Derived q-Hodge complexes and refined TC^-

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Abstract. — As a consequence of Efimov's proof of rigidity of the ∞-category of localising motives [Efi-Rig], Scholze and Efimov have constructed refinements of localising invariants such as TC^-. In this article we compute the homotopy groups of the refined invariants $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$. The computation involves a surprising connection to q-de Rham cohomology. In particular, it suggests a construction of a *q-Hodge filtration* on q-de Rham complexes in certain situations, which can be used to construct a functorial *derived q-Hodge complex* for many rings. This is in contrast to the no-go result from [Wag24], which showed that such a q-Hodge complex (and thus also a q-Hodge filtration) cannot exist in full generality.

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§1. Introduction

This article studies q-de Rham cohomology, refined TC^-, and a surprising connection between the two. This connection enables us to construct, for a certain full subcategory of rings, a functorial q-Hodge filtration on derived q-de Rham cohomology, and to compute the homotopy groups of $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes\mathbb{Q})/\mathrm{KU})$.

We'll first introduce the two main characters in §1.1 and §1.2, and then explain their relation in §1.3. In §1.4, we'll speculate on how this work should give rise to an interesting cohomology theory for varieties over $\mathbb{Q}$.

§1.1. q-de Rham and q-Hodge complexes

In the following, we'll work over $\mathbb{Z}$ for simplicity. Everything in this subsection can (and will, in the main body of the text) be developed over a Λ-ring A which is *perfectly covered*, meaning that the Adams operations $\psi^m\colon A\rightarrow A$ are faithfully flat; equivalently, A admits a faithfully flat Λ-map $A\rightarrow A_\infty$ into a perfect Λ-ring (see [Wag24, Remark 2.46]).

1.1. The q-de Rham complex. — For a polynomial ring $\mathbb{Z}[x]$, one can define a *q-derivative* (or *Jackson derivative* after [Jac10]) $q\text{-}\partial\colon \mathbb{Z}[x,q]\rightarrow\mathbb{Z}[x,q]$ via

$$q\text{-}\partial f(x,q)\coloneqq\frac{f(qx,q)-f(x,q)}{qx-x}\,.$$

For example, $q\text{-}\partial(x^m)=[m]_qx^{m-1}$, where $[m]_q\coloneqq 1+q+\dots+q^{m-1}$ denotes the *Gaussian q-analogue of m*. For a polynomial ring in several variables $\mathbb{Z}[x_1,\dots,x_n]$, one can similarly define *partial q-derivatives* $q\text{-}\partial_i$ for $i=1,\dots,n$ and organise them into a *q-de Rham complex*, as was first done by Aomoto [Aom90].

In [Sch17], Scholze observed that, upon completing at $(q-1)$, this construction can be extended to more general situations as follows: Define a *framed smooth $\mathbb{Z}$-algebra* to be a pair $(S,\square)$ of a smooth algebra S over $\mathbb{Z}$ and an étale map $\square\colon\mathbb{Z}[T_1,\dots,T_n]\rightarrow S$. Scholze shows that the partial q-derivatives can be extended to maps $q\text{-}\partial_i\colon S\llbracket q-1\rrbracket\rightarrow S\llbracket q-1\rrbracket$. Putting $q\text{-}\nabla\coloneqq\sum_{i=1}^n q\text{-}\partial_i\,\mathrm{d}x_i$, one can then construct a $(q-1)$-complete q-de Rham complex

$$q\text{-}\Omega^*_{S/\mathbb{Z},\square}\coloneqq\left(S\llbracket q-1\rrbracket\xrightarrow{q\text{-}\nabla}\Omega^1_{S/\mathbb{Z}}\llbracket q-1\rrbracket\xrightarrow{q\text{-}\nabla}\cdots\xrightarrow{q\text{-}\nabla}\Omega^n_{S/\mathbb{Z}}\llbracket q-1\rrbracket\right).$$

This is a q-deformation of the usual de Rham complex in the sense that $q\text{-}\Omega^*_{S/\mathbb{Z},\square}/(q-1)\cong\Omega^*_{S/\mathbb{Z}}$. In general, $q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ is not just a completed base change $\Omega^*_{S/\mathbb{Z}}\mathbin{\widehat{\otimes}_{\mathbb{Z}}}\mathbb{Z}\llbracket q-1\rrbracket$.

As a complex, $q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ depends on the choice of étale coordinates $\square$. However, Bhatt and Scholze proved that as an object in the derived category $D(\mathbb{Z}\llbracket q-1\rrbracket)$, $q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ is independent of $\square$. More precisely, they showed that there exists a functor

$$q\text{-}\Omega_{-/\mathbb{Z}}\colon\mathrm{Sm}_{\mathbb{Z}}\longrightarrow\mathrm{CAlg}\left(\widehat{\mathcal{D}}_{(q-1)}\bigl(\mathbb{Z}\llbracket q-1\rrbracket\bigr)\right)$$

such that $q\text{-}\Omega_{S/\mathbb{Z}}\simeq q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ for all framed smooth $\mathbb{Z}$-algebras $(S,\square)$. The essential step [BS19, Theorem 16.22] is to identify the p-completion $(q\text{-}\Omega^*_{S/\mathbb{Z},\square})^\wedge_p$ with the prismatic cohomology $\mathbb{\Delta}_{\widehat{S}_p[\zeta_p]/\mathbb{Z}_p\llbracket q-1\rrbracket}$ of $\widehat{S}_p[\zeta_p]$ relative to the q-de Rham prism $(\mathbb{Z}_p\llbracket q-1\rrbracket,([p]_q))$. Here ζ_p denotes a p^{th} root of unity and $q\mapsto\zeta_p$. Using the p-complete result, one can construct the global q-de Rham complex functor $q\text{-}\Omega_{-/\mathbb{Z}}$ in a more or less formal way; we'll explain this in the appendix to this paper, in §A.

1.2. The q-Hodge filtration. — It's a natural question whether the q-de Rham complex can be equipped with a q-analogue of the Hodge filtration. For a framed smooth $\mathbb{Z}$-algebra $(S,\square)$, there's an obvious guess: We could define $\mathrm{Fil}^i_{q\text{-Hdg}}\, q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ to be the complex

$$\left((q-1)^i S\llbracket q-1\rrbracket \to (q-1)^{i-1}\Omega^1_{S/\mathbb{Z}}\llbracket q-1\rrbracket \to \dots \to \Omega^i_{S/\mathbb{Z}}\llbracket q-1\rrbracket \to \dots \to \Omega^n_{S/\mathbb{Z}}\llbracket q-1\rrbracket\right).$$

This *q-Hodge filtration* is a q-deformation of the Hodge filtration on the usual de Rham complex in the sense that $\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\Omega^*_{S/\mathbb{Z},\square}/(q-1) \cong \mathrm{Fil}^*_{\mathrm{Hdg}}\,\Omega^*_{S/\mathbb{Z}}$, where the quotient is taken in filtered abelian groups, with $(q-1)$ in filtration degree 1.

The question then becomes: Can this filtration be made functorial as well? As it turns out, this is most likely *not* the case. To formulate a precise objection, let us introduce another construction: The *q-Hodge complex of* $(S,\square)$ is the complex

$$q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square} \coloneqq \left(S\llbracket q-1\rrbracket \xrightarrow{(q-1)\,q\text{-}\nabla} \Omega^1_{S/\mathbb{Z}}\llbracket q-1\rrbracket \xrightarrow{(q-1)\,q\text{-}\nabla} \cdots \xrightarrow{(q-1)\,q\text{-}\nabla} \Omega^n_{S/\mathbb{Z}}\llbracket q-1\rrbracket\right)$$

given by multiplying all the differentials in the de Rham complex by $(q-1)$. Observe that if the q-Hodge filtration could be made functorial, then the same would be true for the q-Hodge complex, as it can also be obtained as

$$q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square} \cong \operatorname{colim}\left(\mathrm{Fil}^0_{q\text{-Hdg}}\, q\text{-}\Omega^*_{S/\mathbb{Z},\square} \xrightarrow{(q-1)} \mathrm{Fil}^1_{q\text{-Hdg}}\, q\text{-}\Omega^*_{S/\mathbb{Z},\square} \xrightarrow{(q-1)} \cdots\right)^\wedge_{(q-1)}.$$

The q-Hodge complex was first introduced (with different notation) by Pridham [Pri19] and was studied by the second author in [Wag24], where the following result was shown:

1.3. Theorem (see [Wag24, Theorems 4.28 and 5.1])**.** — *Let $(S,\square)$ be a framed smooth $\mathbb{Z}$-algebra and let $q\text{-}\mathbb{W}_m\Omega^*_{S/\mathbb{Z}}$ denote the m^{th} q-de Rham–Witt complex as introduced in [Wag24, Definition 3.13].*

(*a*) *For every $m \in \mathbb{N}$ the cohomology of $q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square}/(q^m-1)$ is independent of the choice of $\square$. More precisely, there's an isomorphism of differential-graded $\mathbb{Z}[q]$-algebras*

$$\mathrm{H}^*\left(q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square}/(q^m-1)\right) \cong \left(q\text{-}\mathbb{W}_m\Omega^*_{S/\mathbb{Z}}\right)^\wedge_{(q-1)},$$

where the differential on the left-hand side is the Bockstein differential. Under this isomorphism, for all $d \mid m$ the projection $q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square}/(q^m-1) \to q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square}/(q^d-1)$ induces the Frobenius on q-de Rham–Witt complexes.

(*b*) *$(S,\square) \mapsto q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square}$ cannot be extended to a functor $q\text{-}\mathrm{Hdg}_{-/\mathbb{Z}}\colon \mathrm{Sm}_{\mathbb{Z}} \to \widehat{\mathcal{D}}_{(q-1)}(\mathbb{Z}\llbracket q-1\rrbracket)$ in such a way that the identifications from* (*a*) *also become functorial.*

Theorem 1.3 is by all means a weird result. Part (*a*) promises functoriality and a wealth of extra structure. But then part (*b*) shows that functoriality of the q-Hodge complex (and thus of the q-Hodge filtration) is impossible, at least not in a way compatible with the extra structure.

1.4. Remark. — It's not known to the authors whether the q-Hodge complex can be made functorial in a way that's *incompatible* with the extra structure, but we consider this unlikely.

In this article, we would like to propose the following partial fix for this lack of functoriality.

1.5. The q-Hodge filtration, revisited. — In the following, $\mathrm{dR}_{-/\mathbb{Z}_p}$ denotes the p-complete derived de Rham complex (that is, the p-completed non-abelian derived functor of the de Rham complex), $\mathrm{Fil}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{-/\mathbb{Z}_p}$ its derived Hodge filtration, and $q\text{-}\mathrm{dR}_{-/\mathbb{Z}_p}$ denotes the p-complete derived q-de Rham complex.

