TORSION IN p-ADIC ÉTALE COHOMOLOGY: REMARKS AND CONJECTURES

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Abstract. Let C be a complete algebraically closed extension of $\mathbb{Q}_p$, and let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_C$. By the work of Bhatt–Morrow–Scholze, it is known that when $\mathfrak{X}$ is proper, the length of the torsion in the integral p-adic étale cohomology of the generic fiber $\mathfrak{X}_C$ is bounded above by the length of the torsion in the crystalline cohomology of its special fiber. In this note, we focus on the non-proper case and observe that when $\mathfrak{X}$ is affine, the torsion in the integral p-adic étale cohomology of $\mathfrak{X}_C$ can even be expressed as a functor of the special fiber, unlike in the proper case. As a consequence, we show that, surprisingly, if $\mathfrak{X}$ is affine, the integral p-adic étale cohomology groups of $\mathfrak{X}_C$ have finite torsion subgroups. We discuss further applications and propose conjectures predicting the torsion in the integral p-adic étale cohomology of a broader class of rigid-analytic varieties over C.

Contents

1. Introduction

Let p be a prime number. We denote by C an algebraically closed complete non-archimedean extension of $\mathbb{Q}_p$ and we write $\mathcal{O}_C$ for its ring of integers.

In recent years, there has been increasing interest in the study of the integral/rational p-adic (pro-)étale cohomology of rigid-analytic varieties over C that are not necessarily proper, such as local Shimura varieties. See e.g. the works [CDN20a], [CDN20b], [CDN21], [Bos21], [Bos23]. One difficulty in the latter works is that, in general, the rational p-adic étale cohomology groups of non-proper rigid-analytic varieties over C are not finite-dimensional $\mathbb{Q}_p$-vector spaces. Consequently, it becomes important to understand whether such cohomology groups have a reasonable topological structure, or, at the integral level, to understand the torsion in the integral p-adic étale cohomology of such spaces.

Let us begin with the building blocks of rigid-analytic varieties, namely affinoid spaces. While writing [Bos23], and in particular thinking about [Bos23, Conjecture 1.15] (cf. [CDN22, Remarque 0.3]), we arrived from several directions at the following conjecture.

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Conjecture 1.1. *Let X be a smooth affinoid rigid space over C.*

(i) For all $i \in \mathbb{Z}$, the torsion subgroup of the pro-étale cohomology group $H^i_{\text{proét}}(X, \mathbb{Z}_p)$ is finite.
(ii) For all $n \in \mathbb{Z}_{\geq 0}$, the p^n-torsion subgroup of the Picard group $\text{Pic}(X)$ is finite.

We note that Conjecture 1.1(i) implies in particular that, for X a smooth affinoid rigid space over C, the cohomology group $H^i_{\text{proét}}(X, \mathbb{Q}_p)$ has a natural structure of $\mathbb{Q}_p$-Banach space (see Proposition 4.8), therefore it is not pathological as one might think.

The first goal of this note is to prove Conjecture 1.1 in the good reduction case. More precisely:

Theorem 1.2 (Theorem 4.1, Corollary 4.7)**.** *Conjecture 1.1 holds true in the case X has an affine smooth p-adic formal model over $\mathcal{O}_C$.*

We prove Theorem 1.2 by reducing it to a problem in characteristic p. For the first part of the statement, i.e. in order to prove Conjecture 1.1(i) in the good reduction case, we will rely on the following result expressing the p^n-torsion in the integral p-adic étale cohomology in terms of the special fiber.

We denote by k the residue field of C.

Theorem 1.3 (Theorem 2.1, Theorem 4.12)**.** *Let $\mathfrak{X}$ be a smooth p-adic formal scheme over $\mathcal{O}_C$ that is affine, or, more generally, can be written as a countable increasing union of affine opens. Then, for all $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ we have a natural isomorphism*

$$H^{i+1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)[p^n] \cong H^1(\mathfrak{X}_k, W\Omega^i_{\log})[p^n].$$

Here, we denote by $W\Omega^i_{\log}$ the logarithmic de Rham–Witt sheaf (see Notation 2.4).

Remark 1.4 (Comparison to the proper case)**.** We note that Theorem 1.3 cannot hold for any smooth p-adic formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$. In fact, in the case $\mathfrak{X}$ is proper, Bhatt–Morrow–Scholze proved that (cf. [BMS18, Theorem 1.1, (ii)]), for all $i, n \in \mathbb{Z}_{\geq 0}$, we have the inequality

$$\text{length}_{\mathbb{Z}_p}(H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)[p^n]) \leq \text{length}_{W(k)}(H^i_{\text{crys}}(\mathfrak{X}_k/W(k))[p^n]),$$

but, the torsion in $H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)$ is not a functor of the special fiber, [BMS18, Remark 2.4]. Therefore, Theorem 1.3 is somewhat orthogonal to the cited results in the proper case.

The proof of Theorem 1.3 in the affine case relies on the work [BMS19], in particular on the comparison between p-adic étale and syntomic cohomology, and crucially uses a rigidity result for the syntomic cohomology proved by Antieau–Mathew–Morrow–Nikolaus, [AMMN22]. Using Theorem 1.3 in the affine case, we prove Theorem 1.2 by employing results on the structure of logarithmic de Rham–Witt cohomology due to Illusie–Raynaud, [IR83]. On the other hand, the extension of Theorem 1.3 beyond the affine case relies on the same finiteness results shown in the proof of Theorem 1.2.

We expect that one can prove Conjecture 1.1 in the semistable reduction case using a logarithmic variant of the results in this paper.

