{X}_{C,\infty}^\square := \mathrm{Spa}(R_\infty^\square[1/p], R_\infty^\square)$. We set

$$R_\infty := (R \otimes_{R^\square} R_\infty^\square)_p^\wedge$$

and we note that (see also [CK19, §3.2])

$$\mathfrak{X}_{C,\infty} := \mathrm{Spa}(R_\infty[1/p], R_\infty) \rightarrow \mathfrak{X}_C \quad (4.3)$$

is an affinoid perfectoid pro-étale cover of \mathfrak{X}_C with Galois group

$$\Gamma := \mathbb{Z}_p(1)^d \cong \mathbb{Z}_p^d$$

where the latter isomorphism is given by the choice of a compatible system of p -th power roots of unity in \mathcal{O}_C (see 2.31). We denote by $\gamma_1, \dots, \gamma_d$ the generators of Γ defined by

$$\begin{aligned} \gamma_i &:= (\varepsilon^{-1}, 1, \dots, 1, \varepsilon, 1, \dots, 1) \quad \text{for } i = 1, \dots, r \\ \gamma_i &:= (1, \dots, 1, \varepsilon, 1, \dots, 1) \quad \text{for } i = r+1, \dots, d \end{aligned}$$

where ε sits on the i -th entry.

4.1.1. The condensed ring $\mathbb{B}_I(R_\infty)$. In the setting of Notation 4.4, given \mathbb{M} any pro-étale period sheaf of §2.3.1, we put

$$\mathbb{M}(R_\infty^\square) := \mathbb{M}(\mathfrak{X}_{C,\infty}^\square) \quad \mathbb{M}(R_\infty) := \mathbb{M}(\mathfrak{X}_{C,\infty})$$

which we regard as condensed rings.

Remark 4.5. We recall from [CK19, §3.14] that we have the following decomposition of $\mathbb{A}_{\mathrm{inf}}(R_\infty^\square)$

$$\mathbb{A}_{\mathrm{inf}}(R_\infty^\square) \cong A_{\mathrm{inf}}(R^\square) \oplus \mathbb{A}_{\mathrm{inf}}(R_\infty^\square)^{\mathrm{n-int}} \quad (4.4)$$

where $A_{\mathrm{inf}}(R^\square)$ denotes the “integral” part, and $\mathbb{A}_{\mathrm{inf}}(R_\infty^\square)^{\mathrm{n-int}}$ the “nonintegral part”. We have

$$A_{\mathrm{inf}}(R^\square) \cong A_{\mathrm{inf}}\{X_0, \dots, X_r, X_{r+1}^{\pm 1}, \dots, X_d^{\pm 1}\}/(X_0 \cdots X_r - [p^b]^q) \quad (4.5)$$

where $X_i := [t_i^b]$, and the convergence is (p, μ) -adic. Such decomposition lifts to $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ as follows

$$\mathbb{A}_{\mathrm{inf}}(R_\infty) \cong A_{\mathrm{inf}}(R) \oplus \mathbb{A}_{\mathrm{inf}}(R_\infty)^{\mathrm{n-int}} \quad (4.6)$$

where $A_{\mathrm{inf}}(R)$ is the unique lift of the étale (R^\square/p) -algebra R/p , along $\theta : A_{\mathrm{inf}}(R^\square) \twoheadrightarrow R^\square$, to a (p, μ) -adically complete, formally étale $A_{\mathrm{inf}}(R^\square)$ -algebra.

Remark 4.6. Given a compact interval $I \subset (0, \infty)$ with rational endpoints, by [Bos21, Proposition 4.7(ii)], we have

$$\mathbb{A}_I(R_\infty) \cong \mathbb{A}_{\mathrm{inf}}(R_\infty) \widehat{\otimes}_{A_{\mathrm{inf}}} A_I \quad (4.7)$$

where the completion $\widehat{\otimes}_{A_{\mathrm{inf}}}$ is p -adic. Then, one has similar decompositions as (4.6) replacing $\mathbb{A}_{\mathrm{inf}}$ with \mathbb{A}_I , resp. \mathbb{B}_I , and $A_{\mathrm{inf}}(R)$ with $A_I(R) := A_{\mathrm{inf}}(R) \widehat{\otimes}_{A_{\mathrm{inf}}} A_I$, resp. $B_I(R) := A_I(R)[1/p]$ (where the completion $\widehat{\otimes}_{A_{\mathrm{inf}}}$ is p -adic).

Remark 4.7. Let $I \subset (0, \infty)$ be a compact interval with rational endpoints. We claim that we have a natural isomorphism

$$\mathbb{A}_I(R_\infty) \cong \mathbb{A}_{\text{inf}}(R_\infty) \otimes_{\mathbb{A}_{\text{inf}}}^{\blacksquare} A_I \quad (4.8)$$

In particular, inverting p , using that the solid tensor product commutes with filtered colimits, we have an isomorphism

$$\mathbb{B}_I(R_\infty) \cong \mathbb{A}_{\text{inf}}(R_\infty) \otimes_{\mathbb{A}_{\text{inf}}}^{\blacksquare} B_I.$$

To show (4.8), up to twisting by the Frobenius, we can assume that $I \subset [1/(p-1), \infty)$. Now, we use the isomorphism (4.7), and then we apply Proposition A.3 taking $M = \mathbb{A}_{\text{inf}}(R_\infty)$ and $N = A_{\text{inf}, I}$ (see 2.31 for the notation), regarded as objects of $\text{Mod}_{\mathbb{A}_{\text{inf}}}^{\text{solid}}$, thus obtaining that

$$\mathbb{A}_{\text{inf}}(R_\infty) \otimes_{\mathbb{A}_{\text{inf}}}^{\text{L}\blacksquare} (A_{\text{inf}, I})_p^\wedge \cong (\mathbb{A}_{\text{inf}}(R_\infty) \otimes_{\mathbb{A}_{\text{inf}}} A_{\text{inf}, I})_p^\wedge \quad (4.9)$$

where $(-)_p^\wedge$ denotes the derived p -adic completion. Since $A_{\text{inf}, I}$ is p -torsion-free, thanks to Lemma A.5 the derived p -adic completion $(A_{\text{inf}, I})^\wedge$ identifies with A_I . Then, it remains to show that the derived p -adic completion appearing on the right-hand side of (4.9) is underived: by [BMS18, Lemma 12.2] and Remark 2.32, we have $\mu^{p-1}/p \in A_{\text{cris}} \subset A_I$, and therefore, for any integer $n \geq 1$, we have that $A_{\text{inf}, I}/p^n = A_I/p^n \cong A_I/(p^n, \mu^{n'})$ for a large enough integer n' ;³⁰ now, it suffices to observe that, by [CK19, Lemma 3.13], $(p^n, \mu^{n'})$ is an $\mathbb{A}_{\text{inf}}(R_\infty)$ -regular sequence and $\mathbb{A}_{\text{inf}}(R_\infty)/(p^n, \mu^{n'})$ is flat over $A_{\text{inf}}/(p^n, \mu^{n'})$,³¹ hence

$$\mathbb{A}_{\text{inf}}(R_\infty) \otimes_{\mathbb{A}_{\text{inf}}}^{\text{L}} A_{\text{inf}, I}/(p^n, \mu^{n'})$$

is concentrated in degree 0, and the claim follows.

4.1.2. Local computations. Next, by a standard argument, we express locally the B -cohomology and the B_{dR}^+ -cohomology in terms of Koszul complexes.

Lemma 4.8. *Let $I \subset (0, \infty)$ be a compact interval with rational endpoints, and let $m \geq 1$ be an integer. Given*

$$\mathbf{B} \in \{\mathbb{B}_I, \mathbb{B}, \mathbb{B}_{\text{dR}}^+, \mathbb{B}_{\text{dR}}^+/\text{Fil}^m\}$$

we write $\mathcal{B} = \mathbf{B}_{\text{Spa}(C)_{\text{proét}}}$ for the corresponding condensed period ring. In the setting of Notation 4.4, we have a natural isomorphism in $D(\text{Mod}_{\mathcal{B}}^{\text{solid}})$

$$R\Gamma_{\mathcal{B}}(\mathfrak{X}_C) \simeq L\eta_t \text{Kos}_{\mathbf{B}(R_\infty)}(\gamma_1 - 1, \dots, \gamma_d - 1) \quad (4.10)$$

compatible with the filtration décalée of Definition 2.9.

Proof. Using Proposition 2.42(ii), it remains to check that

$$R\Gamma_{\text{proét}}(\mathfrak{X}_C, \mathbf{B}) \simeq \text{Kos}_{\mathbf{B}(R_\infty)}(\gamma_1 - 1, \dots, \gamma_d - 1).$$

Considering the Cartan–Leray spectral sequence associated to the affinoid perfectoid pro-étale cover $\mathfrak{X}_{C, \infty} \rightarrow \mathfrak{X}_C$ of (4.3) with Galois group Γ ([Bos21, Proposition 4.12]), we have the following natural isomorphism in $D(\text{Mod}_{\mathcal{B}}^{\text{solid}})$

$$R\Gamma_{\text{cond}}(\Gamma, \mathbf{B}(R_\infty)) \xrightarrow{\sim} R\Gamma_{\text{proét}}(\mathfrak{X}_C, \mathbf{B}).$$

Then, the statement follows from [Bos21, Proposition B.3]. \square

³⁰In fact, one can take $n' := (p-1)n$.

³¹In fact, *loc. cit.* translates to the condensed setting, for the ideal sheaf $(p^n, \mu^{n'})$ in the condensed ring $\mathbb{A}_{\text{inf}}(R_\infty)$, observing that both $\mathbb{A}_{\text{inf}}(R_\infty)/(p^n, \mu^{n'})$ and $A_{\text{inf}}/(p^n, \mu^{n'})$ are discrete.

Our next goal is to express the right-hand side of (4.10) in terms of differential forms. For this, recalling the notation introduced in Remark 4.5 and Remark 4.6, in the setting of Notation 4.4 we denote the dual basis of the log A_{inf} -derivations (see [CK19, §5.10]) as follows:

$$\partial_i := \frac{\partial}{\partial \log(X_i)} : A_{\text{inf}}(R) \rightarrow A_{\text{inf}}(R) \quad (4.11)$$

for $1 \leq i \leq d$. Given $I \subset (0, \infty)$ a compact interval with rational endpoints, by slight abuse of notation, we will also denote by ∂_i the extension of the derivatives (4.11) to $A_I(R)$ or $B_I(R)$.

Lemma 4.9. *Let $I \subset [1/(p-1), \infty)$ be a compact interval with rational endpoints. In the setting of Notation 4.4, we have a B_I -linear quasi-isomorphism*

$$\text{Kos}_{A_{\text{inf}}(R)}(\partial_1, \dots, \partial_d) \otimes_{A_{\text{inf}}}^{\mathbf{L}} B_I \xrightarrow{\sim} \text{Kos}_{B_I(R)}(\partial_1, \dots, \partial_d) \xrightarrow{\sim} L\eta_t \text{Kos}_{\mathbb{B}_I(R_\infty)}(\gamma_1 - 1, \dots, \gamma_d - 1) \quad (4.12)$$

compatible with the action of Frobenius φ .

Proof. We will generalize the proof of [Bos21, Proposition 7.13]. Since μ divides $\gamma_i - 1$ in $B_I(R)$ for all i , i.e. Γ acts trivially on $B_I(R)/\mu$, and since, by the choice of I , the elements μ and t differ by a unit in B_I , by [Bos21, Lemma 7.8] we have that

$$\eta_t \text{Kos}_{B_I(R)}(\gamma_1 - 1, \dots, \gamma_d - 1) \simeq \text{Kos}_{B_I(R)}\left(\frac{\gamma_1 - 1}{t}, \dots, \frac{\gamma_d - 1}{t}\right). \quad (4.13)$$

Using that $A_{\text{cris}} \subset B_I$, by the choice of I , the arguments in [BMS18, Lemma 12.5] and [CK19, Lemma 5.15] show that, for each i , we have the following Taylor expansion in $B_I(R)$

$$\frac{\gamma_i - 1}{t} = \frac{\partial}{\partial \log(X_i)} \cdot h, \quad \text{with } h := 1 + \sum_{j \geq 1} \frac{t^j}{(j+1)!} \left(\frac{\partial}{\partial \log(X_i)} \right)^j$$

where $h - 1$ is topologically nilpotent, in particular the factor h is an automorphism of $B_I(R)$; furthermore, the latter automorphism is φ -equivariant.³² Then, recalling the notation (4.11), we deduce that the maps

$$\left((B_I(R) \xrightarrow{\partial_i} B_I(R)) \xrightarrow{(\text{id}, h)} (B_I(R) \xrightarrow{\gamma_i - 1} B_I(R)) \right)$$

for $1 \leq i \leq d$, induce a φ -equivariant quasi-isomorphism

$$\text{Kos}_{B_I(R)}(\partial_1, \dots, \partial_d) \xrightarrow{\sim} \eta_t \text{Kos}_{B_I(R)}(\gamma_1 - 1, \dots, \gamma_d - 1). \quad (4.14)$$

Next, we show that the natural map

$$\eta_t \text{Kos}_{B_I(R)}(\gamma_1 - 1, \dots, \gamma_d - 1) \rightarrow \eta_t \text{Kos}_{\mathbb{B}_I(R_\infty)}(\gamma_1 - 1, \dots, \gamma_d - 1) \quad (4.15)$$

is a quasi-isomorphism. For this, recalling Remark 4.6, we have

$$\mathbb{A}_I(R_\infty) \cong A_I(R) \oplus \mathbb{A}_I(R_\infty)^{\text{n-int}}$$

where $\mathbb{A}_I(R_\infty)^{\text{n-int}}$ denotes the “nonintegral part” of $\mathbb{A}_I(R_\infty)$. Then, as $L\eta_\mu(-)$ commutes with filtered colimits (hence with inverting p), it suffices to show that

$$L\eta_\mu \text{Kos}_{\mathbb{A}_I(R_\infty)^{\text{n-int}}}(\gamma_1 - 1, \dots, \gamma_d - 1) \simeq 0. \quad (4.16)$$

³²To check this one can argue as in the proof of [Bos21, Proposition 7.13].

In order to show (4.16) we need to prove that μ kills $H_{\text{cond}}^i(\Gamma, \mathbb{A}_I(R_\infty)^{\text{n-int}})$ for all $i \in \mathbb{Z}$. By [CK19, Proposition 3.25] (and [Bos21, Proposition B.3]), the element μ kills $H_{\text{cond}}^i(\Gamma, \mathbb{A}_{\text{inf}}(R_\infty)^{\text{n-int}})$ for all $i \in \mathbb{Z}$, then we conclude by Corollary 4.11 below.

Now, combining (4.14) with (4.15), to prove the statement it remains to check that we have an isomorphism

$$\text{Kos}_{A_{\text{inf}}(R)}(\partial_1, \dots, \partial_d) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_I \xrightarrow{\sim} \text{Kos}_{B_I(R)}(\partial_1, \dots, \partial_d). \quad (4.17)$$

Since the solid tensor product commutes with filtered colimits, it suffices to show (4.17) replacing $B_I = A_I[1/p]$ with A_I . Then, using Proposition A.3, we reduce to showing that we have an isomorphism

$$\text{Kos}_{A_{\text{inf}}(R)}(\partial_1, \dots, \partial_d) \widehat{\otimes}_{A_{\text{inf}}}^{\text{L}} A_I \xrightarrow{\sim} \text{Kos}_{A_I(R)}(\partial_1, \dots, \partial_d)$$

where the completion $\widehat{\otimes}_{A_{\text{inf}}}^{\text{L}}$ is derived p -adic. For this, we observe that the latter completion is an underived (termwise) p -adic completion: as recalled in Remark 4.7, for any integer $n \geq 1$, we have $A_I/p^n \cong A_I/(p^n, \mu^{n'})$ for a large enough integer n' , and, by [CK19, Lemma 3.13], $(p^n, \mu^{n'})$ is an $A_{\text{inf}}(R)$ -regular sequence with $A_{\text{inf}}(R)/(p^n, \mu^{n'})$ flat over $A_{\text{inf}}/(p^n, \mu^{n'})$. \square

We used crucially the following result.

Lemma 4.10. *Let A be a condensed ring, and let $(f) \subset A$ be a principal ideal sheaf. Let M be an (f) -adically complete A -module in CondAb . Consider the following condition on a given A -module P in CondAb :*

$$\text{for every } j, n \geq 1 \text{ the map } \text{Tor}_j^A(P, M/f^{n'}) \rightarrow \text{Tor}_j^A(P, M/f^n) \text{ vanishes for some } n' > n. \quad (4.18)$$

For any bounded complex P^\bullet of A -modules in CondAb , with each P^i and $H^i(P^\bullet)$ satisfying (4.18), for all $i \in \mathbb{Z}$ we have a natural isomorphism in CondAb

$$H^i(P^\bullet \widehat{\otimes}_A M) \cong H^i(P^\bullet) \widehat{\otimes}_A M$$

where the completion $\widehat{\otimes}_A$ is (f) -adic.

Proof. See the proof of [CK19, Lemma 3.30]. \square

Corollary 4.11. *Let $I \subset [1/(p-1), \infty)$ be a compact interval with rational endpoints. Let us denote $N_\infty := \mathbb{A}_{\text{inf}}(R_\infty)^{\text{n-int}}$. Then, for every $i \in \mathbb{Z}$, we have a natural isomorphism in CondAb*

$$H_{\text{cond}}^i(\Gamma, N_\infty \widehat{\otimes}_{A_{\text{inf}}} A_I) \cong H_{\text{cond}}^i(\Gamma, N_\infty) \widehat{\otimes}_{A_{\text{inf}}} A_I$$

where the completion $\widehat{\otimes}_{A_{\text{inf}}}$ is p -adic.

Proof. We will show that the condition (4.18) of Lemma 4.10 holds for $A = A_{\text{inf}}$, $f = p$, $M = A_I$, and each $P \in \{\mathbb{A}_{\text{inf}}(R_\infty), H_{\text{cond}}^i(\Gamma, N_\infty), H_{\text{cond}}^i(\Gamma, N_\infty/\mu)\}$, adapting the argument of [CK19, Lemma 3.31] to our setting.

For $P = \mathbb{A}_{\text{inf}}(R_\infty)$, we even have that $P \otimes_{A_{\text{inf}}}^{\text{L}} A_I/p^n \in D(\text{CondAb})$ is concentrated in degree 0 for all $n \geq 1$: in fact, recalling that $A_I/p^n \cong A_I/(p^n, \mu^{n'})$ for a large enough integer n' (see Remark 4.7), it suffices to apply [CK19, Lemma 3.13], which implies that $(p^n, \mu^{n'})$ is an $\mathbb{A}_{\text{inf}}(R_\infty)$ -regular sequence with $\mathbb{A}_{\text{inf}}(R_\infty)/(p^n, \mu^{n'})$ flat over $A_{\text{inf}}/(p^n, \mu^{n'})$ (noting that the latter two condensed rings are discrete).

Next, we claim that the case $P = H_{\text{cond}}^i(\Gamma, N_\infty)$ follows from the case $P = H_{\text{cond}}^i(\Gamma, N_\infty/\mu)$. In fact, by [CK19, Proposition 3.25] (and [Bos21, Proposition B.3]), μ kills every $H_{\text{cond}}^j(\Gamma, N_\infty)$,

therefore, the long exact sequence in condensed group cohomology associated to the short exact sequence $0 \rightarrow N_\infty \xrightarrow{\mu} N_\infty \rightarrow N_\infty/\mu \rightarrow 0$, gives the short exact sequences

$$0 \rightarrow H_{\text{cond}}^i(\Gamma, N_\infty) \rightarrow H_{\text{cond}}^i(\Gamma, N_\infty/\mu) \rightarrow H_{\text{cond}}^{i+1}(\Gamma, N_\infty) \rightarrow 0.$$

By [Bos21, Proposition B.3], $H_{\text{cond}}^i(\Gamma, N_\infty)$ vanishes for a large enough integer i , hence the claim follows by descending induction on i .

Finally, for $P = H_{\text{cond}}^i(\Gamma, N_\infty/\mu)$, we claim that P satisfies the following conditions that imply the condition (4.18) of Lemma 4.10:

- (i) P is p -torsion-free;
- (ii) for every $n \geq 1$, P/p^n is a filtered colimit of A_{inf} -modules isomorphic to $A_{\text{inf}}/(\varphi^{-r}(\mu), p^n)$ for variable $r \geq 0$.

In fact, recalling [Bos21, Proposition B.3], condition (i) follows from [CK19, Proposition 3.19], and condition (ii) follows from [CK19, Proposition 3.19, Corollary 3.23] and Lazard's theorem ([Sta20, 058G]), observing that P/p^n is discrete. It remains to show that such conditions on P imply the condition (4.18) of Lemma 4.10 in our case. For this, by condition (i) and Lemma A.5, we have that $P \otimes_{A_{\text{inf}}}^{\mathbb{L}} A_{\text{inf}}/p^n = P \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n$ is concentrated in degree 0, and then by condition (ii) and the μ -torsion-freeness of A_{inf}/p^n , we can reduce to checking that, for every $n \geq 1$ and $r \geq 0$, the map

$$\text{Tor}_1^{A_{\text{inf}}/p^{n'}}(A_{\text{inf}}/(\varphi^{-r}(\mu), p^{n'}), A_I/p^{n'}) \rightarrow \text{Tor}_1^{A_{\text{inf}}/p^n}(A_{\text{inf}}/(\varphi^{-r}(\mu), p^n), A_I/p^n) \quad (4.19)$$

vanishes for some $n' > n$. In order to check this, we observe that the source of the map (4.19) identifies with $(A_I/p^{n'})[\varphi^{-r}(\mu)]$ (and the target with $(A_I/p^n)[\varphi^{-r}(\mu)]$), and we conclude observing that $\varphi^{-r}(\mu)$ divides μ in A_{inf} ,³³ and, by Lemma 2.33, the map $(A_I/p^{n'})[\mu] \rightarrow A_I/p^n$ vanishes for some $n' > n$. \square

4.1.3. The functorial isomorphism. As we will see in this subsection, the source of the quasi-isomorphism (4.12) of Lemma 4.9 computes

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^\times) \otimes_{A_{\text{cris}}}^{\mathbb{L}\blacksquare} B_I.$$

Therefore, Lemma 4.9, combined with Lemma 4.8, already provides a local version of the desired Theorem 4.3. However, such quasi-isomorphism depends on the choice of the coordinates $\mathfrak{X} \rightarrow \text{Spf}(R^\square)$, introduced in Notation 4.4. To have a functorial quasi-isomorphism, we will rely on a modification of the method of “all possible coordinates” pioneered by Bhatt–Morrow–Scholze in [BMS18, §12.2], and used by Česnavičius–Koshikawa in [CK19, §5] to prove the functoriality with respect to étale maps of the absolute crystalline comparison isomorphism for the A_{inf} -cohomology in the semistable case.

Let us begin resuming the setting of [CK19, §5.17].

Notation 4.12. We denote by k_C the residue field of \mathcal{O}_C .

- Let $\mathfrak{X} = \text{Spf}(R)$ be an affine p -adic formal scheme over $\text{Spf}(\mathcal{O}_C)$, such that every two irreducible components of $\text{Spec}(R \otimes_{\mathcal{O}_C} k_C)$ intersect, and such that there exist finite sets Σ and $\Lambda \neq \emptyset$, and a closed immersion of p -adic formal schemes over $\text{Spf}(\mathcal{O}_C)$

$$\mathfrak{X} \hookrightarrow \text{Spf}(R_\Sigma^\square) \times_{\text{Spf}(\mathcal{O}_C)} \prod_{\lambda \in \Lambda} \text{Spf}(R_\lambda^\square) =: \text{Spf}(R_{\Sigma, \Lambda}^\square) \quad (4.20)$$

³³In fact, for any $r \geq 1$, one has $\mu = (\prod_{j=0}^{r-1} \varphi^{-j}(\xi)) \cdot \varphi^{-r}(\mu)$.

where

$$R_\Sigma^\square := \mathcal{O}_C\{t_\sigma^{\pm 1} : \sigma \in \Sigma\}$$

the induced map $\mathfrak{X} \rightarrow \mathrm{Spf}(R_\Sigma^\square)$ is a closed immersion, and, for each $\lambda \in \Lambda$,

$$R_\lambda^\square := \mathcal{O}_C\{t_{\lambda,0}, \dots, t_{\lambda,r_\lambda}, t_{\lambda,r_\lambda+1}^{\pm 1}, \dots, t_{\lambda,d}^{\pm 1}\} / (t_{\lambda,0} \cdots t_{\lambda,r_\lambda} - p^{q_\lambda})$$

for some $0 \leq r_\lambda \leq d$, and $q_\lambda \in \mathbb{Q}_{>0}$, and the induced map $\mathfrak{X} \rightarrow \mathrm{Spf}(R_\lambda^\square)$ is étale.

We will denote by M_R the canonical log structure on R , Notation 3.8.

- We define

$$A_{\mathrm{inf},\Sigma,\Lambda}^\square := A_{\mathrm{inf}}(R_\Sigma^\square) \widehat{\otimes}_{A_{\mathrm{inf}}} \widehat{\bigotimes}_{\lambda \in \Lambda} A_{\mathrm{inf}}(R_\lambda^\square)$$

where the completions are (p, μ) -adic.

We will denote by $M_{\mathrm{inf},\Sigma,\Lambda}^\square$ the log structure on $A_{\mathrm{inf},\Sigma,\Lambda}^\square$ associated to the log structures on $A_{\mathrm{inf}}(R_\Sigma^\square)$ and $A_{\mathrm{inf}}(R_\lambda^\square)$, for varying $\lambda \in \Lambda$, defined in [CK19, §5.9].

- Similarly to Notation 4.4, we define $R_{\Sigma,\Lambda,\infty}^\square$ a perfectoid $R_{\Sigma,\Lambda}^\square$ -algebra such that

$$\mathrm{Spa}(R_{\Sigma,\Lambda,\infty}^\square[1/p], R_{\Sigma,\Lambda,\infty}^\square) \rightarrow \mathfrak{X}_C \quad (4.21)$$

is an affinoid perfectoid pro-étale cover of \mathfrak{X}_C with Galois group

$$\Gamma_{\Sigma,\Lambda} := \Gamma_\Sigma \times \prod_{\lambda \in \Lambda} \Gamma_\lambda \cong \mathbb{Z}_p^{|\Sigma|} \times \prod_{\lambda \in \Lambda} \mathbb{Z}_p^d$$

(see also [CK19, §5.18]).

We denote by $(\gamma_\sigma)_{\sigma \in \Sigma}, (\gamma_{\lambda,i})_{\lambda \in \Lambda, 1 \leq i \leq d}$ the generators of $\Gamma_{\Sigma,\Lambda}$ defined by

$$\gamma_\sigma := (1, \dots, 1, \varepsilon, 1, \dots, 1) \quad \text{for } \sigma \in \Sigma$$

where ε sits on the σ -th entry, and

$$\gamma_{\lambda,i} := (\varepsilon^{-1}, 1, \dots, 1, \varepsilon, 1, \dots, 1) \quad \text{for } i = 1, \dots, r_\lambda$$

$$\gamma_{\lambda,i} := (1, \dots, 1, \varepsilon, 1, \dots, 1) \quad \text{for } i = r_\lambda + 1, \dots, d$$

where ε sits on the i -th entry.

The base change of (4.21) along the generic fiber of (4.20) defines an affinoid perfectoid pro-étale cover of \mathfrak{X}_C

$$\mathfrak{X}_{C,\Sigma,\Lambda,\infty} := \mathrm{Spa}(R_{\Sigma,\Lambda,\infty}^\square[1/p], R_{\Sigma,\Lambda,\infty}^\square) \rightarrow \mathfrak{X}_C$$

with Galois group $\Gamma_{\Sigma,\Lambda}$.

- Given \mathbb{M} any pro-étale period sheaf of §2.3.1, we set

$$\mathbb{M}(R_{\Sigma,\Lambda,\infty}^\square) := \mathbb{M}(\mathfrak{X}_{C,\Sigma,\Lambda,\infty})$$

and we regard it as a condensed ring.

Remark 4.13. In Notation 4.12, the assumption on the special fiber $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k_C)$ guarantees that each irreducible component of such special fiber is cut out by a unique $t_{\lambda,i}$ with $0 \leq i \leq r_\lambda$ (see also [CK19, §5.17]).

We note that Remark 4.6 and Remark 4.7 hold in the setting of Notation 4.12, i.e. with R_Σ^\square and R_λ^\square in place of R^\square . In particular, to fix the notation, we let

$$A_{\mathrm{inf}}(R_\Sigma^\square) \cong A_{\mathrm{inf}}\{X_\sigma^{\pm 1} \mid \sigma \in \Sigma\}$$

$$A_{\mathrm{inf}}(R_\lambda^\square) \cong A_{\mathrm{inf}}\{X_{\lambda,0}, \dots, X_{\lambda,r_\lambda}, X_{\lambda,r_\lambda+1}^{\pm 1}, \dots, X_{\lambda,d}^{\pm 1}\} / (X_{\lambda,0} \cdots X_{\lambda,r_\lambda} - [p^b]^{q_\lambda})$$

be the isomorphisms (4.5) for R_Σ^\square and R_λ^\square .

For the proof of Theorem 4.3, we will need the next result from [CK19, §5.22] on the log-crystalline cohomology over A_{cris} , that we translate here in the condensed setting.

Lemma 4.14. *In the setting of Notation 4.12, we have a φ -equivariant identification*

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^\times) \simeq \Omega_{D_{\Sigma,\Lambda}(R)}^\bullet := \text{Kos}_{D_{\Sigma,\Lambda}(R)}((\partial_\sigma)_{\sigma \in \Sigma}, (\partial_{\lambda,i})_{\lambda \in \Lambda, 1 \leq i \leq d}) \quad (4.22)$$

where $D_{\Sigma,\Lambda}(R)$ is an A_{cris} -algebra in CondAb characterized by the following properties: $D_{\Sigma,\Lambda}(R)$ is p -adically complete, and, for each integer $n \geq 1$, $D_{\Sigma,\Lambda}(R)/p^n$ is the log PD envelope of ³⁴

$$(A_{\text{inf},\Sigma,\Lambda}^\square \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n, M_{\text{inf},\Sigma,\Lambda}^\square) \twoheadrightarrow (R/p, M_R) \quad \text{over} \quad A_{\text{cris}}^\times/p^n \twoheadrightarrow \mathcal{O}_C^\times/p.$$

Here, $\partial_\sigma := \frac{\partial}{\partial \log(X_\sigma)}$, resp. $\partial_{\lambda,i} := \frac{\partial}{\partial \log(X_{\lambda,i})}$, are as in (4.11) with R_Σ^\square , resp. R_λ^\square , in place of R .

Proof. This is the content of [CK19, Proposition 5.23], which relies on [Bei13, (1.8.1)]. The characterization of $D_{\Sigma,\Lambda}(R)$ follows from [CK19, §5.22, Lemma 5.29]. \square

We note that the action of $\Gamma_{\Sigma,\Lambda}$ on $A_{\text{inf},\Sigma,\Lambda}^\square$ induces a natural action of $\Gamma_{\Sigma,\Lambda}$ on $D_{\Sigma,\Lambda}(R)$.

The next lemma, which is a semistable version of [BMS18, Corollary 12.7], expresses the complex $\Omega_{D_{\Sigma,\Lambda}(R)}^\bullet$ of Lemma 4.14, which computes the log-crystalline cohomology over A_{cris} , in terms of condensed group cohomology $R\Gamma_{\text{cond}}(\Gamma_{\Sigma,\Lambda}, D_{\Sigma,\Lambda}(R))$ via passage through Lie algebra cohomology. Cf. [CN17, §4.3].

Lemma 4.15. *In the setting of Notation 4.12, we denote by $\text{Lie } \Gamma_{\Sigma,\Lambda}$ the Lie algebra of $\Gamma_{\Sigma,\Lambda}$, and we write $\exp : \text{Lie } \Gamma_{\Sigma,\Lambda} \cong \Gamma_{\Sigma,\Lambda}$ for the exponential isomorphism. Then, there is a natural action of $\text{Lie } \Gamma_{\Sigma,\Lambda}$ on $D_{\Sigma,\Lambda}(R)$ defined for $g \in \text{Lie } \Gamma_{\Sigma,\Lambda}$, with $\exp(g) = \gamma \in \Gamma_{\Sigma,\Lambda}$, by*

$$g = \log(\gamma) := \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (\gamma - 1)^n. \quad (4.23)$$

We write $U_{\Sigma,\Lambda}$ for the universal enveloping algebra of $\text{Lie } \Gamma_{\Sigma,\Lambda}$, and we denote by

$$R\Gamma(\text{Lie } \Gamma_{\Sigma,\Lambda}, D_{\Sigma,\Lambda}(R)) := R\text{Hom}_{U_{\Sigma,\Lambda}}(\mathbb{Z}_p, D_{\Sigma,\Lambda}(R)) \in D(\text{CondAb})$$

the Lie group cohomology.