Let R be a p-torsion free p-complete ring which is a quasiregular quotient over $\mathbb{Z}_p$ and such that the Frobenius on R/p is semiperfect. We'll introduce the technical terms in 3.1; for now, we remark that these assumptions ensure that $\mathrm{dR}_{R/\mathbb{Z}_p}$ and $q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ are actual rings and $\mathrm{Fil}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{R/\mathbb{Z}_p}$ is a filtration of $\mathrm{dR}_{R/\mathbb{Z}_p}$ by a descending chain of ideals. After $(q-1)$-completed rationalisation, the q-de Rham complex becomes a base change of the de Rham complex (see Lemma A.3) and so

$$q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}\,\widehat{\otimes}_{\mathbb{Z}[\![q-1]\!]}\,\mathbb{Q}[\![q-1]\!]\simeq\big(\mathrm{dR}_{R/\mathbb{Z}_p}\otimes_{\mathbb{Z}}\mathbb{Q}\big)[\![q-1]\!]\,.$$

Observe that the left-hand side carries two filtrations: The Hodge filtration and the $(q-1)$-adic filtration. We can combine both of them into one and then define:

1.6. Definition. — The *q-Hodge filtration* $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ is the preimage of the combined Hodge- and $(q-1)$-adic filtration under $q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}\to(\mathrm{dR}_{R/\mathbb{Z}_p}\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$. The *derived q-Hodge complex of R over $\mathbb{Z}_p$* is the ring

$$q\text{-}\mathrm{Hdg}_{R/\mathbb{Z}_p}\coloneqq\operatorname{colim}\Big(\mathrm{Fil}^0_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}\xrightarrow{(q-1)}\mathrm{Fil}^1_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}\xrightarrow{(q-1)}\cdots\Big)^\wedge_{(p,q-1)}\,.$$

A priori, it seems outrageous to hope that $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ would be a well-behaved construction, and indeed, usually, it's not true that the q-Hodge filtration is a q-deformation of the Hodge filtration. For example, this provably fails if R is a p-torsion free perfectoid ring (that's essentially how Theorem 1.3(b) is proved). However, we'll show the following:

1.7. Theorem (Theorem 3.10(b))**.** — *Suppose that, in addition to the assumptions of 1.5, R admits a lift to a p-complete $\mathbb{E}_1$-ring spectrum $\mathbb{S}_R$ such that $R\simeq\mathbb{S}_R\,\widehat{\otimes}_{\mathbb{S}_p}\,\mathbb{Z}_p$. Then $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ is a q-deformation of the Hodge filtration $\mathrm{Fil}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{R/\mathbb{Z}_p}$.*

1.8. Remark. — Let us stress that $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ is completely functorial in R. The choice of a lift $\mathbb{S}_R$ is *not* part of the functoriality and the condition from Theorem 1.7 is really just an existence condition.

1.9. Remark. — Thanks to Burklund's breakthrough on the existence of $\mathbb{E}_1$-structures on quotients [Bur22], it's easy to construct examples of rings R that satisfy the condition. For example, if B is a p-complete perfect δ-ring and $(x_1,\dots,x_r)$ is a Koszul-regular sequence in B such that $B/(x_1,\dots,x_r)$ is p-torsion free, then $R\cong B/(x_1^{\alpha_1},\dots,x_r^{\alpha_r})$ admits such a lift $\mathbb{S}_R$ as soon as $\alpha_i\geqslant 2$ for all i.[(1.1)] We'll explain how this works in 3.11.

Let us also remark the following curious consequence: If $R\cong\mathrm{A}_{\mathrm{inf}}/d$ is a perfectoid ring, then we've claimed above that the q-Hodge is ill-behaved. However, it becomes well-behaved as soon as one passes to the nil-thickenings $\mathrm{A}_{\mathrm{inf}}/d^\alpha$, $\alpha\geqslant 2$.

[(1.1)] This assumes $p\geqslant 3$. In the case $p=2$, we need all α_i to be even and $\geqslant 4$ for Burklund's result to apply. Interestingly, the conclusion of Theorem 1.7 is still true for $p=2$ if we only assume $\alpha_i\geqslant 2$. This will be shown in Theorem 3.10(a).

In the situation of Remark 1.9, everything can be made explicit: The derived q-de Rham complex is the prismatic envelope

$$q\text{-dR}_{R/\mathbb{Z}_p} \simeq B\llbracket q-1\rrbracket\left\{\frac{\phi(x_1^{\alpha_1})}{[p]_q},\dots,\frac{\phi(x_r^{\alpha_r})}{[p]_q}\right\}^\wedge_{(p,q-1)}$$

and the q-Hodge filtration is the preimage of the $(x_1^{\alpha_1},\dots,x_r^{\alpha_r},q-1)$-adic filtration on the completed rationalisation. If $\gamma(-)\coloneqq(-)^p/p$ denotes the p-adic divided power, then $q\text{-dR}_{R/\mathbb{Z}_p}$ contains lifts of the iterated divided powers $\gamma^{(n)}(x_i^{\alpha_i})$ for all $n\geqslant 0$. What Theorem 1.7 essentially says in this case is that if $\alpha_i\geqslant 2$ then we can always find such a lift that also lies in the ideal $(x_1^{\alpha_1},\dots,x_r^{\alpha_r},q-1)^{p^n}$ after completed rationalisation.

It's possible to give an elementary proof of Theorem 1.7 in this case and we'll do so in §5.3. For general R, we'll derive Theorem 1.7 instead from a homotopy-theoretical result. We let ku and KU denote the connective and the periodic complex K-theory spectrum. We also let $\beta\in\pi_2(\mathrm{ku}^{\mathrm{h}S^1})$ is the Bott element and $t\in\pi_{-2}(\mathrm{ku}^{\mathrm{h}S^1})$ the complex orientation generator for which $1+\beta t$ classifies the standard representation of S^1 on $\mathbb{C}$.

1.10. Theorem (Theorem 3.22)**.** — *Let* $\mathrm{ku}_R\coloneqq(\mathrm{ku}\otimes\mathbb{S}_R)_p^\wedge$ *and* $\mathrm{KU}_R\coloneqq(\mathrm{KU}\otimes\mathbb{S}_R)_p^\wedge$*. Then the spectra* $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$ *and* $\mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU})_p^\wedge$ *are concentrated in even degrees and their even homotopy groups are given as follows:*

$$\pi_{2*}\bigl(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge\bigr)\cong\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}\,,\quad \pi_{2*}\bigl(\mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU})_p^\wedge\bigr)\cong q\text{-Hdg}_{R/\mathbb{Z}_p}[\beta^{\pm1}]\,.$$

Here $q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$ *denotes the completion of the q-de Rham complex at the q-Hodge filtration and the isomorphisms identify* $q-1=\beta t$*.*

1.11. The q-de Rham complex and ku. — The connection between ku and q-de Rham cohomology[(1.2)], of which Theorem 1.10 is an instance, was discovered by Arpon Raksit. In unpublished work, reviewed in [DM23, Remark 4.3.24], he showed

$$\mathrm{gr}^0_{\mathrm{ev}}\,\mathrm{TP}\bigl(\mathrm{ku}[x]/\mathrm{ku}\bigr)\simeq q\text{-}\Omega^*_{\mathbb{Z}[x]/\mathbb{Z},\square}\quad\text{and}\quad\mathrm{gr}^0_{\mathrm{ev}}\,\mathrm{TC}^-\bigl(\mathrm{KU}[x]/\mathrm{KU}\bigr)\simeq q\text{-Hdg}^*_{\mathbb{Z}[x]/\mathbb{Z},\square}\,.$$

Here $\mathrm{gr}^0_{\mathrm{ev}}$ refers to the 0^{th} graded piece of the *even filtration* as introduced in [HRW22] and $\square\colon\mathbb{Z}[x]\rightarrow\mathbb{Z}[x]$ is the identical framing.

To the authors' knowledge, this computation doesn't imply Theorem 1.10 yet. Instead, the heart of our proof is another unpublished result, which we learned from Arpon Raksit, who in turn learned it from Thomas Nikolaus: There exists an S^1-equivariant equivalence of $\mathbb{E}_1$-ring spectra

$$\mathrm{THH}\bigl(\mathbb{Z}_p[\zeta_p]/\mathbb{S}\llbracket q-1\rrbracket\bigr)_p^\wedge\xrightarrow{\simeq}\tau_{\geqslant0}\bigl(\mathrm{ku}^{\mathrm{t}C_p}\bigr)$$

(see Theorem 3.18). This equivalence allows us to construct a map

$$\psi_R^{(1)}\colon\mathrm{TC}^-\bigl(R[\zeta_p]/\mathbb{S}\llbracket q-1\rrbracket\bigr)_p^\wedge\longrightarrow\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$$

(see 3.19). Via the [BMS19]-approach to prismatic cohomology, one can show that the left-hand side computes the q-de Rham cohomology of R, except with a Frobenius twist and completed at

(1.2) We wish to propose the following slogan for this connection: "*q-de Rham cohomology is the de Rham cohomology of a lift along* $\mathrm{ku}\rightarrow\mathbb{Z}$", similar to how crystalline cohomology is the de Rham cohomology of a lift along $\mathbb{Z}_p\rightarrow\mathbb{F}_p$. Here "de Rham cohomology over ku" is realised as $\mathrm{TC}^-(-/\mathrm{ku})$ or $\mathrm{TP}(-/\mathrm{ku})$, equipped with their even filtrations.

the Nygaard filtration. The rest of the proof of Theorem 1.10 proceeds as follows: We massage this map until its domain is really $q\text{-dR}_{R/\mathbb{Z}_p}$, then we verifiy that it is an equivalence up to completion at some filtration (which can be done modulo $(q-1)$), and finally we identify the filtration (via comparison with the rationalisation).

1.12. Higher chromatic deformations? — In fact, for the purpose of Raksit's computation, there's nothing special about ku. For any even periodic $\mathbb{E}_\infty$-ring spectrum E with connective cover $e \simeq \tau_{\geqslant 0}(E)$, Raksit computes $\mathrm{gr}^0_{\mathrm{ev}}\,\mathrm{TP}(e[x]/e)$ to be a certain deformation of $\Omega^*_{\mathbb{Z}[x]/\mathbb{Z}}$ attached to the formal group law of E, and $\mathrm{gr}^0_{\mathrm{ev}}\,\mathrm{TC}^-(E[x]/E)$ is again a “Hodge-variant” of this. As for the q-de Rham complex and the q-Hodge complex in 1.1 and 1.2, these deformations depend on a choice of coordinates, so it's not clear whether they can be defined functorially. But we suspect that they show up in the picture that we'll sketch in §1.4 (see 1.22 specifically).

§1.2. Refined THH and TC$^-$

In the following, we use the notation and conventions for symmetric monoidal ∞-categories from 1.25. Recall the following notion due to Gaitsgory and Rozenblyum (see [GR17, Definition I.9.1.2], which is different but equivalent):

1.13. Definition. — A presentable stable symmetric monoidal ∞-category $\mathcal{E}$ is *rigid* if the following two conditions are satisfied:

(*a*) The tensor unit $\mathbb{1} \in \mathcal{E}$ is compact.

(*b*) $\mathcal{E}$ is generated under colimits by objects of the form $X \simeq \operatorname{colim}(X_1 \to X_2 \to \cdots)$, where each $X_n \to X_{n+1}$ is *trace-class*, that is, induced by a morphism $\mathbb{1} \to X_{n+1}^\vee \otimes X_n$ (see §2.2).

A deep result of Efimov [Efi-Rig] shows that whenever $\mathcal{E}$ is rigid, the ∞-category $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}}$ of *localising motives over* $\mathcal{E}$ (introduced in [BGT16]) is rigid as well. The following construction, due to Scholze and Efimov, uses this to construct refinements of localising invariants. As we'll see, these refinements often contain a lot more information than the original invariant.

1.14. Refined localising invariants. — Suppose $E\colon \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}} \to \mathcal{D}$ is a symmetric monoidal localising invariant whose target is *not* rigid. Then there exists a unique localising invariant E^{ref} that fits into a diagram

$$\begin{array}{ccc} \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}} & \xrightarrow{\;E\;} & \mathcal{D} \\ & \overset{E^{\mathrm{ref}}}{\searrow} & \uparrow \\ & & \mathcal{D}^{\mathrm{rig}} \end{array}$$

where $\mathcal{D}^{\mathrm{rig}} \to \mathcal{D}$ is the universal functor from a rigid symmetric monoidal ∞-category. We call E^{ref} the *refined invariant* of E.