Remark 1.5 (On Drinfeld's upper half-spaces)**.** In light of the above, it is natural to conjecture, in particular, an extension of Theorem 1.3 to the semistable reduction case; see Conjecture 4.14 for a precise statement. More generally, one could ask for a version of Theorem 1.3 for $\mathfrak{X}$ a semistable p-adic formal scheme over $\mathcal{O}_C$ whose generic fiber $\mathfrak{X}_C$ can be written as a countable increasing union of open affinoids. The appeal of a positive answer to the latter question lies in the fact that it could be applied, for example, to Stein rigid spaces over C admitting a semistable formal model over $\mathcal{O}_C$, such as the d-dimensional Drinfeld's upper half-space $\mathbb{H}^d_C$ over C, for $d \in \mathbb{Z}_{\geq 1}$. In this example, this would lead to a conceptual proof of a result by Colmez–Dospinescu–Nizioł, [CDN21], which states that

$$H^i_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Z}_p) \text{ is torsion-free for all } i \geq 0. \tag{1.1}$$

In fact, denoting by $\mathfrak{H}$ the standard semistable formal model over $\mathcal{O}_C$ of $\mathbb{H}^d_C$, [GK05, §6.1], [CDN20b, §5.1],[1] by a result of Grösse-Klonne (see [CDN20b, Corollary 6.25(2)]), we have

$$H^j(\mathfrak{H}_k/k^0, W\Omega^i_{\log}) = 0 \text{ for all } j > 0 \text{ and } i \geq 0$$

(see Notation 4.13 for the log structures we are considering here). In [CDN21], the result (1.1) is deduced from an explicit computation of $H^i_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Z}_p)$, [CDN21, Theorem 1.1]; on the other hand, as explained in the introduction of *op. cit.*, using (1.1), this computation can be immediately deduced from its rational variant, previously treated in [CDN20b, Theorem 6.55].

We conclude recalling that, according to a folklore conjecture, any smooth affinoid rigid space over C should have a semistable p-adic formal model over $\mathcal{O}_C$. However, it would be interesting to find a proof of Conjecture 1.1 working purely in terms of rigid-analytic varieties. We hope that, in the future, tools from integral p-adic Hodge theory of rigid-analytic varieties will become available to help address these questions.

Acknowledgments. The main ideas behind this note emerged during conversations with Matthew Morrow in the fall of 2021. However, I was only able to complete the proof of Theorem 1.2 recently, after coming across Illusie–Raynaud's result [IR83, II, Corollaire 3.11]. I warmly thank Matthew for his help and encouragement. I am grateful to Alexander Petrov for many helpful comments and for suggesting Lemma 4.3, which led to a simplification of my previous argument. I also thank Gabriel Dospinescu, Akhil Mathew, and Peter Scholze for comments or corrections on a previous version of this note and for subsequent discussions. Additionally, I would like to thank Marco D'Addezio, Ian Gleason, and Wiesława Nizioł for helpful conversations.

Notation and conventions 1.6. Let p be a prime number. Unless otherwise stated, we denote by C an algebraically closed complete non-archimedean extension of $\mathbb{Q}_p$, and we write $\mathcal{O}_C$ for its ring of integers.

All formal schemes are p-adic and locally of finite type over the base. Given a formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$, we denote by $\mathfrak{X}_C$ its generic fiber regarded as an adic space. In general, all rigid-analytic varieties will be regarded as adic spaces.

We fix an uncountable cardinal κ as in [Sch21, Lemma 4.1]. We denote by CondAb the category of κ-condensed abelian groups (and the prefix "κ" will often be tacit), [CS19, Definition 2.1].

For an adic space X over $\mathbb{Q}_p$, we denote by $X_{\text{proét}}$ its κ-small pro-étale site. Given a sheaf of abelian groups $\mathcal{F}$ on $X_{\text{proét}}$, we regard, especially in §4.0.3, the pro-étale cohomology complex $R\Gamma_{\text{proét}}(X, \mathcal{F}) := R\Gamma(X_{\text{proét}}, \mathcal{F})$ in the derived category $D(\text{CondAb})$ (see also [Bos21, Definition 2.7]). We adopt similar notation and conventions for the κ-small pro-étale site of a (formal) scheme.

[1] More precisely, $\mathfrak{H}$ is the base change to $\mathcal{O}_C$ of the semistable formal model over $\mathbb{Z}_p$ of the d-dimensional Drinfeld's upper half-space over $\mathbb{Q}_p$ considered in *loc. cit.*.

For an abelian group A and $m \in \mathbb{Z}_{\geq 1}$, we denote by $A[m] \subseteq A$ the kernel of the multiplication by m on A

2. Rigidity of torsion

Our starting point is the following result.

Theorem 2.1. *Let $\mathfrak{X}$ be an affine smooth formal scheme over $\mathcal{O}_C$. Let k denote the residue field of C. Then, for all $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ we have a natural isomorphism*

$$H^{i+1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)[p^n] \cong H^1(\mathfrak{X}_k, W\Omega^i_{\log})[p^n].$$

Here, we denote by $W\Omega^i_{\log}$ the logarithmic de Rham–Witt sheaf (see Notation 2.4).

We will prove Theorem 2.1 via comparison with syntomic cohomology, using crucially a rigidity result for the latter, due to Antieau–Mathew–Morrow–Nikolaus, [AMMN22, Theorem 5.2].

2.0.1. **p-adic Tate twists**. We begin by recalling some crucial results, due to Bhatt–Morrow–Scholze [BMS19], on the quasisyntomic sheaves $\mathbb{Z}_p(i)^{\text{syn}}$.

Notation 2.2. Let QSyn denote the quasisyntomic site, [BMS19, §4], and let QRSPerfd $\subset$ QSyn denote its basis consisting of the quasiregular semiperfectoid rings, [BMS19, Proposition 4.21].

