Refined TC^{-} over ku and derived q-Hodge complexes

Ferdinand Wagner (joint work with Samuel Meyer)

September 10, 2024

Goal: Sketch a computation of $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$.

Reminder on refined $\mathrm{THH}/\mathrm{TC}^{-}$

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If k is complex orientable, then $k^{\mathrm{h}S^1} \simeq k[\![t]\!]$, $|t| = -2$, and $(-)^{\mathrm{h}S^1}$ induces an equivalence of ∞-categories $\mathrm{Mod}_k(\mathrm{Sp})^{BS^1} \simeq \mathrm{Mod}_{k[\![t]\!]}(\mathrm{Sp})_t^{\wedge}$.

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Via this equivalence, we can define $\mathrm{TC}^{-,\mathrm{ref}}(-/k) \in \mathrm{Nuc}(k[\![t]\!])$ (closely related to nuclear solid $k[\![t]\!]$-modules á la Clausen–Scholze).

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Strategy: Compute $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{MU} \otimes \mathbb{Q}/\mathrm{MU})$ and then use Adams–Novikov descent.

So far, we can describe $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$.

Main result

For simplicity, replace ku by $\mathrm{KU}_p^\wedge$.

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Theorem (Meyer–W. 2024)**.**

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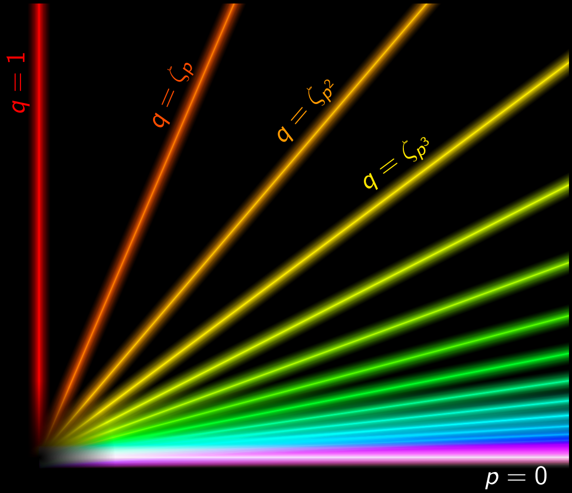
where

- $\mathcal{O}^\dagger(-)$ denotes rings of overconvergent functions,
- $Z \subseteq \operatorname{Spa} \mathbb{Z}_p[\![q-1]\!]$ denotes the subset

$$\operatorname{Spa}\left(\mathbb{F}_p(\!(q-1)\!), \mathbb{F}_p[\![q-1]\!]\right) \cup \bigcup_{\alpha \geqslant 0} \operatorname{Spa}\left(\mathbb{Q}_p(\zeta_{p^\alpha}), \mathbb{Z}_p[\zeta_{p^\alpha}]\right).$$

An image of Z



Proof sketch

Formal part: There's a cofibre sequence

$$\text{“}\underset{\alpha\geqslant 2}{\mathrm{colim}}\text{”}\ \mathrm{TC}^{-}\big((\mathrm{KU}/p^{\alpha})/\mathrm{KU}\big)^{\vee}\longrightarrow \mathrm{KU}_p^{\wedge}[\![t]\!]\longrightarrow \mathrm{TC}^{-,\mathrm{ref}}\big(\mathrm{KU}_p^{\wedge}\otimes\mathbb{Q}/\mathrm{KU}_p^{\wedge}\big)\,.$$

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Essential calculation: Compute $\pi_*\,\mathrm{TC}^{-}((\mathrm{ku}/p^{\alpha})/\mathrm{ku})$, or equivalently $\pi_0\,\mathrm{TP}((\mathrm{ku}/p^{\alpha})/\mathrm{ku})$ together with the filtration coming from the Tate spectral sequence.

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Essential calculation: Compute $\pi_*\,\mathrm{TC}^{-}((\mathrm{ku}/p^{\alpha})/\mathrm{ku})$, or equivalently $\pi_0\,\mathrm{TP}((\mathrm{ku}/p^{\alpha})/\mathrm{ku})$ together with the filtration coming from the Tate spectral sequence. Here we equip ku/p^{α} with an $\mathbb{E}_1$-ku-algebra structure obtained via base change from a fixed tower of $\mathbb{E}_1$-algebras

$$\cdots \longrightarrow \mathbb{S}/p^4 \longrightarrow \mathbb{S}/p^3 \longrightarrow \mathbb{S}/p^2$$

(Burklund [Bur22] showed that such a tower exists).

The essential calculation I

Theorem (Devalapurkar–Raksit).