(i) *There is a quasi-isomorphism*

$$L\eta_\mu R\Gamma(\text{Lie } \Gamma_{\Sigma,\Lambda}, D_{\Sigma,\Lambda}(R)) \simeq \Omega_{D_{\Sigma,\Lambda}(R)}^\bullet.$$

(ii) *There is a quasi-isomorphism*

$$R\Gamma(\text{Lie } \Gamma_{\Sigma,\Lambda}, D_{\Sigma,\Lambda}(R)) \simeq R\Gamma_{\text{cond}}(\Gamma_{\Sigma,\Lambda}, D_{\Sigma,\Lambda}(R)).$$

Proof. We first need to check that, for $g \in \text{Lie } \Gamma_{\Sigma,\Lambda}$, the series (4.23) converges to an endomorphism of $D_{\Sigma,\Lambda}(R)$. For this, it suffices to prove that the action of $\gamma - 1$ on $D_{\Sigma,\Lambda}(R)$ takes values in $([\varepsilon] - 1)D_{\Sigma,\Lambda}(R)$: in fact, by [BMS18, Lemma 12.2], this implies that the action of $\frac{(\gamma-1)^n}{n}$ on $D_{\Sigma,\Lambda}(R)$ has values in $D_{\Sigma,\Lambda}(R)$, and such values converge to 0 as $n \rightarrow \infty$. Thus, following the proof of [BMS18, Lemma 12.6] and using [CK19, Lemma 5.29], we can reduce to checking that the action

³⁴Here, R/p is equipped with the pullback of the canonical log structure M_R on R , and $A_{\text{inf},\Sigma,\Lambda}^\square \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n$ is endowed with the pullback of the log structure $M_{\text{inf},\Sigma,\Lambda}^\square$ on $A_{\text{inf},\Sigma,\Lambda}^\square$, Notation 4.12.

of $\gamma - 1$ on $A_{\text{inf}, \Sigma, \Lambda}^{\square}$ takes values in $([\varepsilon] - 1)A_{\text{inf}, \Sigma, \Lambda}^{\square}$: this is clear for γ being one of the generators $(\gamma_{\sigma})_{\sigma \in \Sigma}, (\gamma_{\lambda, i})_{\lambda \in \Lambda, 1 \leq i \leq d}$ of $\Gamma_{\Sigma, \Lambda}$.³⁵

Next, to prove both part (i) and part (ii), we denote $T(\Sigma, \Lambda) := \Sigma \cup \{(\lambda, i) : \lambda \in \Lambda, 1 \leq i \leq d\}$. By (the proof of) [CK19, Lemma 5.33], the element $\gamma_{\tau} \in \Gamma_{\Sigma, \Lambda}$ acts on $D_{\Sigma, \Lambda}(R)$ as the endomorphism $\exp(\log([\varepsilon]) \cdot \partial_{\tau})$, for varying $\tau \in T(\Sigma, \Lambda)$. Therefore, denoting $g_{\tau} := \log([\varepsilon]) \cdot \partial_{\tau} \in \text{Lie } \Gamma_{\Sigma, \Lambda}$, we have a quasi-isomorphism

$$R\Gamma(\text{Lie } \Gamma_{\Sigma, \Lambda}, D_{\Sigma, \Lambda}(R)) \simeq \text{Kos}_{D_{\Sigma, \Lambda}(R)}(g_{\tau \in T(\Sigma, \Lambda)}) \quad (4.24)$$

Since $\log([\varepsilon])$ and μ differ by a unit in A_{cris} , part (i) follows from the quasi-isomorphism (4.24) and [Bos21, Lemma 7.8(ii)]. For part (ii), we note that $g_{\tau} = \log(\gamma_{\tau}) = (\gamma_{\tau} - 1) \cdot h_{\tau}$ for $\tau \in T(\Sigma, \Lambda)$, with h_{τ} automorphisms of $D_{\Sigma, \Lambda}(R)$ commuting with each other and with $\gamma_{\tau} - 1$, therefore

$$\text{Kos}_{D_{\Sigma, \Lambda}(R)}(g_{\tau \in T(\Sigma, \Lambda)}) \cong \text{Kos}_{D_{\Sigma, \Lambda}(R)}((\gamma_{\tau} - 1)_{\tau \in T(\Sigma, \Lambda)}) \quad (4.25)$$

(cf. the proof of [BMS18, Lemma 12.5]). Then, the statement of part (ii) follows combining (4.24), (4.25), and [Bos21, Proposition B.3]. \square

We deduce the following result.

Corollary 4.16. *In the setting of Notation 4.12, we have a natural (in \mathfrak{X} and the datum (4.20)) quasi-isomorphism*

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^{\times}) \xrightarrow{\sim} L\eta_{\mu}R\Gamma_{\text{cond}}(\Gamma_{\Sigma, \Lambda}, D_{\Sigma, \Lambda}(R)) \quad (4.26)$$

Proof. We first construct the desired natural morphism (4.26). By the proof of [Bei13, (1.8.1)] and by [CK19, Lemma 5.29], we have the following Čech-Alexander computation of the log-crystalline cohomology over A_{cris} :

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^{\times}) \simeq (D_{\Sigma, \Lambda}(R)(0) \rightarrow D_{\Sigma, \Lambda}(R)(1) \rightarrow D_{\Sigma, \Lambda}(R)(2) \rightarrow \cdots) \quad (4.27)$$

where $D_{\Sigma, \Lambda}(R)(n) := \varprojlim_{m \geq 1} D_{\Sigma, \Lambda, m}(R)(n)$ with $\text{Spec}(D_{\Sigma, \Lambda, m}(R)(n))$ the $(n+1)$ -fold product of $\text{Spec}(D_{\Sigma, \Lambda}(R)/p^m)$ in $(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}, m}^{\times})_{\text{cris}}$ (we recall 3.1 for the notation). On the other hand, by [Bos21, Proposition B.2(i)], the condensed group cohomology is computed by

$$R\Gamma_{\text{cond}}(\Gamma_{\Sigma, \Lambda}, D_{\Sigma, \Lambda}(R)) \simeq (D_{\Sigma, \Lambda}(R) \rightarrow \underline{\text{Hom}}(\mathbb{Z}[\Gamma_{\Sigma, \Lambda}], D_{\Sigma, \Lambda}(R)) \rightarrow \underline{\text{Hom}}(\mathbb{Z}[\Gamma_{\Sigma, \Lambda}^2], D_{\Sigma, \Lambda}(R)) \rightarrow \cdots). \quad (4.28)$$

Under the identifications (4.27) and (4.28), we define the morphism

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^{\times}) \rightarrow R\Gamma_{\text{cond}}(\Gamma_{\Sigma, \Lambda}, D_{\Sigma, \Lambda}(R)) \quad (4.29)$$

induced, in degree $n \geq 0$, by the composite of the termwise action $\Gamma_{\Sigma, \Lambda}^n \times D_{\Sigma, \Lambda}(R)(n) \rightarrow D_{\Sigma, \Lambda}(R)(n)$ with the co-diagonal map $D_{\Sigma, \Lambda}(R)(n) \rightarrow D_{\Sigma, \Lambda}(R)$. By [BMS18, Lemma 6.10] there is a natural map

$$L\eta_{\mu}R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^{\times}) \rightarrow R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^{\times}) \quad (4.30)$$

which is a quasi-isomorphism, as it follows combining Lemma 4.14 and Lemma 4.15(i). Using the quasi-isomorphism (4.30), applying the décalage functor $L\eta_{\mu}(-)$ to (4.29), we obtain the desired morphism (4.26), which is a quasi-isomorphism thanks to Lemma 4.14 (which relies on [Bei13, (1.8.1)], as (4.27)) and Lemma 4.15. \square

³⁵Note that, for $1 \leq i \leq d$, the element $\gamma_{\lambda, i} - 1$ acts as follows on X_j : it sends $X_i \mapsto ([\varepsilon] - 1)X_i$; $X_j \mapsto 0$ if $0 < j \neq i$; $X_0 \mapsto ([\varepsilon^{-1}] - 1)X_0 = -([\varepsilon] - 1)[\varepsilon^{-1}]X_0$ if $i \leq r_{\lambda}$; and $X_0 \mapsto 0$ if $i > r_{\lambda}$.

As a next step toward Theorem 4.3, we want to construct a comparison morphism from the log-crystalline cohomology over A_{cris} to the B_I -cohomology. For this, we will rely on the morphism (4.26), defined in Corollary 4.16, and on the following construction (which is inspired from the proof of [Kos22, Theorem 8.1]), that will allow us to compare the target of (4.26) with the B_I -cohomology.

Lemma 4.17. *In the setting of Notation 4.12, there is a natural (in R and the datum (4.20)) $\Gamma_{\Sigma,\Lambda}$ -equivariant map*

$$D_{\Sigma,\Lambda}(R) \rightarrow \mathbb{A}_{\text{cris}}(R_{\Sigma,\Lambda,\infty}) \quad (4.31)$$

where $\mathbb{A}_{\text{cris}}(R_{\Sigma,\Lambda,\infty}) := \mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty}) \widehat{\otimes}_{A_{\text{inf}}} A_{\text{cris}}$ and the completion $\widehat{\otimes}_{A_{\text{inf}}}$ is p -adic.

Proof. Since both $\mathbb{A}_{\text{cris}}(R_{\Sigma,\Lambda,\infty})$ and $D_{\Sigma,\Lambda}(R)$ are p -adically complete (for the latter, recall [CK19, Lemma 5.29]), it suffices to construct, for each $n \geq 1$, a natural map $D_{\Sigma,\Lambda}(R)/p^n \rightarrow \mathbb{A}_{\text{cris}}(R_{\Sigma,\Lambda,\infty})/p^n$. We observe that $\mathbb{A}_{\text{cris}}(R_{\Sigma,\Lambda,\infty})/p^n = \mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty}) \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n$. Then, we consider the following commutative diagram of log rings

$$\begin{array}{ccc} (A_{\text{inf},\Sigma,\Lambda}^{\square} \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n, M_{\text{inf},\Sigma,\Lambda}^{\square}) & \longrightarrow & (R/p, M_R) \\ \downarrow & & \downarrow \\ (\mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty}) \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n, N)^a & \longrightarrow & (R_{\Sigma,\Lambda,\infty}/p, M_R). \end{array} \quad (4.32)$$

Here, R/p and $R_{\Sigma,\Lambda,\infty}/p$ are equipped with the pullback of the canonical log structure M_R on R , and $A_{\text{inf},\Sigma,\Lambda}^{\square} \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n$ is endowed with the pullback of the log structure $M_{\text{inf},\Sigma,\Lambda}^{\square}$ on $A_{\text{inf},\Sigma,\Lambda}^{\square}$, as in Lemma 4.14. Moreover, $\mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty}) \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n$ is equipped with the pullback of the log structure on $\mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty})$ associated to the pre-log structure defined as follows: we set $N := (h^{\text{gp}})^{-1}(M_{\text{inf},\Sigma,\Lambda}^{\square})$ where h^{gp} denotes the morphism of groups associated to the natural morphism of monoids $h : M_{\text{inf},\Sigma,\Lambda}^{\square} \rightarrow M_R$; the argument of [CK19, Lemma 5.37] shows that the natural map $M_{\text{inf},\Sigma,\Lambda}^{\square} \rightarrow \mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty})$ uniquely extends to a map $N \rightarrow \mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty})$.³⁶ The resulting surjective map of log rings at the bottom of the diagram (4.32) is exact by construction, hence, the universal property of the log PD envelope $D_{\Sigma,\Lambda}(R)/p^n$ gives the desired natural map $D_{\Sigma,\Lambda}(R)/p^n \rightarrow \mathbb{A}_{\text{cris}}(R_{\Sigma,\Lambda,\infty})/p^n$. \square

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. It suffices to prove the statement Zariski locally on \mathfrak{X} in a functorial way, as \mathfrak{X} is assumed to be qcqs and the derived tensor product $\otimes_{A_{\text{cris}}}^{\mathbf{L}\square}$ commutes with finite limits.

Thus, let $\mathfrak{X} = \text{Spf}(R)$ as in Notation 4.12, with fixed finite sets Σ and Λ . We will denote $T(\Sigma, \Lambda) := \Sigma \cup \{(\lambda, i) : \lambda \in \Lambda, 1 \leq i \leq d\}$.

First, we note that, similarly to Lemma 4.8, we have a φ -equivariant identification

$$R\Gamma_{B_I}(\mathfrak{X}_C) \simeq L\eta_t R\Gamma_{\text{cond}}(\Gamma_{\Sigma,\Lambda}, \mathbb{B}_I(R_{\Sigma,\Lambda,\infty})) \simeq L\eta_t \text{Kos}_{\mathbb{B}_I(R_{\Sigma,\Lambda,\infty})}((\gamma_{\tau} - 1)_{\tau \in T(\Sigma,\Lambda)}). \quad (4.33)$$

Then, using the natural map (4.26) constructed in Corollary 4.16, together with the natural morphism (4.31) constructed in Lemma 4.17, we have a natural (in \mathfrak{X} and the datum (4.20)) morphism

$$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^{\times}) \xrightarrow{\sim} L\eta_{\mu} R\Gamma_{\text{cond}}(\Gamma_{\Sigma,\Lambda}, D_{\Sigma,\Lambda}(R)) \rightarrow L\eta_{\mu} R\Gamma_{\text{cond}}(\Gamma_{\Sigma,\Lambda}, \mathbb{B}_I(R_{\Sigma,\Lambda,\infty})). \quad (4.34)$$

³⁶In fact, in the notation of [CK19, §5.25, §5.26], it suffices to prove that $\mathbb{A}_{\text{inf}}(R_{\Sigma,\Lambda,\infty})$ is naturally an $(A_{\text{inf},\Sigma,\Lambda}^{\square} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$ -algebra compatibly with the change of $\lambda_0 \in \Lambda$, which is shown in [CK19, Lemma 5.37].

Using that μ and t differ by a unit in B_I , (4.34) induces a natural morphism

$$f_{\mathfrak{X}, \Sigma, \Lambda} : R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^\times) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} B_I \rightarrow R\Gamma_{B_I}(\mathfrak{X}_C) \quad (4.35)$$

that we claim to be a quasi-isomorphism. For this, it suffices to show that, for a fixed $\lambda \in \Lambda$, we have a commutative diagram as follows, whose arrows are quasi-isomorphisms

$$\begin{array}{ccc} \text{Kos}_{A_{\text{cris}}(R)_\lambda}((\partial_{\lambda, i})_{1 \leq i \leq d}) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} B_I & \longrightarrow & \eta_t \text{Kos}_{\mathbb{B}_I(R_{\lambda, \infty})}((\gamma_{\lambda, i} - 1)_{1 \leq i \leq d}) \\ \downarrow & & \downarrow \\ \text{Kos}_{D_{\Sigma, \Lambda}(R)}((\partial_\tau)_{\tau \in T(\Sigma, \Lambda)}) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} B_I & \longrightarrow & \eta_t \text{Kos}_{\mathbb{B}_I(R_{\Sigma, \Lambda, \infty})}((\gamma_\tau - 1)_{\tau \in T(\Sigma, \Lambda)}) \end{array} \quad (4.36)$$

with the bottom arrow of (4.36) compatible with the morphism (4.35), under the identifications (4.22) and (4.33). Here, we denote $A_{\text{cris}}(R)_\lambda := A_{\text{inf}}(R)_\lambda \widehat{\otimes}_{A_{\text{inf}}} A_{\text{cris}}$, where the completion $\widehat{\otimes}_{A_{\text{inf}}}$ is p -adic, and $A_{\text{inf}}(R)_\lambda$ is the unique lift of the étale (R_λ^\square/p) -algebra R/p , along $\theta : A_{\text{inf}}(R_\lambda^\square) \rightarrow R_\lambda^\square$, to a (p, μ) -adically complete, formally étale $A_{\text{inf}}(R_\lambda^\square)$ -algebra.

- (i) The right vertical map of (4.36) is a quasi-isomorphism since, by Lemma 4.8 and (4.33), both the target and the source are quasi-isomorphic to $R\Gamma_{B_I}(\mathfrak{X}_C)$.
- (ii) The top horizontal arrow of (4.36) is induced by the quasi-isomorphism constructed in Lemma 4.9, observing that we have the following identifications:

$$\text{Kos}_{A_{\text{inf}}(R)_\lambda}((\partial_{\lambda, i})_{1 \leq i \leq d}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_I = (\text{Kos}_{A_{\text{inf}}(R)_\lambda}((\partial_{\lambda, i})_{1 \leq i \leq d}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} A_{\text{cris}}) \otimes_{A_{\text{cris}}}^{\text{L}\blacksquare} B_I$$

and, by Proposition A.3,

$$\text{Kos}_{A_{\text{inf}}(R)_\lambda}((\partial_{\lambda, i})_{1 \leq i \leq d}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} A_{\text{cris}} \simeq \text{Kos}_{A_{\text{cris}}(R)_\lambda}((\partial_{\lambda, i})_{1 \leq i \leq d}) \quad (4.37)$$

as we now explain. In fact, Proposition A.3 implies that the derived solid tensor product $\otimes_{A_{\text{inf}}}^{\text{L}\blacksquare}$ appearing in (4.37) can be replaced by the derived p -adic completion $\widehat{\otimes}_{A_{\text{inf}}}^{\text{L}}$, and then it remains to observe that the latter completion identifies with the underived (termwise) p -adic completion: for this, denoting by $A_{\text{cris}}^{(m)} \subset A_{\text{cris}}$ the p -adic completion of the A_{inf} -subalgebra generated by $\xi^j/j!$ for varying $j \leq m$, we note that, for any integer $n \geq 1$, we have $A_{\text{cris}}/p^n \cong \varinjlim_{m \geq p} A_{\text{cris}}^{(m)}/p^n$, with $A_{\text{cris}}^{(m)}/p^n \cong A_{\text{cris}}^{(m)}/(p^n, \mu^{n'})$ for $m \geq p$, and a large enough integer n' (see [CK19, §3.26]);³⁷ then, we recall that $(p^n, \mu^{n'})$ is an $A_{\text{inf}}(R)$ -regular sequence with $A_{\text{inf}}(R)/(p^n, \mu^{n'})$ flat over $A_{\text{inf}}(R)/(p^n, \mu^{n'})$ (see [CK19, Lemma 3.13]).

- (iii) The left vertical map of (4.36) is constructed as follows. Since $D_{\Sigma, \Lambda}(R)$ is a p -adically complete pro-nilpotent thickening of R/p (here we use [CK19, Lemma 5.29]), by the infinitesimal lifting criterion for the p -adic formally étale map $A_{\text{cris}}(R_\lambda^\square) \rightarrow A_{\text{cris}}(R)_\lambda$, we deduce that in the following diagram

$$\begin{array}{ccc} A_{\text{cris}}(R_\lambda^\square) & \longrightarrow & A_{\text{cris}}(R)_\lambda \\ \downarrow & \swarrow \text{---} & \downarrow \\ D_{\Sigma, \Lambda}(R) & \longrightarrow & R/p \end{array}$$

³⁷In fact, for $m \geq p$, one has $\mu^p/p! \in A_{\text{cris}}^{(m)}$, therefore one can take $n' := pn$.

there exists a unique dotted arrow making the diagram commute. The resulting map

$$\mathrm{Kos}_{A_{\mathrm{cris}}(R)_\lambda}((\partial_{\lambda,i})_{1 \leq i \leq d}) \rightarrow \mathrm{Kos}_{D_{\Sigma,\Lambda}(R)}((\partial_\tau)_{\tau \in T(\Sigma,\Lambda)})$$

is a quasi-isomorphism, as both the source and the target compute $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times)$, by [CK19, Proposition 5.13] and Lemma 4.14, respectively.

- (iv) The bottom horizontal arrow of (4.36) is constructed as follows. Similarly to the map (4.1.2) in Lemma 4.9, for varying $\tau \in T(\Sigma, \Lambda)$, we have φ -equivariant maps

$$\left((D_{\Sigma,\Lambda}(R) \xrightarrow{\partial_\tau} D_{\Sigma,\Lambda}(R)) \xrightarrow{(\mathrm{id}, h_\tau)} (D_{\Sigma,\Lambda}(R) \xrightarrow{\gamma_\tau^{-1}} D_{\Sigma,\Lambda}(R)) \right) \quad (4.38)$$

with $h_\tau := 1 + \sum_{j \geq 1} \frac{t^j}{(j+1)!} (\partial_\tau)^j$, which induce a φ -equivariant quasi-isomorphism

$$\mathrm{Kos}_{D_{\Sigma,\Lambda}(R)}((\partial_\tau)_{\tau \in T(\Sigma,\Lambda)}) \xrightarrow{\sim} \eta_\mu \mathrm{Kos}_{D_{\Sigma,\Lambda}(R)}((\gamma_\tau - 1)_{\tau \in T(\Sigma,\Lambda)}) \quad (4.39)$$

(cf. [CK19, Proposition 5.34]). Moreover, the $\Gamma_{\Sigma,\Lambda}$ -equivariant map (4.31) constructed in Lemma 4.17, induces a morphism

$$\eta_\mu \mathrm{Kos}_{D_{\Sigma,\Lambda}(R)}((\gamma_\tau - 1)_{\tau \in T(\Sigma,\Lambda)}) \rightarrow \eta_\mu \mathrm{Kos}_{\mathbb{B}_I(R_{\Sigma,\Lambda,\infty})}((\gamma_\tau - 1)_{\tau \in T(\Sigma,\Lambda)}). \quad (4.40)$$

Then, we define the bottom horizontal arrow of (4.36) as the morphism induced by the composite of (4.39) and (4.40). The so constructed morphism makes the diagram (4.36) commute, it is compatible with the morphism (4.35), under the identifications (4.22) and (4.33), and, by the previous points, it is a quasi-isomorphism.

Taking the filtered colimit $\varinjlim_{\Sigma,\Lambda}$ of the quasi-isomorphisms $f_{\mathfrak{X},\Sigma,\Lambda}$ constructed in (4.35), putting everything together, using that $\otimes_{A_{\mathrm{cris}}}^{\mathbf{L}}$ commutes with filtered colimits, we obtain the desired quasi-isomorphism

$$f_{\mathfrak{X}} : R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times) \otimes_{A_{\mathrm{cris}}}^{\mathbf{L}} B_I \xrightarrow{\sim} \varinjlim_{\Sigma,\Lambda} K(R_{\Sigma,\Lambda,\infty}) \simeq R\Gamma_{B_I}(\mathfrak{X}_C).$$

where we denoted $K(R_{\Sigma,\Lambda,\infty}) := L\eta_t R\Gamma_{\mathrm{cond}}(\Gamma_{\Sigma,\Lambda}, \mathbb{B}_I(R_{\Sigma,\Lambda,\infty}))$.

It remains to show that the morphism $f_{\mathfrak{X}}$ depends functorially on $\mathfrak{X} = \mathrm{Spf}(R)$. For this, it suffices to prove that the filtered colimit $\varinjlim_{\Sigma,\Lambda} K(R_{\Sigma,\Lambda,\infty})$ depends functorially on R , compatibly with the constructed morphism from the complex $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times)$ (i.e. the filtered colimit $\varinjlim_{\Sigma,\Lambda}$ of the morphisms defined in (4.34)). We study separately the latter filtered colimit in the case \mathfrak{X} is smooth or non-smooth, and show that it reduces to a simpler filtered colimit.³⁸

- Suppose \mathfrak{X} is smooth. In this case, given a finite set Λ as in Notation 4.12, for each pair (λ, i) with $\lambda \in \Lambda$ and $0 \leq i \leq d$, we have

$$t_{\lambda,i} = (p^q)^{n_{\lambda,i}} \cdot u_{\lambda,i} \quad \text{for unique } n_{\lambda,i} \in \mathbb{Z}_{\geq 0} \text{ and } u_{\lambda,i} \in R^\times \quad (4.41)$$

where $q \in \mathbb{Q}_{>0}$ is the unique element such that $\mathbb{Z} \cdot q = \sum_{\lambda \in \Lambda} \mathbb{Z} \cdot q_\lambda$ inside \mathbb{Q} (see [CK19, (5.26.2)]). We recall that p -th power roots of p in \mathcal{O}_C are fixed (Notation 3.8). Then, for sufficiently large finite sets $\Sigma \subset R^\times$ (containing the $u_{\lambda,i}$'s as in (4.41)), denoting $R_{\Sigma,\infty} := R_{\Sigma,\emptyset,\infty}$, we have natural surjections

$$R_{\Sigma,\Lambda,\infty} \twoheadrightarrow R_{\Sigma,\infty} \quad (4.42)$$

³⁸We thank Teruhisa Koshikawa for suggesting the following idea.

given by assigning the images of p -th power roots of $u_{\lambda,i}$. The maps (4.42) induce an isomorphism

$$\varinjlim_{\Sigma,\Lambda} K(R_{\Sigma,\Lambda,\infty}) \xrightarrow{\sim} \varinjlim_{\Sigma} K(R_{\Sigma,\infty}) \quad (4.43)$$

as both sides of (4.43) compute $R\Gamma_{B_I}(\mathfrak{X}_C)$.

- Suppose \mathfrak{X} is non-smooth. We choose an étale map $\mathfrak{X} \rightarrow \mathrm{Spf}(R_{\lambda_0}^{\square})$ as in Notation 4.12. Then, by Remark 4.13, given a finite set Λ containing λ_0 , for each pair (λ, i) with $\lambda \in \Lambda$ and $0 \leq i \leq d$, either the image of $t_{\lambda,i}$ in R lies in R^{\times} , or it cuts out an irreducible component of the special fiber of \mathfrak{X} , and there exists a unique $0 \leq i' \leq r_{\lambda_0}$ (depending on i) such that

$$t_{\lambda,i} = u_{\lambda,\lambda_0,i} \cdot t_{\lambda_0,i'} \quad \text{for a unique } u_{\lambda,\lambda_0,i} \in R^{\times} \quad (4.44)$$

(see [CK19, (5.27.1)]). Then, for sufficiently large finite sets $\Sigma \subset R^{\times}$ (containing the invertible $t_{\lambda,i}$'s in R and the $u_{\lambda,\lambda_0,i}$'s as in (4.44)), we have natural surjections

$$R_{\Sigma,\Lambda,\infty} \twoheadrightarrow R_{\Sigma,\{\lambda_0\},\infty} \quad (4.45)$$

given by assigning the images of p -th power roots of $t_{\lambda,i}$ in the case the image of $t_{\lambda,i}$ in R lies in R^{\times} , or by assigning the images of p -th power roots of $u_{\lambda,\lambda_0,i}$ in the complementary case. The maps (4.45) induce an isomorphism

$$\varinjlim_{\Sigma,\Lambda} K(R_{\Sigma,\Lambda,\infty}) \xrightarrow{\sim} \varinjlim_{\Sigma} K(R_{\Sigma,\{\lambda_0\},\infty}). \quad (4.46)$$

For later reference, we note here that for a finite set Λ as in Notation 4.12 containing λ_0 , and for sufficiently large finite sets Σ , for any $\lambda \in \Lambda$ we have a commutative diagram

$$\begin{array}{ccc} & & R_{\Sigma,\{\lambda\},\infty} \\ & \nearrow & \downarrow \wr \\ R_{\Sigma,\Lambda,\infty} & & R_{\Sigma,\{\lambda_0\},\infty} \end{array} \quad (4.47)$$

Here, the bottom diagonal arrow is (4.45), the top diagonal arrow is defined similarly to the latter (with λ in place of λ_0), and the vertical arrow is the isomorphism defined as follows. For (λ, i) such that the image of $t_{\lambda,i}$ in R does not lie in R^{\times} , the relation (4.44) implies that there exists a unique $0 \leq i' \leq r_{\lambda_0}$ (depending on i) such that, for any integer $m \geq 0$, we have

$$t_{\lambda,i}^{1/p^m} = u_{\lambda,\lambda_0,i}^{(m)} \cdot t_{\lambda_0,i'}^{1/p^m} \quad \text{in } R_{\Sigma,\Lambda,\infty} \text{ for a unique } u_{\lambda,\lambda_0,i}^{(m)} \in R_{\Sigma,\Lambda,\infty}^{\times}$$

using that $t_{\lambda_0,i'}^{1/p^m}$ is a unit in $R_{\Sigma,\Lambda,\infty}[1/p]$ and $R_{\Sigma,\Lambda,\infty}$ is integrally closed in $R_{\Sigma,\Lambda,\infty}[1/p]$. Then, the vertical arrow of (4.47) is defined by sending, for $m \geq 0$ and (λ, i) as before, the element $t_{\lambda,i}^{1/p^m}$ in $R_{\Sigma,\{\lambda\},\infty}$ to the image of $u_{\lambda,\lambda_0,i}^{(m)} \cdot t_{\lambda_0,i'}^{1/p^m}$ in $R_{\Sigma,\{\lambda_0\},\infty}$; this defines an isomorphism as the $u_{\lambda,\lambda_0,i}^{(m)}$'s are invertible.

Now, let $g : \mathfrak{X}' = \mathrm{Spf}(R') \rightarrow \mathfrak{X} = \mathrm{Spf}(R)$ be a map of affine p -adic formal schemes over $\mathrm{Spf}(\mathcal{O}_C)$, with \mathfrak{X} and \mathfrak{X}' equipped with data as in Notation 4.12, for some sets Σ and Λ_0 with $\Lambda_0 = \{\lambda_0\}$ in the case \mathfrak{X} is non-smooth and $\Lambda = \emptyset$ in the smooth case, resp. Σ' and Λ'_0 with $\Lambda'_0 = \{\lambda'_0\}$ in the case \mathfrak{X}' is non-smooth and $\Lambda = \emptyset$ in the smooth case. Then, for a sufficiently large finite set $\Sigma' \subset R'^{\times}$,

there exists a canonical map $g^\square : \mathrm{Spf}(R'_{\Sigma', \Lambda'_0}^\square) \rightarrow \mathrm{Spf}(R_{\Sigma, \Lambda_0}^\square)$ of p -adic formal schemes over $\mathrm{Spf}(\mathcal{O}_C)$ making the following diagram commute

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g} & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathrm{Spf}(R'_{\Sigma', \Lambda'_0}^\square) & \xrightarrow{g^\square} & \mathrm{Spf}(R_{\Sigma, \Lambda_0}^\square). \end{array} \quad (4.48)$$

In fact, suppose \mathfrak{X} and \mathfrak{X}' are non-smooth (in the other cases one can argue similarly), then by Remark 4.13, for $0 \leq i \leq d$, we have the following relation in R'

$$t_{\lambda_0, i} = (p^{g\lambda'_0})^{n_{\lambda_0, i}} \cdot u_{\lambda_0, \lambda'_0, i} \cdot \prod_{0 \leq j \leq r_{\lambda'_0}} (t_{\lambda'_0, j})^{a_{\lambda_0, i, j}}$$

for unique $n_{\lambda_0, i} \in \mathbb{Z}_{\geq 0}$, $u_{\lambda_0, \lambda'_0, i} \in R'^{\times}$, and $a_{\lambda_0, i, j} \in \mathbb{Z}_{\geq 0}$ not all positive; hence, we take $\Sigma' \subset R'^{\times}$ containing a copy of Σ and the units $u_{\lambda_0, \lambda'_0, i}$ for $0 \leq i \leq d$.