For $i \in \mathbb{Z}_{\geq 0}$, we denote by $\mathbb{Z}_p(i)^{\text{syn}}$ the sheaf on QSyn introduced in [BMS19, §7.4], defined for $R \in$ QRSPerfd by

$$\mathbb{Z}_p(i)^{\text{syn}}(R) := \text{fib}(\text{Fil}^i_{\mathcal{N}} \widehat{\Delta}_R\{i\} \xrightarrow{\varphi_i - 1} \widehat{\Delta}_R\{i\})$$

where, in terms of [BS22] and [BL22], we denote by $\widehat{\Delta}_R\{i\}$ the Breuil–Kisin twisted Nygaard completed absolute prismatic cohomology of R, we write $\text{Fil}^\star_{\mathcal{N}}$ for the Nygaard filtration ([BMS19, Theorem 1.12(3)], [BS22, Theorem 13.1]), and we denote by φ_i the i-th divided Frobenius map.

For $n \in \mathbb{Z}_{\geq 1}$, we define the sheaf on QSyn

$$\mathbb{Z}/p^n(i)^{\text{syn}} := \mathbb{Z}_p(i)^{\text{syn}}/p^n.$$

In mixed characteristic, the sheaves defined above can be compared to p-adic nearby cycles, as we now recall. In the following, given a smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$, we regard $\mathbb{Z}_p(i)^{\text{syn}}$ on the pro-étale site $\mathfrak{X}_{\text{proét}}$ using [BMS19, Remark 10.4].

Theorem 2.3 ([BMS19, Theorem 10.1])**.** *Let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_C$ with generic fiber X. Denote by $\psi : X_{\text{proét}} \to \mathfrak{X}_{\text{proét}}$ the natural morphism of sites. Then, there is a natural isomorphism of sheaves of complexes on the pro-étale site $\mathfrak{X}_{\text{proét}}$*

$$\mathbb{Z}_p(i)^{\text{syn}} \simeq \tau^{\leq i} R\psi_* \mathbb{Z}_p(i)$$

and, for all $n \in \mathbb{Z}_{\geq 1}$, there is a natural isomorphism

$$\mathbb{Z}/p^n(i)^{\text{syn}} \simeq \tau^{\leq i} R\psi_* \mathbb{Z}/p^n(i).$$

Moreover, in equal characteristic p, the sheaves $\mathbb{Z}_p(i)^{\text{syn}}$ can be compared to the logarithmic de Rham–Witt sheaves. To state this result precisely, we introduce the following notation.

Notation 2.4. Let k be a perfect field of characteristic p, and let X be a smooth scheme over k.

We denote by
$$W\Omega^\bullet = \lim_n W_n\Omega^\bullet \quad \text{on} \quad X_{\text{proét}}$$
the *de Rham–Witt complex* of Bloch–Deligne–Illusie of X, [Ill79, §I.1], regarded as a complex of pro-étale sheaves on X. It comes equipped with a Frobenius operator F and a Verschiebung operator V satisfying $FV = VF = p$. We denote by
$$W\Omega^\bullet_{\log} = \lim_n W_n\Omega^\bullet_{\log} \quad \text{on} \quad X_{\text{proét}}$$
the *logarithmic de Rham–Witt complex* of X, [Ill79, §II.5.7], fitting in the following exact sequence of complexes of pro-étale sheaves on X
$$0 \to W\Omega^\bullet_{\log} \longrightarrow W\Omega^\bullet \xrightarrow{F-1} W\Omega^\bullet \to 0,$$
[Ill79, I, Théorème 5.7.2].

Remark 2.5. We recall from [BMS19, Proposition 8.4] that we have isomorphisms
$$W\Omega^i \cong R\lim_n W_n\Omega^i, \qquad W\Omega^i_{\log} \cong R\lim_n W_n\Omega^i_{\log}$$
on $X_{\text{proét}}$.

We will use several times the following result of Illusie.

Lemma 2.6 ([Ill79, I, Corollaire 5.7.5])**.** *Let k be a perfect field of characteristic p, and let X be a smooth scheme over k. For all $n \in \mathbb{Z}_{\geq 1}$ the natural map*
$$W\Omega^i_{\log}/p^n \to W_n\Omega^i_{\log}$$
is an isomorphism of pro-étale sheaves on X.

In the following, we regard $\mathbb{Z}_p(i)^{\text{syn}}$ on the pro-étale site $X_{\text{proét}}$.

Theorem 2.7 ([BMS19, Corollary 8.21])**.** *Let k be a perfect field of characteristic p, and let X be a smooth scheme over k. Then, there is a natural isomorphism of sheaves of complexes on the pro-étale site $X_{\text{proét}}$*
$$\mathbb{Z}_p(i)^{\text{syn}} \simeq W\Omega^i_{\log}[-i] \tag{2.1}$$
and, for all $n \in \mathbb{Z}_{\geq 1}$, there is a natural isomorphism
$$\mathbb{Z}/p^n(i)^{\text{syn}} \simeq W_n\Omega^i_{\log}[-i]. \tag{2.2}$$

Proof. The isomorphism (2.1) is [BMS19, Corollary 8.21]. Reducing (2.1) modulo p^n, we obtain the isomorphism (2.2), thanks to Lemma 2.6. □

2.0.2. **Rigidity of syntomic cohomology**. Let $i \in \mathbb{Z}_{\geq 0}$. By [AMMN22, Theorem 5.1, (2)], one can extend the definition of the *integral syntomic cohomology*
$$\mathbb{Z}_p(i)^{\text{syn}}(R) = R\Gamma_{\text{syn}}(R, \mathbb{Z}_p(i))$$
to all p-adically complete rings R, via left Kan extension from finitely generated p-adically complete polynomial $\mathbb{Z}_p$-algebras.