If k is an $\mathbb{E}_\infty$-ring spectrum, we can consider $\mathcal{E} \simeq \mathrm{Mod}_k(\mathrm{Sp})$. Then topological Hochschild homology relative to k, $\mathrm{THH}(-/k)\colon \mathrm{Mot}^{\mathrm{loc}}_k \to \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$ is an example of a localising invariant with rigid source but non-rigid target. We let $\mathrm{THH}^{\mathrm{ref}}(-/k)$ denote its refinement. In this case the ridificitation $(\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1})^{\mathrm{rig}}$ is the ∞-category of *nuclear objects* in $\mathrm{Ind}(\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1})$, as introduced in Definition 2.8.[(1.3)] If k is complex orientable and $t \in \pi_{-2}(k^{\mathrm{h}S^1})$ is a complex

[(1.3)] Here we use that $\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$ is *locally rigid*: Its tensor unit isn't compact, but $\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$ still satisfies Definition 1.13(*b*), because it is compactly generated and all compact objects are dualisable. In general, $\mathcal{D}^{\mathrm{rig}}$ is not $\mathrm{Nuc}(\mathrm{Ind}(\mathcal{D}))$, but its full sub-$\infty$-category generated under colimits by $\mathbb{Q}_{\geqslant 0}$-indexed diagrams in which all transition maps are trace-class.

orientation generator, then taking S^1-fixed points induces a symmetric monoidal equivalence

$$(-)^{\mathrm{h}S^1}\colon \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1} \xrightarrow{\simeq} \mathrm{Mod}_{k^{\mathrm{h}S^1}}(\mathrm{Sp})_t^\wedge$$

between k-modules with S^1-action and t-complete $k^{\mathrm{h}S^1}$-modules (see Lemma 4.7). Scholze and Efimov then define the *refined* TC^- *relative to* k, $\mathrm{TC}^{-,\mathrm{ref}}(-/k)$, to be the composition of $\mathrm{THH}^{\mathrm{ref}}(-/k)$ with this equivalence. This identifies the rigidification of $\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$ with Efimov's ∞-category $\mathrm{Nuc}(k^{\mathrm{h}S^1})$ of *nuclear* $k^{\mathrm{h}S^1}$*-modules*, defined as the full sub-∞-category of $\mathrm{Ind}(\mathrm{Mod}_{k^{\mathrm{h}S^1}}(\mathrm{Sp})_t^\wedge)$ generated under colimits by sequential ind-objects of the form "colim"$(M_1 \to M_2 \to \cdots)$ such that each $M_n \to M_{n+1}$ is trace-class.

1.15. Remark. — The refinement procedure from 1.14 offers a lot of flexibility, even if we stick to THH, since the refinement is very sensitive to the choice of $\mathcal{E}$. For example, if C is a complete non-archimedean algebraically closed field, one can look at the refinement $\mathrm{THH}^{\mathrm{ref}}_{/\mathcal{O}_C}(-;\mathbb{Z}_p)$ of the functor

$$\mathrm{THH}(-;\mathbb{Z}_p)\colon \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{O}_C} \longrightarrow \mathrm{Mod}_{\mathrm{THH}(\mathcal{O}_C;\mathbb{Z}_p)}\big(\mathrm{Sp}^{\mathrm{B}S^1}\big)_p^\wedge\,;$$

that is, we refine (p-completed) absolute THH, but only accept motives over $\mathcal{O}_C$ as input.(1.4) Scholze and Efimov [Sch24] have sketched a computation of $\mathrm{THH}^{\mathrm{ref}}_{/\mathcal{O}_C}(C;\mathbb{Z}_p)$ (or $\mathrm{TC}^{-,\mathrm{ref}}_{/\mathcal{O}_C}(C;\mathbb{Z}_p)$, which is equivalent by Lemma 4.7), by reducing the problem to the known computation of $\mathrm{THH}(\mathcal{O}_C/p^\alpha;\mathbb{Z}_p)$ for all $\alpha \geqslant 1$.

The result is still vastly different from $\mathrm{THH}^{\mathrm{ref}}(C;\mathbb{Z}_p)$, where we would allow all localising motives as input—which brings us to the main question that motivated this work.

§1.3. What's $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$?

We'll explain in §1.4 why the answer to this question should be interesting, but let us already remark that it has to be non-trivial: While $\mathrm{THH}(\mathbb{Q}) \simeq \mathbb{Q}$, one has $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})_p^\wedge \not\simeq 0$ for all primes p. Indeed, this follows fromTheorem 1.19, but also from Scholze's and Efimov's result in Remark 1.15. But actually computing $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$, or just its p-completions, is a highly non-trivial task: We'll explain in §4.1 how to reduce this to a computation of $\mathrm{THH}(\mathbb{S}/p^\alpha)$ for all sufficiently large α, but computing these spectra seems out of reach.

Scholze and Efimov have suggested that a more approachable goal would be to compute the base change $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{MU} \simeq \mathrm{THH}^{\mathrm{ref}}((\mathrm{MU}\otimes\mathbb{Q})/\mathrm{MU})$ and then to attack the original question—to the extent in which that's possible—via Adams–Novikov descent. Since MU is complex orientable, this contains the same information as $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{MU}\otimes\mathbb{Q})/\mathrm{MU})$. While we still don't know what happens for MU, the purpose of this article is to give an answer for ku. To formulate the result, we need the notion of *killing a pro-idempotent algebra*, which we review in §2.3, as well as the following ad-hoc construction.

1.16. More q-Hodge filtrations. — We wish to define a q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ (since $\mathbb{Z}/p^\alpha$ is not p-torsion free, it doesn't fall within the scope of 1.5). Let $\mathbb{Z}_p\{x\}_\infty$ be the free p-complete perfect δ-ring on a generator x and let $\mathbb{Z}_p\{x\}_\infty \to \mathbb{Z}_p$ be the unique δ-ring map sending $x \mapsto p$. Then $\mathbb{Z}/p^\alpha \simeq \mathbb{Z}_p\{x\}_\infty/x^\alpha \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}_p\{x\}_\infty} \mathbb{Z}_p$, and so it makes sense to define

$$\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \coloneqq \mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}_p\{x\}_\infty[\![q-1]\!]} \mathbb{Z}_p[\![q-1]\!]\,.$$

(1.4) Historically, $\mathrm{THH}^{\mathrm{ref}}_{/\mathcal{O}_C}(-;\mathbb{Z}_p)$ is the first refined invariant considered by Scholze and Efimov.

For $\alpha \geqslant 2$ (the case $p = 2$ needs α even and $\geqslant 4$), this is a reasonable object, as Theorem 1.7 and Remark 1.9 show that it's a q-deformation of the usual Hodge filtration on $\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. For an arbitrary $m \in \mathbb{N}$ with prime factorisation $m = \prod_p p^{\alpha_p}$, we put

$$\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-dR}_{(\mathbb{Z}/m)/\mathbb{Z}} \coloneqq \prod_p \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-dR}_{(\mathbb{Z}/p^{\alpha_p})/\mathbb{Z}_p}\,.$$

One can then also define the *q-Hodge completed* derived de Rham complex $q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and the *derived q-Hodge complex* $q\text{-Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$.

1.17. Theorem (Theorem 4.20). — *$\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes\mathbb{Q})/\mathrm{KU})$ are concentrated in even degrees. Moreover, their even homotopy groups are described as follows:*

(*a*) $\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku}) \cong \mathrm{A}^*_{\mathrm{ku}}$, *where $\mathrm{A}^*_{\mathrm{ku}}$ is the idempotent nuclear graded $\mathbb{Z}[\beta][\![t]\!]$-algebra obtained by killing the pro-idempotent* "$\lim_{m\in\mathbb{N}}$" $\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$.

(*b*) $\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes\mathbb{Q})/\mathrm{KU}) \cong \mathrm{A}_{\mathrm{KU}}[\beta^{\pm1}]$, *where A_{KU} is the idempotent nuclear $\mathbb{Z}[\![q-1]\!]$-algebra obtained by killing the pro-idempotent* "$\lim_{m\in\mathbb{N}}$" $q\text{-Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$.

1.18. Remark. — Burklund [Bur22] showed that there exists a compatible system of $\mathbb{E}_1$-structures on $\mathbb{S}/m$ for m ranging through a coinitial subset of $\mathbb{N}^{\mathrm{op}}$, where $\mathbb{N}$ is the poset of positive integers ordered by divisibility. Then $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})$ itself is the idempotent nuclear $\mathrm{ku}^{\mathrm{h}S^1}$-algebra obtained by killing the pro-idempotent "lim" $\mathrm{TC}^-((\mathrm{ku}\otimes\mathbb{S}/m)/\mathrm{ku})$. This is more or less formal, except for the following input: For fixed m_0, the base change functor

$$-\otimes_{\mathbb{S}/m}\mathbb{S}/m_0\colon \mathrm{RMod}_{\mathbb{S}/m}(\mathrm{Sp}) \longrightarrow \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp})$$

is trace-class as a morphism in the ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$ of stable compactly generated ∞-categories and left adjoint functors that preserve compact objects. We'll explain the argument in §4.1. An analogous conclusion holds for $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes\mathbb{Q})/\mathrm{KU})$.

After that, the proof of Theorem 1.17 essentially reduces to a computation of the homotopy groups $\pi_*\,\mathrm{TC}^-((\mathrm{ku}\otimes\mathbb{S}/m)/\mathrm{ku})$, and then further to the case where $m = p^\alpha$ is a prime power. This is the origin of the constructions in §1.1: We realised that the S^1-equivariant $\mathbb{E}_1$-equivalence $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!])^\wedge_p \simeq \tau_{\geqslant 0}(\mathrm{ku}^{\mathrm{t}C_p})$ constructed by Thomas Nikolaus could be used to show that $\pi_0\,\mathrm{TP}((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ is the derived q-de Rham complex of $\mathbb{Z}/p^\alpha$, completed at some filtration. An investigation of that filtration and a generalisation of our argument then naturally lead us to Definition 1.6 and Theorem 1.7.

Theorem 1.17 provides a description of the desired homotopy rings, but it relies on the q-Hodge filtrations on $q\text{-dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. In §5.3, we'll show that these guys can be explicitly understood. This leads to a much more explicit description of $\mathrm{A}^*_{\mathrm{ku}}$ and A_{KU} in terms of rings of overconvergent functions on certain adic spaces. For simplicity, we'll work with $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}_p^\wedge\otimes\mathbb{Q})/\mathrm{ku}_p^\wedge)$ and $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}_p^\wedge\otimes\mathbb{Q})/\mathrm{KU}_p^\wedge)$ instead. Let us first formulate the result for $\mathrm{KU}_p^\wedge$, as it is easier to state. We put

$$\mathrm{A}_{\mathrm{KU},p} \coloneqq \pi_0\,\mathrm{TC}^{-,\mathrm{ref}}\big((\mathrm{KU}_p^\wedge\otimes\mathbb{Q})/\mathrm{KU}_p^\wedge\big)\,,$$

so $\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}_p^\wedge\otimes\mathbb{Q})/\mathrm{KU}_p^\wedge) \cong \mathrm{A}_{\mathrm{KU},p}[\beta^{\pm1}]$. Let also $X \coloneqq \operatorname{Spa}\mathbb{Z}_p[\![q-1]\!] \smallsetminus \{p=0, q=1\}$ be the "Tate locus"(1.5) where p or $q-1$ is invertible. Then $\mathrm{A}_{\mathrm{KU},p}$ has the following description, confirming a conjecture of Scholze and Efimov.

(1.5) Following Clausen–Scholze, we'll call an adic space *Tate* (rather than *analytic*) if, locally, there exists a topologically nilpotent unit.

1.19. Theorem. — *Let $Z \subseteq X$ denote the union of the closed subsets* $\operatorname{Spa}(\mathbb{F}_p(\!(q-1)\!), \mathbb{F}_p[\![q-1]\!])$ *and* $\operatorname{Spa}(\mathbb{Q}_p(\zeta_{p^n}), \mathbb{Z}_p[\zeta_{p^n}])$ *for all $n \geqslant 0$. Let $Z^\dagger$ denote the overconvergent neighbourhood of Z in X and $\mathcal{O}(Z^\dagger)$ the nuclear $\mathbb{Z}_p[\![q-1]\!]$-algebra of overconvergent functions on Z. Then*

$$\mathrm{A}_{\mathrm{KU},p} \cong \mathcal{O}(Z^\dagger)\,.$$

In Fig. 1 we show a picture of $Z^\dagger$. It should be reminiscent of Scholze's famous prismatic picture (a nice depiction of which can be found in [HN20, p. 4]), but the rays are "overconvergently blurred" and the "origin" $\{p = 0, q = 1\}$ has been removed.