For a sufficiently large finite set $\Sigma' \subset R'^{\times}$, the diagram (4.48) induces a morphism of Galois covers

$$\begin{array}{ccc} \mathfrak{X}'_{C, \Sigma', \Lambda'_0, \infty} & \longrightarrow & \mathfrak{X}_{C, \Sigma, \Lambda_0, \infty} \\ \downarrow & & \downarrow \\ \mathfrak{X}'_C & \longrightarrow & \mathfrak{X}_C \end{array}$$

which in turn induces a map $\varinjlim_{\Sigma} K(R_{\Sigma, \Lambda_0, \infty}) \rightarrow \varinjlim_{\Sigma} K(R'_{\Sigma', \Lambda'_0, \infty})$. The latter map composed with the isomorphism (4.43) (in the case R is smooth, and similarly for R') or the isomorphism (4.46) (in the case R is non-smooth, and similarly for R'), induces a map

$$\varinjlim_{\Sigma, \Lambda} K(R_{\Sigma, \Lambda, \infty}) \rightarrow \varinjlim_{\Sigma', \Lambda'} K(R'_{\Sigma', \Lambda', \infty}) \quad (4.49)$$

which does not depend on the choices of λ_0 and λ'_0 : this follows from the commutative diagram (4.47) (and a similar diagram for R'). By construction, the map (4.49) fits in the following commutative diagram

$$\begin{array}{ccc} R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times) & \longrightarrow & \varinjlim_{\Sigma, \Lambda} K(R_{\Sigma, \Lambda, \infty}) \\ \downarrow & & \downarrow \\ R\Gamma_{\mathrm{cris}}(\mathfrak{X}'_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times) & \longrightarrow & \varinjlim_{\Sigma', \Lambda'} K(R'_{\Sigma', \Lambda', \infty}) \end{array}$$

where the top, resp. bottom, horizontal arrow is the filtered colimit $\varinjlim_{\Sigma, \Lambda}$, resp. $\varinjlim_{\Sigma', \Lambda'}$, of the morphisms defined in (4.34). This concludes the proof. \square

4.2. The comparison with the Hyodo–Kato cohomology. Now, we have all the ingredients to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. First, let $\mathfrak{X} \in \mathcal{M}_{\mathrm{ss}, \mathrm{qcqs}}$, i.e. \mathfrak{X} a qcqs semistable p -adic formal scheme over $\mathrm{Spf}(\mathcal{O}_C)$. We want to show the statement for $X = \mathfrak{X}_C$. Combining Theorem 4.3 with Theorem 3.2(i), and recalling Remark 2.32, for any compact interval $I \subset [1/(p-1), \infty)$ with rational

endpoints, we have a natural isomorphism

$$R\Gamma_{B_I}(\mathfrak{X}_C) \simeq (R\Gamma_{\mathrm{HK}}(\mathfrak{X}_C) \otimes_{\check{F}}^{\mathrm{L}\square} B_{\log,I})^{N=0}. \quad (4.50)$$

Twisting by the Frobenius, and recalling that φ is a $(\sigma_{\check{F}}$ -semilinear) automorphism on the Hyodo–Kato cohomology, we deduce that (4.50) extends to any compact interval $I \subset (0, \infty)$ with rational endpoints. Then, passing to the derived limit over all such I , by Lemma 2.41 we have a natural isomorphism

$$R\varprojlim_I R\Gamma_{B_I}(\mathfrak{X}_C) \simeq R\Gamma_B(\mathfrak{X}_C)$$

and then, by [Bos21, Corollary A.67(ii)], which applies observing that $B_{\log,I}$ is a nuclear \check{F} -vector space, and recalling that $R\Gamma_{\mathrm{HK}}(\mathfrak{X}_C)$ is representable by a bounded complex of \check{F} -Banach spaces (see Theorem 3.15(ii) and its proof), we obtain a natural isomorphism

$$R\Gamma_B(\mathfrak{X}_C) \simeq (R\Gamma_{\mathrm{HK}}(\mathfrak{X}_C) \otimes_{\check{F}}^{\mathrm{L}\square} \widehat{B}_{\log})^{N=0}$$

where $\widehat{B}_{\log} := R\varprojlim_I B_{\log,I}$. Now, we claim that the natural map

$$(R\Gamma_{\mathrm{HK}}(\mathfrak{X}_C) \otimes_{\check{F}}^{\mathrm{L}\square} B_{\log})^{N=0} \rightarrow (R\Gamma_{\mathrm{HK}}(\mathfrak{X}_C) \otimes_{\check{F}}^{\mathrm{L}\square} \widehat{B}_{\log})^{N=0} \quad (4.51)$$

is an isomorphism. In fact, recalling that $B_{\log} = B[U]$ and $B_{\log,I} = B_I[U]$, both the source and the target of (4.51) identify with the complex $R\Gamma_{\mathrm{HK}}(\mathfrak{X}_C) \otimes_{\check{F}}^{\mathrm{L}\square} B$ via the operator $\exp(N \cdot U)$ on the latter tensor product (cf. Lemma 7.6; note that the monodromy N is nilpotent on $R\Gamma_{\mathrm{HK}}(\mathfrak{X}_C)$, by Theorem 3.15(ii), in particular such operator is well-defined).³⁹

Next, let X be a qcqs rigid-analytic variety over C . Consider a simplicial object \mathfrak{U}_{\bullet} of $\mathcal{M}_{\mathrm{ss},\mathrm{qcqs}}$ such that $\mathfrak{U}_{\bullet,\eta} \rightarrow X$ is a $\acute{\mathrm{e}}\mathrm{h}$ -hypercover. To show that for such X we have an isomorphism as in (4.1), since the B -cohomology satisfies $\acute{\mathrm{e}}\mathrm{h}$ -hyperdescent, we have

$$R\Gamma_B(X) = \lim_{[n] \in \Delta} R\Gamma_B(\mathfrak{U}_{n,C}) \quad (4.52)$$

and then, using that the Hyodo–Kato cohomology satisfies $\acute{\mathrm{e}}\mathrm{h}$ -hyperdescent as well, it suffices to show that the natural map at the top of the following commutative diagram is an isomorphism

$$\begin{array}{ccc} \lim_{[n] \in \Delta} (R\Gamma_{\mathrm{cris}}(\mathfrak{U}_{n,\mathcal{O}_C/p}^0 / \mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \otimes_{\check{F}}^{\mathrm{L}\square} B_{\log})^{N=0} & \longrightarrow & (R\Gamma_{\mathrm{HK}}(X) \otimes_{\check{F}}^{\mathrm{L}\square} B_{\log})^{N=0} \\ \uparrow & & \uparrow \\ \lim_{[n] \in \Delta} (R\Gamma_{\mathrm{cris}}(\mathfrak{U}_{n,\mathcal{O}_C/p}^0 / \mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p} \otimes_{\check{F}}^{\mathrm{L}\square} B) & \longrightarrow & R\Gamma_{\mathrm{HK}}(X) \otimes_{\check{F}}^{\mathrm{L}\square} B. \end{array} \quad (4.53)$$

Here, arguing as above, the vertical arrows are the isomorphisms defined by the operator $\exp(N \cdot U)$ (recall that the monodromy N is nilpotent on $R\Gamma_{\mathrm{HK}}(X)$, by Theorem 3.15(ii), in particular such operator is well-defined).⁴⁰ We note that the bottom horizontal arrow of the diagram (4.53) is an isomorphism thanks to [Bos21, Corollary A.67(ii)] (which applies recalling that B is a Fréchet space

³⁹We observe that, in the case when \mathfrak{X}_C is the base change to C of a rigid-analytic variety defined over K , such identifications may be not \mathcal{G}_K -equivariant. However, in this case, the map (4.51) is \mathcal{G}_K -equivariant.

⁴⁰Similarly to Footnote 39, in the case when X is the base change to C of a rigid-analytic variety defined over K , such isomorphisms may be not \mathcal{G}_K -equivariant. However, in this case, the top horizontal arrow of the diagram above is \mathcal{G}_K -equivariant.

and each $R\Gamma_{\text{cris}}(\mathfrak{U}_{n, \mathcal{O}_C/p}^0/\mathcal{O}_{\check{F}}^0)_{\mathbb{Q}_p}$ is representable by a complex of nuclear \check{F} -vector spaces, and using (3.16)). Then, we deduce that the top horizontal arrow is an isomorphism as well, as desired.⁴¹

Lastly, for a X connected, paracompact, rigid-analytic variety over C , choosing a quasi-compact admissible covering $\{U_n\}_{n \in \mathbb{N}}$ of X such that $U_n \subseteq U_{n+1}$, using Theorem 3.15(ii), the same argument used above to show that the top horizontal arrow in (4.53) is an isomorphism reduces us to the previous case.⁴² \square

5. B_{dR}^+ -COHOMOLOGY

Our first goal in this section is to compare the B_{dR}^+ -cohomology with the de Rham cohomology.

5.1. The comparison with the de Rham cohomology. In the smooth case, the following result is essentially already contained in [Bos21, §6], and relies on Scholze's Poincaré lemma for \mathbb{B}_{dR}^+ .

In the following, we denote by $R\Gamma_{\text{dR}}(X)$ the de Rham cohomology endowed with the Hodge filtration (Definition 2.16).

Theorem 5.1. *Let X be a connected, paracompact, rigid-analytic variety defined over K . Then, we have a natural isomorphism in $D(\text{Mod}_{B_{\text{dR}}^+}^{\text{solid}})$*

$$R\Gamma_{B_{\text{dR}}^+}(X_C) \simeq R\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\blacksquare} B_{\text{dR}}^+ \quad (5.1)$$

compatible with filtrations, and the action of \mathcal{G}_K .

Proof. Assume first that X is smooth. By [Bos21, Lemma 2.17, Corollary 6.12, Lemma 6.13, Lemma 5.6(ii)] we have an isomorphism in $D(\text{Mod}_{B_{\text{dR}}^+}^{\text{solid}})$

$$R\Gamma_{B_{\text{dR}}^+}(X_C) \simeq R\Gamma(X, \Omega_X^\bullet \otimes_K^\blacksquare B_{\text{dR}}^+)$$

compatible with filtrations. Then, in this case, the statement follows from [Bos21, Theorem 5.20].

The general case follows by éh-hyperdescent, using [Bos21, Corollary A.67(ii)], and observing, for the compatibility with filtrations part, that the filtration on $R\Gamma_{\text{dR}}(X)$ is finite, Proposition 2.18 (as X is also assumed to be of finite dimension, §1.7). \square

The same argument and references used in the proof of Theorem 5.1 also show the following result, which generalizes [Bos21, Theorem 1.8(ii)] to the singular case:

Theorem 5.2. *Let X be a connected, paracompact, rigid-analytic variety defined over K . Then, we have a natural isomorphism in $D(\text{Mod}_{B_{\text{dR}}^+}^{\text{solid}})$*

$$R\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{dR}}^+) \simeq \text{Fil}^0(R\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\blacksquare} B_{\text{dR}}).$$

⁴¹In the case X is a qcqs smooth rigid-analytic variety over C , we can use the following alternative argument. We first note that the complex $R\Gamma_B(X)$ is bounded: for this, as X is qcqs, we can reduce to the case X is a smooth affinoid over C , in which case the claim follows from the identification $R\Gamma_B(X) = L\eta_t R\Gamma_{\text{proét}}(X, \mathbb{B})$ (Lemma 2.42(ii)) and the vanishing $H_{\text{proét}}^i(X, \mathbb{B}) = 0$ for $i > \dim X$ (see the proof of Lemma 2.40). Therefore, we can replace the limit $\lim_{[n] \in \Delta}$ appearing in (4.52) by a finite limit $\lim_{[n] \in \Delta_{\leq m}}$ for a sufficiently big integer m . Then, using that also the complex $R\Gamma_{\text{HK}}(X)$ is bounded (Theorem 3.15(ii)), the top horizontal arrow in (4.53) is an isomorphism using that $\otimes_{\check{F}}^{\text{L}\blacksquare}$ commutes with finite limits.

⁴²Recall that any rigid-analytic variety is assumed to be quasi-separated and of finite dimension, §1.7.

5.1.1. *Compatibility.* Our next goal is to prove that the comparison of Theorem 5.1 is compatible with the comparison of Theorem 4.1.

Theorem 5.3. *Let X be a connected, paracompact, rigid-analytic variety defined over K .*

(i) (Hyodo–Kato isomorphism over B_{dR}^+) *We have a natural isomorphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$*

$$R\Gamma_{\mathrm{HK}}(X_C) \otimes_{\mathbb{F}}^{\mathrm{L}\square} B_{\mathrm{dR}}^+ \xrightarrow{\sim} R\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} B_{\mathrm{dR}}^+. \quad (5.2)$$

(ii) (Compatibility) *The isomorphism (5.1) of Theorem 5.1 is compatible with the isomorphism (4.1) of Theorem 4.1, i.e. we have a commutative diagram as follows*

$$\begin{array}{ccc} R\Gamma_B(X_C) & \xrightarrow{\sim} & (R\Gamma_{\mathrm{HK}}(X_C) \otimes_{\mathbb{F}}^{\mathrm{L}\square} B_{\mathrm{log}})^{N=0} \\ \downarrow & & \downarrow \\ R\Gamma_{B_{\mathrm{dR}}^+}(X_C) & \xrightarrow{\sim} & R\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} B_{\mathrm{dR}}^+ \end{array}$$

where the left vertical map is induced by the inclusion $\mathbb{B} \hookrightarrow \mathbb{B}_{\mathrm{dR}}^+$, and the right vertical map is induced by (5.2).

We refer the reader to Theorem 5.20 for a version of Theorem 5.3 for rigid-analytic varieties defined over C .

5.2. **The comparison with the infinitesimal cohomology over B_{dR}^+ .** To prove Theorem 5.3 we will relate the B_{dR}^+ -cohomology of Definition 2.38 to the infinitesimal cohomology over B_{dR}^+ in a way that is compatible with the comparison between the B -cohomology and the log-crystalline cohomology over A_{cris} (following from Theorem 4.3).

5.2.1. *The infinitesimal cohomology over B_{dR}^+ and its filtrations.* We begin with some recollections on the infinitesimal cohomology over B_{dR}^+ of rigid-analytic varieties, introduced in [Guo21].

Definition 5.4 (Infinitesimal cohomology over B_{dR}^+). Let X be a rigid-analytic variety over C . Given an integer $m \geq 1$, denote $B_{\mathrm{dR},m}^+ = B_{\mathrm{dR}}^+/\xi^m$.

The *infinitesimal site* $(X/B_{\mathrm{dR},m}^+)_{\mathrm{inf}}$ of X over $B_{\mathrm{dR},m}^+$ is defined as follows:

- the underlying category has objects the pairs (U, T) where U is an open subspace of X and $U \rightarrow T$ is an infinitesimal thickening of adic spaces with T an adic space of topological finite presentation over $B_{\mathrm{dR},m}^+$, and morphisms $(U, T) \rightarrow (U', T')$ with $U \rightarrow U'$ an open immersion and $T \rightarrow T'$ a compatible map of adic spaces over $B_{\mathrm{dR},m}^+$;
- the coverings are given by the families of morphisms $\{(U_i, T_i) \rightarrow (U, T)\}$ with $U_i \rightarrow U$ and $T_i \rightarrow T$ coverings for the analytic topology.

The *infinitesimal site* of X over B_{dR}^+ is defined as the direct limit of sites (in the sense of [AGV71, Exposé VI, §8])

$$(X/B_{\mathrm{dR}}^+)_{\mathrm{inf}} := \varinjlim_m (X/B_{\mathrm{dR},m}^+)_{\mathrm{inf}}.$$

The *infinitesimal structure sheaf* $\mathcal{O}_{\mathrm{inf}}$ on $(X/B_{\mathrm{dR}}^+)_{\mathrm{inf}}$ is the sheaf with values in $\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{cond}}$ sending (U, T) to $\mathcal{O}_T(T)$, and the *infinitesimal cohomology* of X over B_{dR}^+ is defined as

$$R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) := R\Gamma((X/B_{\mathrm{dR}}^+)_{\mathrm{inf}}, \mathcal{O}_{\mathrm{inf}}) \in D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{cond}}).$$

One can also define a *big infinitesimal site* version of the (small) infinitesimal sites defined above (cf. [Guo21, Definition 2.1.2]).

Definition 5.5 (Infinitesimal filtration). Let X be a rigid-analytic variety over C . We define the *infinitesimal filtration on the infinitesimal cohomology of X over B_{dR}^+* as the \mathbb{N}^{op} -indexed filtration

$$\mathrm{Fil}_{\mathrm{inf}}^{\star} R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) := R\Gamma((X/B_{\mathrm{dR}}^+)_{\mathrm{inf}}, \mathcal{J}_{\mathrm{inf}}^{\star})$$

induced on the i -th level by the subsheaf $\mathcal{J}_{\mathrm{inf}}^i \subset \mathcal{O}_{\mathrm{inf}}$ on $(X/B_{\mathrm{dR}}^+)_{\mathrm{inf}}$, where $\mathcal{J}_{\mathrm{inf}}$ is the kernel ideal of the natural map from the infinitesimal structure sheaf $\mathcal{O}_{\mathrm{inf}}$ to the pullback, on the infinitesimal site, of the analytic structure sheaf of X .

We recall that the infinitesimal cohomology over B_{dR}^+ satisfies $\acute{\mathrm{e}}\mathrm{h}$ -hyperdescent:

Lemma 5.6. *The presheaf*

$$\mathrm{Rig}_C^{\mathrm{op}} \rightarrow D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{cond}}) : Y \mapsto R\Gamma_{\mathrm{inf}}(Y/B_{\mathrm{dR}}^+)$$

satisfies $\acute{\mathrm{e}}\mathrm{h}$ -hyperdescent.

Proof. The statement follows from [Guo21, Theorem 6.2.5]. \square

On the other hand, the pieces of the infinitesimal filtration on the infinitesimal cohomology over B_{dR}^+ (Definition 5.5) do not satisfy $\acute{\mathrm{e}}\mathrm{h}$ -hyperdescent in general.⁴³ For this reason, we introduce the following filtration on the infinitesimal cohomology over B_{dR}^+ that is closer to the Hodge filtration on the de Rham cohomology (Definition 2.16) and it will be crucial in the formulation of the semistable conjecture for proper (possibly singular) rigid-analytic varieties over C (see Theorem 7.4). The following definition is based on Proposition 2.14 (and Remark 2.15).

Definition 5.7 (Hodge filtration). Let X be a rigid-analytic variety over C . We define the *Hodge filtration on the infinitesimal cohomology of X over B_{dR}^+* as the \mathbb{N}^{op} -indexed filtration

$$\mathrm{Fil}_{\mathrm{Hdg}}^{\star} R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+)$$

given on the i -th level by the cohomology on X of the hypersheaf on $\mathrm{Rig}_{C, \acute{\mathrm{e}}\mathrm{h}}$ associated to the presheaf

$$\mathrm{RigSm}_C^{\mathrm{op}} \rightarrow D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{cond}}) : Y \mapsto \mathrm{Fil}_{\mathrm{inf}}^i R\Gamma_{\mathrm{inf}}(Y/B_{\mathrm{dR}}^+).$$

In the smooth case the Hodge filtration on the infinitesimal cohomology over B_{dR}^+ (Definition 5.7) agrees with the infinitesimal filtration (Definition 5.5).

⁴³Similarly, the pieces of the infinitesimal filtration on the infinitesimal cohomology over C do satisfy $\acute{\mathrm{e}}\mathrm{h}$ -hyperdescent. In fact, supposing the contrary, by [Guo21, Theorem 1.2.1(i)] and Proposition 2.14 we would have, for any rigid-analytic variety X over C , a natural filtered isomorphism between the infinitesimal cohomology of X over C and the de Rham cohomology of X over C (Definition 2.16). Now, assuming that X is a complete intersection affinoid, there is a natural filtered isomorphism between the infinitesimal cohomology of X over C and the cohomology of the analytic derived de Rham complex of X over C , [Guo21, Corollary 5.5.2]; but, the graded pieces of the latter do not vanish if X is not smooth (recalling that the i -th graded piece of the analytic derived de Rham complex of X over C can be identified with a shift of the i -fold wedge product of the analytic cotangent complex of X over C), instead, by Proposition 2.18, the graded pieces of the de Rham cohomology of X over C eventually vanish.

Proposition 5.8. (Hodge filtration in the smooth case) *Let X be a smooth rigid-analytic variety over C . We have a natural isomorphism of filtered objects*

$$\mathrm{Fil}_{\mathrm{inf}}^* R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \xrightarrow{\sim} \mathrm{Fil}_{\mathrm{Hdg}}^* R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+).$$

We will prove Proposition 5.8 in the next subsection, together with the following already announced comparison of the B_{dR}^+ -cohomology with the infinitesimal cohomology over B_{dR}^+ .

Theorem 5.9. *For any rigid-analytic variety X over C , we have a natural isomorphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{cond}})$*

$$R\Gamma_{B_{\mathrm{dR}}^+}(X) \simeq R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \quad (5.3)$$

compatible with filtrations, endowing the B_{dR}^+ -cohomology with the filtration décalée (Definition 2.9) and the infinitesimal cohomology over B_{dR}^+ with the Hodge filtration (Definition 5.7).

5.2.2. *Proofs.* We want to prove Theorem 5.3, Proposition 5.8 and Theorem 5.9.

As a first step toward Theorem 5.3, we need to construct a natural map from the log-crystalline cohomology over A_{cris} to the infinitesimal cohomology over B_{dR}^+ .

Lemma 5.10. *Let \mathfrak{X} be a semistable p -adic formal scheme over \mathcal{O}_C , and let $X = \mathfrak{X}_C$ denote its generic fiber. Then, there exists a natural morphism*

$$R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times) \rightarrow R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+). \quad (5.4)$$

Proof. We can assume that \mathfrak{X} is affine. We note that we have a natural isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times) \simeq R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A_{\mathrm{cris}}^\times).$$

Then, it suffices to construct, for each integer $m \geq 1$, a morphism of big sites

$$f : (X/B_{\mathrm{dR},m}^+)_{\mathrm{INF}} \rightarrow (\mathfrak{X}/A_{\mathrm{cris}}^\times)_{\mathrm{CRIS}}$$

recalling that the restriction functor from the big topos to the small one preserves cohomology (see [Guo21, Corollary 2.2.8] for the infinitesimal topos). We define f via the continuous functor sending $\mathfrak{U} \rightarrow \mathrm{Spf}(A)$ in the big log-crystalline site $(\mathfrak{X}/A_{\mathrm{cris}}^\times)_{\mathrm{CRIS}}$ to $\mathfrak{U}_C \rightarrow \mathrm{Spa}(A \otimes_{A_{\mathrm{cris}}} B_{\mathrm{dR},m}^+)$ in the big infinitesimal site $(X/B_{\mathrm{dR},m}^+)_{\mathrm{INF}}$ (forgetting the log structures). One checks that f is a well-defined morphism of sites, with the help of [Sta20, 00X4, 00X5]. \square

Before proving Theorem 5.3, we will show the following intermediate compatibility result.

Proposition 5.11. *Let X be the generic fiber of a qcqs semistable formal scheme \mathfrak{X} defined over $\mathrm{Spf}(\mathcal{O}_C)$. Let $I = [1, r] \subset (0, \infty)$ be an interval with rational endpoints. Then, the isomorphism (4.2) is compatible with the isomorphism (5.3), i.e. we have a commutative diagram as follows*

$$\begin{array}{ccc} R\Gamma_{B_I}(X) & \xrightarrow{\sim} & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times) \otimes_{A_{\mathrm{cris}}}^{\mathrm{L}\blacksquare} B_I \\ \downarrow & & \downarrow \\ R\Gamma_{B_{\mathrm{dR}}^+}(X) & \xrightarrow{\sim} & R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \end{array}$$

where the left vertical map is induced by the inclusion $\mathbb{B}_I \hookrightarrow \mathbb{B}_{\mathrm{dR}}^+$, and the right vertical map is induced by the morphism (5.4) constructed in Lemma 5.10.

To show Proposition 5.11, we will prove Theorem 5.9 going over the same steps as in the proof of Theorem 4.3. We begin with the first step, corresponding to Lemma 4.9.

Lemma 5.12. *In the setting of Notation 4.4, for any integer $m \geq 1$, we have a $B_{\mathrm{dR},m}^+$ -linear quasi-isomorphism*

$$\mathrm{Kos}_{A_{\mathrm{inf}}(R)}(\partial_1, \dots, \partial_d) \otimes_{A_{\mathrm{inf}}}^L B_{\mathrm{dR}}^+/\xi^m \xrightarrow{\sim} \mathrm{Kos}_{B_{\mathrm{dR}}^+/\xi^m(R)}(\partial_1, \dots, \partial_d) \xrightarrow{\sim} L\eta_t \mathrm{Kos}_{(\mathbb{B}_{\mathrm{dR}}^+/\mathrm{Fil}^m)(R_\infty)}(\gamma_1-1, \dots, \gamma_d-1)$$

compatible with the quasi-isomorphism (4.12).

Proof. Since μ and t differ by a unit in B_{dR}^+/ξ^m , as in the proof of Lemma 4.9 we can reduce to showing that the element μ kills $H_{\mathrm{cond}}^i(\Gamma, \mathbb{A}_{\mathrm{inf}}(R_\infty)^{\mathrm{n-int}} \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+/\xi^m)$ for all $i \in \mathbb{Z}$. Let $N_\infty := \mathbb{A}_{\mathrm{inf}}(R_\infty)^{\mathrm{n-int}}$. We proceed by induction on $m \geq 1$. Since ξ is a non-zero-divisor in $\mathbb{A}_{\mathrm{inf}}(R_\infty) \supset N_\infty$, we have the following exact sequence

$$0 \rightarrow N_\infty \otimes_{A_{\mathrm{inf}}} C(m) \rightarrow N_\infty \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+/\xi^{m+1} \rightarrow N_\infty \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+/\xi^m \rightarrow 0$$

which allows us to reduce to the case $m = 1$. Then, it suffices to show that the element $\varepsilon_p - 1$ kills $H_{\mathrm{cond}}^i(\Gamma, \widehat{\mathcal{O}}^+(R_\infty)^{\mathrm{n-int}})$ for all $i \in \mathbb{Z}$, which follows from [Sch13a, Lemma 5.5] (and [Bos21, Proposition B.3]). \square

The following byproduct of Lemma 5.12 will be useful later on.

Corollary 5.13. *In the setting of Notation 4.4, for any $i \geq 0$, we have a B_{dR}^+ -linear quasi-isomorphism*

$$\xi^{\max(i-\bullet, 0)} \Omega_{B_{\mathrm{dR}}^+(R)}^\bullet \xrightarrow{\sim} \mathrm{Fil}^i R\Gamma_{B_{\mathrm{dR}}^+}(X) \quad (5.5)$$

where $\Omega_{B_{\mathrm{dR}}^+(R)}^\bullet := \mathrm{Kos}_{B_{\mathrm{dR}}^+(R)}(\partial_1, \dots, \partial_d)$.

Proof. The statement follows combining Lemma 4.8, Lemma 5.12, and Lemma 2.41. In fact, by induction on $i \geq 0$, we can reduce to showing (5.5) on graded pieces. \square

As done in §4.1.3, in order to construct a functorial isomorphism, we need to introduce more general coordinates. For this, we resume here the setting of [CK19, §6.3].⁴⁴

Notation 5.14.

- Let $X = \mathrm{Spa}(A, A^\circ)$ an affinoid space over $\mathrm{Spa}(C, \mathcal{O}_C)$ that is the base change to $\mathrm{Spa}(C, \mathcal{O}_C)$ of an affinoid space $X_0 = \mathrm{Spa}(A_0, A_0^\circ)$ defined over $\mathrm{Spa}(L, \mathcal{O}_L)$, for some finite subextension $\check{F} \subset L \subset C$, and admitting an étale $\mathrm{Spa}(L, \mathcal{O}_L)$ -morphism

$$X_0 \rightarrow \mathrm{Spa}(L\langle T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - p^q), \mathcal{O}_L\langle T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - p^q))$$

for some $0 \leq r \leq d$, and $q \in \mathbb{Q}_{>0}$.

Assume, in addition, that there are finite subsets $\Psi_0 \subset (A_0^\circ)^\times$ and $\Xi_0 \subset A_0^\circ \cap A_0^\times$ such that the L -linear map

$$L\langle (X_u^{\pm 1})_{u \in \Psi_0}, (X_a)_{a \in \Xi_0} \rangle \rightarrow A_0, \quad X_u \mapsto u, X_a \mapsto a, \quad (5.6)$$

is surjective. In particular, there are finite subsets $\Psi \subset (A^\circ)^\times$ and $\Xi \subset A^\circ \cap A^\times$ such that the C -linear map

$$C\langle (X_u^{\pm 1})_{u \in \Psi}, (X_a)_{a \in \Xi} \rangle \rightarrow A, \quad X_u \mapsto u, X_a \mapsto a, \quad (5.7)$$

⁴⁴We warn the reader that our notation slightly differs from *loc. cit.*

is surjective.⁴⁵

- We consider the affinoid perfectoid cover

$$C\langle (X_u^{\pm 1})_{u \in \Psi}, (X_a)_{a \in \Xi} \rangle \rightarrow C\langle (X_u^{\pm 1/p^\infty})_{u \in \Psi}, (X_a^{1/p^\infty})_{a \in \Xi} \rangle \quad (5.8)$$

with Galois group

$$\Gamma_{\Psi, \Xi} := \prod_{\Psi \sqcup \Xi} \mathbb{Z}_p(1) \cong \mathbb{Z}_p^{|\Psi \sqcup \Xi|}.$$

We denote by $(\gamma_u)_{u \in \Psi \sqcup \Xi}$ the canonical generators of $\Gamma_{\Psi, \Xi}$.

The base change of (5.8) along the surjection (5.7) gives the following affinoid perfectoid pro-étale cover of X

$$X_{\Psi, \Xi, \infty} := \mathrm{Spa}(A_{\Psi, \Xi, \infty}, A_{\Psi, \Xi, \infty}^+) \rightarrow X$$

with Galois group $\Gamma_{\Psi, \Xi}$.

- Given \mathbb{M} any pro-étale period sheaf of §2.3.1, we set

$$\mathbb{M}(A_{\Psi, \Xi, \infty}^+) := \mathbb{M}(X_{\Psi, \Xi, \infty})$$

and we regard it as a condensed ring.

Remark 5.15. As observed in [CK19, §6.2 and §6.3], for any smooth rigid-analytic variety X over C , the affinoid spaces $\mathrm{Spa}(A, A^\circ)$ of Notation 5.14 form an basis for the analytic topology of X .

The following result should be regarded as the analogue of Lemma 4.14 over B_{dR}^+ .

Lemma 5.16. *In the setting of Notation 5.14, we have a filtered quasi-isomorphism*

$$R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \simeq \Omega_{D_{\Psi, \Xi}(A)/B_{\mathrm{dR}}^+}^\bullet := \mathrm{Kos}_{D_{\Psi, \Xi}(A)}((\partial_u)_{u \in \Psi}, (\partial_a)_{a \in \Xi}) \quad (5.9)$$

where $D_{\Psi, \Xi}(A) = \varprojlim_{m \geq 1} D_{\Psi, \Xi, m}(A)$, and, for each $m \geq 1$, $D_{\Psi, \Xi, m}(A)$ is the B_{dR}^+/ξ^m -algebra representing the envelope of

$$\mathrm{Spa}(A) \hookrightarrow \mathrm{Spa}(B_{\mathrm{dR}}^+/\xi^m \langle (X_u^{\pm 1})_{u \in \Psi}, (X_a)_{a \in \Xi} \rangle) \text{ over } \mathrm{Spa}(C) \hookrightarrow \mathrm{Spa}(B_{\mathrm{dR}}^+/\xi^m). \quad (5.10)$$

Here, $\partial_u := \frac{\partial}{\partial \log(X_u)} = X_u \cdot \frac{\partial}{\partial X_u}$ for $a \in \Psi \cup \Xi$, and the right-hand side of (5.9) is endowed with the infinitesimal filtration, defined on the i -th level, for $i \geq 0$, as follows:

$$\mathrm{Fil}^i \Omega_{D_{\Psi, \Xi}(A)/B_{\mathrm{dR}}^+}^\bullet := J^{\max(i-\bullet, 0)} \Omega_{D_{\Psi, \Xi}(A)/B_{\mathrm{dR}}^+}^\bullet \quad (5.11)$$

where $J := \varprojlim_m J_m$ with J_m the ideal corresponding to the closed immersion (5.10).

Proof. The statement follows from [Guo21, Theorem 4.1.1, Theorem 7.2.3]. \square

Keeping the notation of the above lemma, we state the following result.

Lemma 5.17. *In the setting of Notation 5.14, for any sufficiently large Ψ and Ξ , we have an isomorphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$*

$$\Omega_{D_{\Psi, \Xi}(A)/B_{\mathrm{dR}}^+}^\bullet \simeq \Omega_{A_0/K}^\bullet \otimes_L^\square B_{\mathrm{dR}}^+ \quad (5.12)$$

compatible with filtrations, where the left-hand side is endowed with the infinitesimal filtration (5.11), and the right-hand side is endowed with the tensor product filtration.

⁴⁵The descent (5.6) of (5.7) to L is needed in [CK19, Lemma 6.3.8], which we will use in Lemma 5.17 below.