We can then state the following rigidity result.

Theorem 2.8 ([AMMN22, Theorem 5.2]). *Let (R, I) be a henselian pair where R and R/I are p-adically complete rings. Then, the homotopy fiber*

$$\mathrm{fib}\,(\mathbb{Z}_p(i)^{\mathrm{syn}}(R) \to \mathbb{Z}_p(i)^{\mathrm{syn}}(R/I))$$

is concentrated in cohomological degrees $\leq i$.

As an immediate consequence, we have the following. We will switch from algebraic to geometric notation to denote syntomic cohomology.

Corollary 2.9. *Let $\mathfrak{X} = \mathrm{Spf}(R)$ be an affine formal scheme over $\mathcal{O}_C$. Denote by k the residue field of C. Then, the homotopy fiber*

$$\mathrm{fib}\,(\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X}) \to \mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X}_k))$$

is concentrated in cohomological degrees $\leq i$.

Proof. Denote by $\mathfrak{m}_C$ the maximal ideal of $\mathcal{O}_C$, so that $k = \mathcal{O}_C/\mathfrak{m}_C$ is the residue field of C. We observe that the pair $(R, \mathfrak{m}_C R)$ is henselian: by [Sta24, Tag 0DYD], we can reduce to checking that (R, pR) and $(R/p, \mathfrak{m}_C R/p)$ are henselian pairs; for the first one, by [Sta24, Tag 0ALJ], it suffices to note that R is p-adically complete; for the second pair, we observe that the ideal $\mathfrak{m}_C R/p$ of R/p is locally nilpotent, and then we apply [Sta24, Tag 0ALI]. Thus, the statement follows by applying Theorem 2.8 to the henselian pair $(R, \mathfrak{m}_C R)$. □

We are ready to prove the main result of this section.

Proof of Theorem 2.1. Considering the short exact sequences

$$0 \to H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p(i))/p^n \to H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}/p^n(i)) \to H^{i+1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p(i))[p^n] \to 0$$

and

$$0 \to H^i(\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X}))/p^n \to H^i(\mathbb{Z}/p^n(i)^{\mathrm{syn}}(\mathfrak{X})) \to H^{i+1}(\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X}))[p^n] \to 0,$$

using Theorem 2.3, we deduce that we have a natural isomorphism

$$H^{i+1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p(i))[p^n] \cong H^{i+1}(\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X}))[p^n]. \tag{2.3}$$

On the other hand, by Corollary 2.9, we have a natural isomorphism

$$H^{i+1}(\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X})) \cong H^{i+1}(\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X}_k)). \tag{2.4}$$

Therefore, combining (2.3), (2.4), and using Theorem 2.7, we deduce that

$$H^{i+1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p(i))[p^n] \cong H^1(\mathfrak{X}_k, W\Omega^i_{\log})[p^n]. \tag{2.5}$$

We obtain the statement observing that, as C is algebraically closed, we can ignore the twist (i) on the left hand side of (2.5). □

3. Preliminaries on log de Rham–Witt cohomology

Notation 3.1. In this section, we denote by k an algebraically closed field of characteristic p.

Our goal here is to collect a number of known results on the logarithmic de Rham–Witt cohomology of smooth proper schemes over k that will be useful in the next section, when combined with Theorem 2.1.

We start with the following results of Illusie–Raynaud on the structural properties of the logarithmic de Rham–Witt cohomology.

Lemma 3.2 ([IR83, IV, Corollaire 3.5(a)])**.** *Let X be a smooth proper scheme over k. For all $i, j \in \mathbb{Z}$, we have a short exact sequence*

$$0 \to H^j(X, W\Omega^i_{\log}) \longrightarrow H^j(X, W\Omega^i) \xrightarrow{F-1} H^j(X, W\Omega^i) \to 0.$$

Remark 3.3. Given a smooth proper scheme X over k, in [IR83, §IV.3] the cohomology group $H^j(X, W\Omega^i_{\log})$ denotes $\lim_n H^j(X, W_n\Omega^i_{\log})$. However, it is also shown in the proof of [IR83, IV, Corollaire 3.5(a)], that the inverse system $\{H^{j-1}(X, W_n\Omega^i_{\log})\}_n$ is Mittag-Leffler, therefore, in our notation $H^j(X, W\Omega^i_{\log}) := H^j(R\Gamma(X, W\Omega^i_{\log}))$, we have $H^j(X, W\Omega^i_{\log}) \xrightarrow{\sim} \lim_n H^j(X, W_n\Omega^i_{\log})$.

Lemma 3.4 ([IR83])**.** *Let X be a smooth proper scheme over k. Let $i, j \in \mathbb{Z}$.*

(i) We have a natural (in X) short exact sequence

$$0 \to M^{ij}(X) \to H^j(X, W\Omega^i_{\log}) \to N^{ij}(X) \to 0$$

where $N^{ij}(X)$ is a finitely generated $\mathbb{Z}_p$-module and $M^{ij}(X)$ is killed by a power of p.

(ii) The following facts are equivalent:
- *(a) $M^{ij}(X) = 0$,*
- *(b) $d : H^j(X, W\Omega^{i-1}) \to H^j(X, W\Omega^i)$ is the zero map,*
- *(c) $H^j(X, W\Omega^i)^{V=0}$ is finite-dimensional over k, where V denotes the Verschiebung operator.*

(iii) The conditions in (ii) are satisfied for $j = 0, 1$.