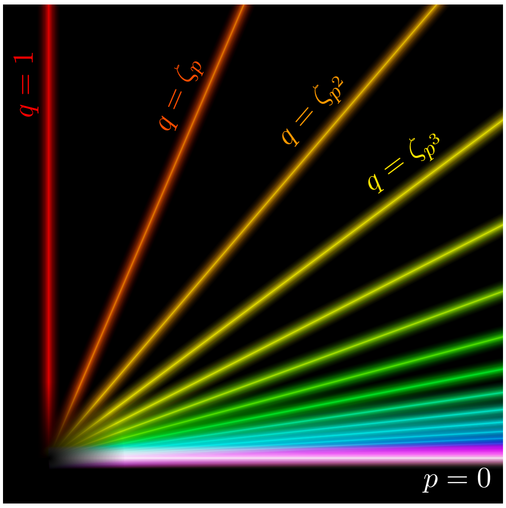

Figure 1: The analytic spectrum of $\mathrm{A}_{\mathrm{KU},p} \cong \mathcal{O}(Z^\dagger)$.

Since $Z^\dagger$ visibly contains the entire infinitesimal neighbourhood of $\{p = 0\}$ except for the "origin", we see that $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}_p^\wedge \otimes \mathbb{Q})/\mathrm{KU}_p^\wedge)_p^\wedge \neq 0$. In particular, it follows that $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})_p^\wedge \not\simeq 0$, as we've claimed above.

To formulate a similar geometric result for $\mathrm{ku}_p^\wedge$, consider the ungraded ring $\mathbb{Z}[\beta, t]_{(p,t)}^\wedge$ with its (p,t)-adic topology. We wish to encode the graded (p,t)-complete ring $\mathbb{Z}_p[\beta][\![t]\!]$ in terms of an action of $\mathbb{G}_m$ on $\operatorname{Spa} \mathbb{Z}[\beta, t]_{(p,t)}^\wedge$, as usual—but we have to be careful: Since we wish that t is a topologically nilpotent elements in non-zero graded degree, we can only act by units u "of norm $|u| = 1$". More precisely, we have to replace $\mathbb{G}_m$ by the "adic unit circle" $\mathbb{T} \coloneqq \operatorname{Spa}(\mathbb{Z}[u^{\pm 1}], \mathbb{Z}[u^{\pm 1}])$.

With this modification, everything works (as we'll elaborate in §5.2): Declaring β and t to have degree 2 and -2, respectively, determines an action of $\mathbb{T}$ on $\operatorname{Spa} \mathbb{Z}[\beta, t]_{(p,t)}^\wedge$, and we can identify $\mathbb{Z}_p[\beta][\![t]\!]$ with the structure sheaf on $(\operatorname{Spa} \mathbb{Z}[\beta, t]_{(p,t)}^\wedge)/\mathbb{T}$, where the quotient is always taken in the derived (or "stacky") sense. We also let $X^* \coloneqq \operatorname{Spa} \mathbb{Z}[\beta, t]_{(p,t)}^\wedge \smallsetminus \{p = 0, \beta t = 0\}$. Since p and βt are homogeneous, X^* inherits an action of $\mathbb{T}$. Putting

$$\mathrm{A}_{\mathrm{ku},p}^* \coloneqq \pi_{2*} \mathrm{TC}^{-,\mathrm{ref}}\big((\mathrm{ku}_p^\wedge \otimes \mathbb{Q})/\mathrm{ku}_p^\wedge\big)\,,$$

we see that $\mathrm{A}_{\mathrm{ku},p}^*$ is a graded $\mathbb{Z}_p[\beta][\![t]\!]$-module, hence we can regard it as a quasi-coherent sheaf on $(\operatorname{Spa} \mathbb{Z}[\beta, t]_{(p,t)}^\wedge)/\mathbb{T}$. As we'll see, it is already a sheaf on the open substack $X^*/\mathbb{T}$. Then $\mathrm{A}_{\mathrm{ku},p}^*$ can be described as follows:

1.20. Theorem. — *Let $Z^* \subseteq X^*$ be union of the $\mathbb{T}$-equivariant closed subsets $\{p=0\}$ and $\{[p^n]_{\mathrm{ku}}(t)=0\}$ for all $n \geqslant 0$, where $[p^n]_{\mathrm{ku}}(t) \coloneqq ((1+\beta t)^{p^n}-1)/\beta$ denotes the p^n-series of the formal group law of* ku*. Let $Z^{*,\dagger}$ denote the overconvergent neighbourhood of Z^*. Then $Z^{*,\dagger}$ inherits a $\mathbb{T}$-action and*

$$\mathrm{A}_{\mathrm{ku},p}^* \cong \mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}\,.$$

§1.4. Synthesis: A new cohomology theory for $\mathbb{Q}$-varieties

Let us end with a bit of speculation. It should be possible to adapt the formalism of even filtrations from [HRW22] to $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{ku})$. For a smooth variety X over $\mathbb{Q}$, the graded pieces $\mathrm{gr}_{\mathrm{ev}}^*\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes X)/\mathrm{ku})$ should define a cohomology theory $\mathrm{R\Gamma}_{\mathrm{ku}}(X)$ together with a filtration $\mathrm{Fil}^*\mathrm{R\Gamma}_{\mathrm{ku}}(X)$. Morally, this should be the "q-Hodge-filtered q-de Rham cohomology of X".[(1.6)] Similarly, we can also put a filtration on $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{KU})$. By 2-periodicity, the information contained in the associated graded is already fully captured by the 0th graded piece $\mathrm{gr}_{\mathrm{ev}}^0\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes X)/\mathrm{KU}) \eqqcolon \mathrm{R\Gamma}_{\mathrm{KU}}(X)$. Morally, $\mathrm{R\Gamma}_{\mathrm{KU}}(X)$ should be the "q-Hodge cohomology of X".

These cohomology theories could be very interesting: Even though the input X is rational, their output will also contain non-trivial p-complete information. This article can be viewed as a computation of the coefficients of $\mathrm{Fil}^*\mathrm{R\Gamma}_{\mathrm{ku}}(-)$ and $\mathrm{R\Gamma}_{\mathrm{KU}}(-)$.

1.21. Relation to q-de Rham/q-Hodge cohomology — We expect that for a framed smooth $\mathbb{Z}$-algebra $(S,\square)$ as in 1.1, the value of $\mathrm{Fil}^*\mathrm{R\Gamma}_{\mathrm{ku}}(-)$ on the generic fibre $S_{\mathbb{Q}} \coloneqq S\otimes_{\mathbb{Z}}\mathbb{Q}$ should be

$$\mathrm{Fil}^*\mathrm{R\Gamma}_{\mathrm{ku}}(\operatorname{Spec} S_{\mathbb{Q}}) \simeq \mathrm{Fil}_{q\text{-Hdg}}^*\,q\text{-}\Omega_{S/\mathbb{Z},\square}^* \otimes_{\mathbb{Z}[\beta]⟦t⟧}^{\mathrm{L}\blacksquare} \mathrm{A}_{\mathrm{ku}}^*$$

where the coordinate-dependent q-Hodge filtration is defined as in 1.2. Similarly, we expect

$$\mathrm{R\Gamma}_{\mathrm{KU}}(\operatorname{Spec} S_{\mathbb{Q}}) \simeq q\text{-}\mathrm{Hdg}_{S/\mathbb{Z},\square} \otimes_{\mathbb{Z}⟦q-1⟧}^{\mathrm{L}\blacksquare} \mathrm{A}_{\mathrm{KU}}$$

The philosophy is that after base change to the "period rings" $\mathrm{A}_{\mathrm{ku}}^*$ and A_{KU}, the constructions from 1.2 become functorial, but also only depend on the generic fibre. We hope to study these questions in future work.

But let us already point out that $\mathrm{Fil}^*\mathrm{R\Gamma}_{\mathrm{ku}}(-)$ is still far from the finest possible information that one can squeeze out of refined THH.

1.22. Higher chromatic refinements. — First off, a finer cohomology theory should arise from working with $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{MU})$, or directly with the absolute $\mathrm{THH}^{\mathrm{ref}}(-)$. We would be very interested in seeing the calculation for MU or any higher chromatic base like $\mathrm{BP}\langle n\rangle$ or E_n. The heart of our approach is the S^1-equivariant $\mathbb{E}_1$-equivalence $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}⟦q-1⟧)_p^\wedge \simeq \tau_{\geqslant 0}(\mathrm{ku}^{\mathrm{t}C_p})$ from 1.11. This has conjectural higher chromatic analogues (see [Dev23, Conjecture 2.2.18]), but even assuming those its not clear to us how to proceed. It would also be nice to see Raksit's deformed de Rham complexes from 1.12 appear.

We should also point out that $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$ is an $\mathbb{E}_\infty$-algebra over the K-theory spectrum $\mathrm{K}(\mathbb{Q})$, which vanishes upon $\mathrm{K}(n)$-localisation for $n \geqslant 2$. Due to the delicate nature of the refinement, this doesn't mean that the answer over a higher chromatic base would be trivial,

(1.6) Note that the naive q-de Rham cohomology of $\mathbb{Q}$-varieties (obtained, for example, by applying Theorem A.1 for $A=\mathbb{Q}$) would just be a $(q-1)$-completed base change of ordinary de Rham cohomology. So $\mathrm{R\Gamma}_{\mathrm{ku}}(X)$ also gives a non-trivial answer to what the q-de Rham cohomology of X is supposed to be.

and $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{MU})$ should still contain strictly more information than $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{ku})$, but that information will necessarily be rather subtle.

1.23. Habiro refinements. — There's another direction in which the theory can be refined: $\mathrm{THH}(-)$ naturally takes values in the ∞-category CycSp of *cyclotomic spectra*, so we can lift its refinement $\mathrm{THH}^{\mathrm{ref}}(-)$ to a functor with values in the rigidification $(\mathrm{CycSp})^{\mathrm{rig}}$. This doesn't work over ku, since ku is not a cyclotomic base.[(1.7)]

However, already over ku, we can take into account that the S^1-action on $\mathrm{THH}(-/\mathrm{ku})$ can be enhanced to a genuine action with respect to the finite subgroups $C_m \subseteq S^1$. In particular, for all $m \in \mathbb{N}$, we can consider the symmetric monoidal functor

$$\left(\mathrm{THH}(-/\mathrm{ku})^{\Phi C_m}\right)^{\mathrm{h}(S^1/C_m)} \colon \mathrm{Mot}_{\mathrm{ku}}^{\mathrm{loc}} \longrightarrow \mathrm{Mod}_{(\mathrm{ku}^{\Phi C_m})^{\mathrm{h}(S^1/C_m)}}(\mathrm{Sp})^\wedge_{(q^m-1)/\beta}$$

given by first taking *geometric* C_m-fixed points and then the usual homotopy fixed points for the residual S^1/C_m-action. These functors can then be fed into the refinement machine from 1.14. Via the resulting refinements, we expect that $\mathrm{R\Gamma}_{\mathrm{KU}}(-)$ naturally descends from $\mathbb{Z}[\![q-1]\!]$ to the *Habiro ring* $\mathcal{H} \coloneqq \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^\wedge_{(q^m-1)}$. In particular, its ring of coefficients A_{KU} should admit a Habiro descent $\mathcal{A}_{\mathrm{KU}}$ satisfying $\mathrm{A}_{\mathrm{KU}} \cong \mathcal{A}_{\mathrm{KU}} \otimes^{\blacksquare}_{\mathcal{H}} \mathbb{Z}[\![q-1]\!]$.