Proof. We consider the B_{dR}^+ -algebra $\mathcal{O}D_{\Psi,\Xi}(A)$ defined as the completion of

$$(A_0 \otimes_L^{\blacksquare} B_{\text{dR}}^+) \otimes_{B_{\text{dR}}^+}^{\blacksquare} D_{\Psi,\Xi}(A) \rightarrow A_0$$

along its kernel. Then, we have the following natural

$$D_{\Psi,\Xi}(A) \rightarrow \mathcal{O}D_{\Psi,\Xi}(A) \leftarrow A_0 \otimes_L^{\blacksquare} B_{\text{dR}}^+ \quad (5.13)$$

and, denoting by $\Omega_{\mathcal{O}D_{\Psi,\Xi}(A)/B_{\text{dR}}^+}^{\bullet}$ the de Rham complex associated to $\mathcal{O}D_{\Psi,\Xi}(A)$ over B_{dR}^+ , we consider the natural maps of complexes

$$\Omega_{D_{\Psi,\Xi}(A)/B_{\text{dR}}^+}^{\bullet} \rightarrow \Omega_{\mathcal{O}D_{\Psi,\Xi}(A)/B_{\text{dR}}^+}^{\bullet} \leftarrow \Omega_{A_0/K}^{\bullet} \otimes_L^{\blacksquare} B_{\text{dR}}^+. \quad (5.14)$$

We claim that the zigzag (5.14) is a filtered quasi-isomorphism. As in [BMS18, Lemma 13.13], it suffices to check that, for sufficiently large Ψ and Ξ , we have

$$D_{\Psi,\Xi}(A) \cong (A_0 \otimes_L^{\blacksquare} B_{\text{dR}}^+) \llbracket (X_u - \tilde{u})_{u \in (\Psi \sqcup \Xi) \setminus \{T_1, \dots, T_d\}} \rrbracket, \quad \mathcal{O}D_{\Psi,\Xi}(A) \cong (A_0 \otimes_L^{\blacksquare} B_{\text{dR}}^+) \llbracket (X_u - \tilde{u})_{u \in \Psi \sqcup \Xi} \rrbracket.$$

The first isomorphism follows from [CK19, Lemma 6.3.8], observing that the completed tensor product of Tate L -algebras, appearing in *loc. cit.*, agrees with the solid tensor product \otimes_L^{\blacksquare} . The second isomorphism follows from the first one, using that the completion of $A_0 \otimes_L^{\blacksquare} A_0 \rightarrow A_0$ along its kernel is isomorphic to $A_0 \llbracket (u \otimes 1 - 1 \otimes u)_{u \in \{T_1, \dots, T_d\}} \rrbracket$. \square

We proceed by constructing *in coordinates* a natural map from the log-crystalline cohomology over A_{cris} to the infinitesimal cohomology over B_{dR}^+ , with an eye to the compatibility between (4.2) and (5.3) that we want to prove. For this, we first need to relate the setting of Notation 4.12 to the one of Notation 5.14.

Notation 5.18. In the setting of Notation 4.12, we put $\text{Spa}(A, A^\circ) = X := \mathfrak{X}_C$, and

$$\Psi := \{t_\sigma\}_{\sigma \in \Sigma} \cup \bigcup_{\lambda \in \Lambda} \{t_{\lambda, r_\lambda + 1}, \dots, t_{\lambda, d}\}, \quad \Xi := \bigcup_{\lambda \in \Lambda} \{t_{\lambda, 1}, \dots, t_{\lambda, r_\lambda}\}.$$

These choices satisfy the assumptions of Notation 5.14, therefore in the following we can retain the notation of *loc. cit.* using such choices.

Lemma 5.19. *In the setting of Notation 5.18, there is a natural morphism*

$$\Omega_{D_{\Sigma, \Lambda}(R)/A_{\text{cris}}}^{\bullet} \rightarrow \Omega_{D_{\Psi, \Xi}(A)/B_{\text{dR}}^+}^{\bullet}$$

which, under the isomorphisms (4.22) and (5.9), is compatible with the morphism (5.4) constructed in Lemma 5.10.

Proof. We need to construct, in the setting of Notation 5.18, for each integer $m \geq 1$, a natural map

$$D_{\Sigma, \Lambda}(R) \rightarrow D_{\Psi, \Xi, m}(A) \quad (5.15)$$

compatible with the natural maps to A .⁴⁶ By [CK19, §6.5], there exist a p -adically complete ring of definition $D_{\Psi, \Xi, m}(A)_0$ of $D_{\Psi, \Xi, m}(A)$, and a commutative diagram of log rings as follows

$$\begin{array}{ccc} (A_{\text{inf}, \Sigma, \Lambda}^{\square} \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n, M_{\text{inf}, \Sigma, \Lambda}^{\square}) & \twoheadrightarrow & (R/p, M_R) \\ \downarrow & & \parallel \\ (D_{\Psi, \Xi, m}(A)_0/p^n, N)^a & \twoheadrightarrow & (R/p, M_R). \end{array}$$

⁴⁶This is done in [CK19, §6.5], which we rephrase here in a slightly different way, proceeding as in the proof of Lemma 4.17.

Here, R/p is equipped with the pullback of the canonical log structure M_R on R , and the ring $A_{\text{inf},\Sigma,\Lambda}^{\square} \otimes_{A_{\text{inf}}} A_{\text{cris}}/p^n$ is endowed with the pullback of the log structure $M_{\text{inf},\Sigma,\Lambda}^{\square}$ on $A_{\text{inf},\Sigma,\Lambda}^{\square}$. Moreover, $D_{\Psi,\Xi,m}(A)_0/p^n$ is equipped with the pullback of the log structure on $D_{\Psi,\Xi,m}(A)_0$ associated to the following pre-log structure: as in Lemma 4.17, we set $N := (h^{\text{gp}})^{-1}(M_{\text{inf},\Sigma,\Lambda}^{\square})$ where h^{gp} denotes the morphism of groups associated to the natural morphism of monoids $h : M_{\text{inf},\Sigma,\Lambda}^{\square} \rightarrow M_R$; the argument of [CK19, §6.5] shows that the natural map $M_{\text{inf},\Sigma,\Lambda}^{\square} \rightarrow D_{\Psi,\Xi,m}(A)_0$ uniquely extends to a map $N \rightarrow D_{\Psi,\Xi,m}(A)_0$. The resulting surjective map of log rings at the bottom of the diagram (4.32) is exact by construction, hence, the universal property of the log PD envelope $D_{\Sigma,\Lambda}(R)/p^n$ gives the desired natural map $D_{\Sigma,\Lambda}(R)/p^n \rightarrow D_{\Psi,\Xi,m}(A)_0/p^n$. Since both $D_{\Psi,\Xi,m}(A)_0$ and $D_{\Sigma,\Lambda}(R)$ are p -adically complete (for the latter, recall [CK19, Lemma 5.29]), we obtain a map (5.15) as desired. \square

We can finally prove the main results of this section.

Proof of Theorem 5.9. The argument will be analogous to the one in the proof of Theorem 4.3, in order to prove the compatibility stated in Proposition 5.11.

As both the B_{dR}^+ -cohomology, together with its filtration decalée, and the infinitesimal cohomology over B_{dR}^+ , together with its Hodge filtration, satisfy éh-hyperdescent (for the infinitesimal cohomology, see Lemma 5.6 and Definition 5.7), it suffices to prove the statement éh-locally on X in a functorial way. Thus, by Proposition 3.10, we can place ourselves in the setting of Notation 5.18, with sufficiently large Ψ and Ξ .

Fix $m \geq 1$. As in Lemma 4.8, we have a quasi-isomorphism

$$R\Gamma(\mathfrak{X}_C, L\eta_t R\nu_*(\mathbb{B}_{\text{dR}}^+/\text{Fil}^m)) \simeq L\eta_t \text{Kos}_{(\mathbb{B}_{\text{dR}}^+/\text{Fil}^m)(A_{\Psi,\Xi,\infty}^+)}((\gamma_u - 1)_{u \in \Psi \sqcup \Xi}). \quad (5.16)$$

Next, we claim that we have a commutative diagram as follows, whose arrows are natural quasi-isomorphisms

$$\begin{array}{ccc} \text{Kos}_{A_{\text{cris}}(R)_\lambda}((\partial_{\lambda,i})_{1 \leq i \leq d}) \otimes_{A_{\text{cris}}^{\text{L}\blacksquare}} B_{\text{dR}}^+/\xi^m & \longrightarrow & \eta_t \text{Kos}_{(\mathbb{B}_{\text{dR}}^+/\text{Fil}^m)(R_{\lambda,\infty})}((\gamma_{\lambda,i} - 1)_{1 \leq i \leq d}) \\ \downarrow & & \downarrow \\ \text{Kos}_{D_{\Psi,\Xi,m}(A)}((\partial_u)_{u \in \Psi \sqcup \Xi}) & \longrightarrow & \eta_t \text{Kos}_{(\mathbb{B}_{\text{dR}}^+/\text{Fil}^m)(A_{\Psi,\Xi,\infty}^+)}((\gamma_u - 1)_{u \in \Psi \sqcup \Xi}). \end{array} \quad (5.17)$$

- (i) The right vertical map of (5.17) is a quasi-isomorphism since, by Lemma 4.8, both the target and the source are quasi-isomorphic to the complex $R\Gamma(\mathfrak{X}_C, L\eta_t R\nu_*(\mathbb{B}_{\text{dR}}^+/\text{Fil}^m))$.
- (ii) The top horizontal arrow of (5.17) is the quasi-isomorphism obtained combining Lemma 5.12 with (4.37).
- (iii) The left vertical map of (5.17) is induced by the one constructed in Lemma 5.19. We observe that both the target and the source of this map are derived ξ -adically complete.⁴⁷ Then, by the derived Nakayama lemma, it suffices to show that such a map is a quasi-isomorphism for $m = 1$. In this case, both the target and the source compute the de Rham cohomology $R\Gamma_{\text{dR}}(\mathfrak{X}_C)$. The former follows reducing the quasi-isomorphism (5.12) mod ξ . For the latter, we observe that (for $m = 1$) by [CK19, Proposition 5.13], and Proposition A.3, the source

⁴⁷For the latter, one can use for example the quasi-isomorphism of part (ii) combined with [BS15, Proposition 3.4.4, Lemma 3.4.14].

computes $(R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^\times) \widehat{\otimes}_{A_{\text{cris}}}^L \mathcal{O}_C)_{\mathbb{Q}_p}$, and by [Bei13, (1.11.1) and (1.8.1)] we have a quasi-isomorphism

$$(R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}^\times) \widehat{\otimes}_{A_{\text{cris}}}^L \mathcal{O}_C)_{\mathbb{Q}_p} \xrightarrow{\sim} R\Gamma_{\log \text{dR}}(\mathfrak{X})_{\mathbb{Q}_p} \xrightarrow{\sim} R\Gamma_{\text{dR}}(\mathfrak{X}_C).$$

(iv) To construct the bottom horizontal arrow of (5.17), we first note that, similarly to (4.39), we have a quasi-isomorphism

$$\text{Kos}_{D_{\Psi, \Xi, m}(A)}((\partial_u)_{u \in \Psi \sqcup \Xi}) \xrightarrow{\sim} \eta_t \text{Kos}_{D_{\Psi, \Xi, m}(A)}((\gamma_u - 1)_{u \in \Psi \sqcup \Xi}). \quad (5.18)$$

Next, we define a natural $\Gamma_{\Psi, \Xi}$ -equivariant B_{dR}^+/ξ^m -linear map

$$D_{\Psi, \Xi, m}(A) \rightarrow (B_{\text{dR}}^+/\text{Fil}^m)(A_{\Psi, \Xi, \infty}^+) \quad (5.19)$$

via sending X_u to $[(u, u^{1/p}, \dots)]$, which induces a morphism

$$\text{Kos}_{D_{\Psi, \Xi, m}(A)}((\gamma_u - 1)_{u \in \Psi \sqcup \Xi}) \rightarrow \eta_t \text{Kos}_{(B_{\text{dR}}^+/\text{Fil}^m)(A_{\Psi, \Xi, \infty}^+)}((\gamma_u - 1)_{u \in \Psi \sqcup \Xi}). \quad (5.20)$$

Then, we define the bottom horizontal arrow of (5.17) as the composite of (5.18) and (5.20). The so constructed map makes the diagram (5.17) commute and, by the previous points, it is a quasi-isomorphism.

Now, taking the filtered colimit $\varinjlim_{\Psi, \Xi}$, over all sufficiently large Ψ and Ξ , of the constructed bottom horizontal quasi-isomorphism of (5.17), and then passing to the limit $R\varprojlim_m$, using the quasi-isomorphism (5.16) combined with Lemma 2.41, and recalling Lemma 5.16, we obtain the desired quasi-isomorphism: such morphism is functorial since taking instead the filtered colimit of \varinjlim_{Ψ} , over all sufficiently large Ψ (and $\Xi = \emptyset$), of the bottom horizontal quasi-isomorphism of (5.17), we obtain the same morphism. This finishes the proof of (5.3).

For the compatibility with filtrations part, we need to use in addition the compatibility with filtration stated in Lemma 5.16, Lemma 5.17, and Corollary 5.13. \square

Proof of Proposition 5.11. By the proof of Theorem 5.9 above, we can reduce to checking the commutativity of the following diagram

$$\begin{array}{ccc} D_{\Lambda, \Sigma}(R) & \longrightarrow & \mathbb{A}_{\text{cris}}(R_{\Sigma, \Lambda, \infty}) \\ \downarrow & & \downarrow \\ D_{\Psi, \Xi, m}(A) & \longrightarrow & (B_{\text{dR}}^+/\text{Fil}^m)(A_{\Psi, \Xi, \infty}^+) \end{array}$$

where the top horizontal arrow is (4.31), the left vertical arrow is (5.15), the bottom horizontal arrow is (5.19), and the right vertical arrow is induced by the composition $A_{\text{cris}} \hookrightarrow B_{\text{dR}}^+ \twoheadrightarrow B_{\text{dR}}^+/\text{Fil}^m$. In turn, we can reduce to verifying the commutativity of the following diagram

$$\begin{array}{ccc} A_{\text{inf}, \Sigma, \Lambda}^{\square} & \longrightarrow & \mathbb{A}_{\text{inf}}(R_{\Sigma, \Lambda, \infty}) \\ \downarrow & & \downarrow \\ A_{\text{inf}}\langle (X_u^{\pm 1})_{u \in \Psi}, (X_a)_{a \in \Xi} \rangle & \longrightarrow & \mathbb{A}_{\text{inf}}(A_{\Psi, \Xi, \infty}^+). \end{array}$$

This is clear as both the composition maps from $A_{\text{inf}, \Sigma, \Lambda}^{\square}$ to $\mathbb{A}_{\text{inf}}(A_{\Psi, \Xi, \infty}^+)$ send X_τ to $[(X_\tau, X_\tau^{1/p}, \dots)]$, for any $\tau \in \Sigma \cup \{(\lambda, i) : \lambda \in \Lambda, 1 \leq i \leq d\}$. \square

Proof of Proposition 5.8. We may assume X qcqs. We want to show that, given $i \geq 0$, for any éh-hypercover $Y_\bullet \rightarrow X$ of qcqs smooth rigid-analytic varieties over C , the natural map

$$\mathrm{Fil}_{\mathrm{inf}}^i R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \rightarrow \lim_{[n] \in \Delta} \mathrm{Fil}_{\mathrm{inf}}^i R\Gamma_{\mathrm{inf}}(Y_n/B_{\mathrm{dR}}^+) \quad (5.21)$$

is an isomorphism. For this, we observe that, by the proof of Theorem 5.9 (recalling Remark 5.15), for smooth rigid-analytic varieties over C the infinitesimal filtration on the infinitesimal cohomology over B_{dR}^+ naturally identifies with the filtration décalée on the B_{dR}^+ -cohomology, and the latter satisfies éh-hyperdescent. \square

Before proving Theorem 5.3, we state and prove a version of the latter for rigid-analytic varieties defined over C .

Theorem 5.20. *Let X be a connected, paracompact, rigid-analytic variety defined over C .*

(i) (Hyodo–Kato isomorphism over B_{dR}^+) *We have a natural isomorphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$*

$$R\Gamma_{\mathrm{HK}}(X) \otimes_{\mathbb{F}}^{\mathrm{L}\square} B_{\mathrm{dR}}^+ \xrightarrow{\sim} R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+). \quad (5.22)$$

(ii) (Compatibility) *The isomorphism (5.1) of Theorem 5.1 is compatible with the isomorphism (4.1) of Theorem 4.1, i.e. we have a commutative diagram as follows*

$$\begin{array}{ccc} R\Gamma_B(X) & \xrightarrow{\sim} & (R\Gamma_{\mathrm{HK}}(X) \otimes_{\mathbb{F}}^{\mathrm{L}\square} B_{\mathrm{log}})^{N=0} \\ \downarrow & & \downarrow \\ R\Gamma_{B_{\mathrm{dR}}^+}(X) & \xrightarrow{\sim} & R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \end{array}$$

where the left vertical map is induced by the inclusion $\mathbb{B} \hookrightarrow \mathbb{B}_{\mathrm{dR}}^+$, and the right vertical map is induced by (5.22).

Proof. For part (i), as in the proof of Theorem 3.15(iii), we can reduce to showing the statement locally for X the generic fiber of $\mathfrak{X} \in \mathcal{M}_{\mathrm{ss}, \mathrm{qcqs}}$. For this, by Theorem 3.2, which applies thanks to Remark 3.9, we have a natural morphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$

$$R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}^0/\mathcal{O}_{\mathbb{F}}^0)_{\mathbb{Q}_p} \otimes_{\mathbb{F}}^{\mathrm{L}\square} B_{\mathrm{dR}}^+ \xrightarrow{\sim} R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}^\times) \otimes_{A_{\mathrm{cris}}}^{\mathrm{L}\square} B_{\mathrm{dR}}^+ \rightarrow R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \quad (5.23)$$

which is an isomorphism modulo ξ . Here, the right arrow of (5.23) is the one induced by (5.4). Since both the source and the target of (5.23) are derived ξ -adically complete, we conclude, by the derived Nakayama lemma, that (5.23) is an isomorphism, as desired.

Now, part (ii) is clear from Proposition 5.11 and the construction of (5.22) in part (i). \square

We are ready to prove Theorem 5.3.

Proof of Theorem 5.3. For part (i), we first observe that we have a natural isomorphism

$$R\Gamma_{\mathrm{inf}}(X_C/B_{\mathrm{dR}}^+) \simeq R\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} B_{\mathrm{dR}}^+. \quad (5.24)$$

For this, with an eye to the compatibility of part (ii), by the same ingredients used in the proof of Theorem 5.1, we can reduce to showing (5.24) éh-locally, using Lemma 5.16 and Lemma 5.17. Then, part (i) follows from Theorem 5.20(i)⁴⁸ combined with the isomorphism (5.24).

⁴⁸To avoid confusion, we warn the reader that in Theorem 5.3 the rigid-analytic variety X is defined over K , instead in Theorem 5.20 the rigid-analytic variety X is defined over C .

For part (ii), by Theorem 5.20(ii), we are reduced to showing the compatibility between (5.1) and (5.3), under the isomorphism (5.24). For this, in the setting of Notation 4.4, one readily reduces to check the commutativity of the following diagram

$$\begin{array}{ccccc}
 D_{\Psi, \Xi}(A) & \longrightarrow & \mathcal{O}D_{\Psi, \Xi}(A) & \longleftarrow & A_0 \otimes_K^{\blacksquare} B_{\mathrm{dR}}^+ \\
 \downarrow & & \downarrow & & \parallel \\
 \mathbb{B}_{\mathrm{dR}}^+(A_{\Psi, \Xi, \infty}^+) & \longrightarrow & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(A_{\Psi, \Xi, \infty}^+) & \longleftarrow & A_0 \otimes_K^{\blacksquare} B_{\mathrm{dR}}^+.
 \end{array} \tag{5.25}$$

Here, the top row is the zigzag (5.13) of Lemma 5.17, and the bottom row, coming from Scholze's Poincaré lemma, is constructed, in the condensed setting, in [Bos21, Corollary 6.12]. \square

6. SYNTOMIC FARGUES–FONTAINE COHOMOLOGY

In this section, we define a cohomology theory for rigid-analytic varieties over C , called *syntomic Fargues–Fontaine cohomology*, which is close in spirit to Bhatt–Morrow–Scholze's syntomic cohomology theory for smooth p -adic formal schemes over \mathcal{O}_C . In a stable range, we compare it with the rational p -adic pro-étale cohomology. Our definition is global in nature, it doesn't require neither the existence of nice formal models, nor smoothness, and it extends to coefficients. Moreover, it has a close relationship with the Fargues–Fontaine curve, as we show in §6.2.

Definition 6.1. Let X be a rigid-analytic variety over C . Let $i \geq 0$ be an integer. We define the *syntomic Fargues–Fontaine cohomology of X with coefficients in $\mathbb{Q}_p(i)$* as the complex of $D(\mathrm{Mod}_{\mathbb{Q}_p}^{\mathrm{cond}})$ given by the fiber

$$R\Gamma_{\mathrm{syn}, \mathrm{FF}}(X, \mathbb{Q}_p(i)) := \mathrm{Fil}^i R\Gamma_B(X)^{\varphi=p^i} := \mathrm{fib}(\mathrm{Fil}^i R\Gamma_B(X) \xrightarrow{\varphi^{p^{-i}-1}} \mathrm{Fil}^i R\Gamma_B(X))$$

where $R\Gamma_B(X)$ is endowed with the filtration décalée from Definition 2.9.

6.1. The comparison with the p -adic pro-étale cohomology. The announced comparison between the syntomic Fargues–Fontaine cohomology and the p -adic pro-étale cohomology will rely on the following result.

Lemma 6.2. *Let X an analytic adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$. Let $i \geq 0$ be an integer. We have the following exact sequence of sheaves on $X_{\mathrm{proét}}$*

$$0 \rightarrow \mathbb{Q}_p(i) \rightarrow \mathrm{Fil}^i \mathbb{B} \xrightarrow{\varphi^{p^{-i}-1}} \mathrm{Fil}^i \mathbb{B} \rightarrow 0 \tag{6.1}$$

where $\mathrm{Fil}^i \mathbb{B} = t^i \mathbb{B}$.

Proof. The statement follows from the combination of (2.1) and (2.2) for $i = 0$, recalling that $\varphi(t^i) = p^i t$. \square

Now, let X be a rigid-analytic variety over C . Via the exact sequence (6.1) of sheaves on the pro-étale site $X_{\mathrm{proét}}$ we have a natural morphism

$$R\Gamma_{\mathrm{syn}, \mathrm{FF}}(X, \mathbb{Q}_p(i)) \rightarrow R\Gamma_{\mathrm{proét}}(X, \mathbb{Q}_p(i)). \tag{6.2}$$

where we are implicitly using v -descent, Proposition 2.37.

Theorem 6.3. *Let X be a rigid-analytic variety over C . Let $i \geq 0$ be an integer.*

(i) The truncation $\tau^{\leq i}$ of (6.2) is an isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$, i.e. we have

$$\tau^{\leq i} R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_{\text{proét}}(X, \mathbb{Q}_p(i)).$$

(ii) We have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$

$$R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \simeq \text{fib}(R\Gamma_B(X)^{\varphi=p^i} \rightarrow R\Gamma_{B_{\text{dR}}^+}(X)/\text{Fil}^i).$$

Proof. For part (i), recalling Definition 2.9, we have a natural isomorphism

$$\tau^{\leq i} \text{Fil}^i L\eta_t R\alpha_* \mathbb{B} \xrightarrow{\sim} \tau^{\leq i} R\alpha_* \text{Fil}^i \mathbb{B}$$

which, taking cohomology, induces

$$\tau^{\leq i} \text{Fil}^i R\Gamma_B(X) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_v(X, \text{Fil}^i \mathbb{B}).$$

Then, the statement follows from the exact sequence (6.1).

For part (ii), considering the natural map of fiber sequences

$$\begin{array}{ccccc} R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) & \longrightarrow & \text{Fil}^i R\Gamma_B(X) & \xrightarrow{\varphi^{p^{-i}-1}} & \text{Fil}^i R\Gamma_B(X) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_B(X)^{\varphi=p^i} & \longrightarrow & R\Gamma_B(X) & \xrightarrow{\varphi^{p^{-i}-1}} & R\Gamma_B(X) \end{array} \quad (6.3)$$

we deduce that it suffices to construct a natural map

$$R\Gamma_{B_{\text{dR}}^+}(X)/\text{Fil}^i \rightarrow \text{fib}(R\Gamma_B(X)/\text{Fil}^i \xrightarrow{\varphi^{p^{-i}-1}} R\Gamma_B(X)/\text{Fil}^i) \quad (6.4)$$

and show that it is an isomorphism. Endowing the source, resp. the target, of (6.4) with its natural (finite) filtration, induced by the filtration décalée on $R\Gamma_{B_{\text{dR}}^+}(X)$, resp. $R\Gamma_B(X)$, we see that we can reduce to showing that, for $0 \leq j < i$, there is a natural map

$$\text{gr}^j L\eta_t R\alpha_* \mathbb{B}_{\text{dR}}^+ \rightarrow \text{fib}(\text{gr}^j L\eta_t R\alpha_* \mathbb{B} \xrightarrow{\varphi^{p^{-i}-1}} \text{gr}^j L\eta_t R\alpha_* \mathbb{B})$$

that is an isomorphism, or equivalently, applying Proposition 2.7(ii), that there is a natural map

$$\tau^{\leq j} R\alpha_* \text{gr}^j \mathbb{B}_{\text{dR}}^+ \rightarrow \text{fib}(\tau^{\leq j} R\alpha_* \text{gr}^j \mathbb{B} \xrightarrow{\varphi^{p^{-i}-1}} \tau^{\leq j} R\alpha_* \text{gr}^j \mathbb{B}) \quad (6.5)$$

that is an isomorphism. For this, combining the exact sequences of sheaves (2.1), (2.2), and (6.1), we have the following exact sequence of sheaves on $X_{\text{proét}}$

$$0 \rightarrow \mathbb{B}_{\text{dR}}^+ / \text{Fil}^i \mathbb{B}_{\text{dR}}^+ \rightarrow \mathbb{B} / \text{Fil}^i \mathbb{B} \xrightarrow{\varphi^{p^{-i}-1}} \mathbb{B} / \text{Fil}^i \mathbb{B} \rightarrow 0 \quad (6.6)$$

where $\text{Fil}^i \mathbb{B}_{\text{dR}}^+ = t^i \mathbb{B}_{\text{dR}}^+$ and $\text{Fil}^i \mathbb{B} = t^i \mathbb{B}$. Then, observing that (6.6) is compatible with filtrations, we have, for $0 \leq j < i$, the following exact sequence of sheaves on $X_{\text{proét}}$

$$0 \rightarrow \text{gr}^j \mathbb{B}_{\text{dR}}^+ \rightarrow \text{gr}^j \mathbb{B} \xrightarrow{\varphi^{p^{-i}-1}} \text{gr}^j \mathbb{B} \rightarrow 0. \quad (6.7)$$

Now, the exact sequence (6.7) induces a natural map (6.5) as desired, that we want to prove to be an isomorphism. Thus, we want to show that the map $\varphi^{p^{-i}-1} : R^j \alpha_* \text{gr}^j \mathbb{B} \rightarrow R^j \alpha_* \text{gr}^j \mathbb{B}$ is surjective, or equivalently that the map $R^{j+1} \alpha_* \text{gr}^j \mathbb{B}_{\text{dR}}^+ \rightarrow R^{j+1} \alpha_* \text{gr}^j \mathbb{B}$, induced by (6.7), is injective: up to

twisting, it suffices to observe that, for $j = 0$, the left map in (6.7) is given by the inclusion in the direct product (recall (2.13))

$$\mathrm{gr}^0 \mathbb{B}_{\mathrm{dR}}^+ \hookrightarrow \mathrm{gr}^0 \mathbb{B} = \prod_{y \in |Y_{\mathrm{FF}}|^{\mathrm{cl}}} \mathbb{B}_{\mathrm{dR}}^+ / t^{\mathrm{ord}_y(t)} \mathbb{B}_{\mathrm{dR}}^+$$

corresponding to the classical point $V(\xi) \in |Y_{\mathrm{FF}}|^{\mathrm{cl}}$. This concludes the proof of part (ii). \square

6.2. Fargues–Fontaine cohomology and nuclear complexes on the curve. In this subsection, we first reinterpret the main comparison theorems proved in the previous sections in terms of the Fargues–Fontaine curve (Theorem 6.17): such results were conjectured by Le Bras, and proven by him in some special cases, cf. [LB18b, Remark 1.2, Conjecture 6.3], and are related to work of Le Bras–Vezzani, [LBV21]. Then, in Theorem 6.19, we naturally attach to any qcqs rigid-analytic variety over C a quasi-coherent complex on the Fargues–Fontaine curve FF , whose cohomology is the syntomic Fargues–Fontaine cohomology (Definition 6.1); we conclude by showing that in the proper case such complex is perfect.

6.2.1. Quasi-coherent, nuclear, and perfect complexes on FF . We will rely on results of Clausen–Scholze, [CS20a], [CS], and Andreychev, [And21], to talk about quasi-coherent, nuclear, and perfect complexes on the adic Fargues–Fontaine curve FF .

Let us start by recalling some notations from §A.3.

Notation 6.4. Given a condensed ring R , we denote by $\mathrm{Perf}_R \subset D(\mathrm{Mod}_R^{\mathrm{cond}})$ the ∞ -subcategory of perfect complexes over R , [And21, Definition 5.1].

Given a pair (A, A^+) with A a complete Huber ring and A^+ a subring of A° , we denote by $(A, A^+)_{\blacksquare}$ the associated analytic ring, [And21, §3.3]. We write $D((A, A^+)_{\blacksquare})$ for the derived ∞ -category of $(A, A^+)_{\blacksquare}$ -complete complexes, and $\mathrm{Nuc}((A, A^+)_{\blacksquare})$ for the ∞ -subcategory of nuclear complexes. Note that, in the case $A^+ = \mathbb{Z}$, we have $D((A, \mathbb{Z})_{\blacksquare}) = D(\mathrm{Mod}_A^{\mathrm{solid}})$ in the notation of §1.7.

Given Y an analytic adic space, we denote by $\mathrm{QCoh}(Y)$ the ∞ -category of quasi-coherent complexes on Y , we write $\mathrm{Nuc}(Y)$ for the ∞ -subcategory of nuclear complexes on Y , and $\mathrm{Perf}(Y)$ for the ∞ -category of perfect complexes on Y , §A.3.

The following construction will be used to define a lift of the B -cohomology theory to a cohomology theory with values in the ∞ -category of quasi-coherent complexes on the adic Fargues–Fontaine curve $\mathrm{FF} = Y_{\mathrm{FF}}/\varphi^{\mathbb{Z}}$, called *Fargues–Fontaine cohomology*.

Construction 6.5. We write

$$Y_{\mathrm{FF}} = \bigcup_{I \subset (0, \infty)} Y_{\mathrm{FF}, I}$$

with $Y_{\mathrm{FF}, I} = \mathrm{Spa}(B_I, B_I^+)$ for varying $I \subset (0, \infty)$ compact intervals with rational endpoints.

By [And21, Theorem 3.27], we have natural maps of analytic rings

$$(B_I, \mathbb{Z})_{\blacksquare} \rightarrow (B_I, B_I^+)_{\blacksquare}$$

for varying $I \subset (0, \infty)$ compact intervals with rational endpoints. Such maps induce base change functors

$$- \otimes_{(B_I, \mathbb{Z})_{\blacksquare}}^{\mathbb{L}} (B_I, B_I^+)_{\blacksquare} : D((B_I, \mathbb{Z})_{\blacksquare}) \rightarrow D((B_I, B_I^+)_{\blacksquare})$$

and then, by analytic descent for quasi-coherent complexes, Theorem A.20, they induce a functor

$$\mathrm{CoAd}(\mathrm{Solid}_B) := \lim_{I \subset (0, \infty)} D((B_I, \mathbb{Z})_{\blacksquare}) \longrightarrow \lim_{I \subset (0, \infty)} \mathrm{QCoh}(Y_{\mathrm{FF}, I}) = \mathrm{QCoh}(Y_{\mathrm{FF}}) \quad (6.8)$$

with source the ∞ -category of *coadmissible solid modules over B* ,⁴⁹ and target the ∞ -category of quasi-coherent complexes on Y_{FF} .

In order to move to the study of quasi-coherent complexes on the Fargues–Fontaine curve, we need to make the following definitions.

Definition 6.6. We define the ∞ -category of *quasi-coherent φ -complexes over Y_{FF}* as the equalizer

$$\mathrm{QCoh}(Y_{\mathrm{FF}})^{\varphi} := \mathrm{eq}(\mathrm{QCoh}(Y_{\mathrm{FF}}) \xrightarrow[\mathrm{id}]{\varphi^*} \mathrm{QCoh}(Y_{\mathrm{FF}}))$$

that is the ∞ -category of the pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$, where \mathcal{E} is a quasi-coherent complex on Y_{FF} , and $\varphi_{\mathcal{E}} : \varphi^* \mathcal{E} \simeq \mathcal{E}$ is an isomorphism.