Proof. Part (i) and (ii) follow from [IR83, IV, Théorème 3.3(b), Corollaire 3.5(b) & II, Corollaire 3.8]. For part (iii), we claim that, for $j = 0, 1$, we have that $H^j(X, W\Omega^i)^{V=0}$ is finite-dimensional over k. In fact, for $j = 0$, by [Ill79, II, Corollaire 2.17], we even have that $H^0(X, W\Omega^i)$ is a finite free module over $W(k)$. For $j = 1$, this is the content of (the proof of) [IR83, II, Corollaire 3.11]. □

Remark 3.5. We note that some of the previous results were generalized to the log-smooth case by Lorenzon, [Lor02].

Let us now collect some consequences. For part (i) of the result below, see also [Pet24, Lemma A.2].

Corollary 3.6 ([IR83])**.** *Let X be a smooth proper scheme over k. Let $i \in \mathbb{Z}$.*

(i) For all $j \in \mathbb{Z}$, we have an isomorphism of $\mathbb{Z}_p$-modules

$$H^j(X, W\Omega^i_{\log}) \cong \mathbb{Z}_p^{\oplus r} \oplus T$$

for some $r \in \mathbb{Z}_{\geq 0}$, with T killed by a power of p.

(ii) For $j = 0, 1$, the cohomology group $H^j(X, W\Omega^i_{\log})$ is a finitely generated $\mathbb{Z}_p$-module, and for $j = 0$ it is also a free $\mathbb{Z}_p$-module.

Proof. Part (i) readily follows from Lemma 3.4(i) and the classification of finitely generated $\mathbb{Z}_p$-modules. The first assertion of part (ii) follows combining part (i) and (iii) of Lemma 3.4. The last assertion follows recalling that $H^0(X, W\Omega^i_{\log}) = H^0(X, W\Omega^i)^{F=1}$ and, by [Ill79, II, Corollaire 2.17], $H^0(X, W\Omega^i)$ is a finite free module over $W(k)$. □

Remark 3.7. Keeping the assumptions of Corollary 3.6, by [Ill79, II, Proposition 5.9], we also have that the cohomology group $H^2(X, W\Omega^1_{\log})$ is a finitely generated $\mathbb{Z}_p$-module. On the other hand, for X a supersingular K3 surface we have that, $H^3(X, W\Omega^1_{\log}) \cong k$, [Ill79, §II.7.2], thus the torsion subgroup T in Corollary 3.6(i) can be infinite in general.

4. Finiteness of torsion and applications

4.0.1. **Finiteness**. Our first goal here is to prove the following surprising finiteness result, Theorem 4.1, which thanks to Theorem 2.1 can be reduced to a problem in characteristic p.

Theorem 4.1. *Let $\mathfrak{X}$ be an affine smooth formal scheme over $\mathcal{O}_C$. Then, for all $i \in \mathbb{Z}$, the cohomology group $H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)$ has finite torsion subgroup.*

Proof. The statement follows combining Theorem 2.1 and Proposition 4.2(ii) below. □

The following is the crucial technical result that we use for the proof of Theorem 4.1. We will prove it by reduction to the proper case, via refined alterations and purity of the logarithmic de Rham–Witt sheaves.

Proposition 4.2. *Let k be an algebraically closed field of characteristic p. Let Y be a smooth variety over k.*[2] *Let $i \in \mathbb{Z}$.*

(i) For all $n \in \mathbb{Z}_{\geq 1}$, the cohomology group $H^0(Y, W_n\Omega^i_{\log})$ is a finite abelian group.
(ii) The cohomology group $H^1(Y, W\Omega^i_{\log})$ has finite torsion subgroup.

Proof. Part (i) in the case Y is smooth proper over k follows from Corollary 3.6(ii), using the short exact sequence

$$0 \to H^0(Y, W\Omega^i_{\log})/p^n \to H^0(Y, W_n\Omega^i_{\log}) \to H^1(Y, W\Omega^i_{\log})[p^n] \to 0, \tag{4.1}$$

which follows from Lemma 2.6.

Next, we want to show that we can reduce to the proper case. Let Y be a smooth variety over k. Using a refinement of de Jong's alterations, [BS, Theorem 1.2], there exist an étale morphism $X \to Y$ over k, and a dense open immersion $X \hookrightarrow \overline{X}$ into a smooth projective variety $\overline{X}$ over k whose complement is a strict normal crossing divisor $D \subset \overline{X}$. Since $W_n\Omega^i_{\log}$ satisfies étale descent, we have that $H^0(Y, W_n\Omega^i_{\log})$ injects into $H^0(X, W_n\Omega^i_{\log})$, hence it suffices to prove the statement with X in place of Y. For this, we consider the localization exact sequence

$$H^0_D(\overline{X}, W_n\Omega^i_{\log}) \to H^0(\overline{X}, W_n\Omega^i_{\log}) \to H^0(X, W_n\Omega^i_{\log}) \to H^1_D(\overline{X}, W_n\Omega^i_{\log})$$

where $H^\bullet_D$ denotes the cohomology with support in the closed subscheme $D \subset \overline{X}$. Then, using the purity of the logarithmic de Rham–Witt sheaves, [Gro85, II, Théorème 3.5.8, (3.5.19)], [Shi07, Theorem 3.1], we deduce the exact sequence

$$0 \to H^0(\overline{X}, W_n\Omega^i_{\log}) \to H^0(X, W_n\Omega^i_{\log}) \to \bigoplus_{r=1}^m H^0(D_r, W_n\Omega^{i-1}_{\log}) \tag{4.2}$$

where we wrote $D = \bigcup_{r=1}^m D_r$ as union of its irreducible components D_r. As $\overline{X}$ and D_r, for $r = 1, \ldots, m$, are smooth proper varieties over k, part (i) follows from the previous case and the exact sequence (4.2).