In [W-Hab], we'll show that whenever the q-Hodge filtration is a q-deformation of the Hodge filtration, $q\text{-}\mathrm{Hdg}_{R/\mathbb{Z}}$ admits a functorial descent to the Habiro ring. This descent satisfies a derived analogue of Theorem 1.7 and a version of Theorem 1.10 for the functors above (more precisely, a version for *genuine* C_m-fixed points).

§1.5. Overview of this article

1.24. Leitfaden. — §2 is a collection of technical results that will be needed later. In §3 we study the q-Hodge filtration on q-de Rham cohomology and the derived q-Hodge complex: §3.1 contains the constructions. In §3.2 we show that the q-Hodge filtration is often a q-deformation of the Hodge filtration and explain the connection to ku. In §§3.3–3.4 we study some formal properties of the q-Hodge filtration/complex and explain how to globalise the constructions (that is, define them over $\mathbb{Z}$ rather than $\mathbb{Z}_p$).

In §4, we compute $\pi_*\,\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})$ and $\pi_*\,\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes\mathbb{Q})/\mathrm{KU})$. In §4.1 we explain how, in general, the computation of $\mathrm{THH}^{\mathrm{ref}}((k\otimes\mathbb{Q})/k)$ reduces to a computation of $\mathrm{THH}((k\otimes\mathbb{S}/p^\alpha)/k)$. In §4.2 we do the computation for $k = \mathrm{ku}$ and $k = \mathrm{KU}$. Finally, in §5, we make the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ explicit enough to derive the simpler descriptions of Theorems 1.19 and 1.20.

1.25. Notation and conventions. — Throughout the article, we'll use the following notation and conventions:

(*a*) **∞-categories.** We freely use the language of ∞-categories. We let Sp denote the ∞-category of spectra. For an ordinary ring R, we let $\mathcal{D}(R)$ denote the derived ∞-category of R. We often implicitly regard objects of $\mathcal{D}(R)$ as spectra via the Eilenberg–MacLane functor H, but we'll always suppress this functor in our notation. For a stable ∞-category $\mathcal{C}$, we let $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denote the mapping spectra in $\mathcal{C}$. The shift functor and its inverse will always be denoted by Σ and Σ^{-1} (even for $\mathcal{D}(R)$), to avoid confusion with shifts in graded or filtered objects.

[(1.7)] But it could be interesting over MU.

(*b*) **Symmetric monoidal ∞-categories.** If no confusion can occur, we denote the tensor unit by $\mathbb{1}$ and the tensor product by $\otimes$. If $\mathcal{C}$ is symmetric monoidal, we let $\operatorname{Alg}(\mathcal{C})$ and $\operatorname{CAlg}(\mathcal{C})$ denote the ∞-categories of $\mathbb{E}_1$-algebras and $\mathbb{E}_\infty$-algebras in $\mathcal{C}$, respectively.

Whenever we consider a symmetric monoidal ∞-category $\mathcal{C}$ which is stable or presentable, we always implicitly assume that the tensor product commutes with finite colimits or arbitrary colimits, respectively. In the presentable case, we let $\underline{\operatorname{Hom}}_{\mathcal{C}}(-,-)$ denote the *internal Hom* in $\mathcal{C}$ and $X^\vee \coloneqq \underline{\operatorname{Hom}}_{\mathcal{C}}(X,\mathbb{1})$ the *dual* of an object $X\in\mathcal{C}$.

(*c*) **Graded and filtered objects.** For a stable ∞-category $\mathcal{C}$, we let $\operatorname{Gr}(\mathcal{C})$ and $\operatorname{Fil}(\mathrm{Sp})$ denote the ∞-categories of *graded* and (*descendingly*) *filtered objects in* $\mathcal{C}$. An object with a descending filtration is typically denoted $\operatorname{Fil}^* X = (\cdots \leftarrow \operatorname{Fil}^n X \leftarrow \operatorname{Fil}^{n+1} X \leftarrow \cdots)$ and we let $\operatorname{gr}^* X$ denote the associated graded. We say that a filtered object $\operatorname{Fil}^* X$ is *complete* if $0 \simeq \lim_{n\to\infty} \operatorname{Fil}^n X$, and an *exhaustive filtration on* X if $X \simeq \operatorname{colim}_{n\to-\infty} \operatorname{Fil}^n X$. The shift in graded or filtered objects is denoted $(-)(1)$, to avoid confusion with the shift in homotopical/homological direction. We'll always try to distinguish between *graded/filtered degree* and *homotopical/homological degree*.

Sometimes we also consider *ascending* filtrations; these will typically be denoted $\operatorname{Fil}_* X = (\cdots \to \operatorname{Fil}^n X \to \operatorname{Fil}^{n+1} X \to \cdots)$ and the associated graded by $\operatorname{gr}_* X$.

(*d*) **Condensed mathematics.** Whenever we use condensed mathematics, we work in the light condensed setting. We'll distinguish between the words *static* ("un-animated") for a spectrum concentrated in degree 0, and *discrete* ("un-condensed") for a condensed spectrum with the discrete topology.

(*e*) **Derived quotients.** For an $\mathbb{E}_1$-ring spectrum R, a homotopy class $x\in\pi_n(R)$, and a left or right R-module M, we let M/x denote the cofibre of the multiplication map $x\colon \Sigma^n M\to M$. For several homotopy classes $x_1,\dots,x_r$, we let $M/(x_1,\dots,x_r) \coloneqq (\cdots(M/x_1)/x_2\cdots)/x_r$. Observe that if M is an ordinary module over an ordinary ring R, then $M/(x_1,\dots,x_r)$ agrees with the usual quotient only if $(x_1,\dots,x_r)$ is a Koszul-regular sequence on M, but we'll never use the notation in a case where this is not satisfied.

(*f*) **Completions.** For an $\mathbb{E}_\infty$-ring spectrum R, finitely many homogeneous homotopy classes $x_1,\dots,x_r\in\pi_*(R)$, and and an R-module spectrum M, we let

$$\widehat{M}_{(x_1,\dots,x_r)} \coloneqq \lim_{n\geqslant 1} M/(x_1^n,\dots,x_r^n)$$

denote the $(x_1,\dots,x_r)$-*adic completion* of M. Since it only depends on the ideal $I=(x_1,\dots,x_r)\subseteq\pi_*(R)$, we often just write $\widehat{M}_I$ (or $(-)^\wedge_I$ for longer arguments). If R is an ordinary ring, this recovers the notion of *derived I-completion*; in particular, all completions in this article will be derived. For the p-completions of $\mathbb{Z}$ and the sphere spectrum $\mathbb{S}$ we omit the hat and just write $\mathbb{Z}_p$ and $\mathbb{S}_p$.

We let $\operatorname{Mod}_R(\mathrm{Sp})^\wedge_I \subseteq \operatorname{Mod}_R(\mathrm{Sp})$, or $\widehat{\mathcal{D}}_I(R)\subseteq\mathcal{D}(R)$ for ordinary rings R, denote the full sub-∞-category spanned by the *I-complete objects*, that is, those M for which $M\simeq\widehat{M}_I$. The following fact will be used countless times: If M is $(x_1,\dots,x_r)$-complete, and the homotopy groups of $M/(x_1,\dots,x_r)$ vanish in some degree d, then also the homotopy groups of M must vanish in degree d.

(*g*) **Completed tensor products.** To ease notation, we introduce the convention that $-\widehat{\otimes}_R-$ denotes a p-, $(q-1)$-, or $(p,q-1)$-completed tensor product depending on whether R satisfies the same kind of completeness. Also $-\widehat{\otimes}_{\mathrm{ku}^{\mathrm{h}S^1}}-$ denotes a t-completed tensor product, and $-\widehat{\otimes}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}-$ denotes a t-completed graded tensor product.

If R is an ordinary ring and $I \subseteq R$ a finitely generated ideal, then $M \in \mathcal{D}(R)$ is called *I-completely flat* if $M \otimes^{\mathrm{L}}_R R/I$ is static and a flat R/I-module. Similarly we define *p-completely étale/smooth* ring maps.

(*h*) **Completed de Rham complexes.** Again, to ease notation, we introduce the convention that whenever R is p-complete, all de Rham, q-de Rham, or q-Hodge complexes of R will be implicitly completed at p. We also let $\widehat{\mathrm{dR}}_{R/k}$ denote the Hodge-completed derived de Rham complex of R over k. If R is a p-complete ring, then according to our convention, $\widehat{\mathrm{dR}}_{R/k}$ will additionally be completed at p.

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§2. Technical preliminaries

In this section, we prove a few technical statements that will be needed later in the text. We advise the reader to skip this section and only read the necessary parts as they come up later.

§2.1. Lifting properties of spherical Witt vectors

Recall that Lurie's spherical Witt vector construction [L-Ell$_{\mathrm{II}}$, Example 5.2.7] shows that any p-complete perfect δ-ring $A \cong \mathrm{W}(A^\flat)$ admits a unique lift to a p-complete connective $\mathbb{E}_\infty$-ring $\mathbb{S}_A$ satisfying $A \simeq \mathbb{S}_A \mathbin{\widehat{\otimes}}_{\mathbb{S}_p} \mathbb{Z}_p$.

2.1. Lemma. — *The Tate-valued Frobenius (see [TC18, Definition IV.1.1]) $\mathbb{S}_A \rightarrow \mathbb{S}_A^{\mathrm{t}C_p}$ can be equipped with an S^1-equivariant structure, where $\mathbb{S}_A$ receives the trivial S^1-action and $\mathbb{S}_A^{\mathrm{t}C_p}$ the residual $S^1 \simeq S^1/C_p$-action. In particular, the augmentation $\mathrm{THH}(\mathbb{S}_A) \rightarrow \mathbb{S}_A$ can be upgraded to a map of $\mathbb{E}_\infty$-algebras in cyclotomic spectra and $\mathrm{THH}(-/\mathbb{S}_A)$ carries a natural cyclotomic structure.*

Proof. Equivalently, using the Tate-fixed point lemma [TC18, Lemma II.4.2], we must construct a factorisation of the Tate-valued Frobenius through an $\mathbb{E}_\infty$-map

$$\mathbb{S}_A \longrightarrow \left(\mathbb{S}_A^{\mathrm{t}C_p}\right)^{\mathrm{h}(S^1/C_p)} \simeq \left(\mathbb{S}_A^{\mathrm{t}S^1}\right)_p^\wedge .$$

The δ-ring Frobenius on A lifts uniquely to an $\mathbb{E}_\infty$-map $\phi_A\colon \mathbb{S}_A \rightarrow \mathbb{S}_A$. From the trivial S^1-action we obtain an $\mathbb{E}_\infty$-map $\mathbb{S}_A \rightarrow \mathbb{S}_A^{\mathrm{h}S^1}$ splitting the usual limit projection. Now we claim that the composition

$$\mathbb{S}_A \xrightarrow{\phi_A} \mathbb{S}_A \longrightarrow \mathbb{S}_A^{\mathrm{h}S^1} \longrightarrow \left(\mathbb{S}_A^{\mathrm{t}S^1}\right)_p^\wedge$$

provides the desired factorisation. By the universal property of spherical Witt vectors [L-Ell$_{\mathrm{II}}$, Definition 5.2.1(*c*)], this can be checked on $\pi_0(-)/p$. But then both the Tate-valued Frobenius and our map are given by the usual Frobenius $(-)^p\colon A \rightarrow A/p$.