We define $\mathrm{Nuc}(Y_{\mathrm{FF}})^{\varphi}$ (resp. $\mathrm{Perf}(Y_{\mathrm{FF}})^{\varphi}$) as the full ∞ -subcategory of $\mathrm{QCoh}(Y_{\mathrm{FF}})^{\varphi}$ spanned by the pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$, with \mathcal{E} a nuclear (resp. perfect) complex on Y_{FF} .

Definition 6.7. We define the ∞ -category of *coadmissible solid φ -modules over B* as the equalizer

$$\mathrm{CoAd}(\mathrm{Solid}_B)^{\varphi} := \mathrm{eq}(\mathrm{CoAd}(\mathrm{Solid}_B) \xrightarrow[\mathrm{id}]{\varphi^*} \mathrm{CoAd}(\mathrm{Solid}_B))$$

that is the ∞ -category of the pairs $(\mathcal{M}, \varphi_{\mathcal{M}})$, where \mathcal{M} is a coadmissible solid module over B , and $\varphi_{\mathcal{M}} : \varphi^* \mathcal{M} \simeq \mathcal{M}$ is an isomorphism.

We define $\mathrm{CoAd}(\mathrm{Nuc}_B)^{\varphi}$ (resp. $\mathrm{CoAd}(\mathrm{Perf}_B)^{\varphi}$) the ∞ -category of *coadmissible nuclear* (resp. *perfect*) φ -modules over B as the full ∞ -subcategory of $\mathrm{CoAd}(\mathrm{Solid}_B)^{\varphi}$ spanned by the pairs $(\mathcal{M}, \varphi_{\mathcal{M}})$ with \mathcal{M} in $\lim_{I \subset (0, \infty)} \mathrm{Nuc}((B_I, \mathbb{Z})_{\blacksquare})$ (resp. in $\lim_{I \subset (0, \infty)} \mathrm{Perf}_{B_I}$).

Now, recalling that the action of φ on Y_{FF} is free and totally discontinuous, [FS21, Proposition II.1.16], it follows formally from the analytic descent for quasi-coherent complexes, Theorem A.20, that we have an equivalence of ∞ -categories

$$\mathrm{QCoh}(Y_{\mathrm{FF}})^{\varphi} \simeq \mathrm{QCoh}(Y_{\mathrm{FF}}/\varphi^{\mathbb{Z}}). \quad (6.9)$$

Thus, from (6.8) we obtain a functor

$$\mathcal{E}_{\mathrm{FF}}(-) : \mathrm{CoAd}(\mathrm{Solid}_B)^{\varphi} \rightarrow \mathrm{QCoh}(\mathrm{FF}) \quad (6.10)$$

with target the ∞ -category of quasi-coherent complexes on the Fargues–Fontaine curve.

Next, we focus on nuclear and perfect complexes on FF. We invite the reader to compare the following result with [FF14, Theorem 7.18].

Proposition 6.8 (Nuclear complexes on the curve as coadmissible φ -modules over B).

⁴⁹The terminology adopted here comes from [ST03, §3].

(i) The functor $\mathcal{E}_{\text{FF}}(-)$, defined in (6.10), induces an equivalence of ∞ -categories

$$\text{CoAd}(\text{Nuc}_B)^\varphi \xrightarrow{\sim} \text{Nuc}(\text{FF}) \quad (6.11)$$

which restricts to an equivalence of ∞ -categories

$$\text{CoAd}(\text{Perf}_B)^\varphi \xrightarrow{\sim} \text{Perf}(\text{FF}). \quad (6.12)$$

(ii) Given $\mathcal{E} \in \text{Nuc}(\text{FF})$, let $((M_I(\mathcal{E}))_{I \subset (0, \infty)}, \varphi)$ be the coadmissible nuclear φ -module over B corresponding to \mathcal{E} via the equivalence (6.11). Let $M(\mathcal{E}) := R \lim_{I \subset (0, \infty)} M_I(\mathcal{E})$. Then, there is a natural identification in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$

$$R\Gamma(\text{FF}, \mathcal{E}) = \text{fib}(M(\mathcal{E}) \xrightarrow{\varphi-1} M(\mathcal{E})).$$

Proof. For part (i), we first observe that, for any compact interval $I \subset (0, \infty)$ with rational endpoints, the base change functor $-\otimes_{(B_I, \mathbb{Z})_\bullet}^L (B_I, B_I^+)_\bullet : D((B_I, \mathbb{Z})_\bullet) \rightarrow D((B_I, B_I^+)_\bullet)$ induces an equivalence of ∞ -categories⁵⁰

$$\text{Nuc}((B_I, \mathbb{Z})_\bullet) \xrightarrow{\sim} \text{Nuc}((B_I, B_I^+)_\bullet). \quad (6.13)$$

In fact, by Theorem A.17(ii), $\text{Nuc}((B_I, \mathbb{Z})_\bullet)$ is generated, under shifts and colimits, by the objects $\underline{\text{Hom}}(\mathbb{Z}[S], B_I)$, for varying S profinite sets; then, as, thanks to Corollary A.12, nuclearity is preserved under base change, by Proposition A.15 we deduce that such objects also generate, under shifts and colimits, $\text{Nuc}((B_I, B_I^+)_\bullet)$. Then, we have an equivalence of ∞ -categories

$$\text{CoAd}(\text{Nuc}_B)^\varphi \xrightarrow{\sim} \text{Nuc}(Y_{\text{FF}})^\varphi. \quad (6.14)$$

Now, by analytic descent for nuclear complexes, Theorem A.21, we have

$$\text{Nuc}(Y_{\text{FF}})^\varphi \simeq \text{Nuc}(Y_{\text{FF}}/\varphi^\mathbb{Z}) \quad (6.15)$$

which combined with (6.14) implies the equivalence (6.11). Such equivalence restricts to (6.12) by analytic descent for perfect complexes, Theorem A.23.

Part (ii) follows from part (i), as we now explain. In fact, for any $\mathcal{E} \in \text{QCoh}(\text{FF})$, using the equivalence (6.9), we have the following identification

$$\begin{aligned} R\Gamma(\text{FF}, \mathcal{E}) &= \text{Hom}_{\text{QCoh}(\text{FF})}(\mathcal{O}, \mathcal{E}) \\ &= \text{fib}(\text{Hom}_{\text{QCoh}(Y_{\text{FF}})}(\mathcal{O}, \mathcal{E}|_{Y_{\text{FF}}}) \xrightarrow{\varphi-1} \text{Hom}_{\text{QCoh}(Y_{\text{FF}})}(\mathcal{O}, \mathcal{E}|_{Y_{\text{FF}}})) \\ &= \text{fib}(R\Gamma(Y_{\text{FF}}, \mathcal{E}|_{Y_{\text{FF}}}) \xrightarrow{\varphi-1} R\Gamma(Y_{\text{FF}}, \mathcal{E}|_{Y_{\text{FF}}})) \end{aligned}$$

where

$$R\Gamma(Y_{\text{FF}}, \mathcal{E}|_{Y_{\text{FF}}}) = R \lim_{I \subset (0, \infty)} \text{Hom}_{\text{QCoh}(Y_{\text{FF}, I})}(\mathcal{O}, \mathcal{E}|_{Y_{\text{FF}, I}}).$$

Now, assuming $\mathcal{E} \in \text{Nuc}(\text{FF})$ as in the statement, thanks to the equivalence (6.13), we have

$$\text{Hom}_{\text{QCoh}(Y_{\text{FF}, I})}(\mathcal{O}, \mathcal{E}|_{Y_{\text{FF}, I}}) = \text{Hom}_{\text{Nuc}(Y_{\text{FF}, I})}(\mathcal{O}, \mathcal{E}|_{Y_{\text{FF}, I}}) = \text{Hom}_{\text{Nuc}((B_I, \mathbb{Z})_\bullet)}(B_I, M_I(\mathcal{E})) = M_I(\mathcal{E}).$$

Hence, the statement follows, observing that all the ∞ -categories appearing above are naturally enriched over $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$. \square

Perfect complexes on FF are well-understood. Let us recall the following characterization due to Anschutz–Le Bras.

⁵⁰Such equivalence follows from Theorem A.22, however we give here a self-contained proof.

Proposition 6.9 ([ALB21, Proposition 2.6]). *Each $\mathcal{E} \in \text{Perf}(\text{FF})$ is quasi-isomorphic to bounded complex of vector bundles on FF .*

In addition, perfect complexes on FF are closely related to Banach–Colmez spaces, as we now recall.

Definition 6.10 ([LB18a, Définition 2.11]). The category \mathcal{BC} of *Banach–Colmez spaces over C* (for short, *BC spaces*) is the smallest abelian subcategory of sheaves of $\underline{\mathbb{Q}_p}$ -modules on the v -site $\text{Spa}(C, \mathcal{O}_C)_v$ stable under extensions and containing the v -sheaves $\underline{\mathbb{Q}_p}$ and $(\mathbb{A}_C^1)^\diamond$.

In the following, we denote by

$$\tau : \text{FF}_v \rightarrow \text{Spa}(C, \mathcal{O}_C)_v$$

the natural morphism of v -sites, sending an affinoid perfectoid $S \in \text{Spa}(C, \mathcal{O}_C)_v$ to the relative Fargues–Fontaine curve $\text{FF}_S \in \text{FF}_v$.

Proposition 6.11 ([ALB21, Corollary 3.10, Remark 3.12]). *We have an equivalence of categories*

$$R\tau_* : \text{Perf}(\text{FF}) \xrightarrow{\sim} D^b(\mathcal{BC}) \tag{6.16}$$

where the right-hand side denotes the bounded derived category of Banach–Colmez spaces over C .

6.2.2. Fargues–Fontaine cohomology. We are almost ready to define the Fargues–Fontaine cohomology together with its filtration. First, in order to prove [LB18b, Conjecture 6.3], we want to extend the definition of the B -cohomology to dagger varieties over C .

In the following, we abbreviate $D(B) = D(\text{Mod}_B^{\text{cond}})$.

Definition 6.12 (B -cohomology of dagger varieties). Let X be a dagger variety over C . Denote

$$\mathcal{F}_B \in \text{Shv}^{\text{hyp}}(\text{Rig}_{C, \text{eh}}, D(B))$$

the hypersheaf defined by the B -cohomology $R\Gamma_B(-)$. Via Construction 3.18, we define

$$R\Gamma_B(X) := R\Gamma(X, \mathcal{F}_B^\dagger) \in D(B)$$

and we endow it with the filtration induced by the filtration décalée on the B -cohomology of rigid-analytic varieties over C . We give analogous definitions replacing the B -cohomology with the B_I -cohomology, for varying $I \subset (0, \infty)$ compact intervals with rational endpoints.

Remark 6.13 (Comparison with the Hyodo–Kato cohomology in the dagger case). We note that the main comparison theorems proved in §4 for the B -cohomology (and the B_I -cohomology) of rigid-analytic varieties extend to the dagger case, thanks to the properties of the solid tensor product. In particular, a version of Theorem 4.1 holds true for X a qcqs dagger variety over C : in fact, by the proof of *loc. cit.* we can reduce to the case X is a smooth dagger affinoid over C , which follows from the statement of *loc. cit.* using Lemma 3.24 (which applies thanks to Proposition 2.42(i)) and the fact that the solid tensor product commutes with filtered colimits.

In order to define the Fargues–Fontaine cohomology, we will apply the functor (6.10) in the following situation.

Lemma 6.14 (Coadmissibility of B -cohomology). *Let X be a qcqs rigid-analytic/dagger variety over C . Let $i \geq 0$. The pair*

$$((\text{Fil}^i R\Gamma_{B_I}(X))_{I \subset (0, \infty)}, \varphi) \tag{6.17}$$

defines a coadmissible nuclear φ -module over B . Here, the complexes $R\Gamma_{B_I}(X)$, for $I \subset (0, \infty)$ compact intervals with rational endpoints, are endowed with the filtration décalée (Definition 2.9).

Proof. First, we recall that $\varphi(t) = pt$, and p is invertible in B_I . Next, let us assume $i = 0$. To show that the pair (6.17) is a coadmissible solid φ -module over B , we need to check that

$$R\Gamma_{B_J}(X) \otimes_{B_J}^{\mathbf{L}\blacksquare} B_I \simeq R\Gamma_{B_I}(X) \quad (6.18)$$

for any $I \subset J \subset (0, \infty)$ compact intervals with rational endpoints: this follows for example from Theorem 4.1 (and Remark 6.13).⁵¹

In order to prove the nuclearity of (6.17) for $i = 0$, we can use again Theorem 4.1, Theorem 3.15(ii), and the fact that nuclearity is preserved under base change, Corollary A.12. Alternatively, and more directly, the nuclearity can be checked as follows. By Theorem A.17(i) and éh-hyperdescent, we can reduce to the case when X is a smooth affinoid rigid space over C . Then, by Proposition 2.42, and the fact that nuclearity can be checked on cohomology groups thanks to Theorem A.17(iii), using [BMS18, Lemma 6.4] we can reduce to the assertion that each complex of solid B_I -modules $R\Gamma_{\text{proét}}(X, \mathbb{B}_I)$ is nuclear, which was shown in Lemma 6.15.

It remains to show that the isomorphism (6.18) is compatible with the filtration décalée, and to prove the nuclearity of the filtered pieces. Proceeding by induction on $i \geq 0$, we can check this on graded pieces using Proposition 2.7(ii), again with the help of Theorem A.17, Proposition 2.42, and Lemma 6.15 for the nuclearity. \square

The following lemma on the nuclearity of the cohomology of the rational pro-étale period sheaves was used above.

Lemma 6.15. *Let X be a qcqs analytic adic space over $\text{Spa}(C, \mathcal{O}_C)$. Let $I \subset (0, \infty)$ be a compact interval with rational endpoints, and let $m \geq 1$ be an integer. Given $\mathbf{B} \in \{\mathbb{B}_I, \mathbb{B}_{\text{dR}}^+ / \text{Fil}^m\}$ we write $\mathcal{B} = \mathbf{B}_{\text{Spa}(C)_{\text{proét}}}$ for the corresponding condensed period ring. Then, we have*

$$R\Gamma_{\text{proét}}(X, \mathbf{B}) \in \text{Nuc}((\mathcal{B}, \mathbb{Z})_{\blacksquare}).$$

Proof. Picking a simplicial proét-hypercover $U_{\bullet} \rightarrow X$ such that all U_n are affinoid perfectoid spaces over $\text{Spa}(C, \mathcal{O}_C)$, by proét-hyperdescent and Theorem A.17(i) (applied to the Banach \mathbb{Q}_p -algebra \mathcal{B}), it suffices to check that for any affinoid perfectoid space U over $\text{Spa}(C, \mathcal{O}_C)$ we have that $R\Gamma_{\text{proét}}(U, \mathbf{B}) \in \text{Nuc}((\mathcal{B}, \mathbb{Z})_{\blacksquare})$. We note that, by [Bos21, Proposition 4.7] and [Bos21, Corollary 4.9], we have $R\Gamma_{\text{proét}}(U, \mathbf{B}) = \mathbf{B}(U)[0]$ concentrated in degree 0 (as it can be checked on S -valued points, for any κ -small extremally disconnected set S). Thus, by Remark A.18, we are reduced to prove that $\mathbf{B}(U)[0] \in \text{Nuc}((\mathbb{Q}_p, \mathbb{Z})_{\blacksquare})$, which follows [Bos21, Corollary A.50] observing that $\mathbf{B}(U)$ is a \mathbb{Q}_p -Banach space. \square

Finally, we can give the following definition.

Definition 6.16. Let X be a qcqs rigid-analytic/dagger variety over C . We define the *Fargues–Fontaine cohomology* of X , denoted

$$\mathcal{H}_{\text{FF}}(X) \in \text{QCoh}(\text{FF})$$

as the quasi-coherent complex on the Fargues–Fontaine curve FF , endowed with filtration $\text{Fil}^* \mathcal{H}_{\text{FF}}(X)$, associated, via the functor $\mathcal{E}_{\text{FF}}(-)$ defined in (6.10), to the filtered coadmissible solid φ -module over B given by (6.17). For $i \in \mathbb{Z}$, we denote by $\mathcal{H}_{\text{FF}}^i(X)$ its i -th cohomology group.

⁵¹Trivializing the monodromy action.

Using Definition 6.16 we can now reformulate, in terms of the Fargues–Fontaine curve, the comparison theorems proven in the previous sections.

Theorem 6.17 (cf. [LB18b, Conjecture 6.3]). *Let X be a qcqs dagger variety over C . Let $i \geq 0$.*

- (i) *The quasi-coherent complex $\mathcal{H}_{\text{FF}}(X)$ on FF is perfect, and its cohomology groups are vector bundles on FF . We have a natural isomorphism*

$$\mathcal{H}_{\text{FF}}^i(X) \cong \mathcal{E}(H_{\text{HK}}^i(X)) \quad (6.19)$$

where $\mathcal{E}(H_{\text{HK}}^i(X))$ is the vector bundle on FF associated to the finite (φ, N) -module $H_{\text{HK}}^i(X)$ over \check{F} . If X is the base change to C of a rigid-analytic variety defined over K , then (6.19) is \mathcal{G}_K -equivariant.

- (ii) *The completion at ∞ of (6.19) gives a natural isomorphism*

$$\mathcal{H}_{\text{FF}}^i(X)_{\infty}^{\wedge} \cong H_{\text{inf}}^i(X/B_{\text{dR}}^+) \quad (6.20)$$

where $R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+)$ is defined, via Construction 3.18, from Definition 5.4.

Proof. Part (i) follows from Theorem 4.1 (combined with Remark 6.13), Theorem 3.30(i) and Lemma 7.6. Part (ii) follows from Theorem 5.20 (which extends to the dagger case similarly to Remark 6.13). \square

Remark 6.18. Let X be a qcqs dagger variety over C and let $i \geq 0$. Recalling that the functor from finite φ -modules over \check{F} to vector bundles on FF induces a bijection on isomorphism classes, [FS21, Theorem II.0.3], we deduce from Theorem 6.17(i) that the vector bundle $\mathcal{H}_{\text{FF}}^i(X)$ determines, up to isomorphisms, the φ -module structure on $H_{\text{HK}}^i(X)$.

Using Theorem 6.17(i), we can also recover from $\mathcal{H}_{\text{FF}}^i(X)$ the (φ, N) -module structure on $H_{\text{HK}}^i(X)$, up to isomorphisms: in fact, denoting $\mathcal{G}_{\check{F}} = \text{Gal}(C/\check{F})$, by [FF18, Proposition 10.3.20(2)] we have a natural isomorphism of (φ, N) -modules over \check{F}

$$H_{\text{HK}}^i(X) \cong (H^0(\text{FF} \setminus \{\infty\}, \mathcal{H}_{\text{FF}}^i(X)) \otimes_{B_e} B_{\log}[1/t])^{\mathcal{G}_{\check{F}}}$$

observing that, since (6.19) is an isomorphism of $\mathcal{G}_{\check{F}}$ -equivariant vector bundles on FF , we have a $\mathcal{G}_{\check{F}}$ -equivariant isomorphism $H^0(\text{FF} \setminus \{\infty\}, \mathcal{H}_{\text{FF}}^i(X)) \cong (H_{\text{HK}}^i(X) \otimes_{\check{F}} B_{\log}[1/t])^{\varphi=1, N=0}$.

Next, we state the main result of this subsection.

Theorem 6.19. *Let X be a qcqs rigid-analytic/dagger variety over C . Let $i \geq 0$. Consider the quasi-coherent complex on FF defined by*

$$\mathcal{H}_{\text{syn}}(X)(i) := \text{Fil}^i \mathcal{H}_{\text{FF}}(X) \otimes \mathcal{O}(i).$$

We have

$$R\Gamma(\text{FF}, \mathcal{H}_{\text{syn}}(X)(i)) = R\Gamma_{\text{syn}, \text{FF}}(X, \mathbb{Q}_p(i)). \quad (6.21)$$

If X is a proper rigid-analytic variety over C , the complex $\mathcal{H}_{\text{syn}}(X)(i)$ is perfect, in particular the complex $R\Gamma_{\text{syn}, \text{FF}}(X, \mathbb{Q}_p(i))$ identifies with the C -points of a bounded complex of Banach–Colmez spaces.

Proof. By Proposition 6.8(ii), which applies thanks to Lemma 6.14, we have

$$\begin{aligned} R\Gamma(\mathrm{FF}, \mathcal{H}_{\mathrm{syn}}(X)(i)) &= \mathrm{fib}(R\Gamma(Y_{\mathrm{FF}}, \mathcal{H}_{\mathrm{syn}}(X)(i)|_{Y_{\mathrm{FF}}}) \xrightarrow{\varphi^{-1}} R\Gamma(Y_{\mathrm{FF}}, \mathcal{H}_{\mathrm{syn}}(X)(i)|_{Y_{\mathrm{FF}}})) \\ &= \mathrm{fib}(R\Gamma(Y_{\mathrm{FF}}, \mathrm{Fil}^i \mathcal{H}_{\mathrm{FF}}(X)|_{Y_{\mathrm{FF}}}) \xrightarrow{\varphi^{p^{-i}-1}} R\Gamma(Y_{\mathrm{FF}}, \mathrm{Fil}^i \mathcal{H}_{\mathrm{FF}}(X)|_{Y_{\mathrm{FF}}})) \\ &= \mathrm{fib}(\mathrm{Fil}^i R\Gamma_B(X) \xrightarrow{\varphi^{p^{-i}-1}} \mathrm{Fil}^i R\Gamma_B(X)) \end{aligned}$$

where in the last step we used in addition Lemma 2.41. This shows (6.21).

Next, assume X proper. To show that $\mathcal{H}_{\mathrm{syn}}(X)(i)$ is a perfect complex, using the derived Beauville–Laszlo gluing, Lemma 6.20, we can reduce to showing that, for $I \subset (0, \infty)$ a compact interval with rational endpoints, the complexes $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}_I[1/t])$ and $\mathrm{Fil}^i R\Gamma_{B_{\mathrm{dr}}^+}(X)$ are perfect. First, we note that such complexes are bounded thanks to Proposition 2.40. Then, to prove that the complex $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}_I[1/t])$ is perfect, we can apply Theorem 4.1 combined with Theorem 3.30(i) (or, alternatively, [Sch13b, Theorem 3.17]), and for the complex $\mathrm{Fil}^i R\Gamma_{B_{\mathrm{dr}}^+}(X)$ we use Proposition 6.21 below.

Finally, we note that, by Proposition 6.11, the complex $R\Gamma_{\mathrm{syn}, \mathrm{FF}}(X, \mathbb{Q}_p(r))$ identifies with the C -points of a bounded complex of Banach–Colmez spaces, as desired (in fact, with the same notation as in Proposition 6.11, for $\mathcal{E} \in \mathrm{Perf}(\mathrm{FF})$, we have $R\tau_*(\mathcal{E})(\mathrm{Spa}(C, \mathcal{O}_C)) = R\Gamma(\mathrm{FF}, \mathcal{E})$, by the v -descent result [FS21, Proposition II.2.1] combined with Proposition 6.9). \square

We used the following derived version of the classical Beauville–Laszlo gluing.

Lemma 6.20 (Derived Beauville–Laszlo gluing). *Let R be a commutative condensed ring, let $f \in R$ be a non-zero-divisor, and denote by \widehat{R} the f -adic completion of R . We have a natural equivalence of ∞ -categories*

$$\mathrm{Perf}_R \simeq \mathrm{Perf}_{R[1/f]} \times_{\mathrm{Perf}_{\widehat{R}[1/f]}} \mathrm{Perf}_{\widehat{R}}.$$

Proof. By [Bha16, Proposition 5.6(2), Example 5.10], we have a natural equivalence of ∞ -categories $\mathrm{Perf}_{R(*)} \simeq \mathrm{Perf}_{R[1/f](*)} \times_{\mathrm{Perf}_{\widehat{R}[1/f](*)}} \mathrm{Perf}_{\widehat{R}(*)}$. In order to carry such equivalence into the condensed setting, we recall that, for any commutative condensed ring A , we define the *condensification functor*, [And21, Definition 5.8], as the composite

$$\mathrm{Cond}_A : D(\mathrm{Mod}_{A(*)}) \hookrightarrow D(\mathrm{Mod}_{A(*)}^{\mathrm{cond}}) \xrightarrow{\otimes_{A(*)}^L A} D(\mathrm{Mod}_A^{\mathrm{cond}}).$$

Then, the statement follows observing that Cond_A preserves perfect complexes, and applying [And21, Lemma 5.10]. \square

We also used the following finiteness/degeneration result on the B_{dr}^+ -cohomology of proper rigid-analytic varieties over C . As we will see, part (i) follows from results of Guo, instead part (ii) relies crucially on a combination of Conrad–Gabber’s spreading out for proper rigid-analytic varieties and a generic smoothness result recently proved by Bhatt–Hansen, [BH22], which allow us to reduce the statement to the case when X is the base change to C of a proper smooth rigid-analytic variety over a discretely valued subfield of C .

Proposition 6.21. *Let X be a proper rigid-analytic variety over C .*

(i) *For all $i \in \mathbb{Z}$, the cohomology group $H_{B_{\mathrm{dr}}^+}^i(X)$ is a finite free module over B_{dr}^+ .*

(ii) For all $i, r \in \mathbb{Z}$, the natural map

$$H^i(\mathrm{Fil}^r R\Gamma_{B_{\mathrm{dR}}^+}(X)) \rightarrow H_{B_{\mathrm{dR}}^+}^i(X) \quad (6.22)$$

is injective. Equivalently, for all $i, r \in \mathbb{Z}$, the natural map

$$H_{B_{\mathrm{dR}}^+}^i(X)/\mathrm{Fil}^r \rightarrow H^i(R\Gamma_{B_{\mathrm{dR}}^+}(X)/\mathrm{Fil}^r)$$

is an isomorphism, where $\mathrm{Fil}^r H_{B_{\mathrm{dR}}^+}^i(X) := \mathrm{im}(H^i(\mathrm{Fil}^r R\Gamma_{B_{\mathrm{dR}}^+}(X)) \rightarrow H_{B_{\mathrm{dR}}^+}^i(X))$.

Proof. Part (i) follows from [Guo21, Theorem 1.2.7(vi)] combined with Theorem 5.9. Next, we prove part (ii) and give at the same time an alternative proof of part (i). We first note that the equivalence of the two assertions in part (ii) immediately follows considering the long exact sequence in cohomology associated to the exact triangle

$$\mathrm{Fil}^r R\Gamma_{B_{\mathrm{dR}}^+}(X) \rightarrow R\Gamma_{B_{\mathrm{dR}}^+}(X) \rightarrow R\Gamma_{B_{\mathrm{dR}}^+}(X)/\mathrm{Fil}^r.$$

We will use that, by [BMS18, Corollary 13.16], there exists a proper flat morphism $f : \mathcal{X} \rightarrow S$ of rigid-analytic varieties over a discretely valued subfield $L \subset C$ such that X is the fibre of f over a point $\eta \in S(C)$; we note that we may assume $S = \mathrm{Spa}(A, A^\circ)$ to be a smooth affinoid over L . Then, the map η corresponds to a map $A \rightarrow C$ of affinoid L -algebras, and, by the (formal) smoothness of A over L , the latter map lifts to a map $A \rightarrow B_{\mathrm{dR}}^+$. Now, we divide the argument in several cases.

(a) In the case S is a point, we have that X is the base change to C of a proper rigid-analytic variety \mathcal{X} defined over a discretely valued subfield $L \subset C$. Then, by Theorem 5.1, we have a natural filtered quasi-isomorphism

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}) \otimes_L^{\mathbb{L}} B_{\mathrm{dR}}^+ \simeq R\Gamma_{B_{\mathrm{dR}}^+}(X). \quad (6.23)$$

Here, we used that, as \mathcal{X} is proper, the cohomology groups of $R\Gamma_{\mathrm{dR}}(X)$ and its filtered pieces, are finite-dimensional over L : in fact, by Proposition 2.14, there exists a proper éh-hypercover $\mathcal{X}_\bullet \rightarrow \mathcal{X}$ with each \mathcal{X}_n smooth over L ; then, by cohomological descent, we can reduce to the case \mathcal{X} is smooth, which follows from [Kie67]. Then, part (i) is clear from the quasi-isomorphism (6.23), and part (ii) follows from the compatibility with filtrations of (6.23) and the degeneration of the Hodge-de Rham spectral sequence for \mathcal{X}

$$H^{i-j}(\mathcal{X}, \Omega_{\mathcal{X}_{\mathrm{eh}}}^j) \implies H_{\mathrm{dR}}^i(\mathcal{X}) \quad (6.24)$$

(see [Sch13a, Corollary 1.8] for the case \mathcal{X} is smooth, and [Guo19, Proposition 8.0.8] for the case \mathcal{X} is singular).

(b) In the case $f : \mathcal{X} \rightarrow S$ is smooth, denoting by $Rf_{\mathrm{dR}*}\mathcal{O}_{\mathcal{X}} := Rf_*\Omega_{\mathcal{X}/S}^\bullet$ the relative de Rham cohomology of f endowed with its Hodge filtration $Rf_*\Omega_{\mathcal{X}/S}^{\geq \star}$, we claim that we have a filtered quasi-isomorphism

$$Rf_{\mathrm{dR}*}\mathcal{O}_{\mathcal{X}} \otimes_A^{\mathbb{L}} B_{\mathrm{dR}}^+ \simeq R\Gamma_{B_{\mathrm{dR}}^+}(X) \quad (6.25)$$

compatible with (6.23) in the case S is a point. For this, by Theorem 5.9 and Proposition 5.8, it suffices to show that we have a B_{dR}^+ -linear map

$$Rf_{\mathrm{dR}*}\mathcal{O}_{\mathcal{X}} \otimes_A^{\mathbb{L}} B_{\mathrm{dR}}^+ \rightarrow R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \quad (6.26)$$

which is a quasi-isomorphism compatible with filtrations, where the right-hand side is endowed with the infinitesimal filtration. Arguing as in the proof of [BMS18, Theorem 13.19], we can

construct (6.26) on a hypercover of \mathcal{X} by *very small* smooth affinoid spaces over L in the sense of [BMS18, Definition 13.5],⁵² using in addition Lemma 5.16.⁵³ We note that the constructed map (6.26) is a quasi-isomorphism after applying the functor $-\otimes_{B_{\mathrm{dR}}^+}^L B_{\mathrm{dR}}^+/\xi$, recalling that we have a natural quasi-isomorphism

$$R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \otimes_{B_{\mathrm{dR}}^+}^L B_{\mathrm{dR}}^+/\xi \simeq R\Gamma_{\mathrm{dR}}(X)$$

by [Guo21, Theorem 1.2.7(i), Theorem 1.2.1(i)]; then, in order to show that (6.26) is a quasi-isomorphism, using the derived Nakayama lemma, it suffices to check that both the source and the target of (6.26) are derived ξ -adically complete: for the source, we note that each $R^i f_{\mathrm{dR}*} \mathcal{O}_{\mathcal{X}}$ is a coherent \mathcal{O}_S -module with (integrable) connection, in particular it is a locally free \mathcal{O}_S -module (see [And01, Corollaire 2.5.2.2]); for the target, we can use for example the Čech-Alexander complex computing the infinitesimal cohomology over B_{dR}^+ , [Guo21, Proposition 4.1.3]. To prove that the quasi-isomorphism (6.26) is compatible with filtrations, proceeding by induction on the index $i \geq 0$ of the filtrations, it suffices to check the compatibility on graded pieces, which follows from Lemma 6.22.

Now, to prove part (i), using the quasi-isomorphism (6.25), it suffices to recall that each $R^i f_{\mathrm{dR}*} \mathcal{O}_{\mathcal{X}}$ is a finite projective A -module, as we have seen above. For part (ii), using the filtered quasi-isomorphism (6.25), it suffices to recall that, thanks to [Sch13a, Theorem 8.8] and [SW20, Theorem 10.5.1], for all $i, j \in \mathbb{Z}$ the relative Hodge cohomology $R^{i-j} f_* \Omega_{\mathcal{X}/S}^j$ is a finite projective A -module, and that the relative Hodge-de Rham spectral sequence

$$R^{i-j} f_* \Omega_{\mathcal{X}/S}^j \implies R^i f_{\mathrm{dR}*} \mathcal{O}_{\mathcal{X}}$$

degenerates.

- (c) In the general case, we will use that $Rf_{\mathrm{proét}*} \mathbb{Z}_p$ is a bounded Zariski-constructible complex of sheaves on S , in the sense of [BH22, Definition 3.32, Remark 3.38], thanks to [BH22, Theorem 3.10, Theorem 3.36].