For part (ii), let $M := H^1(Y, W\Omega^i_{\log})$. By part (i) and the exact sequence (4.1) for $n = 1$, we have that $M[p]$ is a finite abelian group. Observing that M is derived p-adically complete,[3] we deduce the statement from Lemma 4.3 below. □

[2] Here, a *variety over k* is a separated, integral, scheme of finite type over $\operatorname{Spec} k$.

[3] In fact, $R\Gamma(Y, W\Omega^i_{\log}) = R\lim_n R\Gamma(Y, W_n\Omega^i_{\log})$ is derived p-adically complete, as it is limit of p^n-torsion complexes. Hence, we can use [BS15, Proposition 3.4.4].

We used the following simple lemma.

Lemma 4.3. *Let M be a derived p-adically complete $\mathbb{Z}_p$-module. Suppose that the p-torsion submodule $M[p]$ is finite, then the torsion submodule $M_{\mathrm{tors}} = \mathrm{colim}_n\, M[p^n]$ is finite.*

Proof. We denote by $T_p(M) := \lim_n M[p^n]$ the p-adic Tate module of M, where the transition maps $M[p^n] \to M[p^{n-1}]$ are given by multiplication by p. Since M is derived p-adically complete (and concentrated in degree 0), by [Sta24, Tag 0BKG] we have that

$$T_p(M) = H^{-1}(M^\wedge) = H^{-1}(M) = 0$$

where $(-)^\wedge$ denotes the derived p-adic completion. As $M[p]$ is finite, we deduce that there exists a sufficiently big integer m such that the map $M[p^{m+1}] \to M[p]$, given by multiplication by p^m, is the zero map. In other words, $M_{\mathrm{tors}} = M[p^m]$. Now, using the exact sequence

$$0 \to M[p^{m-1}] \to M[p^m] \to M[p]$$

where the right map is given by multiplication by p^{m-1}, arguing by induction we deduce that $M[p^m]$ is finite. Putting everything together, we conclude that M_{tors} is finite, as desired. □

Remark 4.4. After a first draft of this note was distributed, Alexander Petrov informed us that Proposition 4.2(i) for $n = 1$ can also be deduced from [Ill79, Corollaire 2.5.6 (Page 534), Théorème 2.4.2 (Page 528)]. We note also that Proposition 4.2(i) for a general $n \in \mathbb{Z}_{\geq 1}$ can be deduced from the case $n = 1$. For this, we can argue by induction on n, using the exact sequence of pro-étale sheaves on Y, which follows from Lemma 2.6:

$$0 \to W_{n-1}\Omega^i_{\log} \to W_n\Omega^i_{\log} \to \Omega^i_{\log} \to 0$$

where the right map is given by reduction modulo p.

We observe the following easy consequence of Theorem 4.1.

Corollary 4.5. *Let $\mathfrak{X}$ be an affine smooth formal scheme over $\mathcal{O}_C$. Then, for all $i \in \mathbb{Z}$, we have a natural isomorphism*

$$H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p) \xrightarrow{\sim} \lim_n H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}/p^n).$$

Proof. Considering the short exact sequence

$$0 \to H^{i-1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)/p^n \to H^{i-1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}/p^n) \to H^i_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)[p^n] \to 0,$$

by Theorem 4.1 and the Mittag-Leffler criterion, [Gro61, Proposition 13.2.2], we have that

$$R^1 \lim_n H^{i-1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}/p^n) = 0.$$

This implies the statement. □

4.0.2. **p-torsion in the Picard group**. Let us now collect another interesting consequence of Proposition 4.2(i). A proof of the following result was sketched in [GJRW96, Lemma 2.8], assuming resolution of singularities in characteristic p.

Proposition 4.6. *Let k be an algebraically closed field of characteristic p, and let Y be a smooth variety over k. Then, for all $n \in \mathbb{Z}_{\geq 0}$,*

$$\mathrm{Pic}(Y)[p^n] \textit{ is finite.}$$

Proof. By [BMS19, Proposition 7.17], we have a natural isomorphism

$$\mathbb{Z}_p(1)^{\mathrm{syn}}(Y) \simeq R\Gamma_{\text{ét}}(Y, \mathbb{G}_m)^{\wedge}[-1]$$

where $(-)^{\wedge}$ denotes the derived p-adic completion. Combining this with Theorem 2.4, we deduce that we have a short exact sequence

$$0 \to H^0(Y, \mathbb{G}_m)/p^n \to H^0(Y, W_n\Omega^1) \to \mathrm{Pic}(Y)[p^n] \to 0.$$

In particular, $H^0(Y, W_n\Omega^1)$ surjects onto $\mathrm{Pic}(Y)[p^n]$, and then the statement follows from Proposition 4.2(i). □

Corollary 4.7. *Let $\mathfrak{X}$ be an affine smooth formal scheme over $\mathcal{O}_C$. Then, for all $n \in \mathbb{Z}_{\geq 0}$,*

$$\mathrm{Pic}(\mathfrak{X}_C)[p^n] \text{ is finite.}$$

Proof. Let k denote the residue field of C. By [Lüt16, Lemma 6.2.4], [Heu21, Lemma 3.6], we have natural isomorphisms of Picard groups

$$\mathrm{Pic}(\mathfrak{X}_C) \cong \mathrm{Pic}(\mathfrak{X}) \cong \mathrm{Pic}(\mathfrak{X}_k).$$

Then, the statement follows from Proposition 4.6. □

4.0.3. **Rational p-adic étale cohomology of affinoids**. As a corollary of Theorem 4.1, we will deduce that, for X an affinoid rigid space over C having an affine smooth formal model over $\mathcal{O}_C$, the rational p-adic pro-étale cohomology groups of X are $\mathbb{Q}_p$-Banach spaces.