The upgrade of the augmentation $\mathrm{THH}(\mathbb{S}_A) \rightarrow \mathbb{S}_A$ to a map of $\mathbb{E}_\infty$-algebras in cyclotomic spectra follows immediately from the universal property of THH on $\mathbb{E}_\infty$-ring spectra. Then $\mathrm{THH}(-/\mathbb{S}_A) \simeq \mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{S}_A)} \mathbb{S}_A$ obtaines a cyclotomic structure, as desired. □

2.2. Lemma. — *Suppose R is a p-complete p-torsion free ring and $\mathbb{S}_R$ is a connective p-complete $\mathbb{E}_1$-ring spectrum satisfying $R \simeq \mathbb{S}_R \mathbin{\widehat{\otimes}}_{\mathbb{S}_p} \mathbb{Z}_p$. If $A \rightarrow R$ is map from a perfect δ-ring, then the $\mathbb{E}_1$-structure on $\mathbb{S}_R$ refines to an $\mathbb{E}_1$-structure in $\mathbb{S}_A$-modules.*

The proof needs two preliminaries.

2.3. Lemma. — *We have $\pi_*(\mathbb{S}_R) \cong \pi_*(\mathbb{S}) \otimes_{\mathbb{Z}} R$ as graded rings.*

Proof. As graded abelian groups this easily follows from $\mathbb{S}_R \mathbin{\widehat{\otimes}}_{\mathbb{S}_p} \mathbb{Z}_p$. Indeed, choose a two-term resolution $0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0$ by free abelian groups. Let $\mathbb{S}_P$ be a free $\mathbb{S}$-module on a basis of P and define $\mathbb{S}_Q$ similarly. Then we can lift the above short exact sequence to a sequence $\mathbb{S}_P \rightarrow \mathbb{S}_Q \rightarrow \mathbb{S}_R$ with nullhomotopic composition. Choosing a nullhomotopy gives a map $\operatorname{cofib}(\mathbb{S}_P \rightarrow \mathbb{S}_Q)_p^\wedge \rightarrow \mathbb{S}_R$, which is an equivalence after $- \mathbin{\widehat{\otimes}}_{\mathbb{S}_p} \mathbb{Z}_p$. Since that functor is conservative on p-complete connective spectra, we deduce $\operatorname{cofib}(\mathbb{S}_P \rightarrow \mathbb{S}_Q)_p^\wedge \simeq \mathbb{S}_R$, whence the desired description of $\pi_*(\mathbb{S}_R)$.

To pin down the multiplicative structure, observe that $\pi_*(\mathbb{S}_R)$ is a graded $\pi_*(\mathbb{S})$-algebra and a graded $\pi_0(\mathbb{S}_R) \cong R$-algebra, so its enough to check that any $x \in R \cong \pi_0(\mathbb{S}_R)$ commutes with $\pi_*(\mathbb{S})$. Fix such an x and consider the induced map $\mathbb{S}[x] \to \mathbb{S}_R$ from the free $\mathbb{E}_1$-algebra on a generator x. Then it suffices to check that x commutes with $\pi_*(\mathbb{S})$ in $\pi_*(\mathbb{S}[x])$, which is clear since $\mathbb{S}[x]$ refines to an $\mathbb{E}_\infty$-ring spectrum. □

2.4. Lemma. — *Let* $\mathrm{L}^{\mathbb{E}_1}_{-/\mathbb{S}}$ *and* $\mathrm{L}^{\mathbb{E}_1}_{-/\mathbb{S}_A}$ *denote the cotangent complex for* $\mathbb{E}_1$*-algebras in* Sp *and* $\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp})$*, respectively, as defined by Lurie. If* T *is any* $\mathbb{E}_1$*-algebra in* $\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp})$*, then the canonical base change map*

$$\mathrm{L}^{\mathbb{E}_1}_{T/\mathbb{S}} \otimes_{\mathbb{S}_A^{\mathrm{op}} \otimes \mathbb{S}_A} \mathbb{S}_A \longrightarrow \mathrm{L}^{\mathbb{E}_1}_{T/\mathbb{S}_A}\,,$$

induced by $- \otimes_{\mathbb{S}_A^{\mathrm{op}} \otimes \mathbb{S}_A} \mathbb{S}_A \colon {}_T\mathrm{BiMod}_T(\mathrm{Sp}) \to {}_T\mathrm{BiMod}_T(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}))$*, becomes an equivalence after* p*-completion.*

Proof. The idea is that A being a perfect δ-ring implies that $\mathbb{S}_A$ is p-completely formally étale over $\mathbb{S}$ in the sense that $\mathrm{THH}(\mathbb{S}_A)_p^\wedge \simeq \mathbb{S}_A$. Indeed, this follows from the same argument as [BMS19, Proposition 11.7].

To turn this idea into a proof, first recall from [L-HA, Theorems 7.3.4.13 and 7.3.5.1] that the cotangent complex $\mathrm{L}^{\mathbb{E}_1}_{T/\mathbb{S}}$ can be described as the fibre of the multiplication map for T,

$$\mathrm{L}^{\mathbb{E}_1}_{T/\mathbb{S}} \simeq \mathrm{fib}(T \otimes T \to T)\,,$$

regarded as an object in the bimodule ∞-category $\mathrm{Mod}^{\mathbb{E}_1}_T(\mathrm{Sp}) \simeq {}_T\mathrm{BiMod}_T(\mathrm{Sp})$. An analogous description exists for cotangent complexes of $\mathbb{E}_1$-algebras in $\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp})$. So we must show that

$$\mathrm{fib}\big(T \otimes T \to T\big) \otimes_{\mathbb{S}_A^{\mathrm{op}} \otimes \mathbb{S}_A} \mathbb{S}_A \longrightarrow \mathrm{fib}\big(T \otimes_{\mathbb{S}_A} T \to T\big)$$

is an equivalence upon p-completion. The base change functor $- \otimes_{\mathbb{S}_A^{\mathrm{op}} \otimes \mathbb{S}_A} \mathbb{S}_A$ transforms tensor products over $\mathbb{S}$ into tensor products over $\mathbb{S}_A$, so it will be enough to check that $T \otimes_{\mathbb{S}_A^{\mathrm{op}} \otimes \mathbb{S}_A} \mathbb{S}_A \to T$ is a p-complete equivalence. But we've seen that

$$\Big(\mathbb{S}_A \otimes_{\mathbb{S}_A^{\mathrm{op}} \otimes \mathbb{S}_A} \mathbb{S}_A\Big)_p^\wedge \simeq \mathrm{THH}(\mathbb{S}_A)_p^\wedge \xrightarrow{\ \simeq\ } \mathbb{S}_A$$

is an equivalence, and so tensoring with T finishes the proof. □

Proof of Lemma 2.2. By Lemma 2.3, $\mathbb{S}_R$ is quasi-commutative in the sense of [L-HA, Definition 7.5.1.1]. We'll now inductively lift the truncations $\tau_{\leqslant n}\mathbb{S}_R$ to an $\mathbb{E}_1$-algebra in $\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp})$. For $n = 0$ this clearly works since R is a A-algebra. Now suppose we've constructed a lift for some $n \geqslant 0$. We can write $\tau_{\leqslant n+1}\mathbb{S}_R$ as a square-zero extension of $\tau_{\leqslant n}\mathbb{S}_R$, which is classified by a map of $\tau_{\leqslant n}\mathbb{S}_R$-$\tau_{\leqslant n}\mathbb{S}_R$-bimodules

$$\mathrm{L}^{\mathbb{E}_1}_{\tau_{\leqslant n}\mathbb{S}_R/\mathbb{S}} \longrightarrow \Sigma^{n+2}\pi_{n+1}(\mathbb{S}_R)\,.$$

Since $\mathbb{S}_R$ is quasi-commutative, the bimodule structure on $\Sigma^{n+2}\pi_{n+1}(\mathbb{S}_R)$ factors through the multiplication map $\tau_{\leqslant n}\mathbb{S}_R^{\mathrm{op}} \otimes \tau_{\leqslant n}\mathbb{S}_R \to R^{\mathrm{op}} \otimes R \to R$ (which is $\mathbb{E}_1$). Moreover, $\pi_{n+1}(\mathbb{S}_R)$ is derived p-complete. Thus, the extension is equivalently described by an R-module map

$$\Big(\mathrm{L}^{\mathbb{E}_1}_{\tau_{\leqslant n}\mathbb{S}_R/\mathbb{S}} \otimes_{\tau_{\leqslant n}\mathbb{S}_R^{\mathrm{op}} \otimes \tau_{\leqslant n}\mathbb{S}_R} R\Big)_p^\wedge \longrightarrow \Sigma^{n+2}\pi_{n+1}(\mathbb{S}_R)\,.$$

Using Lemma 2.4, this map is equivalently given by a morphism $\mathrm{L}^{\mathbb{E}_1}_{\tau_{\leqslant n}\mathbb{S}_R/\mathbb{S}_A} \to \Sigma^{n+2}\pi_n(\mathbb{S}_R)$ of $\tau_{\leqslant n}\mathbb{S}_R$-$\tau_{\leqslant n}\mathbb{S}_R$-bimodules in $\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp})$. Thus we've lifted $\tau_{\leqslant n+1}\mathbb{S}_R \to \tau_{\leqslant n}\mathbb{S}_R$ to a square-zero extension of $\mathbb{E}_1$-algebras in $\mathbb{S}_A$-modules, as desired. □

§2.2. Nuclear objects

The notion of *nuclearity* plays an important role in our main results (Theorems 1.17, 1.19, and 1.20). In this subsection we briefly recall the necessary definitions and compare the nuclear categories of Clausen–Scholze and Efimov.

2.5. Setup — Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. By Lurie's adjoint functor theorem[(2.1)], $\mathcal{C}$ admits an internal Hom $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)$, characterised by the "Hom-tensor adjunction" $\mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z)) \simeq \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z)$ for all $X, Y, Z \in \mathcal{C}$. We denote by $X^\vee \coloneqq \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbb{1})$ the dual of X and by $\mathrm{ev}_X \colon X \otimes X^\vee \to \mathbb{1}$ the natural evaluation morphism.

2.6. Definition. — A morphism $\varphi\colon X \to Y$ in $\mathcal{C}$ is called *trace-class* if there exists a morphism $\eta\colon \mathbb{1} \to X^\vee \otimes Y$ in such a way that φ is the composition

$$X \simeq X \otimes \mathbb{1} \xrightarrow{X \otimes \eta} X \otimes X^\vee \otimes Y \xrightarrow{\mathrm{ev}_X \otimes Y} Y\,.$$

We'll often call η the *classifier of* φ.

Trace-class morphism have a number of nice properties. We'll often use the properties from [CS22, Lemma 8.2] as well as the following lemma.

2.7. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between presentable symmetric monoidal ∞-categories. By abuse of notation, we use $(-)^\vee$ to denote both the dual $\underline{\mathrm{Hom}}_{\mathcal{C}}(-, \mathbb{1}_{\mathcal{C}})$ in $\mathcal{C}$ and the dual $\underline{\mathrm{Hom}}_{\mathcal{D}}(-, \mathbb{1}_{\mathcal{D}})$ in $\mathcal{D}$. Then:*

(a) *There exists a natural transformation $F((-)^\vee) \Rightarrow F(-)^\vee$.*

(b) *If $X \to Y$ is a trace-class morphism in $\mathcal{C}$, then $F(X) \to F(Y)$ as well as $Y^\vee \to X^\vee$ are trace-class again and the morphisms $F(X^\vee) \to F(X)^\vee$ and $F(Y^\vee) \to F(Y)^\vee$ from (a) fit into a commutative diagram in $\mathcal{D}$ of the following form:*

$$\begin{array}{ccc} F(Y^\vee) & \longrightarrow & F(X^\vee) \\ \big\downarrow & \nearrow & \big\downarrow \\ F(Y)^\vee & \longrightarrow & F(X)^\vee \end{array}$$

Proof. The natural transformation from (a) is adjoint to $F((-)^\vee) \otimes_{\mathcal{D}} F(-) \Rightarrow \mathbb{1}_{\mathcal{D}}$, which is in turn given by applying F to the evaluation $(-)^\vee \otimes_{\mathcal{C}} (-) \Rightarrow \mathbb{1}_{\mathcal{C}}$.