There's a p-complete S^1-equivariant $\mathbb{E}_1$-equivalence

$$\psi\colon\ \mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!]\big)\xrightarrow{\simeq}\tau_{\geqslant 0}(\mathrm{ku}^{\mathrm{t}C_p})$$

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After taking $(-)^{\mathrm{h}S^1}$, this yields a map

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After taking $(-)^{\mathrm{h}S^1}$, this yields a map

$$\underset{\text{[BMS2]: Nygaard-completed Frobenius-twisted prismatic cohomology}}{\mathrm{TC}^-\big((\mathbb{Z}/p^\alpha)[\zeta_p]/\mathbb{S}[\![q-1]\!]\big)} \longrightarrow \mathrm{TP}\big((\mathrm{ku}/p^\alpha)/\mathrm{ku}\big)\,.$$

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As in Bhatt–Morrow–Scholze [BMS2, §11.3], we can get rid of the Frobenius twist and obtain a map

$$q\text{-}\Omega_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \longrightarrow \pi_0\,\mathrm{TP}\big((\mathrm{ku}/p^\alpha)/\mathrm{ku}\big)\,.$$

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$$\mathrm{L}\Omega_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \longrightarrow \pi_0\,\mathrm{HP}\big((\mathbb{Z}/p^\alpha)/\mathbb{Z}\big) \simeq \widehat{\mathrm{L}\Omega}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\,.$$

The q-Hodge filtration I

Instead of $\mathbb{Z}/p^\alpha$, consider R which is p-torsion free, p-complete, and a quasi-regular quotient over $\mathbb{Z}_p$ (this ensures that $\mathrm{L}\Omega_{R/\mathbb{Z}_p}$ and its Hodge filtration are concentrated in homological degree 0 and p-torsion free).

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If R admits an $\mathbb{E}_1$-lift $\mathbb{S}_R$, then the q-Hodge filtration is really a q-deformation of the Hodge filtration and

$$\pi_{2*}\,\mathrm{TC}^-\big((\mathrm{ku}\otimes\mathbb{S}_R)/\mathrm{ku}\big) \cong \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\Omega}_{R/\mathbb{Z}_p}\,.$$

The q-Hodge filtration II

Proof. Repeat the calculation for $\mathbb{S}_R$ instead of $\mathbb{S}/p^\alpha$ to get

$$q\text{-}\Omega_{R/\mathbb{Z}_p} \longrightarrow \pi_0\,\mathrm{TP}\big((\mathrm{ku}\otimes\mathbb{S}_R)/\mathrm{ku}\big)$$

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$$\operatorname{TP}(-/\mathrm{ku} \otimes \mathbb{Q}) \simeq \operatorname{HP}(-/\mathbb{Q}[\beta])\,. \qquad \square$$

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Note: We do *not* expect that the Hodge filtration in general can be q-deformed to a filtration on q-de Rham cohomology.

The q-Hodge filtration III

Definition (continued)**.**

Let $W \coloneqq \mathbb{Z}_p\{x\}_{\mathrm{perf}}$ and consider $W \to \mathbb{Z}_p$ given by $x \mapsto p$. Define:

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The q-Hodge filtration III

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Definition (continued)**.**

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- $\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\Omega_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} := \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\Omega_{(W/x^\alpha)/\mathbb{Z}_p} \otimes_W \mathbb{Z}_p$.
- The *derived q-Hodge complex* of $\mathbb{Z}/p^\alpha$ is

$$q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} := q\text{-}\Omega_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\left[\qquad\qquad\qquad \middle|\; n \geqslant 0\right]^{\wedge}_{(p,q-1)} .$$

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We obtain:

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Back to the main result

Recall:

Theorem (Meyer–W. 2024)**.**

$$\pi_* \, \mathsf{TC}^{-,\mathsf{ref}}(\mathsf{KU}_p^\wedge \otimes \mathbb{Q}/\mathsf{KU}_p^\wedge) \cong \mathcal{O}^\dagger(Z)[\beta^{\pm 1}]\,.$$

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$$\left\{q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\right\}_{\alpha\geqslant 2} \cong \left\{\mathcal{O}^+(U_i)\right\}_{i\in I}\ ,$$

where $\{U_i\}_{i\in I}$ is a system of open subsets of $\operatorname{Spa}\mathbb{Z}_p[\![q-1]\!]$ that exhausts the complement of Z.

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where $\{U_i\}_{i\in I}$ is a system of open subsets of $\operatorname{Spa}\mathbb{Z}_p[\![q-1]\!]$ that exhausts the complement of Z. Work of Samuel Meyer makes the q-Hodge filtration on $q\text{-}\Omega_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ sufficiently explicit to obtain such a pro-isomorphism.

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