We denote by $\nu : S_{\mathrm{proét}} \rightarrow S_{\mathrm{ét}}$ the natural morphism of sites, and we set

$$D_{\mathrm{dR}}(Rf_{\mathrm{proét}*} \mathbb{Z}_p) := R\nu_*(Rf_{\mathrm{proét}*} \mathbb{Z}_p \otimes_{\mathbb{Z}_p}^L \mathcal{O}_{\mathbb{B}_{\mathrm{pdR},S}})$$

where the sheaf $\mathcal{O}_{\mathbb{B}_{\mathrm{pdR},S}}$ on $S_{\mathrm{proét}}$ is defined as follows:⁵⁴ we consider Scholze's sheaf $\mathcal{O}_{\mathbb{B}_{\mathrm{dR},S}}^+$ on $S_{\mathrm{proét}}$ (see e.g. [Bos21, Definition 6.6]) and we put $\mathcal{O}_{\mathbb{B}_{\mathrm{pdR},S}}^+ := \mathcal{O}_{\mathbb{B}_{\mathrm{dR},S}}^+[\log t]$; then we define the sheaf $\mathcal{O}_{\mathbb{B}_{\mathrm{pdR}}}$ as the completion of the sheaf $\mathcal{O}_{\mathbb{B}_{\mathrm{pdR}}^+}[1/t]$ with respect to the filtration given by $\mathrm{Fil}^r(\mathcal{O}_{\mathbb{B}_{\mathrm{pdR}}^+}[1/t]) := \sum_{j \in \mathbb{Z}} t^{-j} (\ker \theta)^{r+j} \mathcal{O}_{\mathbb{B}_{\mathrm{pdR}}^+}$, and we equip $\mathcal{O}_{\mathbb{B}_{\mathrm{pdR}}}$ with the induced filtration; we endow $\mathcal{O}_{\mathbb{B}_{\mathrm{pdR}}}$ with an action of \mathcal{G}_L , extending the one on $\mathcal{O}_{\mathbb{B}_{\mathrm{dR},S}}^+$, by setting $g(\log t) = \log(\chi(g)) + \log t$ for all $g \in \mathcal{G}_L$, where $\chi : \mathcal{G}_L \rightarrow \mathbb{Z}_p^\times$ denotes the cyclotomic character. We will use crucially that

$$R^0 \nu_* \mathcal{O}_{\mathbb{B}_{\mathrm{pdR},S}} = \mathcal{O}_S \quad \text{and} \quad R^i \nu_* \mathcal{O}_{\mathbb{B}_{\mathrm{pdR},S}} = 0 \quad \text{for all } i > 0 \quad (6.27)$$

as it formally follows from [Sch13a, Proposition 6.16(ii)].

⁵²I.e. the affinoid spaces $\mathrm{Spa}(A_0, A_0^\circ)$ of Notation 5.14 for $r = 0$, $q = 1$, and $\Xi_0 = \emptyset$.

⁵³In the case S is a point, the stated compatibility with (6.23) follows from the proof of Theorem 5.3 (see in particular the commutative diagram (5.25)).

⁵⁴Cf. Fontaine's definition of the *presque de Rham period ring* B_{pdR} , [Fon04, §4.3]. Cf. also the upcoming work of Bhatt–Lurie, [BL].

We claim that we have a natural filtered quasi-isomorphism

$$D_{\mathrm{dR}}(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p) \otimes_A^L B_{\mathrm{dR}}^+ \simeq R\Gamma_{B_{\mathrm{dR}}^+}(X). \quad (6.28)$$

For this, we argue by induction on $\dim(S)$. For the base case $\dim(S) = 0$, we have that S is a disjoint union of points, hence we can reduce to the case S is a point; then, by the de Rham comparison theorem for proper (possibly singular) rigid-analytic varieties defined over L , [Guo19, Theorem 1.1.4], together with (6.27), we have a natural filtered quasi-isomorphism $D_{\mathrm{dR}}(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p) \simeq R\Gamma_{\mathrm{dR}}(\mathcal{X})$, and the claim follows from the filtered quasi-isomorphism (6.23) of part (a).

For the inductive step, using Proposition 2.14, picking a proper $\acute{e}h$ -hypercover $\mathcal{X}_\bullet \rightarrow \mathcal{X}$ with each \mathcal{X}_n smooth over L , we can reduce to the case \mathcal{X} is smooth over L . In the latter case, by [BH22, Theorem 2.29], the maximal open subset $S' \subset S$ such that $f' : f^{-1}(S') \rightarrow S'$ is smooth is a dense Zariski-open subset; in particular, the Zariski-closed complement $Z := S \setminus S'$ is nowhere dense in S , and hence we have $\dim(Z) < \dim(S)$. Then, if η is contained in Z , restricting f over Z , the claim follows from the inductive hypothesis, by proper base change [BH22, Theorem 3.15, Theorem 3.36]; if η is contained in S' , by the smoothness of the restriction f' of f over S' , the claim follows again by proper base change from the filtered quasi-isomorphism (6.25) of part (b), observing that in this case, by the relative de Rham comparison ([Sch13a, Theorem 8.8] combined with [SW20, Theorem 10.5.1]), and (6.27), we have a filtered quasi-isomorphism

$$D_{\mathrm{dR}}(Rf'_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p) \simeq R\nu_*(Rf'_{\mathrm{dR}*}\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_S}^L \mathcal{O}_{\mathbb{B}_{\mathrm{pdR},S}}) \simeq Rf'_{\mathrm{dR}*}\mathcal{O}_{\mathcal{X}}.$$

Now, to show part (i), using the quasi-isomorphism (6.28), it suffices to prove that, up to replacing S by a suitable Zariski locally closed subset containing η , the cohomology groups of $D_{\mathrm{dR}}(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p)$ are finite projective A -modules. For this, recalling that $Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p$ is a bounded Zariski-constructible complex of sheaves on S , and using [BH22, Lemma 3.35], we can suppose that $R^n f_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p$ is a \mathbb{Z}_p -local system on $S_{\mathrm{pro\acute{e}t}}$ for all $n \in \mathbb{Z}$; next, we note that the latter \mathbb{Z}_p -local systems are de Rham (in the sense of [Sch13a, Definition 8.3]), in fact, by [LZ17, Theorem 3.9, (iv)], to show this condition it suffices to check the de Rhamness (in the sense of Fontaine) of the stalks $(R^n f_{\mathrm{pro\acute{e}t}*}\mathbb{Q}_p)_{\bar{y}}$ at geometric points \bar{y} over classical points y of S ,⁵⁵ which follows from the de Rham comparison theorem for proper rigid-analytic varieties defined over L , [Guo19, Theorem 1.1.4]. Then, recalling (6.27), by [LZ17, Theorem 3.9, (iv) and (i)], the cohomology groups of $D_{\mathrm{dR}}(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p)$ are finite projective A -modules, as desired.

For part (ii), using the filtered quasi-isomorphism (6.28), it suffices to check that, up to replacing S by a suitable Zariski locally closed subset containing η , the spectral sequence associated to the filtered complex $D_{\mathrm{dR}}(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p)$, i.e.

$$H^{i-j}(\mathrm{gr}^j D_{\mathrm{dR}}(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p)) \implies H^i(D_{\mathrm{dR}}(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p)),$$

degenerates. For this, by the same argument used above to show part (i), using in addition [LZ17, Theorem 3.9, (iii)], we can suppose that all the terms of the spectral sequence above are vector bundles on S , in which case the degeneration can be checked on stalks at classical points, where it follows from the degeneration of the spectral sequence (6.24) of part (a). □

⁵⁵We note that, since $f : \mathcal{X} \rightarrow S$ is proper, we have $(R^n f_{\mathrm{pro\acute{e}t}*}\mathbb{Q}_p)_{\bar{y}} \cong H^n_{\mathrm{pro\acute{e}t}}(\mathcal{X} \times_S \bar{y}, \mathbb{Q}_p)$: in fact, the natural map $(Rf_{\mathrm{pro\acute{e}t}*}\mathbb{Z}_p)_{\bar{y}} \rightarrow Rf_{\mathrm{pro\acute{e}t}*}(\mathcal{X} \times_S \bar{y}, \mathbb{Z}_p)$ is an isomorphism, as it follows from [Hub96, Proposition 2.6.1] reducing the latter map mod p .

The following lemma was used in the proof above.

Lemma 6.22. *Let X be a smooth rigid-analytic variety over C . For any $r \geq 0$, we have a natural quasi-isomorphism*

$$\mathrm{gr}^r R\Gamma_{B_{\mathrm{dR}}^+}(X) \simeq \bigoplus_{0 \leq i \leq r} R\Gamma(X, \Omega_X^i)(r-i)[-i].$$

Proof. First, we note that, by Proposition 2.7(ii), we have $\mathrm{gr}^r R\Gamma_{B_{\mathrm{dR}}^+}(X) \simeq R\Gamma(X, \tau^{\leq r} R\alpha_* \widehat{\mathcal{O}}(r))$. We want to show that we have a natural identification

$$\tau^{\leq r} R\alpha_* \widehat{\mathcal{O}}(r) \simeq \bigoplus_{0 \leq i \leq r} \Omega_X^i(r-i)[-i]. \quad (6.29)$$

We may assume that X is a smooth affinoid over C , and then, by [Elk73, Theorem 7 and Remark 2], we can further assume that X is the base change to C of a smooth affinoid defined over a finite extension of K . In this case, (6.29) follows from [Bos21, Corollary 6.12]. \square

6.3. Comparison with [BMS19]. In this subsection, we compare the syntomic Fargues–Fontaine cohomology for rigid-analytic varieties over C , Definition 6.1, and the syntomic cohomology for semistable p -adic formal schemes over \mathcal{O}_C defined (in the smooth case) by Bhatt–Morrow–Scholze, [BMS19, §10]. In particular, we show that the syntomic Fargues–Fontaine cohomology can be locally recovered from the A_{inf} -cohomology together with its Nygaard filtration. The results proved here are not used in the rest of the paper, but we hope they will be useful for future reference.

As the main comparison results of this subsection will be proven in the semistable case, we begin by recalling the definition of the A_{inf} -cohomology, as well as the Nygaard filtration on it, in the latter setting. We will phrase the definition of the Nygaard filtration in terms of the décalage functors of Definition 2.2, as this will be convenient for the desired comparison.

Definition 6.23 (Nygaard filtration on A_{inf} -cohomology). Let \mathfrak{X} be a semistable p -adic formal scheme over \mathcal{O}_C , and let X denote its generic fiber, regarded as an adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$. Denote by $\nu' : X_{\mathrm{proét}} \rightarrow \mathfrak{X}_{\mathrm{ét,cond}}$ the natural morphism of sites.⁵⁶

(i) We define the A_{inf} -cohomology of \mathfrak{X} as the complex of $D(\mathrm{Mod}_{A_{\mathrm{inf}}}^{\mathrm{cond}})$

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}}), \quad \text{where} \quad A\Omega_{\mathfrak{X}} := L\eta_{\mu} R\nu'_* \mathbb{A}_{\mathrm{inf}}.$$

(ii) Given an integer $i \geq 0$, consider the function $\delta_i : \mathbb{Z} \rightarrow \mathbb{Z}, j \mapsto \max(i-j, 0)$. We endow the A_{inf} -cohomology of \mathfrak{X} with the Nygaard filtration whose i -th level is given by

$$\mathrm{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}, \mathrm{Fil}_{\mathcal{N}}^i A\Omega_{\mathfrak{X}}), \quad \text{where} \quad \mathrm{Fil}_{\mathcal{N}}^i A\Omega_{\mathfrak{X}} := L(\eta_{\delta_i, \xi} \circ \eta_{\mu}) R\nu'_* \mathbb{A}_{\mathrm{inf}}.$$

Remark 6.24. In the Definition 6.23 above we implicitly used that the functor $\eta_{\delta_i, \xi} \circ \eta_{\mu}(-)$ preserves quasi-isomorphisms. To check the latter assertion we observe that, as $\xi = \mu/\varphi^{-1}(\mu)$, we have

$$\eta_{\delta_i, \xi} \circ \eta_{\mu} = \eta_{\delta_i, \mu} \circ \eta_{-\delta_i, \varphi^{-1}(\mu)} \circ \eta_{\mu} = \eta_{\varepsilon_i, \mu} \circ \eta_{-\delta_i, \varphi^{-1}(\mu)}$$

where $\varepsilon_i : \mathbb{Z} \rightarrow \mathbb{Z}, j \mapsto \max(i, j)$. Then, since both ε_i and $-\delta_i$ are non-decreasing functions, we conclude by Proposition 2.4.

⁵⁶Here, the site $\mathfrak{X}_{\mathrm{ét,cond}}$ is defined similarly to [Bos21, Definition 2.13].

The lemma below, combined with [BMS19, Proposition 9.10], shows that the Nygaard filtration defined above is equivalent to the one of [BMS19] (in the smooth case).

First, we recall that $A\Omega_{\mathfrak{X}}$ comes equipped with a Frobenius, [CK19, (2.2.5)]: in fact, the Frobenius automorphism of \mathbb{A}_{inf} induces a $\varphi_{A_{\text{inf}}}$ -semilinear map

$$\varphi : A\Omega_{\mathfrak{X}} \rightarrow A\Omega_{\mathfrak{X}}.$$

Lemma 6.25. *Let $\tilde{\xi} := \varphi(\xi)$. Under the notation of Definition 6.23, the Frobenius $\varphi : A\Omega_{\mathfrak{X}} \rightarrow A\Omega_{\mathfrak{X}}$ factors functorially over a $\varphi_{A_{\text{inf}}}$ -semilinear quasi-isomorphism*

$$A\Omega_{\mathfrak{X}} \xrightarrow{\sim} L\eta_{\tilde{\xi}} A\Omega_{\mathfrak{X}}$$

sending the Nygaard filtration (Definition 6.23) on the source to the filtration décalée (Definition 2.9) on the target.

Proof. Let $A\Omega_{\mathfrak{X}}^{\text{psh}}$ denote the presheaf version of $A\Omega_{\mathfrak{X}}$, [CK19, §4.1]. Given an integer $i \geq 0$, it suffices to observe that the Frobenius automorphism of \mathbb{A}_{inf} induces a quasi-isomorphism

$$\varphi^* \text{Fil}_{\mathcal{N}}^i A\Omega_{\mathfrak{X}}^{\text{psh}} \simeq L(\eta_{\delta_i, \tilde{\xi}} \circ \eta_{\tilde{\xi}}) A\Omega_{\mathfrak{X}}^{\text{psh}} \simeq L\eta_{\varepsilon_i, \tilde{\xi}} A\Omega_{\mathfrak{X}}^{\text{psh}}$$

where we used that $\varphi(\mu) = \tilde{\xi} \cdot \mu$ and we denoted $\varepsilon_i : \mathbb{Z} \rightarrow \mathbb{Z}, j \mapsto \max(i, j)$. \square

Remark 6.26 (Frobenius action on Nygaard filtration). The Frobenius automorphism of \mathbb{A}_{inf} induces a $\varphi_{A_{\text{inf}}}$ -semilinear map

$$\text{Fil}_{\mathcal{N}}^*(\varphi) : \text{Fil}_{\mathcal{N}}^* A\Omega_{\mathfrak{X}} \rightarrow \tilde{\xi}^* \otimes A\Omega_{\mathfrak{X}}.$$

We are almost ready to define the syntomic cohomology of Bhatt–Morrow–Scholze in the semistable reduction case. We shall use the following notation.

Notation 6.27. We consider the Breuil–Kisin–Fargues module over A_{inf}

$$A_{\text{inf}}\{1\} := \frac{1}{\mu}(A_{\text{inf}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))$$

([BMS18, Example 4.24]) and, given $i \in \mathbb{Z}$, for any A_{inf} -module M we denote by

$$M\{i\} := M \otimes_{A_{\text{inf}}} A_{\text{inf}}\{1\}^{\otimes i}$$

its i -th Breuil–Kisin–Fargues twist.

Definition 6.28 (Bhatt–Morrow–Scholze’s syntomic cohomology, cf. [BMS19, §10]). Fix notation as in Definition 6.23. Let $i \geq 0$ be an integer. We define

$$R\Gamma_{\text{syn, BMS}}(\mathfrak{X}, \mathbb{Z}_p(i)) := \text{fib}(\text{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\text{inf}}}(\mathfrak{X})\{i\} \xrightarrow{\varphi\{i\}-1} R\Gamma_{A_{\text{inf}}}(\mathfrak{X})\{i\})$$

where $\text{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\text{inf}}}(\mathfrak{X})\{i\}$ denotes the Breuil–Kisin–Fargues twisted i -th level of the Nygaard filtration on the A_{inf} -cohomology of \mathfrak{X} , and we write $\varphi\{i\}$ for the tensor product of $\text{Fil}_{\mathcal{N}}^i(\varphi)$ (Remark 6.26) with the Frobenius of $A_{\text{inf}}\{i\}$.

Next, we want to compare Definition 6.28 with Definition 6.1. Recalling that the Fargues–Fontaine curve FF has a presentation given by the quotient of $Y_{\text{FF}, [1, p]}$ via the identification $\varphi : Y_{\text{FF}, S, [1, 1]} \cong Y_{\text{FF}, S, [p, p]}$, we will define a Nygaard filtration on the $B_{[1, p]}$ -cohomology of rigid-analytic varieties over C , and we will explain how to recover the syntomic Fargues–Fontaine cohomology from the latter.

Similarly to Definition 6.23, we can give the following.

Definition 6.29 (Nygaard filtration on B_I -cohomology). Let X be a rigid-analytic variety over C and denote by $\alpha : X_v \rightarrow X_{\text{éh,cond}}$ the natural morphism of sites. Let $I = [1, r] \subset (0, \infty)$ be an interval with rational endpoints. Given $\mathbf{B} \in \{\mathbb{B}_I, \mathbb{B}_{\text{dR}}^+\}$, we write $\mathcal{B} = \mathbf{B}_{\text{Spa}(C)_{\text{proét}}}$.

We endow the \mathcal{B} -cohomology of X with the *Nygaard filtration* whose i -th level is given by

$$\text{Fil}_{\mathcal{N}}^i R\Gamma_{\mathcal{B}}(X) := R\Gamma(X, \text{Fil}_{\mathcal{N}}^i L\eta_t R\alpha_* \mathbf{B}), \quad \text{where} \quad \text{Fil}_{\mathcal{N}}^i L\eta_t R\alpha_* \mathbf{B} := L(\eta_{\delta_i, \xi} \circ \eta_t) R\alpha_* \mathbf{B}.$$

Remark 6.30. In the Definition 6.29 above, we used Remark 6.24 together with the fact that, by the choice of the interval I , the elements t and μ differ by a unit in B_I .

We note that the Nygaard filtration on the B_{dR}^+ -cohomology agrees with its filtration décalée, since the elements t and ξ generate the same ideal in B_{dR}^+ . Moreover, we recall that the latter filtration has a more explicit expression in coordinates, as shown in Corollary 5.13. In a similar vein, we have the following result.

Lemma 6.31. *Let $\mathfrak{X} = \text{Spf}(R)$ be a semistable p -adic formal scheme over \mathcal{O}_C as in Notation 4.4, and let $X = \mathfrak{X}_C$ denote its generic fiber. Let $I = [1, r] \subset (0, \infty)$ be an interval with rational endpoints. For any $i \geq 0$, we have a B_I -linear quasi-isomorphism, compatible with Frobenius,*

$$\xi^{\max(i-\bullet, 0)} \Omega_{B_I(R)}^\bullet \xrightarrow{\sim} \text{Fil}_{\mathcal{N}}^i R\Gamma_{B_I}(X) \quad (6.30)$$

where $\Omega_{B_I(R)}^\bullet := \text{Kos}_{B_I(R)}(\partial_1, \dots, \partial_d)$, in the notation of §4.1.2.

Proof. For $i = 0$, the statement follows combining Lemma 4.8 and Lemma 4.9. Arguing by induction on $i \geq 0$, to show the statement in general it suffices to check (6.30) on graded pieces.

We begin by observing that the natural map

$$(L\eta_t R\nu_* \mathbb{B}_I) / \text{Fil}_{\mathcal{N}}^i \rightarrow (L\eta_t R\nu_* \mathbb{B}_{\text{dR}}^+) / \text{Fil}_{\mathcal{N}}^i \quad (6.31)$$

is an isomorphism. In fact, we can reduce to showing that, for each $j \geq 0$, the natural map

$$\text{gr}_{\mathcal{N}}^j L\eta_t R\nu_* \mathbb{B}_I \rightarrow \text{gr}_{\mathcal{N}}^j L\eta_t R\nu_* \mathbb{B}_{\text{dR}}^+ \quad (6.32)$$

is an isomorphism (here, the graded pieces $\text{gr}_{\mathcal{N}}^*$ refer to the Nygaard filtration $\text{Fil}_{\mathcal{N}}^*$). For this, since we can replace $L\eta_t$ with $L\eta_\xi$ on both sides of (6.32) (recall that we have an isomorphism $B_I/\xi \xrightarrow{\sim} B_{\text{dR}}^+/\xi$, and the elements t and ξ generate the same ideal in B_{dR}^+), by Proposition 2.7(ii), the claim reduces to the isomorphism

$$\text{gr}_{\mathcal{N}}^j \mathbb{B}_I \xrightarrow{\sim} \text{gr}^j \mathbb{B}_{\text{dR}}^+$$

where we denote by $\text{gr}_{\mathcal{N}}^*$ the graded pieces for the ξ -adic filtration $\text{Fil}_{\mathcal{N}}^*$ on \mathbb{B}_I .

Now, the desired statement, i.e. the quasi-isomorphism (6.30) on graded pieces, follows from the quasi-isomorphisms (5.5) of Corollary 5.13 using that, for any $j \geq 0$, the natural map $B_I(R)/\xi^j \rightarrow B_{\text{dR}}^+(R)/\xi^j$ is an isomorphism. \square

On the other hand, we have the following local description of the Nygaard filtration on the A_{inf} -cohomology.

Lemma 6.32. *Let $\mathfrak{X} = \text{Spf}(R)$ be a semistable p -adic formal scheme over \mathcal{O}_C as in Notation 4.4. For any $i \geq 0$, there is an A_{inf} -linear quasi-isomorphism, compatible with the Frobenius,*

$$\text{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \simeq \xi^{\max(i-\bullet, 0)} q\text{-}\Omega_{A_{\text{inf}}(R)}^\bullet$$

where $q\text{-}\Omega_{A_{\text{inf}}(R)}^\bullet$ denotes the logarithmic q -de Rham complex defined as⁵⁷

$$q\text{-}\Omega_{A_{\text{inf}}(R)}^\bullet := \text{Kos}_{A_{\text{inf}}(R)} \left(\frac{\partial_q}{\partial_q \log(X_1)}, \dots, \frac{\partial_q}{\partial_q \log(X_d)} \right)$$

with $q = [\varepsilon] \in A_{\text{inf}}$.

Proof. For $i = 0$ the statement follows from [Kos22, Theorem 8.1, Theorem 7.17]. The general case follows by induction on $i \geq 0$, computing the graded pieces of the filtrations, using Lemma 6.25 and Proposition 2.7(ii) (cf. [BMS19, Remark 9.11]). \square

The following proposition can be regarded as a refinement of Lemma 6.31.

Proposition 6.33 (cf. [LB18b, Proposition 3.12]). *Let \mathfrak{X} be a qcqs semistable p -adic formal scheme over \mathcal{O}_C , and let $X = \mathfrak{X}_C$ denote its generic fiber. Denote by*

$$\nu : X_{\text{proét}} \rightarrow X_{\text{ét,cond}} \quad \lambda : X_{\text{ét,cond}} \rightarrow \mathfrak{X}_{\text{ét,cond}}$$

the natural morphisms of sites, and let $\nu' : X_{\text{proét}} \rightarrow \mathfrak{X}_{\text{ét,cond}}$ be their composition.

Let $I = [1, r] \subset (0, \infty)$ be an interval with rational endpoints. For any $i \geq 0$, the natural map

$$\text{Fil}_{\mathcal{N}}^i L\eta_\mu R\nu'_* \mathbb{A}_{\text{inf}} \rightarrow R\lambda_* \text{Fil}_{\mathcal{N}}^i L\eta_\mu R\nu_* \mathbb{B}_I$$

induces a quasi-isomorphism⁵⁸

$$\text{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_I \xrightarrow{\sim} \text{Fil}_{\mathcal{N}}^i R\Gamma_{B_I}(X).$$

Proof. We can reduce to the case $\mathfrak{X} = \text{Spf}(R)$ is a semistable p -adic formal scheme over \mathcal{O}_C as in Notation 4.4. The statement for $i = 0$ is essentially contained in Lemma 4.9. In fact, combining Lemma 4.8 and the quasi-isomorphism (4.13) in the proof of Lemma 4.9, we have a quasi-isomorphism

$$R\Gamma_{B_I}(X) \simeq \text{Kos}_{B_I(R)} \left(\frac{\gamma_1 - 1}{t}, \dots, \frac{\gamma_d - 1}{t} \right).$$

Therefore, using Lemma 6.32 for $i = 0$, and recalling that t and $\mu = q - 1$ differ by a unit in B_I , it suffices to check that the natural map

$$\text{Kos}_{A_{\text{inf}}(R)} \left(\frac{\gamma_1 - 1}{q - 1}, \dots, \frac{\gamma_d - 1}{q - 1} \right) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_I \rightarrow \text{Kos}_{B_I(R)} \left(\frac{\gamma_1 - 1}{q - 1}, \dots, \frac{\gamma_d - 1}{q - 1} \right)$$

is a quasi-isomorphism; this can be done as in the proof of (4.17) in Lemma 4.9. Finally, the statement in general follows arguing by induction on $i \geq 0$, calculating again the graded pieces of the filtrations. \square

We can finally state and prove the main result of this subsection, which in particular tells us how the syntomic Fargues–Fontaine cohomology can be locally recovered from the A_{inf} -cohomology together with its Nygaard filtration.

Proposition 6.34. *Let X be a rigid-analytic variety over C . Denote $I = [1, p]$ and $I' = [1, 1]$. Let $i \geq 0$ be an integer.*

(i) *We have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$*

$$R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \simeq \text{fib}(\text{Fil}_{\mathcal{N}}^i R\Gamma_{B_I}(X) \xrightarrow{\varphi^{p^{-i}-1}} R\Gamma_{B_{I'}}(X)).$$

⁵⁷See also [Kos22, §7.3] for the construction of the logarithmic q -de Rham complex in a more general setting.

⁵⁸Here, we use the fact that μ and t differ by a unit in B_I .

(ii) Assume that X is the generic fiber of a qcqs semistable p -adic formal scheme \mathfrak{X} over \mathcal{O}_C . Then, we have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$

$$R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \simeq \text{fib}(\text{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\text{inf}}}(\mathfrak{X})\{i\} \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_I \xrightarrow{\varphi^{\{i\}-1}} R\Gamma_{A_{\text{inf}}}(\mathfrak{X})\{i\} \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_{I'}).$$

In particular, there is a natural morphism

$$R\Gamma_{\text{syn,BMS}}(\mathfrak{X}, \mathbb{Z}_p(i)) \longrightarrow R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)).$$

Proof. For part (i), using the isomorphism

$$R\Gamma_{B_I}(X)/\text{Fil}_{\mathcal{N}}^i \xrightarrow{\sim} R\Gamma_{B_{\text{dR}}^+}(X)/\text{Fil}^i$$

coming from (6.31), we have a natural isomorphism

$$\text{fib}(\text{Fil}_{\mathcal{N}}^i R\Gamma_{B_I}(X) \xrightarrow{\varphi^{p^{-i}-1}} R\Gamma_{B_{I'}}(X)) \simeq \text{fib}(R\Gamma_{B_I}(X)^{\varphi=p^i} \rightarrow R\Gamma_{B_{\text{dR}}^+}(X)/\text{Fil}^i) \quad (6.33)$$

where

$$R\Gamma_{B_I}(X)^{\varphi=p^i} := \text{fib}(R\Gamma_{B_I}(X) \xrightarrow{\varphi^{p^{-i}-1}} R\Gamma_{B_{I'}}(X)).$$

Then, combining (6.33) with Theorem 6.3(ii), it remains to show that the natural map

$$R\Gamma_B(X)^{\varphi=p^i} \rightarrow R\Gamma_{B_I}(X)^{\varphi=p^i} \quad (6.34)$$

is an isomorphism. For this, in the notation of §6.2, we observe that by Proposition 6.8 the source of (6.34) is computed by the cohomology of $\mathcal{H}_{\text{FF}}(X) \otimes \mathcal{O}(i) \in \text{Nuc}(\text{FF})$. But the latter cohomology also computes the target of (6.34), using the presentation of the curve FF as the quotient of $Y_{\text{FF},[1,p]}$ via the identification $\varphi : Y_{\text{FF},[1,1]} \cong Y_{\text{FF},[p,p]}$. This concludes the proof of part (i).

For part (ii), by part (i) and Proposition 6.33 we have a natural isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{cond}})$

$$R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \simeq \text{fib}(\text{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_I \xrightarrow{\varphi^{p^{-i}-1}} R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_{I'}). \quad (6.35)$$

On the other hand, trivializing the Breuil–Kisin–Fargues twists we can rewrite the fiber in the statement of part (ii) as

$$\text{fib}(\text{Fil}_{\mathcal{N}}^i R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_I \xrightarrow{\varphi^{\tilde{\xi}^{-i}-1}} R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}}^{\text{L}\blacksquare} B_{I'}) \quad (6.36)$$

where $\tilde{\xi} := \varphi(\xi)$. We conclude observing that, writing $\mu = ut$ with u unit in B_I , the multiplication by u^i map induces an isomorphism between the fiber in (6.36) and the fiber in (6.35). \square

6.4. Comparison with [CN20]. In this subsection, we show that, in high degrees, the syntomic Fargues–Fontaine cohomology does not agree with the syntomic cohomology for smooth rigid-analytic varieties over C defined by Colmez–Nizioł.

Notation 6.35. Let X be a smooth rigid-analytic variety over C . Let $i \geq 0$ be an integer. We denote by $R\Gamma_{\text{syn,CN}}(X, \mathbb{Q}_p(i))$ the *syntomic cohomology of X with coefficients in $\mathbb{Q}_p(i)$ of Colmez–Nizioł*, defined in [CN20, §4.1].

Example 6.36. Let $X = \mathbb{P}_C^1$ be the rigid-analytic projective line over C . We claim that

$$H_{\text{syn,FF}}^3(X, \mathbb{Q}_p(0)) \cong C/\mathbb{Q}_p, \quad H_{\text{syn,CN}}^3(X, \mathbb{Q}_p(0)) \cong 0. \quad (6.37)$$

For this, applying Theorem 6.3(i), combined with Theorem 4.1, for the syntomic Fargues–Fontaine cohomology, and [CN21a, Corollary 5.5] for the syntomic cohomology of Colmez–Nizioł, we obtain respectively

$$H_{\text{syn,FF}}^3(X, \mathbb{Q}_p(0)) \cong (H_{\text{HK}}^2(X) \otimes_{\check{F}} B)/(\varphi - 1), \quad H_{\text{syn,CN}}^3(X, \mathbb{Q}_p(0)) \cong (H_{\text{HK}}^2(X) \otimes_{\check{F}} B_{\text{cris}}^+)/(\varphi - 1)$$

where we used that $H_{\text{HK}}^3(X) \cong 0$. Now, the map $\varphi - 1$ is surjective on $H_{\text{HK}}^2(X) \otimes_{\check{F}} B_{\text{cris}}^+$ (see e.g. [CN17, Remark 2.30]). Instead, observing that $H_{\text{HK}}^2(X)$ is a one-dimensional φ -module over \check{F} with slope 1 (by Theorem 3.29(i), and [CLS99, Théorème 3.1.2]⁵⁹), we deduce that the vector bundle on the Fargues–Fontaine curve FF associated to $H_{\text{HK}}^2(X)$ is isomorphic to $\mathcal{O}(-1)$; hence, we can identify $(H_{\text{HK}}^2(X) \otimes_{\check{F}} B)/(\varphi - 1)$ with $H^1(\text{FF}, \mathcal{O}(-1))$, thus showing (6.37).

7. APPLICATIONS

In this section, we gather the results we have obtained so far, giving some applications.