Now let $X \to Y$ be trace-class in $\mathcal{C}$ with classifier $\mathbb{1}_{\mathcal{C}} \to X^\vee \otimes_{\mathcal{C}} Y$. If we apply F to the classifier and compose with the morphism $F(X^\vee) \to F(X)^\vee$ from (a), we obtain a morphism $\mathbb{1}_{\mathcal{D}} \to F(X^\vee) \otimes_{\mathcal{D}} F(Y) \to F(X)^\vee \otimes_{\mathcal{D}} F(Y)$, which shows that $F(X) \to F(Y)$ is trace-class. If we compose the classifier with $Y \to Y^{\vee\vee}$ instead, we obtain $\mathbb{1}_{\mathcal{C}} \to X^\vee \otimes_{\mathcal{C}} Y \to X^\vee \otimes_{\mathcal{C}} Y^{\vee\vee}$, which shows that $Y^\vee \to X^\vee$ is trace-class. The diagonal dashed arrow in the diagram is given as follows:

$$F(Y)^\vee \otimes_{\mathcal{D}} F(\mathbb{1}_{\mathcal{C}}) \longrightarrow F(Y)^\vee \otimes_{\mathcal{D}} F(X^\vee \otimes_{\mathcal{C}} Y) \simeq F(Y)^\vee \otimes_{\mathcal{D}} F(Y) \otimes_{\mathcal{D}} F(X^\vee) \longrightarrow F(X^\vee)\,.$$

Here we use the classifier $\mathbb{1}_{\mathcal{C}} \to X^\vee \otimes_{\mathcal{C}} Y$ and the evaluation map for $F(Y)$. □

[(2.1)] Recall from 1.25 that we always assume $-\otimes-$ commutes with tensor products in both variables, so the adjoint functor theorem is applicable.

2.8. Definition. — Assume additionally that $\mathcal{C}$ is stable, compactly generated, and the tensor unit $\mathbb{1} \in \mathcal{C}$ is compact.

(*a*) An object $X \in \mathcal{C}$ is called *nuclear* if every morphism $P \to X$ from a compact object P is trace-class.

(*b*) An object $X \in \mathcal{C}$ is called *basic nuclear* if $X \simeq \operatorname{colim}(X_1 \to X_2 \to \cdots)$, where each $X_n \to X_{n+1}$ is trace-class.

We denote by $\operatorname{Nuc}(\mathcal{C}) \subseteq \mathcal{C}$ the full sub-∞-category spanned by the nuclear objects.

2.9. Theorem. — *Let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category such that $\mathcal{C}$ is compactly generated and the tensor unit $\mathbb{1} \in \mathcal{C}$ is compact.*

(*a*) *$\operatorname{Nuc}(\mathcal{C})$ is stable and closed under colimits and tensor products in $\mathcal{C}$.*

(*b*) *$\operatorname{Nuc}(\mathcal{C})$ is ω_1-compactly generated and the ω_1-compact objects are precisely the basic nuclears.*

(*c*) *If $F\colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal colimit-preserving functor into another presentable symmetric monoidal ∞-category, then F restricts to a functor $F\colon \operatorname{Nuc}(\mathcal{C}) \to \operatorname{Nuc}(\mathcal{D})$.*

Proof. Parts (*a*) and (*b*) are [CS22, Theorem 8.6]. For (*c*), just note that since F preserves trace-class maps by Lemma 2.7(*b*), it follows that F preserves basic nuclear objects and thus all nuclear objects by (*b*). □

2.10. Remark. — If $\mathcal{C}$ is a small stable symmetric monoidal ∞-category, then Theorem 2.9 can be applied to $\operatorname{Ind}(\mathcal{C})$. Since every trace-class map in $\operatorname{Ind}(\mathcal{C})$ factors through a compact object by [CS22, Lemma 8.4], we see that the basic nuclear objects in $\operatorname{Ind}(\mathcal{C})$ are of the form "colim"$(X_1 \to X_2 \to \cdots)$, where each $X_n \to X_{n+1}$ is trace-class in $\mathcal{C}$.

If $\mathcal{C}$ is a presentable stable symmetric monoidal ∞-category, one can still make sense of $\operatorname{Nuc}(\operatorname{Ind}(\mathcal{C}))$ without running into set-theoretic problems. Indeed, if κ is a sufficiently large regular cardinal such that $\mathcal{C}$ is κ-compactly generated and $\mathbb{1}$ is κ-compact, the same argument as in [CS22, Lemma 8.4] shows that every trace-class morphism in $\mathcal{C}$ factors through a κ-compact object. Then every basic nuclear object is equivalent to one in which each X_n is κ-compact and so the basic nuclear objects in form an essentially small ∞-category. We may then define $\operatorname{Nuc}(\operatorname{Ind}(\mathcal{C}))$ as $\operatorname{Ind}_{\omega_1}(-)$ of the ∞-category of basic nuclear objects.

Finally, let us compare Efimov's and Clausen–Scholze's notions of nuclear modules.

2.11. Nuclear modules à la Efimov and à la Clausen–Scholze. — If R is an $\mathbb{E}_\infty$-ring spectrum and $I \subseteq \pi_*(R)$ a finitely generated homogeneous ideal, Efimov defines the ∞-category of *nuclear $\widehat{R}_I$-modules* to be

$$\operatorname{Nuc}(\widehat{R}_I) \coloneqq \operatorname{Nuc}\bigl(\operatorname{Ind}\bigl(\operatorname{Mod}_R(\mathrm{Sp})_I^\wedge\bigr)\bigr)$$

(which is set-theoretically ok thanks to Remark 2.10). We're mainly interested in the case $R \simeq k^{\mathrm{h}S^1} \simeq (k^{\mathrm{h}S^1})_t^\wedge$, where k is complex orientable and $t \in \pi_{-2}(k^{\mathrm{h}S^1})$ is any orientation generator, and in the case where R is an ordinary ring.

In the latter case, there's another candidate for a well-behaved category of nuclear $\widehat{R}_I$-modules: To the Huber pair $(\widehat{R}_I, \widehat{R}_I)$, Clausen and Scholze associate a *derived ∞-category of solid $\widehat{R}_I$-modules* $\mathcal{D}(\widehat{R}_{I,\blacksquare})$ (see 5.1), which satisfies the assumptions of Theorem 2.9. In [Efi-Lim], Efimov constructs a fully faithful strongly continuous functor

$$\operatorname{Nuc}\bigl(\mathcal{D}(\widehat{R}_{I,\blacksquare})\bigr) \longrightarrow \operatorname{Nuc}(\widehat{R}_I)\,.$$

§2.3. Killing pro-idempotent algebras

There is a well-established notion of *killing an idempotent algebra object* in a symmetric monoidal ∞-category (see for example [CS24, Lecture 13] for a nice review). Here we would like to do the same, but in an ind-setting.

2.12. Setup. — Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category. In the following, we'll ignore the set-theoretic difficulties that come with applying $\operatorname{Pro}(-)$ and $\operatorname{Ind}(-)$ to the large ∞-category $\mathcal{C}$.[(2.2)]

A *pro-idempotent algebra object in* $\mathcal{C}$ is a pro-object $A \coloneqq \text{“}\lim_{i\in I}\text{”} A_i \in \operatorname{Pro}(\mathcal{C})$, where I is always assumed to be cofiltered, equipped with a morphism $\mathbb{1} \to A$ such that the induced morphism $A \to A \otimes A$ is a pro-equivalence. Here $A \otimes A \simeq \text{“}\lim_{(i,j)\in I\times I}\text{”}(A_i \otimes A_j)$ denotes the extension of the tensor product on $\mathcal{C}$ to $\operatorname{Pro}(\mathcal{C})$. Since $\mathcal{C}$ is presentable, it admits an internal Hom, which we denote $\underline{\operatorname{Hom}}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$. By passing to ind-$\infty$-categories, this extends to a functor

$$\operatorname{Pro}(\mathcal{C})^{\mathrm{op}} \otimes \operatorname{Ind}(\mathcal{C}) \simeq \operatorname{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes \operatorname{Ind}(\mathcal{C}) \xrightarrow{\operatorname{Ind}(\underline{\operatorname{Hom}}_{\mathcal{C}})} \operatorname{Ind}(\mathcal{C})\,,$$

which, by abuse of notation, we still denote $\underline{\operatorname{Hom}}_{\mathcal{C}}$. Explicitly,

$$\underline{\operatorname{Hom}}_{\mathcal{C}}\Bigl(\text{“}\lim_{j\in J}\text{”}\, Y_j, \text{“}\operatorname*{colim}_{k\in K}\text{”}\, Z_k\Bigr) \simeq \text{“}\operatorname*{colim}_{(j,k)\in J^{\mathrm{op}}\times K}\text{”}\, \underline{\operatorname{Hom}}_{\mathcal{C}}(Y_j, Z_k)\,.$$

2.13. Lemma. — *Let $j_*\colon \operatorname{Ind}(\mathcal{C})^A \to \operatorname{Ind}(\mathcal{C})$ be the inclusion of the full sub-∞-category of ind-objects X for which $\underline{\operatorname{Hom}}_{\mathcal{C}}(A, X) \simeq 0$. Then j_* admits a left adjoint j^* given by*

$$j^*(X) \simeq \operatorname{cofib}\bigl(\underline{\operatorname{Hom}}_{\mathcal{C}}(A, X) \to \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathbb{1}, X)\bigr)\,.$$

Furthermore, there's a unique symmetric monoidal structure on $\operatorname{Ind}(\mathcal{C})^A$ in such a way that j^* becomes symmetric monoidal and j_* lax symmetric monoidal.

Proof. Since $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathbb{1}, X) \simeq X$, we get a natural transformation $\eta\colon \mathrm{id}_{\operatorname{Ind}(\mathcal{C})} \Rightarrow j^*$. It's clear that η is an equivalence for objects in $\operatorname{Ind}(\mathcal{C})^A$, hence the image of j^* contains $\operatorname{Ind}(\mathcal{C})^A$. Using pro-idempotence, we also see $\underline{\operatorname{Hom}}_{\mathcal{C}}(A, j^*(X)) \simeq \operatorname{cofib}(\underline{\operatorname{Hom}}_{\mathcal{C}}(A\otimes A, X) \to \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)) \simeq 0$, hence the image of j^* is precisely $\operatorname{Ind}(\mathcal{C})^A$. Now in general, if $\mathcal{D}$ is an ∞-category with an endofunctor $L\colon \mathcal{D} \to \mathcal{D}$ and a natural transformation $\eta\colon \mathrm{id}_{\mathcal{D}} \Rightarrow L$ such that $\eta L\colon L \Rightarrow L \circ L$ and $L\eta\colon L \Rightarrow L\circ L$ are both equivalences, then $L\colon \mathcal{D} \to L(\mathcal{D})$ is a left adjoint of $L(\mathcal{D}) \subseteq \mathcal{D}$. In the case at hand, we've already checked that ηj^* is an equivalence. Via the tensor-$\underline{\operatorname{Hom}}_{\mathcal{C}}$-adjunction, this implies the same for $j^*\eta$, finishing the proof that j^* is left adjoint to j_*.