The following result relies crucially on the fact that connected affinoid rigid spaces over a p-adic field are $K(\pi, 1)$ for p-torsion coefficients, as proved by Scholze.

Proposition 4.8. *Let X be an affinoid rigid space over C. Let $i \in \mathbb{Z}$. If $H^i_{\text{proét}}(X, \mathbb{Z}_p)$ has bounded p^∞-torsion, then the condensed $\mathbb{Q}_p$-vector space $H^i_{\text{proét}}(X, \mathbb{Q}_p)$ is a Banach space.*

For the proof, we will use the following general lemma. We refer the reader to [BS15, §3.4] for the definition of derived completion in a general replete topos, as well as its basic properties, which will be used freely in the following.

Lemma 4.9. *Let M be a condensed abelian group, and denote by $M(*)$ its underlying abelian group. Suppose that the following conditions are satisfied:*

(i) the natural maps of condensed abelian groups $M \to M^{\wedge}_p \leftarrow (M(*))^{\wedge}_p$ are isomorphisms; here, we denote $(-)^{\wedge}_p : D(\mathrm{CondAb}) \to D(\mathrm{CondAb})$ the derived p-adic completion, and regard $M(*)$ as a discrete condensed abelian group;

(ii) $M(*)$ has bounded p^∞-torsion.

Then, the condensed $\mathbb{Q}_p$-vector space $M[1/p]$ is a Banach space.

Proof. By condition (i) there exists an exact sequence of condensed abelian groups as follows:

$$\Big(\bigoplus\nolimits_I \mathbb{Z}\Big)^{\wedge}_p \xrightarrow{f} \Big(\bigoplus\nolimits_J \mathbb{Z}\Big)^{\wedge}_p \to M \to 0$$

for some sets I, J. We write N for the target of the map f and L for the image of f, so that we have a short exact sequence of condensed abelian groups

$$0 \to L \to N \to M \to 0. \tag{4.3}$$

Using, in addition, condition (ii) we deduce that the exact sequence (4.3) is given by taking the limit over $n \in \mathbb{Z}_{\geq 0}$ of the exact sequence of inverse systems

$$\{M(*)[p^n]\} \to \{L(*)/p^n\} \to \{N(*)/p^n\} \to \{M(*)/p^n\} \to 0.$$

We deduce in particular that, inverting p in (4.3), $L[1/p]$ is a closed subspace of the $\mathbb{Q}_p$-Banach space $N[1/p]$, and then $M[1/p]$ is a $\mathbb{Q}_p$-Banach space too, as desired. □

Proof of Proposition 4.8. We may assume X connected. Fix $x \in X(C)$ a base point, and let $G := \pi_1(X, x)$ denote the profinite étale fundamental group. We claim that, for any $n \in \mathbb{Z}_{\geq 1}$, there is a natural isomorphism in $D(\mathrm{CondAb})$

$$R\Gamma_{\mathrm{cond}}(G, \mathbb{Z}/p^n) \xrightarrow{\sim} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{Z}/p^n)$$

where the left hand side denotes the condensed group cohomology (see e.g. [Bos21, Definition B.1]). For this, by the proof of [Sch13, Theorem 4.9], for any κ-small extremally disconnected set S (where κ is the cardinal fixed in 1.6), the latter holds true on S-valued points. In particular, we have that $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{Z}_p) \simeq R\Gamma_{\mathrm{cond}}(G, \mathbb{Z}_p)$. Since $\mathbb{Z}_p$ is a solid abelian group and G is profinite, the condensed group cohomology complex $R\Gamma_{\mathrm{cond}}(G, \mathbb{Z}_p)$ is quasi-isomorphic to the complex of solid abelian groups

$$\mathbb{Z}_p \longrightarrow \underline{\mathrm{Hom}}(\mathbb{Z}[G]^{\blacksquare}, \mathbb{Z}_p) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{Z}[G \times G]^{\blacksquare}, \mathbb{Z}_p) \longrightarrow \cdots$$

sitting in non-negative cohomological degrees (see e.g. [Bos21, Proposition B.2, (i)]). By [CS19, Corollary 6.1, (iv)] and [CS19, Corollary 5.5], for any $j \in \mathbb{Z}_{\geq 0}$, using that G^j is profinite, there exist a set J (depending on j), and an isomorphism $\mathbb{Z}[G^j]^{\blacksquare} \cong \prod_J \mathbb{Z}$. Moreover, we recall that $R\underline{\mathrm{Hom}}(\prod_J \mathbb{Z}, \mathbb{Z}) = \bigoplus_J \mathbb{Z}$ concentrated in degree 0 (see the proof of [CS19, Proposition 5.7]); in particular, we have

$$R\underline{\mathrm{Hom}}\left(\prod\nolimits_J \mathbb{Z}, \mathbb{Z}_p\right) = \left(\bigoplus\nolimits_J \mathbb{Z}\right)^{\wedge}_p$$

concentrated in degree 0. Putting everything together, we deduce that $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{Z}_p)$ is quasi-isomorphic to a complex of condensed abelian groups whose terms are of the form $(\bigoplus_I \mathbb{Z})^{\wedge}_p$ for some set I. Therefore, $M := H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{Z}_p)$ satisfies condition (i) of Lemma 4.9: in fact, the objects $(\bigoplus_I \mathbb{Z})^{\wedge}_p$ satisfy this condition, and the condensed abelian groups meeting this condition are stable under kernels and cokernels.[4] Since, by assumption, M also satisfies condition (ii) of Lemma 4.9, the statement follows by applying the latter lemma and observing that $H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{Q}_p) = H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{Z}_p)[1/p]$, as X is quasi-compact and quasi-separated. □

Combining Theorem 4.1 and Proposition 4.8, we obtain the following result.