7.1. Fundamental diagrams of rational p -adic Hodge theory.

Theorem 7.1. *Let X be a qcqs rigid-analytic variety defined over K . We have a \mathcal{G}_K -equivariant pullback square in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

$$\begin{array}{ccc} R\Gamma_{\text{proét}}(X_C, \mathbb{Q}_p) & \longrightarrow & (R\Gamma_{\text{HK}}(X_C) \otimes_{\check{F}}^{\text{L}\blacksquare} B_{\log}[1/t])^{N=0, \varphi=1} \\ \downarrow & & \downarrow \\ \text{Fil}^0(R\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\blacksquare} B_{\text{dR}}) & \longrightarrow & R\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\blacksquare} B_{\text{dR}}. \end{array}$$

Proof. First, we note that we can rewrite the fundamental exact sequence (2.4) of p -adic Hodge theory on the pro-étale site $X_{C, \text{proét}}$ as a pullback square

$$\begin{array}{ccc} \mathbb{Q}_p & \longrightarrow & \mathbb{B}[1/t]^{\varphi=1} \\ \downarrow & & \downarrow \\ \mathbb{B}_{\text{dR}}^+ & \longrightarrow & \mathbb{B}_{\text{dR}}. \end{array}$$

Then, the statement follows combining Theorem 4.1, [Bos21, Theorem 6.5] (together with Theorem 5.2 for the singular case), as well as the compatibility proven in Theorem 5.3. In fact, by Theorem 4.1, using that X is qcqs, we have

$$R\Gamma_{\text{proét}}(X, \mathbb{B}[1/t]^{\varphi=1}) \simeq R\Gamma_B(X)[1/t]^{\varphi=1} \simeq (R\Gamma_{\text{HK}}(X_C) \otimes_{\check{F}}^{\text{L}\blacksquare} B_{\log}[1/t])^{N=0, \varphi=1}$$

where in the last step we used that $\otimes_{\check{F}}^{\text{L}\blacksquare}$ commutes with filtered colimits. \square

We invite the reader to compare the following result with [CN21a].

⁵⁹In this case, in order to deduce that $H_{\text{HK}}^2(X)$ has slope 1, it is sufficient to use the weak Lefschetz theorem for crystalline cohomology, [Ber73].

Theorem 7.2. *Let X be a connected, paracompact, rigid-analytic variety defined over K . For any $i \geq 0$, we have a \mathcal{G}_K -equivariant isomorphism in $D(\text{Mod}_{\mathbb{Q}_p}^{\text{solid}})$*

$$\tau^{\leq i} R\Gamma_{\text{proét}}(X_C, \mathbb{Q}_p(i)) \simeq \tau^{\leq i} \text{fib}((R\Gamma_{\text{HK}}(X_C) \otimes_{\check{F}}^{\mathbf{L}} B_{\log})^{N=0, \varphi=p^i} \rightarrow (R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+)/\text{Fil}^i).$$

Proof. This follows combining Theorem 6.3, Theorem 4.1, Theorem 5.1 and Theorem 5.3. \square

In some special cases, we can explicitly compute the cohomology groups of the de Rham contribution in the fiber sequence of Theorem 7.2.

Proposition 7.3. *Let X be a rigid-analytic variety over K . Let $i \geq 0$, and denote*

$$\text{dR}(X, i) := (R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+)/\text{Fil}^i.$$

(i) *If X is proper, for any $j \geq 0$, we have a \mathcal{G}_K -equivariant isomorphism in $\text{Mod}_K^{\text{solid}}$*

$$H^j(\text{dR}(X, i)) \cong (H_{\text{dR}}^j(X) \otimes_K B_{\text{dR}}^+)/\text{Fil}^i.$$

(ii) *If X is a smooth affinoid or Stein space, for $j \geq i$ we have $H^j(\text{dR}(X, i)) = 0$, and for $0 \leq j < i$ we have a \mathcal{G}_K -equivariant exact sequence in $\text{Mod}_K^{\text{solid}}$*

$$0 \rightarrow (\Omega^j(X_C)/\text{im } d)(i - j - 1) \rightarrow H^j(\text{dR}(X, i)) \rightarrow H_{\text{dR}}^j(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+/t^{i-j-1} \rightarrow 0.$$

Proof. One can argue similarly to the proof of [Bos21, Corollary 6.17]. Part (i) follows from (the proof of) Proposition 6.21(ii). For part (ii), using Tate's acyclicity theorem for affinoid spaces, and Kiehl's acyclicity theorem for Stein spaces (which hold true in the condensed setting, [Bos21, Lemma 5.6(i), Lemma 5.9]), and relying crucially on the flatness of the K -Fréchet space B_{dR}^+ (and its filtered pieces) for the solid tensor product $\otimes_K^{\mathbf{L}}$ ([Bos21, Corollary A.65]), we have

$$\text{dR}(X, i) \simeq [\mathcal{O}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+/t^i \rightarrow \Omega^1(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+/t^{i-1} \rightarrow \dots \rightarrow \Omega^{i-1}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+/t]$$

from which one readily deduces the statement. \square

7.2. Proper spaces. In this subsection, we prove a version of the semistable conjecture for proper (possibly singular) rigid-analytic varieties over C . We remark that in the smooth case the following result is already known, [CN21a, Theorem 5.8]; however, already in the latter case our proof is different from *loc. cit.* as it does not rely on Fontaine–Messing syntomic cohomology.

Theorem 7.4. *Let X be a proper rigid-analytic variety over C . For each $i \geq 0$, we have a natural isomorphism*

$$H_{\text{ét}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\log}[1/t] \cong H_{\text{HK}}^i(X) \otimes_{\check{F}} B_{\log}[1/t] \quad (7.1)$$

compatible with the actions of the Frobenius φ and the monodromy N , and inducing a natural isomorphism

$$H_{\text{ét}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H_{\text{inf}}^i(X/B_{\text{dR}}^+) \otimes_{B_{\text{dR}}^+} B_{\text{dR}} \quad (7.2)$$

compatible with filtrations. In particular, we have a natural isomorphism

$$H_{\text{ét}}^i(X, \mathbb{Q}_p) \cong (H_{\text{HK}}^i(X) \otimes_{\check{F}} B_{\log}[1/t])^{\varphi=1, N=0} \cap \text{Fil}^0(H_{\text{inf}}^i(X/B_{\text{dR}}^+) \otimes_{B_{\text{dR}}^+} B_{\text{dR}}). \quad (7.3)$$

Here, the filtration on $H_{\text{inf}}^i(X/B_{\text{dR}}^+)$ is defined by

$$\text{Fil}^* H_{\text{inf}}^i(X/B_{\text{dR}}^+) := \text{im}(H^i(\text{Fil}^* R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+)) \rightarrow H_{\text{inf}}^i(X/B_{\text{dR}}^+))$$

where $R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+)$ is endowed with the Hodge filtration (Definition 5.7).

Proof. Let us fix $i \geq 0$. By the properness of X , using Scholze's primitive comparison theorem, and the finiteness of the \mathbb{Q}_p -vector space $H_{\text{ét}}^i(X, \mathbb{Q}_p)$, [Sch13b, Theorem 3.17], we have a natural isomorphism of vector bundles on the Fargues–Fontaine curve FF

$$H_{\text{ét}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{FF}} \cong \mathcal{E}(H_{\text{proét}}^i(X, \mathbb{B}_e), H_{\text{proét}}^i(X, \mathbb{B}_{\text{dR}}^+)) \quad (7.4)$$

where the right-hand side of (7.4) denotes the vector bundle on FF associated to the B -pair $(H_{\text{proét}}^i(X, \mathbb{B}_e), H_{\text{proét}}^i(X, \mathbb{B}_{\text{dR}}^+))$ with $\mathbb{B}_e = \mathbb{B}[1/t]^{\varphi=1}$. We claim that (7.4) is naturally isomorphic to the vector bundle on FF associated to the B -pair

$$(H_{\text{HK}}^i(X) \otimes_{\check{F}} B_{\log}[1/t])^{N=0, \varphi=1}, \text{Fil}^0(H^i(X/B_{\text{dR}}^+) \otimes_{B_{\text{dR}}^+} B_{\text{dR}})). \quad (7.5)$$

For this, using Theorem 4.1, and the perfectness of $R\Gamma_{\text{HK}}(X)$ over \check{F} proven in Theorem 3.30(i) (combined with Proposition 3.26), we have a natural isomorphism

$$R\Gamma_{\text{proét}}(X, \mathbb{B}_e) \simeq (R\Gamma_{\text{HK}}(X) \otimes_{\check{F}} B_{\log}[1/t])^{N=0, \varphi=1}. \quad (7.6)$$

Taking cohomology of (7.6), by Lemma 7.5 combined with Lemma 7.6, we have a natural isomorphism

$$H_{\text{proét}}^i(X, \mathbb{B}_e) \cong (H_{\text{HK}}^i(X) \otimes_{\check{F}} B_{\log}[1/t])^{N=0, \varphi=1}. \quad (7.7)$$

Moreover, by Theorem 5.9 and Proposition 6.21, we have a natural isomorphism

$$H_{\text{proét}}^i(X, \mathbb{B}_{\text{dR}}^+) \cong \text{Fil}^0(H^i(X/B_{\text{dR}}^+) \otimes_{B_{\text{dR}}^+} B_{\text{dR}}). \quad (7.8)$$

Hence, the desired claim follows combining the isomorphisms (7.7) and (7.8), and the compatibility shown in Theorem 5.20(i).

Now, we are ready to prove that we have a natural isomorphism (7.1) as in the statement. From what we have shown above, applying $H^0(\text{FF}, -)$ to (7.4) we obtain (7.3), from which we deduce that we have a natural $B_{\log}[1/t]$ -linear injective map

$$H_{\text{ét}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\log}[1/t] \rightarrow H_{\text{HK}}^i(X) \otimes_{\check{F}} B_{\log}[1/t] \quad (7.9)$$

compatible with the actions of the Frobenius φ and the monodromy N . To conclude that (7.9) is an isomorphism, we observe that

$$\dim_{\mathbb{Q}_p} H_{\text{ét}}^i(X, \mathbb{Q}_p) = \dim_{\check{F}} H_{\text{HK}}^i(X).$$

For this, we note that $\dim_{\mathbb{Q}_p} H_{\text{ét}}^i(X, \mathbb{Q}_p)$ is equal to the rank of the vector bundle (7.4) on FF, and hence, from what we have shown above, it is equal to the rank of the vector bundle associated to the B -pair (7.5); but the latter is a modification at ∞ of the vector bundle on FF associated to the finite (φ, N) -module $H_{\text{HK}}^i(X)$ over \check{F} , whose rank is $\dim_{\check{F}} H_{\text{HK}}^i(X)$.

Lastly, we have that the isomorphism (7.1) induces an isomorphism (7.2) which is compatible with filtrations, recalling again Theorem 5.20, Theorem 5.9, and Proposition 6.21. \square

We used the following general results.

Lemma 7.5. *For any finite φ -module (V, φ) over \check{F} , the map*

$$\varphi - 1 : V \otimes_{\check{F}} B[1/t] \rightarrow V \otimes_{\check{F}} B[1/t]$$

is surjective.

Proof. It suffices to show that for any sufficiently big integer m , the map

$$\varphi - 1 : V \otimes_{\check{F}} t^{-m} B \rightarrow V \otimes_{\check{F}} t^{-m} B \quad (7.10)$$

is surjective. For this, we consider $\mathcal{E} := \mathcal{E}(V, \varphi)$ the vector bundle on the Fargues–Fontaine curve FF associated to (V, φ) , and, for any integer m , the vector bundle $\mathcal{E}(m) := \mathcal{E} \otimes \mathcal{O}(m)$ on FF. Note that we have

$$\begin{aligned} R\Gamma(\text{FF}, \mathcal{E}(m)) &= [H^0(Y_{\text{FF}}, \mathcal{E}|_{Y_{\text{FF}}}) \xrightarrow{\varphi p^{-m}-1} H^0(Y_{\text{FF}}, \mathcal{E}|_{Y_{\text{FF}}})] \\ &= [V \otimes_{\check{F}} B \xrightarrow{\varphi p^{-m}-1} V \otimes_{\check{F}} B]. \end{aligned}$$

Then, since for any integer m sufficiently big such that the vector bundle $\mathcal{E}(m)$ has non-negative Harder–Narasimhan slopes, one has $H^1(\text{FF}, \mathcal{E}(m)) = 0$, we deduce that (recalling that $\varphi(t) = pt$) for any such m the map (7.10) is surjective, as desired. \square

Lemma 7.6. *For any finite (φ, N) -module (V, φ, N) over \check{F} , we have a short exact sequence*

$$0 \rightarrow V \otimes_{\check{F}} B \xrightarrow{\alpha} V \otimes_{\check{F}} B_{\log} \xrightarrow{N} V \otimes_{\check{F}} B_{\log} \rightarrow 0 \quad (7.11)$$

where, recalling that $B_{\log} = B[U]$ (Definition 2.27), the morphism α is induced by the isomorphism of finite φ -modules over \check{F}

$$\exp(N \cdot U) : V \otimes_{\check{F}} B \xrightarrow{\sim} (V \otimes_{\check{F}} B_{\log})^{N=0} : x \mapsto \sum_{j \geq 0} \frac{(-1)^j}{j!} N^j(x) \cdot U^j.$$

Proof. For any finite (φ, N) -module (V, φ, N) over \check{F} , the monodromy operator N has finite nilpotency index. If the nilpotency index is 1, i.e. $N = 0$ on V , the statement follows from the exactness of the sequence (7.11) for $V = \check{F}$. The statement in the general case follows by induction on such nilpotency index (cf. the proof of [CDN20b, Lemma 3.20]). \square

7.3. Smooth Stein spaces. In this subsection, our goal is to prove the following result, Theorem 7.7. We remark that it could be deduced from Theorem 7.2, Proposition 7.3(ii) and the theory of Banach–Colmez spaces, as done in [CDN20b]. However, we give here a more direct proof using the relative fundamental exact sequence of p -adic Hodge theory.

Theorem 7.7 (cf. [CDN20b], [CN21b]). *Let X be a smooth Stein space over C . For any $i \geq 0$, we have a commutative diagram in $\text{Mod}_{\mathbb{Q}_p}^{\text{solid}}$ with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^{i-1}(X)/\ker d & \longrightarrow & H_{\text{proét}}^i(X, \mathbb{Q}_p(i)) & \longrightarrow & (H_{\text{HK}}^i(X) \otimes_{\check{F}}^{\blacksquare} B_{\log})^{N=0, \varphi=p^i} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega^{i-1}(X)/\ker d & \longrightarrow & \Omega^i(X)^{d=0} & \longrightarrow & H_{\text{dR}}^i(X) \longrightarrow 0. \end{array}$$

7.3.1. Recollections on Banach–Colmez spaces. As a preparation for the proof of Theorem 7.7, we need further reminders on the category of Banach–Colmez spaces \mathcal{BC} (Definition 6.10).

Recall that we denote by

$$\tau : \text{FF}_v \rightarrow \text{Spa}(C, \mathcal{O}_C)_v$$

the natural morphism of v -sites.

We remark that the equivalence of derived categories (6.16) in Proposition 6.11 does not preserve the natural t -structures. In fact, Le Bras showed that suitably changing t -structure on the source of (6.16) one can pass to the category of BC spaces:

Proposition 7.8 ([LB18a, Théorème 1.2]). *The category of Banach–Colmez spaces \mathcal{BC} is equivalent, via the functor τ_* , to the full subcategory of $\text{Perf}(\text{FF})$ having as objects the perfect complexes \mathcal{F} concentrated in cohomological degrees $[-1, 0]$ such that $H^{-1}(\mathcal{F})$ has negative Harder–Narasimhan slopes and $H^0(\mathcal{F})$ has non-negative Harder–Narasimhan slopes.*

In the following, for \mathcal{E} a vector bundle on FF, we denote the BC spaces

$$\mathcal{H}^0(\text{FF}, \mathcal{E}) := \tau_*\mathcal{E}, \quad \mathcal{H}^1(\text{FF}, \mathcal{E}) := R^1\tau_*\mathcal{E}. \quad (7.12)$$

Let us recall that Colmez defined a *Dimension* function on BC spaces ([Col02, §6])

$$\text{Dim} = (\dim, \text{ht}) : \mathcal{BC} \rightarrow \mathbb{N} \times \mathbb{Z}$$

where \dim is called *dimension* and ht *height*.⁶⁰ In terms of the Fargues–Fontaine curve, the function Dim is characterized by the following properties:

- (i) The function Dim is additive in short exact sequences;
- (ii) For any vector bundle \mathcal{E} on FF, denoting by $\deg(\mathcal{E})$ the degree of \mathcal{E} and by $\text{rk}(\mathcal{E})$ its rank,

$$\chi(\mathcal{E}) := \text{Dim } \mathcal{H}^0(\text{FF}, \mathcal{E}) - \text{Dim } \mathcal{H}^1(\text{FF}, \mathcal{E}) = (\deg(\mathcal{E}), \text{rk}(\mathcal{E})) \quad (7.13)$$

where $\chi(\mathcal{E})$ is the *Euler–Poincaré characteristic* of \mathcal{E} (see e.g. [FF18, Préface, Remarque 4.6]).

Remark 7.9 (BC spaces vs. condensed \mathbb{Q}_p -vector spaces). Let \mathcal{E} be a vector bundle on FF. In the following, we will need to consider the cohomology groups of \mathcal{E} on FF as condensed \mathbb{Q}_p -vector spaces. For this, we denote by

$$f : \text{FF}_{\text{proét}} \rightarrow \text{Spa}(C, \mathcal{O}_C)_{\text{proét}}$$

the natural morphism of (small) sites, and we define

$$H^0(\text{FF}, \mathcal{E}) := f_*\mathcal{E}, \quad H^1(\text{FF}, \mathcal{E}) := R^1f_*\mathcal{E}.$$

Note that the condensed structure on such cohomology groups is just the “shadow” of the structure of BC spaces (7.12). As an example, we have the identification $\mathcal{H}^0(\text{FF}, \mathcal{O}(-1)) = (\mathbb{A}_C^1)^\diamond / \underline{\mathbb{Q}}_p$ as BC spaces, which restricted to the site $\text{Spa}(C, \mathcal{O}_C)_{\text{proét}}$ gives the identification $H^0(\text{FF}, \mathcal{O}(-1)) = C/\mathbb{Q}_p$ as condensed \mathbb{Q}_p -vector spaces.

We are ready to prove the main result of this subsection.

Proof of Theorem 7.7. Let X^\dagger be the smooth Stein dagger space associated with X (via [GK00, Theorem 2.27]). Choose $\{U_n^\dagger\}_{n \in \mathbb{N}}$ a Stein covering of X^\dagger , and denote by $\{U_n\}_{n \in \mathbb{N}}$ the corresponding Stein covering of X (i.e. set $U_n := \widehat{U}_n^\dagger$). Fix $n \in \mathbb{N}$, and let $V^\dagger := U_n^\dagger$. Our first goal is to show that we have a diagram as in the statement replacing X with V^\dagger .

Denoting by $\varinjlim_h V_h$ the presentation of a dagger structure on V corresponding to V^\dagger (recall Lemma 3.22(i)), for \mathbb{M} a sheaf on the big pro-étale site $\text{Rig}_{C, \text{proét}}$, we set

$$R\Gamma_{\text{proét}}(V^\dagger, \mathbb{M}) := \text{colim}_{h \in \mathbb{N}} R\Gamma(V_h, \mathbb{M}).$$

⁶⁰We refer the reader to [LB18a, §7.2] for the relation between Colmez’s original definition of *Espace de Banach de dimension finie* and Definition 6.10.

With this definition, by the relative fundamental exact sequence of p -adic Hodge theory (2.4), we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{\text{proét}}^i(V^\dagger, \mathbb{Q}_p) & \longrightarrow & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_e) & \xrightarrow{\alpha_i} & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \parallel \\ \cdots & \longrightarrow & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_{\text{dR}}^+) & \longrightarrow & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_{\text{dR}}) & \xrightarrow{\beta_i} & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+) \longrightarrow \cdots \end{array}$$

from which we obtain the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{coker } \alpha_{i-1} & \longrightarrow & H_{\text{proét}}^i(V^\dagger, \mathbb{Q}_p) & \longrightarrow & \ker \alpha_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{coker } \beta_{i-1} & \longrightarrow & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_{\text{dR}}^+) & \longrightarrow & \ker \beta_i \longrightarrow 0. \end{array} \quad (7.14)$$

By [Elk73, Theorem 7 and Remark 2], we may assume that V^\dagger is the base change to C of a smooth dagger affinoid V_0^\dagger defined over a finite extension L of K . Then, using Theorem 4.1 together with the perfectness of $R\Gamma_{\text{HK}}(V^\dagger)$ over \check{F} (Theorem 3.30(i)), Lemma 7.5 and Lemma 7.6 in order to compute $H_{\text{proét}}^i(V^\dagger, \mathbb{B}_e)$, and relying on [Bos21, Corollary 6.17(ii)] to determine $H_{\text{proét}}^i(V^\dagger, \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+)$, recalling the compatibility proven in Theorem 5.3(ii), we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & (H_{\text{HK}}^i(V^\dagger) \otimes_{\check{F}} B_{\log}[1/t])^{N=0, \varphi=1} & \xrightarrow{\sim} & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_e) & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow \gamma_i & & \downarrow \alpha_i & & \downarrow \\ 0 & \longrightarrow & H_{\text{dR}}^i(V_0^\dagger) \otimes_L B_{\text{dR}}/t^{-i}B_{\text{dR}}^+ & \longrightarrow & H_{\text{proét}}^i(V^\dagger, \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+) & \longrightarrow & (\Omega^i(V^\dagger)/\ker d)(-i-1) \longrightarrow 0. \end{array}$$

We claim that

$$\ker \gamma_i = (H_{\text{HK}}^i(V^\dagger) \otimes_{\check{F}} B_{\log})^{N=0, \varphi=p^i} \quad \text{and} \quad \text{coker } \gamma_i = 0. \quad (7.15)$$

For this, we consider the vector bundle $\mathcal{E}(H_{\text{HK}}^i(V^\dagger))$ on the Fargues–Fontaine curve FF associated to the finite (φ, N) -module $H_{\text{HK}}^i(V^\dagger)$ over \check{F} . By Theorem 3.30(ii), the vector bundle $\mathcal{E} = \mathcal{E}(H_{\text{HK}}^i(V^\dagger)) \otimes \mathcal{O}(i)$ has non-negative Harder–Narasimhan slopes, in particular $H^1(\text{FF}, \mathcal{E}) = 0$, and we have the short exact sequence

$$0 \rightarrow H^0(\text{FF}, \mathcal{E}) \rightarrow H^0(\text{FF} \setminus \{\infty\}, \mathcal{E}) \rightarrow \mathcal{E}_\infty^\wedge[1/t]/\mathcal{E}_\infty^\wedge \rightarrow 0.$$

Note that such short exact sequence identifies with the short exact sequence

$$0 \rightarrow (H_{\text{HK}}^i(V^\dagger) \otimes_{\check{F}} B_{\log})^{N=0, \varphi=p^i} \rightarrow (H_{\text{HK}}^i(V^\dagger) \otimes_{\check{F}} B_{\log}[1/t])^{N=0, \varphi=1} \xrightarrow{\gamma_i} H_{\text{dR}}^i(V_0^\dagger) \otimes_L B_{\text{dR}}/t^{-i}B_{\text{dR}}^+ \rightarrow 0$$

thus proving the claim (7.15).

Then, twisting by (i) the diagram (7.14), putting everything together we deduce that, for each $n \in \mathbb{N}$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega^{i-1}(U_n^\dagger)/\ker d & \longrightarrow & H_{\text{proét}}^i(U_n^\dagger, \mathbb{Q}_p(i)) & \longrightarrow & (H_{\text{HK}}^i(U_n^\dagger) \otimes_{\check{F}} B_{\log})^{N=0, \varphi=p^i} \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega^{i-1}(U_n^\dagger)/\ker d & \longrightarrow & \Omega^i(U_n^\dagger)^{d=0} & \longrightarrow & H_{\text{dR}}^i(U_n^\dagger) \longrightarrow 0.
\end{array} \tag{7.16}$$

Now, since we have

$$R\Gamma_{\text{proét}}(X, \mathbb{Q}_p(i)) = R\Gamma_{\text{proét}}(X^\dagger, \mathbb{Q}_p(i)) = R\varprojlim_n R\Gamma_{\text{proét}}(U_n^\dagger, \mathbb{Q}_p(i))$$

and similarly

$$R\Gamma_{\text{HK}}(X) = R\varprojlim_n R\Gamma_{\text{HK}}(U_n^\dagger), \quad R\Gamma_{\text{dR}}(X) = R\varprojlim_n R\Gamma_{\text{dR}}(U_n^\dagger)$$

(see Proposition 3.26 and cf. [CN20, Proposition 3.17]), recalling the property [Bos21, Corollary A.67] of the solid tensor product, the statement follows taking the inverse limit of (7.16) over $n \in \mathbb{N}$, observing the following $R^1\varprojlim$ vanishing statements.

- Using that $\{U_n\}_{n \in \mathbb{N}}$ is a Stein covering of X , by the condensed version of the Mittag-Leffler criterion for Banach spaces, [Bos21, Lemma A.37], we have that

$$R^1\varprojlim_n \Omega^i(U_n^\dagger) = R^1\varprojlim_n \Omega^i(U_n) = 0.$$

- Since $H_{\text{dR}}^i(U_n^\dagger)$, for varying $n \in \mathbb{N}$, are finite-dimensional condensed C -vector spaces, we have that

$$R^1\varprojlim_n H_{\text{dR}}^i(U_n^\dagger) = 0.$$

- Lastly, we have

$$R^1\varprojlim_n (H_{\text{HK}}^i(U_n^\dagger) \otimes_{\check{F}} B_{\log})^{N=0, \varphi=p^i} = 0. \tag{7.17}$$

The claim (7.17) is essentially contained in the proof of [CDN20b, Lemma 3.28] (that we write here in slightly different terms). By the Mittag-Leffler criterion for condensed abelian groups,⁶¹ it suffices to show that in the inverse system $\{(H_{\text{HK}}^i(U_n^\dagger) \otimes_{\check{F}} B_{\log})^{N=0, \varphi=p^i}, f_{nm}\}$, for each $n \in \mathbb{N}$ there exists $k \geq m$ such that, for every $m \geq k$, the image of the maps f_{nm} are equal to the image of f_{nk} . For this, recall that, for $\mathcal{E}_n = \mathcal{E}(H_{\text{HK}}^i(U_n^\dagger)) \otimes \mathcal{O}(i)$, we have $H^0(\text{FF}, \mathcal{E}_n) \cong (H_{\text{HK}}^i(U_n^\dagger) \otimes_{\check{F}} B)^{\varphi=p^i}$ (trivializing the monodromy), and $H^1(\text{FF}, \mathcal{E}_n) = 0$. Considering the Euler-Poincaré characteristic of \mathcal{E}_n , by (7.13) we have that

$$\text{Dim } \mathcal{H}^0(\text{FF}, \mathcal{E}_n) = (\text{deg}(\mathcal{E}_n), \text{rk}(\mathcal{E}_n))$$

we deduce that the BC space $\mathcal{H}^0(\text{FF}, \mathcal{E}_n)$ has height ≥ 0 ; moreover, by the characterization of BC spaces in terms of FF, Proposition 7.8, any BC subspace of $\mathcal{H}^0(\text{FF}, \mathcal{E}_n)$ has height ≥ 0 . Thus, in the inverse system $\{\mathcal{H}^0(\text{FF}, \mathcal{E}_n), f_{nm}\}$, for each $n \in \mathbb{N}$ the image of the maps f_{nm} form a chain of BC spaces with decreasing Dimension (for the lexicographic order on $\mathbb{N} \times \mathbb{Z}$) and height ≥ 0 ; in particular, such chain eventually stabilizes, as desired.

⁶¹It follows from the Mittag-Leffler criterion for abelian groups, [Gro61, Proposition 13.2.2], applying the latter to the values on extremally disconnected sets, and using [Sch13a, Lemma 3.18].

□

As a consequence of Theorem 7.7 we have the following result.

Corollary 7.10. *Let X be a smooth Stein space over C . Then, we have*

$$H_{\text{proét}}^i(X, \mathbb{Q}_p) = 0$$

for all $i > \dim X$.

Proof. Using Theorem 7.7, it suffices to prove that $H_{\text{HK}}^i(X) = 0$ for all $i > \dim X$. Thus, by Corollary 3.16, we can reduce to showing that $H_{\text{dR}}^i(X) = 0$ for all $i > \dim X$, which holds true by Kiehl's acyclicity theorem for Stein spaces (cf. [Bos21, Lemma 5.9] for the statement in the condensed setting). □

7.4. Smooth affinoid spaces. In view of Theorem 7.7 and Proposition 7.3(ii), it is natural to formulate the following conjecture.

Conjecture 7.11. *Let X be a smooth affinoid rigid space over C . For any $i \geq 0$, we have a commutative diagram in $\text{Mod}_{\mathbb{Q}_p}^{\text{solid}}$ with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^{i-1}(X)/\ker d & \longrightarrow & H_{\text{proét}}^i(X, \mathbb{Q}_p(i)) & \longrightarrow & (H_{\text{HK}}^i(X) \otimes_{\mathbb{F}}^{\mathbf{L}} B_{\log})^{N=0, \varphi=p^i} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega^{i-1}(X)/\ker d & \longrightarrow & \Omega^i(X)^{d=0} & \longrightarrow & H_{\text{dR}}^i(X) \longrightarrow 0. \end{array}$$

The goal of this subsection is to show Conjecture 7.11 for curves.

Theorem 7.12. *Conjecture 7.11 holds true for X a smooth affinoid rigid space over C of dimension 1.*

Proof. By [Elk73, Theorem 7 and Remark 2], we can assume that X is the base change to C of a smooth affinoid X_0 defined over a finite extension of K , and, without loss of generality, we can further assume that X_0 is defined over K .

We recall that by Theorem 6.3(ii) combined with Theorem 4.1, Theorem 5.1 and Theorem 5.3, we have, for any $i \geq 0$, the following commutative diagram whose rows are exact triangles

$$\begin{array}{ccccc} R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) & \longrightarrow & (R\Gamma_{\text{HK}}(X_C) \otimes_{\mathbb{F}}^{\mathbf{L}} B_{\log})^{N=0, \varphi=p^i} & \longrightarrow & (R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+)/\text{Fil}^i \\ \downarrow & & \downarrow & & \parallel \\ \text{Fil}^i(R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+) & \longrightarrow & R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+ & \longrightarrow & (R\Gamma_{\text{dR}}(X) \otimes_K^{\mathbf{L}} B_{\text{dR}}^+)/\text{Fil}^i \end{array}$$

and, by Theorem 6.3(i), we have an isomorphism $\tau^{\leq i} R\Gamma_{\text{syn,FF}}(X, \mathbb{Q}_p(i)) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_{\text{proét}}(X, \mathbb{Q}_p(i))$. Moreover, by Tate's acyclicity theorem, the bottom exact triangle of the diagram above maps, via Fontaine's morphism $\theta : B_{\text{dR}}^+ \rightarrow C$, to the following exact triangle

$$\Omega^{\geq i}(X)[-i] \longrightarrow \Omega^\bullet(X) \longrightarrow \Omega^{\leq i-1}(X).$$

Then, reducing to the case X is connected, the statement for $i = 0$ follows immediately using that $B_{\log}^{N=0, \varphi=1} = \mathbb{Q}_p$. For the statement in the case $i = 1$, taking cohomology of the diagram above and using Proposition 7.3(ii), it remains to check that

$$H^1(R\Gamma_{\mathrm{HK}}(X) \otimes_{\check{F}}^{\mathrm{L}\blacksquare} B_{\log})^{N=0, \varphi=p} \cong (H_{\mathrm{HK}}^1(X) \otimes_{\check{F}}^{\blacksquare} B_{\log})^{N=0, \varphi=p}. \quad (7.18)$$

For this, we note that, since B_{\log} is a flat solid \check{F} -vector space (by [Bos21, Corollary A.65], as it is a filtered colimit of \check{F} -Fréchet spaces), we have that $H^1(R\Gamma_{\mathrm{HK}}(X) \otimes_{\check{F}}^{\mathrm{L}\blacksquare} B_{\log}) \cong H_{\mathrm{HK}}^1(X) \otimes_{\check{F}}^{\blacksquare} B_{\log}$. Then, the isomorphism (7.18) follows from the exactness of the sequences

$$0 \rightarrow B \rightarrow B_{\log} \xrightarrow{N} B_{\log} \rightarrow 0 \quad 0 \rightarrow B^{\varphi=p} \rightarrow B \xrightarrow{\varphi^{-p}} B \rightarrow 0.$$

Lastly, the statement for $i > 1$ follows from Lemma 7.13 below (which implies the vanishing of $H_{\mathrm{proét}}^i(X, \mathbb{Q}_p(i))$ as X is qcqs), together with the fact that, in this case, $H_{\mathrm{dR}}^i(X)$ vanishes, and, by Corollary 3.16, $H_{\mathrm{HK}}^i(X)$ vanishes as well. \square

We used crucially the following result.

Lemma 7.13 (Rigid-analytic p -adic Artin vanishing). *Let X be a smooth affinoid rigid space over C . Then, we have*

$$H_{\mathrm{proét}}^i(X, \mathbb{Z}_p) = 0$$

for all $i > \dim X$.

Proof. Fix an integer $i > \dim X$. By a result of Bhatt–Mathew, [BM21, Remark 7.4(2)], for any $n \geq 1$, we have the vanishing $H_{\mathrm{proét}}^i(X, \mathbb{Z}/p^n) = 0$.⁶² Thus, using the exact sequence

$$0 \rightarrow R^1 \varprojlim_n H_{\mathrm{proét}}^{i-1}(X, \mathbb{Z}/p^n) \rightarrow H_{\mathrm{proét}}^i(X, \mathbb{Z}_p) \rightarrow \varprojlim_n H_{\mathrm{proét}}^i(X, \mathbb{Z}/p^n) \rightarrow 0$$

it remains to show that

$$R^1 \varprojlim_n H_{\mathrm{proét}}^{i-1}(X, \mathbb{Z}/p^n) = 0. \quad (7.19)$$

Considering the long exact sequence associated to the short exact sequence of sheaves on $X_{\mathrm{proét}}$

$$0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^{n+1} \rightarrow \mathbb{Z}/p^n \rightarrow 0$$

and using again that $H_{\mathrm{proét}}^i(X, \mathbb{Z}/p) = 0$, we deduce that the transition maps of the inverse system $\{H_{\mathrm{proét}}^{i-1}(X, \mathbb{Z}/p^n)\}_n$ are surjective; then, we conclude by the Mittag-Leffler criterion. \square

Now, let us at least indicate a possible direction for proving Conjecture 7.11 in dimension higher than 1.

Remark 7.14 (On the obstruction to proving Conjecture 7.11). Let us assume for simplicity that X is an affinoid rigid space over C having a smooth formal model over \mathcal{O}_C . In this case, by Theorem 3.15(i) we have in particular that the monodromy action N on the Hyodo-Kato cohomology of X is trivial. We claim that to prove Conjecture 7.11 it suffices to show that

$$H^1(\mathrm{FF}, \mathcal{H}_{\mathrm{FF}}^n(X) \otimes \mathcal{O}(m)) = 0 \quad \text{for all } 0 \leq n \leq m \quad (7.20)$$

⁶²The cited result does not keep track of the condensed structure on $H_{\mathrm{proét}}^i(X, \mathbb{Z}/p^n)$, however doing this does not pose any problem.

where $\mathcal{H}_{\text{FF}}^n(X)$ denotes the n -th Fargues–Fontaine cohomology group of X (Definition 6.16). We note that, by Proposition 6.8 (applied to $\mathcal{H}_{\text{FF}}^n(X) \otimes \mathcal{O}(m) \in \text{Nuc}(\text{FF})$ concentrated in degree 0) and (the proof of) Theorem 4.1, the condition (7.20) is equivalent to the following one⁶³

$$\varphi - p^m : H_{\text{HK}}^n(X) \otimes_{\check{F}}^{\blacksquare} B \rightarrow H_{\text{HK}}^n(X) \otimes_{\check{F}}^{\blacksquare} B \quad \text{is surjective for all } 0 \leq n \leq m. \quad (7.21)$$

Before proving the claim, we pause to remark that (7.20) holds replacing X with a dagger structure X^\dagger on X (Remark 3.20), in fact, thanks to Theorem 6.17(i) and Theorem 3.30, the vector bundle $\mathcal{H}_{\text{FF}}^n(X^\dagger)$ on FF has Harder–Narasimhan slopes $\geq -n$. Recalling Theorem 3.15(i), this suggests the following question: can one prove (7.21) using that the Frobenius on the n -th crystalline cohomology group has an inverse up to p^n , [BS22, Theorem 1.8(6)]?

Now, to prove that the condition (7.20) implies Conjecture 7.11, as in the proof of Theorem 7.12, by [Elk73, Theorem 7 and Remark 2], we can assume that X is the base change to C of a smooth affinoid X_0 defined over a finite extension of K , and, without loss of generality, we can further assume that X_0 is defined over K . Then, again as in the proof of Theorem 7.12, combining Theorem 7.2 with Proposition 7.3(ii), for all $i \geq 0$ we obtain the following commutative diagram in $\text{Mod}_{\mathbb{Q}_p}^{\text{solid}}$ with exact rows

$$\begin{array}{ccccccc} (H_{\text{HK}}^{i-1}(X) \otimes_{\check{F}}^{\blacksquare} B)^{\varphi=p^i} & \longrightarrow & \Omega^{i-1}(X)/d\Omega^{i-2}(X) & \longrightarrow & H_{\text{proét}}^i(X, \mathbb{Q}_p(i)) & \xrightarrow{\alpha_i} & (H_{\text{HK}}^i(X) \otimes_{\check{F}}^{\blacksquare} B)^{\varphi=p^i} \longrightarrow 0 \\ \downarrow \gamma_i & & \parallel & & \downarrow & & \downarrow \\ 0 \rightarrow H_{\text{dR}}^{i-1}(X) & \longrightarrow & \Omega^{i-1}(X)/d\Omega^{i-2}(X) & \longrightarrow & \Omega^i(X)^{d=0} & \longrightarrow & H_{\text{dR}}^i(X) \longrightarrow 0. \end{array}$$

Here, we used that, for any $0 \leq j \leq i$, we have

$$H^j((R\Gamma_{\text{HK}}(X) \otimes_{\check{F}}^{\blacksquare} B)^{\varphi=p^i}) \cong (H_{\text{HK}}^j(X) \otimes_{\check{F}}^{\blacksquare} B)^{\varphi=p^i}.$$

In fact, since B is a flat solid \check{F} -vector space (by [Bos21, Corollary A.65]), we have an exact sequence

$$0 \rightarrow (H_{\text{HK}}^{j-1}(X) \otimes_{\check{F}}^{\blacksquare} B)/\text{im}(\varphi - p^i) \rightarrow H^j((R\Gamma_{\text{HK}}(X) \otimes_{\check{F}}^{\blacksquare} B)^{\varphi=p^i}) \rightarrow (H_{\text{HK}}^j(X) \otimes_{\check{F}}^{\blacksquare} B)^{\varphi=p^i} \rightarrow 0$$

and the left term vanishes thanks to (7.21). Now, it remains to show that we have

$$\ker \alpha_i \cong \Omega^{i-1}(X)/\ker d.$$

From the diagram above, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc} (H_{\text{HK}}^{i-1}(X) \otimes_{\check{F}}^{\blacksquare} B)^{\varphi=p^i} & \longrightarrow & \Omega^{i-1}(X)/d\Omega^{i-2}(X) & \longrightarrow & \ker \alpha_i & \longrightarrow & 0 \\ \downarrow \gamma_i & & \parallel & & \downarrow & & \\ 0 \rightarrow H_{\text{dR}}^{i-1}(X) & \longrightarrow & \Omega^{i-1}(X)/d\Omega^{i-2}(X) & \longrightarrow & \Omega^{i-1}(X)/\ker d & \longrightarrow & 0. \end{array} \quad (7.22)$$

Then, using the snake lemma, we deduce that we need to show that the map γ_i is surjective. For this, by [FS21, Proposition II.2.3] we have the following exact triangle in $\text{QCoh}(\text{FF})$

$$\mathcal{H}_{\text{FF}}^{i-1}(X) \otimes \mathcal{O}(i-1) \xrightarrow{t} \mathcal{H}_{\text{FF}}^{i-1}(X) \otimes \mathcal{O}(i) \rightarrow \iota_{\infty,*}(H_{\text{dR}}^{i-1}(X)) \quad (7.23)$$

⁶³Here, we recall [Bos21, Corollary A.65] and we note that we have $R\lim_{I \subset (0, \infty)} H_{\text{HK}}^n(X) \otimes_{\check{F}}^{\blacksquare} B_I \simeq H_{\text{HK}}^n(X) \otimes_{\check{F}}^{\blacksquare} B$ by [Bos21, Corollary A.67(ii)], using that $H_{\text{HK}}^n(X)$ is a quotient of \check{F} -Banach spaces.

where $\iota_{\infty,*} : \mathrm{QCoh}(\mathrm{Spa}(C)) \rightarrow \mathrm{QCoh}(\mathrm{FF})$ denotes the pushforward functor. Here, we used Theorem 3.15(iii) and the flatness of C for the solid tensor product \otimes_F^{\blacksquare} ([Bos21, Corollary A.65]). Taking the long exact sequence in cohomology associated to (7.23) on FF, recalling Proposition 6.8, from (7.20) we deduce that γ_i is surjective, as desired.

7.5. Remarks about coefficients. In this subsection, we indicate a partial extension of Theorem 7.2 to coefficients. For simplicity, we restrict ourselves to smooth rigid-analytic varieties over C .

Definition 7.15. Let X be a smooth rigid-analytic variety over C . Denote by $\nu : X_{\mathrm{pro\acute{e}t}} \rightarrow X_{\acute{e}t,\mathrm{cond}}$ the natural morphism of sites. For \mathbb{M} a \mathbb{B} -local system on $X_{\mathrm{pro\acute{e}t}}$, i.e. a sheaf of \mathbb{B} -modules that is locally on $X_{\mathrm{pro\acute{e}t}}$ free of finite rank, we define the *B-cohomology of X with coefficients in \mathbb{M}* as the complex of $D(\mathrm{Mod}_B^{\mathrm{cond}})$

$$R\Gamma_B(X, \mathbb{M}) := R\Gamma_{\acute{e}t,\mathrm{cond}}(X, L\eta_i R\nu_* \mathbb{M}).$$

and we endow it with the filtration décalée.

With the definition above, given \mathbb{L} a \mathbb{Q}_p -local system on $X_{\mathrm{pro\acute{e}t}}$ with associated \mathbb{B} -local system $\mathbb{M} = \mathbb{L} \otimes_{\mathbb{Q}_p} \mathbb{B}$, tensoring with $\mathbb{L} \otimes_{\mathbb{Q}_p} -$ the exact sequence (6.1), the same argument used in the proof of Theorem 6.3(i) shows that we have a natural isomorphism in $D(\mathrm{Mod}_{\mathbb{Q}_p}^{\mathrm{cond}})$

$$\tau^{\leq i} \mathrm{Fil}^i R\Gamma_B(X, \mathbb{M})^{\varphi=p^i} \xrightarrow{\simeq} \tau^{\leq i} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{L}(i)).$$

Similarly, we have a version with coefficients of Theorem 6.3(ii). Then, combining the same results cited in the proof of Theorem 5.1 (in the smooth case), which rely on Scholze's Poincaré lemma for $\mathbb{B}_{\mathrm{dR}}^+$, we obtain the following result.

Theorem 7.16. *Let X be a connected, paracompact, smooth rigid-analytic variety defined over K . Let \mathbb{L} be a de Rham \mathbb{Q}_p -local system on $X_{\mathrm{pro\acute{e}t}}$, with associated \mathbb{B} -local system $\mathbb{M} = \mathbb{L} \otimes_{\mathbb{Q}_p} \mathbb{B}$, and associated filtered \mathcal{O}_X -module with integrable connection $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$. For any $i \geq 0$, we have a \mathcal{G}_K -equivariant isomorphism in $D(\mathrm{Mod}_{\mathbb{Q}_p}^{\mathrm{solid}})$*

$$\tau^{\leq i} R\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{L}(i)) \simeq \tau^{\leq i} \mathrm{fib}(R\Gamma_B(X_C, \mathbb{M})^{\varphi=p^i}) \rightarrow (R\Gamma_{\mathrm{dR}}(X, \mathcal{E}) \otimes_K^{\mathrm{L}\blacksquare} B_{\mathrm{dR}}^+) / \mathrm{Fil}^i.$$

APPENDIX A. COMPLEMENTS ON CONDENSED MATHEMATICS

This appendix consists of a miscellaneous collection of results on condensed mathematics that we use in the main body of the paper.

Notation and conventions A.1. In this appendix, we adopt the same notation and set-theoretic conventions of [Bos21, Appendix A]. All condensed rings will be assumed to be commutative and unital.

A.1. Derived p -adic completion and solidification. In this section, we compare the derived p -adic completion to the solidification.

Notation A.2. Let p be a prime number. In the following, for $M \in D(\text{CondAb})$, we denote by

$$M_p^\wedge := R \varprojlim_{n \in \mathbb{N}} (M \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n) \in D(\text{CondAb})$$

its *derived p -adic completion*. We say that an object $M \in D(\text{CondAb})$ is *derived p -adically complete* if the natural map

$$M \rightarrow M_p^\wedge$$

is an isomorphism of $D(\text{CondAb})$.

For A a solid ring, we write $\text{Mod}_A^{\text{solid}}$ for the symmetric monoidal category of A -modules in Solid , endowed with the solid tensor product \otimes_A^\blacksquare .

Proposition A.3 (cf. [CS20b]). *Let A be a derived p -adically complete solid ring. For all cohomologically bounded above complexes $M, N \in D(\text{Mod}_A^{\text{solid}})$, we have a natural isomorphism*

$$M_p^\wedge \otimes_A^{\mathbb{L}\blacksquare} N_p^\wedge \xrightarrow{\sim} (M \otimes_A^{\mathbb{L}\blacksquare} N)_p^\wedge. \quad (\text{A.1})$$

Proof. First, we recall that the category $\text{Mod}_A^{\text{solid}}$ is generated under colimits by the compact projective objects $(\prod_I \mathbb{Z}) \otimes_{\mathbb{Z}}^\blacksquare A = (\prod_I \mathbb{Z}_p) \otimes_{\mathbb{Z}_p}^\blacksquare A$, for varying sets I . By hypothesis, we may assume that M and N are connective; then, writing $M = \text{colim}_{[n] \in \Delta^{\text{op}}} M_n$ in $D(\text{Mod}_A^{\text{solid}})$ with M_n a direct sum of objects of the form $(\prod_I \mathbb{Z}_p) \otimes_{\mathbb{Z}_p}^\blacksquare A$, the natural map $\text{colim}_{[n] \in \Delta^{\text{op}}} (M_n)_p^\wedge \rightarrow M_p^\wedge$ is an isomorphism (as it can be checked via a spectral sequence), and similarly for N . Therefore, using that the solid tensor product commutes with colimits, it suffices to prove the statement for $M = M' \otimes_{\mathbb{Z}_p}^\blacksquare A$ and $N = N' \otimes_{\mathbb{Z}_p}^\blacksquare A$ (concentrated in degree 0), with M' and N' objects of $\text{Mod}_{\mathbb{Z}_p}^{\text{solid}}$ of the form $\bigoplus_{j \in J} (\prod_{I_j} \mathbb{Z}_p)$ for varying sets J and I_j , for $j \in J$.

In the case $A = \mathbb{Z}_p$, using Lemma A.4 the statement readily reduces (cf. the proof of [Bos21, Proposition A.49]) to the isomorphism $\prod_I \mathbb{Z}_p \otimes_{\mathbb{Z}_p}^{\mathbb{L}\blacksquare} \prod_{I'} \mathbb{Z}_p = \prod_{I \times I'} \mathbb{Z}_p$ which holds for any sets I and I' (see e.g. [Bos21, Remark A.18]).

In general, we want to show that we can reduce to the case $A = \mathbb{Z}_p$. We will use that $\prod_I \mathbb{Z}_p$ is flat for the tensor product $\otimes_{\mathbb{Z}_p}^\blacksquare$, i.e. for any $Q \in \text{Mod}_{\mathbb{Z}_p}^{\text{solid}}$ we have that $\prod_I \mathbb{Z}_p \otimes_{\mathbb{Z}_p}^{\mathbb{L}\blacksquare} Q$ is concentrated in degree 0: for this, writing Q as a filtered colimit of quotients of objects of the form $\prod_J \mathbb{Z}_p$, we can reduce to the case Q is derived p -adically complete; in this case, using (A.1) for $A = \mathbb{Z}_p$, by the derived Nakayama lemma, we can reduce to checking the claim modulo p , in which case it follows from [Bos21, Lemma A.19], using that any solid \mathbb{F}_p -module can be written as a filtered colimit of

profinite \mathbb{F}_p -vector spaces, [CS20a, Proposition 2.8]. Then, from (A.1) for $A = \mathbb{Z}_p$, using that A is derived p -adically complete, we deduce that

$$M_p^\wedge \cong (M' \otimes_{\mathbb{Z}_p}^{\mathbb{L}\square} A)_p^\wedge \cong (M')_p^\wedge \otimes_{\mathbb{Z}_p}^{\mathbb{L}\square} A$$

and similarly for N . Hence, we have a natural isomorphism

$$M_p^\wedge \otimes_A^{\mathbb{L}\square} N_p^\wedge \cong (M')_p^\wedge \otimes_{\mathbb{Z}_p}^{\mathbb{L}\square} (N')_p^\wedge \otimes_{\mathbb{Z}_p}^{\mathbb{L}\square} A \cong (M' \otimes_{\mathbb{Z}_p}^{\mathbb{L}\square} N' \otimes_{\mathbb{Z}_p}^{\mathbb{L}\square} A)_p^\wedge \cong (M \otimes_A^{\mathbb{L}\square} N)_p^\wedge$$

where we used again (A.1) for $A = \mathbb{Z}_p$. \square

We used crucially the following result.

Lemma A.4. *For $c \in [0, 1]$ we denote $(\mathbb{Z}_p)_{\leq c} := \{x \in \mathbb{Z}_p : |x| \leq c\}$. For any sets J and I_j , for $j \in J$, we have*

$$\left(\bigoplus_{j \in J} \prod_{I_j} \mathbb{Z}_p \right)_p^\wedge = \varinjlim_{f: J \rightarrow [0, 1], f \rightarrow 0} \prod_{j \in J} \prod_{I_j} (\mathbb{Z}_p)_{\leq f(j)}. \quad (\text{A.2})$$

where the colimit runs over the functions $f : J \rightarrow [0, 1]$ tending to 0 (i.e. for every $\varepsilon > 0$, the set $\{j \in J : |f(j)| \geq \varepsilon\}$ is finite) partially ordered by the relation of pointwise inequality $f \leq g$.

Proof. We will adapt the proof of [Bos21, Lemma A.53]. It suffices to prove (A.2) on S -valued points, for all extremally disconnected set S . Let $M := \bigoplus_{j \in J} \prod_{I_j} \mathbb{Z}_p$. We note that, thanks to the flatness of the condensed abelian group M (which follows from Lemma A.5), the derived p -adic completion of M agrees with its underived p -adic completion $\varprojlim_{n \in \mathbb{N}} M/p^n$. Then, we have

$$M_p^\wedge(S) = \varprojlim_{n \in \mathbb{N}} \bigoplus_{j \in J} \text{Hom}(S, \prod_{I_j} \mathbb{Z}_p/p^n)$$

from which we deduce that

$$M_p^\wedge(S) = \{(g_j)_{j \in J} \text{ with } g_j \in \mathcal{C}^0(S, \prod_{I_j} \mathbb{Z}_p) : \forall \varepsilon > 0, g_j(S) \subseteq \prod_{I_j} (\mathbb{Z}_p)_{\leq \varepsilon} \text{ for all but finitely many } g_j\}$$

which, in turn, identifies with

$$\varinjlim_{f: J \rightarrow [0, 1], f \rightarrow 0} \left(\prod_{j \in J} \prod_{I_j} \mathcal{C}^0(S, (\mathbb{Z}_p)_{\leq f(j)}) \right)$$

thus showing (A.2). \square

The following lemma was used above and in the main body of the paper.

Lemma A.5 ([CS22, Proposition 3.4]). *A condensed abelian group $M \in \text{CondAb}$ is flat if and only if, for all extremally disconnected sets S , the abelian group $M(S)$ is torsion-free.*

A.2. ∞ -category of nuclear complexes. In this section, we collect some general properties and characterizations of the ∞ -category of nuclear complexes attached to an analytic ring, which are due to Clausen–Scholze, focusing in particular on a special class of analytic rings relevant to the main body of the paper. This section should be read in conjunction with [Bos21, §A.6].

Notation and conventions A.6. We will denote by ExtrDisc the category of extremally disconnected sets.

We recall that an analytic ring (A, \mathcal{M}) (in the sense of [CS19, Definition 7.4]) is *commutative* if A is commutative (and unital), and *normalized* if the map $A \rightarrow \mathcal{M}[*]$ is an isomorphism. All analytic rings will be assumed to be commutative and normalized.

For an analytic ring (A, \mathcal{M}) , we denote by $D(A)$ the derived ∞ -category $D(\text{Mod}_A^{\text{cond}})$, and we denote by $D(A, \mathcal{M})$ the derived ∞ -category of (A, \mathcal{M}) -complete complexes in $D(A)$, equipped with the symmetric monoidal tensor product $\otimes_{(A, \mathcal{M})}^L$. For $M \in D(A, \mathcal{M})$ we write

$$M^\vee := \underline{\text{Hom}}_{D(A, \mathcal{M})}(M, A)$$

for its *dual*.

Definition A.7 ([CS20a, Definition 13.10]). Let (A, \mathcal{M}) be an analytic ring. A complex $N \in D(A, \mathcal{M})$ is *nuclear* if, for all $S \in \text{ExtrDisc}$, the natural map

$$(\mathcal{M}[S]^\vee \otimes_{(A, \mathcal{M})}^L N)(*) \rightarrow N(S) \tag{A.3}$$

in $D(\text{Ab})$ is an isomorphism. We denote by $\text{Nuc}(A, \mathcal{M})$ the full ∞ -subcategory of $D(A, \mathcal{M})$ spanned by the nuclear complexes.

Remark A.8. We note that the ∞ -subcategory $\text{Nuc}(A, \mathcal{M}) \subset D(A, \mathcal{M})$ is stable under all colimits (as both the source and the target of (A.3) commute with colimits in N), and under finite limits (as $\otimes_{(A, \mathcal{M})}^L$ commutes with finite limits).

In order to recall a useful characterization of nuclear complexes, we need the following definitions.

Definition A.9 ([CS22, Definition 8.1]). Let (A, \mathcal{M}) be an analytic ring. A map $f : M \rightarrow N$ in $D(A, \mathcal{M})$ is called *trace-class* if it lies in the image of the natural map

$$(M^\vee \otimes_{(A, \mathcal{M})}^L N)(*) \rightarrow \text{Hom}_{D(A, \mathcal{M})}(M, N).$$

Definition A.10 ([CS20a, Definition 13.12]). Let (A, \mathcal{M}) be an analytic ring. An object $M \in D(A, \mathcal{M})$ is called *basic nuclear* if it can be written as the colimit of a diagram

$$P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \dots$$

where $P_n \in D(A, \mathcal{M})$ are compact objects and f_n are trace-class maps.

Proposition A.11 ([CS20a, Proposition 13.13]). *Let (A, \mathcal{M}) be an analytic ring. An object in $D(A, \mathcal{M})$ is nuclear if and only if it can be written as a filtered colimit of basic nuclear objects.*

We deduce the following result.

Corollary A.12. *Let $f : (A, \mathcal{M}) \rightarrow (A, \mathcal{N})$ be a morphism of analytic rings. The base change functor*

$$- \otimes_{(A, \mathcal{M})}^L (B, \mathcal{N}) : D(A, \mathcal{M}) \rightarrow D(B, \mathcal{N})$$

preserves nuclear objects.

Proof. It suffices to apply Proposition A.11 observing that the base change functor preserves compact objects and trace-class maps. \square

To further study the ∞ -category of nuclear complexes, we will use the following the definition.

Definition A.13 ([CS]). Let (A, \mathcal{M}) be an analytic ring. We define the *trace-class functor*

$$(-)^{\text{tr}} : D(A, \mathcal{M}) \rightarrow D(A) : M \mapsto M^{\text{tr}}$$

where M^{tr} is defined on S -valued points,⁶⁴ for $S \in \text{ExtrDisc}$, as

$$M^{\text{tr}}(S) = (\mathcal{M}[S]^{\vee} \otimes_{(A, \mathcal{M})}^{\text{L}} M)(*).$$

Now, we collect some basic properties of the trace-class functor.

Lemma A.14. *Let (A, \mathcal{M}) be an analytic ring.*

- (i) *The trace-class functor $(-)^{\text{tr}} : D(A, \mathcal{M}) \rightarrow D(A)$ takes values in $D(A, \mathcal{M})$.*
- (ii) *A map $f : P \rightarrow M$ in $D(A, \mathcal{M})$ with M compact object is trace-class if and only if it factors through M^{tr} .*
- (iii) *An object $M \in D(A, \mathcal{M})$ is nuclear if and only if the natural map $M^{\text{tr}} \rightarrow M$ is an isomorphism.*

Proof. For part (i), one can adapt the argument of [Bos21, Lemma A.47(i)]. For part (ii), as in the proof [Bos21, Lemma A.47(ii)], the case $P = \mathcal{M}[S]$, with S extremally disconnected set, is clear; for a general P compact object of $D(A, \mathcal{M})$, we can reduce to the previous case writing P as a retract of a finite complex whose terms are objects of the form $\mathcal{M}[S]$, with S extremally disconnected set ([Sta20, Tag 094B]). Part (iii) follows immediately from the definitions. \square

Proposition A.15 ([CS], [And]). *Let (A, \mathcal{M}) be an analytic ring. For any $M \in D(A, \mathcal{M})$, the object M^{tr} (and in particular any nuclear object in $D(A, \mathcal{M})$)⁶⁵ can be written as a colimit of shifts of objects of the form $\mathcal{M}[S]^{\vee}$ for S extremally disconnected sets.*

Proof. Recalling that $D(A, \mathcal{M})$ is generated, under shifts and colimits, by $\mathcal{M}[T]$ for varying extremally disconnected sets T , we can apply the same argument of [Bos21, Proposition A.48]. \square

Next, we focus on the categorical properties of the ∞ -category of nuclear complexes for the following special class of analytic rings used in the main body of the paper.

Notation A.16. Let F be a non-archimedean local field, and let A be a solid F -algebra.

For the analytic ring $(A, \mathbb{Z})_{\blacksquare} = (A, \mathcal{M}_A)$ from [Bos21, Proposition A.29] (i.e. the analytic ring structure on A induced from the analytic ring $\mathbb{Z}_{\blacksquare}$), we denote by $\text{Nuc}_A := \text{Nuc}((A, \mathbb{Z})_{\blacksquare})$ the full ∞ -subcategory of nuclear complexes of $\text{Solid}_A := D((A, \mathbb{Z})_{\blacksquare})$, and we write \otimes_A^{\blacksquare} for the symmetric monoidal tensor product $\otimes_{(A, \mathbb{Z})_{\blacksquare}}$.

Theorem A.17 ([CS]). *Let F be a non-archimedean local field, and let A be a nuclear solid F -algebra.*⁶⁶

- (i) *The subcategory $\text{Nuc}_A \subset \text{Solid}_A$ is a stable ∞ -category, closed under the tensor product $\otimes_A^{\text{L}, \blacksquare}$, finite limits, countable products, and all colimits.*
- (ii) *The ∞ -category Nuc_A is generated, under shifts and colimits, by the objects $\underline{\text{Hom}}_A(A[S], A)$, for varying S profinite sets.*
- (iii) *An object $M \in \text{Solid}_A$ lies in Nuc_A if and only if $H^i(M)[0]$ lies in Nuc_A for all i .*

⁶⁴Via the equivalence of ∞ -categories $\text{Cond}(D(\text{Ab})) \cong D(\text{CondAb})$.

⁶⁵Here, we use Lemma A.14(iii).

⁶⁶Given a solid F -algebra A we say that it is *nuclear* if the underlying solid F -module is nuclear in the sense of [Bos21, Definition A.40] with respect to the analytic ring $(F, \mathbb{Z})_{\blacksquare}$ (as we will see in the proof below, this is equivalent to requiring that the complex $A[0] \in \text{Solid}_F$ is nuclear in the sense of Definition A.7). For example, any Fréchet F -algebra is a nuclear F -algebra by [Bos21, Proposition A.64].

Proof. First we note that, by [And21, Proposition 2.11], we have $\mathcal{M}_A[S]^\vee \cong \underline{\mathrm{Hom}}_A(A[S], A)$, for all profinite sets S . Moreover, as the latter objects are flat for the tensor product \otimes_A^\blacksquare by [Bos21, Theorem A.43(ii)], a complex $N[0] \in \mathrm{Solid}_A$ concentrated in degree 0 is nuclear, in the sense of Definition A.7, if and only if the solid A -module N is nuclear, in the sense of [Bos21, Definition A.40].

Thanks to the above observations, part (ii) follows from Proposition A.15 and [Bos21, Theorem A.43(ii)]. For part (i), the closure of $\mathrm{Nuc}_A \subset \mathrm{Solid}_A$ under finite limits and all colimits was observed more generally in Remark A.8; for the closure under the tensor product $\otimes_A^{\mathrm{L}\blacksquare}$ and countable products, taking K -flat resolutions in Nuc_A (which exist thanks to part (ii)), we can reduce to the statement of [Bos21, Theorem A.43(i)].

For part (iii), if $M \in \mathrm{Solid}_A$ lies in Nuc_A , then $H^i(M)[0]$ lies in Nuc_A for all i , using again that the objects $\mathcal{M}_A[S]^\vee$, for varying profinite sets S , are flat for the tensor product \otimes_A^\blacksquare . Conversely, passing to the Postnikov limit, $M \simeq \varprojlim_n \tau^{\geq -n} M$, by part (i), we can suppose that M is cohomologically bounded below, and using $M \simeq \varinjlim_n \tau^{\leq n} M$, we can even suppose that M is bounded, and then concentrated in one degree, in which case the implication is clear. \square

Remark A.18. Let A be a nuclear solid F -algebra and let $M \in \mathrm{Solid}_A$. We have that M lies in Nuc_A if and only if M (regarded in Solid_F) lies in Nuc_F . For this, it suffices to recall Definition A.7, and observe that, by [And21, Proposition 5.35], for all $S \in \mathrm{ExtrDisc}$, we have a natural isomorphism $\mathcal{M}_A[S]^\vee \cong \mathcal{M}_F[S]^\vee \otimes_F^\blacksquare A$.

A.3. Quasi-coherent, nuclear, and perfect complexes on analytic adic spaces. In this section, we recall some results of Andreychev on quasi-coherent, nuclear, and perfect complexes on analytic adic spaces, that we need in the main body of the paper.

Notation A.19. Given a pair (A, A^+) with A a complete Huber ring and A^+ a subring of A° , we denote by $(A, A^+)_{\blacksquare}$ the associated analytic ring, [And21, §3.3].

Theorem A.20 ([And21, Theorem 4.1]). *Let Y an analytic adic space. The association taking any affinoid subspace $U = \mathrm{Spa}(A, A^+) \subset Y$ to the ∞ -category $D((A, A^+)_{\blacksquare})$ defines a sheaf of ∞ -categories on Y . We define the ∞ -category of quasi-coherent complexes on Y as the global sections of such sheaf, and we denote it by $\mathrm{QCoh}(Y)$.*

Next, we recall that also nuclear objects satisfy analytic descent on analytic adic spaces.

Theorem A.21 ([And21, Theorem 5.41]). *Let Y an analytic adic space. The association taking any affinoid subspace $U = \mathrm{Spa}(A, A^+) \subset Y$ to the ∞ -category $\mathrm{Nuc}((A, A^+)_{\blacksquare})$ defines a sheaf of ∞ -categories on Y . We define the ∞ -category of nuclear complexes on Y as the global sections of such sheaf, and we denote it by $\mathrm{Nuc}(Y)$.*

It turns out that the ∞ -category of nuclear objects associated to an analytic complete Huber pair does not depend on the ring of definition. More precisely, we have the following result.

Theorem A.22 ([And]). *Let (A, A^+) a pair with A an analytic complete Huber ring and A^+ a subring of A° . The ∞ -category $\mathrm{Nuc}((A, A^+)_{\blacksquare})$ is generated, under shifts and colimits, by the objects $\underline{\mathrm{Hom}}_A(A[S], A)$ for varying S profinite sets.*

In what follows, given a condensed ring R , we denote by $\mathrm{Perf}_R \subset D(\mathrm{Mod}_R^{\mathrm{cond}})$ the ∞ -subcategory of perfect complexes over R , [And21, Definition 5.1].

Andreychev showed that, passing to dualizable objects, Theorem A.21 implies the following result.

Theorem A.23 ([And21, Theorem 5.3]). *Let Y an analytic adic space. The association taking any affinoid subspace $U = \mathrm{Spa}(A, A^+) \subset Y$ to the ∞ -category Perf_A defines a sheaf of ∞ -categories on Y . We define the ∞ -category of perfect complexes on Y as the global sections of such sheaf, and we denote it by $\mathrm{Perf}(Y)$.*

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