Theorem 4.10. *Let $\mathfrak{X}$ be an affine smooth formal scheme over $\mathcal{O}_C$. Then, for all $i \in \mathbb{Z}$, the condensed $\mathbb{Q}_p$-vector space $H^i_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_C, \mathbb{Q}_p)$ is a Banach space.*

[4] In fact, the full subcategory of CondAb spanned by the objects satisfying condition (i) of Lemma 4.9 can be identified with the essential image of the fully faithful functor from derived p-adically complete abelian groups to condensed abelian groups, sending A to $(A^{\mathrm{disc}})^{\wedge}_p$, where A^{disc} denotes the condensed abelian group associated with A endowed with the discrete topology. Hence, we can use that derived p-adically complete abelian groups are stable under kernels and cokernels, [BS15, Lemma 3.4.14].

4.0.4. **Beyond the affine case**. Our next goal is to extend Theorem 2.1 to a wider class of formal schemes.

Notation 4.11. In this section, we denote by k the residue field of C.

In the proof of the following theorem, it will be crucial once again the finiteness result stated in Proposition 4.2(i).

Theorem 4.12. *Let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_C$ that admits an open affine covering $\{\mathfrak{U}_r\}_{r\in\mathbb{Z}_{\geq 0}}$ with $\mathfrak{U}_r \subseteq \mathfrak{U}_{r+1}$. Then, for all $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ we have a natural isomorphism*

$$H^{i+1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)[p^n] \cong H^1(\mathfrak{X}_k, W\Omega^i_{\log})[p^n].$$

Proof. It suffices to show that Corollary 2.9 extends to $\mathfrak{X}$ a smooth formal scheme over $\mathcal{O}_C$ as in the statement, and then we can use the same argument of the proof of Theorem 2.1. Namely, we want to show that the homotopy fiber

$$\text{fib}\,(\mathbb{Z}_p(i)^{\text{syn}}(\mathfrak{X}) \to \mathbb{Z}_p(i)^{\text{syn}}(\mathfrak{X}_k))$$

is concentrated in cohomological degrees $\leq i$. First, we note that, by the derived Nakayama lemma, the latter assertion is equivalent to its version modulo p, i.e. replacing $\mathbb{Z}_p(i)^{\text{syn}}$ by $\mathbb{F}_p(i)^{\text{syn}}$. In other words, we want to prove that, for all $j \geq i+1$, the natural map

$$H^j(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{X})) \to H^j(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{X}_k))$$

is an isomorphism. Consider the short exact sequence

$$0 \to R^1 \lim_r H^{j-1}(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{U}_r)) \to H^j(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{X})) \to \lim_r H^j(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{U}_r)) \to 0$$

and the analogous short exact sequence with $\mathfrak{X}$ replaced by $\mathfrak{X}_k$ and $\mathfrak{U}_r$ replaced by its special fiber $U_r := (\mathfrak{U}_r)_k$. By Corollary 2.9 (applied to each affine $\mathfrak{U}_r$), we deduce that it is sufficient to show that

$$R^1 \lim_r H^i(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{U}_r)) = 0 \tag{4.4}$$

and

$$R^1 \lim_r H^i(\mathbb{F}_p(i)^{\text{syn}}(U_r)) = 0. \tag{4.5}$$

First of all, we note that (4.4) holds true if and only if (4.5) holds true. In fact, (4.4) is equivalent to $H^{i+1}(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{U}_r)) \xrightarrow{\sim} \lim_r H^{i+1}(\mathbb{F}_p(i)^{\text{syn}}(\mathfrak{U}_r))$, which, in turn, again by Corollary 2.9, is equivalent to (4.5). Next, we observe that, by Theorem 2.7, the vanishing (4.5) is equivalent to

$$R^1 \lim_r H^0(U_r, \Omega^i_{\log}) = 0. \tag{4.6}$$

Therefore, we are reduced to check the vanishing (4.6): this follows from Proposition 4.2(i) and the Mittag-Leffler criterion, [Gro61, Proposition 13.2.2]. □

At this point, it is natural to conjecture an extension of Theorem 4.12 to the semistable reduction case. In order to state the conjecture precisely, we need to introduce some notation.

Notation 4.13. We say that a formal scheme over $\mathcal{O}_C$ is *semistable* if it has étale locally semistable coordinates in the sense of [CK19, §1.5]. Given a semistable formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$ we endow $\mathfrak{X}$ with the canonical log structure, i.e. the one given by the sheafification of the subpresheaf $\mathcal{O}_{\mathfrak{X},\text{ét}} \cap (\mathcal{O}_{\mathfrak{X},\text{ét}}[1/p])^\times \hookrightarrow \mathcal{O}_{\mathfrak{X},\text{ét}}$, and the special fiber $\mathfrak{X}_k$ with the pullback log structure.

We write k^0 for k with the log structure associated with $\mathbb{Z}_{\geq 0} \to k$, $1 \mapsto 0$, $0 \mapsto 1$.

Conjecture 4.14. *Let $\mathfrak{X}$ be a semistable formal scheme over $\mathcal{O}_C$ that admits an open affine covering $\{\mathfrak{U}_r\}_{r\in\mathbb{Z}_{\geq 0}}$ with $\mathfrak{U}_r \subseteq \mathfrak{U}_{r+1}$. Then, for all $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ we have a natural isomorphism*

$$H^{i+1}_{\text{proét}}(\mathfrak{X}_C, \mathbb{Z}_p)[p^n] \cong H^1(\mathfrak{X}_k/k^0, W\Omega^i_{\log})[p^n].$$

Here, we use the logarithmic de Rham-Witt cohomology for log schemes as defined in [Lor02].

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