To show that j^* is symmetric monoidal and j_* lax symmetric monoidal, it's enough to show that η induces equivalences $j^*(X \otimes Y) \simeq j^*(j^*(X)\otimes Y)$ for all $X, Y \in \operatorname{Ind}(\mathcal{C})$; see [L-HA, Proposition 2.2.1.9]. Equivalently, the canonical morphism

$$\underline{\operatorname{Hom}}_{\mathcal{C}}\bigl(A, \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)\otimes Y\bigr) \xrightarrow{\simeq} \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)\otimes Y$$

induced by $\mathbb{1} \to A$ must be an equivalence. To see this, first observe that this morphism has a left inverse given by

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)\otimes Y \simeq \underline{\operatorname{Hom}}_{\mathcal{C}}\bigl(A, \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)\bigr)\otimes Y \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{C}}\bigl(A, \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)\otimes Y\bigr)$$

(2.2) In all cases of interest, we can safely replace $\mathcal{C}$ by its κ-compact objects $\mathcal{C}^\kappa \subseteq \mathcal{C}$ for some large enough regular cardinal κ (usually $\kappa = \omega_1$ is enough).

using idempotence of A and $Y \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathbb{1}, Y)$. Now, in general, let $M \in \mathrm{Ind}(\mathcal{C})$ be an ind-object for which $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, M) \to M$ has a left inverse.[(2.3)] Using this left inverse, we can exhibit $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, M) \to M$ as a retract of $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, \underline{\mathrm{Hom}}_{\mathcal{C}}(A, M)) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, M)$. But the latter is an equivalence by pro-idempotence of A, so already $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, M) \to M$ must be an equivalence. This finishes the proof. □

2.14. Construction. — By the symmetric monoidality statements from Lemma 2.13, we see that $j^*(\mathbb{1})$ is an $\mathbb{E}_\infty$-algebra in $\mathrm{Ind}(\mathcal{C})$. By construction, it sits inside a cofibre sequence

$$\text{“}\operatorname*{colim}_{i \in I^{\mathrm{op}}}\text{”}\, A_i^\vee \longrightarrow \mathbb{1} \longrightarrow j^*(\mathbb{1})\,,$$

where $(-)^\vee \coloneqq \underline{\mathrm{Hom}}_{\mathcal{C}}(-, \mathbb{1})$ denotes the dual in $\mathcal{C}$. We say that $j^*(\mathbb{1})$ *is obtained from* $\mathbb{1}$ *by killing the pro-idempotent algebra* “$\lim_{i \in I}$” A_i.

In general, $j^*(\mathbb{1})$ is not an idempotent $\mathbb{E}_\infty$-algebra in $\mathrm{Ind}(\mathcal{C})$; it is idempotent if and only if $A^\vee \coloneqq$ “$\mathrm{colim}_{i \in I^{\mathrm{op}}}$” $A_i^\vee$ is an *ind-idempotent coalgebra* in the sense that $A^\vee \to \mathbb{1}$ induces an equivalence $A^\vee \otimes A^\vee \simeq A^\vee$ in $\mathrm{Ind}(\mathcal{C})$. In the following lemma we'll study a special situation in which this is the case.

2.15. Lemma. — *Suppose for all $i \in I$ there exists an object $j \to i$ such that $A_j \to A_i$ is trace-class. Let $A^\vee \coloneqq$ “$\mathrm{colim}_{i \in I}$” $A_i^\vee$. Then the canonical map $X \otimes A^\vee \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$ is an equivalence for all $X \in \mathrm{Ind}(\mathcal{C})$. In particular:*

- (a) *$A^\vee$ is a ind-idempotent coalgebra object with trace-class transition maps.*
- (b) *$j^*(\mathbb{1})$ is an idempotent nuclear $\mathbb{E}_\infty$-algebra in $\mathrm{Ind}(\mathcal{C})$, $\mathrm{Ind}(\mathcal{C})^A \subseteq \mathrm{Ind}(\mathcal{C})$ is precisely the full sub-∞-category of $j^*(\mathbb{1})$-modules, and $- \otimes j^*(\mathbb{1}) \simeq j^*(-)$.*
- (c) *If $F\colon \mathcal{C} \to \mathcal{D}$ is any symmetric monoidal functor of presentable symmetric monoidal ∞-categories, then $F(j^*(\mathbb{1}))$ is obtained by killing the pro-idempotent algebra $F(A)$.*

Proof sketch. We can construct an inverse of $X \otimes A^\vee \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$ as follows: Fix some $i \in I$, choose $j \to i$ such that $A_j \to A_i$ is trace-class and let $\mathbb{1} \to A_i \otimes A_j^\vee$ be the corresponding classifier. Then consider the composition

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(A_i, X) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(A_i, X) \otimes A_i \otimes A_j^\vee \longrightarrow X \otimes A_j^\vee\,.$$

In the first map, we tensor $\underline{\mathrm{Hom}}_{\mathcal{C}}(A_i, X)$ with the classifier above. In the second map we use the evaluation $\underline{\mathrm{Hom}}_{\mathcal{C}}(A_i, X) \otimes A_i \to X$. It's straightforward but a little tedious to check that $X \otimes A_i^\vee \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A_i, X) \to X \otimes A_j^\vee$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(A_i, X) \to X \otimes A_j^\vee \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A_j, X)$ agree with the transition maps in the ind-objects $X \otimes A^\vee$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$, respectively; we'll omit the argument. Proving that these maps assemble into an inverse map $X \otimes A^\vee \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$ requires a non-trivial argument, since we're working in an ∞-category, but there's an easier way to show that $X \otimes A^\vee \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$ is an equivalence: Equivalences are detected by $\pi_0 \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(Z, -)$, where Z ranges through all compact objects of $\mathrm{Ind}(\mathcal{C})$; now any morphism from a compact object factors through $X \otimes A_i^\vee$ or $\underline{\mathrm{Hom}}_{\mathcal{C}}(A_i, X)$ for some $i \in I$.

To show (a), plug in $X \simeq A^\vee$: We obtain $A^\vee \otimes A^\vee \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(A, A^\vee) \simeq (A \otimes A)^\vee$. This proves ind-idempotence, because $(A \otimes A)^\vee \simeq A^\vee$ follows by dualising $A \simeq A \otimes A$. The dual transition maps $A_n^\vee \to A_{n+1}^\vee$ are trace-class by Lemma 2.7(b) below. This shows (a).

[(2.3)] So, intuitively, M is an “A-module”.

For (b), since we've shown that $A^\vee$ is an ind-idempotent coalgebra, it follows that $j^*(\mathbb{1})$ is ind-idempotent as well. Also $A^\vee$ is a nuclear object in $\operatorname{Ind}(\mathcal{C})$, since every map $Z \to A^\vee$ factors through a basic nuclear object and is therefore trace-class. Since $\mathbb{1}$ is nuclear too, it follows that $j^*(\mathbb{1})$ is nuclear. $X \otimes j^*(\mathbb{1}) \simeq j^*(X)$ follows immediately from the above equivalence $X \otimes A^\vee \simeq \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)$. Since $j_*\colon \operatorname{Ind}(\mathcal{C})^A \to \operatorname{Ind}(\mathcal{C})$ is lax monoidal by Lemma 2.13, it factors through $\operatorname{Ind}(\mathcal{C})^A \to \operatorname{Mod}_{j^*(\mathbb{1})}(\operatorname{Ind}(\mathcal{C}))$. Since $j^*(\mathbb{1})$ is idempotent, $\operatorname{Mod}_{j^*(\mathbb{1})}(\operatorname{Ind}(\mathcal{C})) \subseteq \operatorname{Ind}(\mathcal{C})$ is the full sub-∞-category spanned by the objects of the form $X \otimes j^*(\mathbb{1})$. Hence we also get an inclusion $\operatorname{Ind}(\mathcal{C})^A \subseteq \operatorname{Mod}_{j^*(\mathbb{1})}(\operatorname{Ind}(\mathcal{C}))$. On the other hand, every object of the form $X \otimes j^*(\mathbb{1}) \simeq j^*(X)$ is contained in $\operatorname{Ind}(\mathcal{C})^A$. This finishes the proof of (b).

To show (c), we only need "$\operatorname{colim}_{i\in I}$" $F(A_i^\vee) \simeq$ "$\operatorname{colim}_{i\in I}$" $F(A_i)^\vee$. If $A_j \to A_i$ is trace-class, Lemma 2.7(b) below provides a map $F(A_i)^\vee \to F(A_j^\vee)$ in the reverse direction. By a formal argument as above, this is enough to show the desired equivalence. □

§2.4. Burklund's $\mathbb{E}_1$-structures on quotients and square-zero extensions

At several points throughout the text, we need to consider $\mathbb{E}_1$-algebras of the form $\mathbb{S}/p^{\alpha_1} \otimes \mathbb{S}/p^{\alpha_2}$. In this subsection we'll prove an abstract result which shows that these guys are often trivial square zero algebras and then deduce some nice consequences.

For the abstract setup, let $\mathcal{C}$ be a stable $\mathbb{E}_2$-monoidal ∞-category and $v\colon \mathcal{I} \to \mathbb{1}$ be a morphism in $\mathcal{C}$ such that $\mathbb{1}/v^n$ admits a right-unital multiplication. Fix $n \geqslant 3$, so that $\mathbb{1}/v^n$ admits a preferred $\mathbb{E}_2$-algebra structure by [Bur22, Theorem 1.5]. The same theorem shows for all $m \geqslant 2$ that $\mathbb{1}/v^n \otimes \mathbb{1}/v^m$ admits a preferred $\mathbb{E}_1$-algebra structure in the $\mathbb{E}_1$-monoidal ∞-category $\operatorname{LMod}_{\mathbb{1}/v^n}(\mathcal{C})$.

2.16. Lemma. — *Suppose $\mathcal{I}$ is $\otimes$-invertible and $m \geqslant 2n$.*

(a) *$\mathbb{1}/v^n \otimes \mathbb{1}/v^m$ agrees with the trivial square-zero extension $\mathbb{1}/v^n \oplus \Sigma(\mathcal{I}^{\otimes m}/v^n)$ as an $\mathbb{E}_1$-algebra in $\operatorname{LMod}_{\mathbb{1}/v^n}(\mathcal{C})$. Under this identification, the multiplication $\mathbb{1}/v^n \otimes \mathbb{1}/v^m \to \mathbb{1}/v^n$ becomes the augmentation map $\mathbb{1}/v^n \oplus \Sigma(\mathcal{I}^{\otimes m}/v^n) \to \mathbb{1}/v^n$.*

(b) *For all $\ell \geqslant m \geqslant 2n$, the map $\mathbb{1}/v^n \otimes \mathbb{1}/v^\ell \to \mathbb{1}/v^n \otimes \mathbb{1}/v^m$ agrees with the map of trivial square-zero extensions induced by $v^{\ell-m}\colon \mathcal{I}^{\otimes \ell}/v^n \to \mathcal{I}^{\otimes m}/v^n$, as maps of $\mathbb{E}_1$-algebras in $\operatorname{LMod}_{\mathbb{1}/v^n}(\mathcal{C})$.*

2.17. Remark. — The bound $m \geqslant 2n$ doesn't seem optimal and the author suspects that Lemma 2.16 might already be true for $m \geqslant n$.

2.18. Remark. — Since the $\mathbb{E}_1$-algebra structures on $\mathbb{1}/v^m$ and $\mathbb{1}/v^n$ refine to $\mathbb{E}_2$-algebra structures, the multplication map in Lemma 2.16(a) is canonically a map of $\mathbb{E}_1$-algebras, and the identification with the augmentation also happens as $\mathbb{E}_1$-algebra maps (as we'll see).

2.19. Corollary. — *If $\mathcal{I}$ is $\otimes$-invertible, $m \geqslant 2n$, and $\ell \geqslant m + n$, then the morphism $\mathbb{1}/v^n \otimes \mathbb{1}/v^\ell \to \mathbb{1}/v^n \otimes \mathbb{1}/v^m$ factors through the tensor unit $\mathbb{1}/v^n$ as a map of $\mathbb{E}_1$-algebras in $\operatorname{LMod}_{\mathbb{1}/v^n}(\mathcal{C})$.*

Proof. By Lemma 2.16(b), it's enough to check that $v^{\ell-m}\colon \mathcal{I}^{\otimes \ell}/v^n \to \mathcal{I}^{\otimes m}/v^n$ is zero in $\operatorname{LMod}_{\mathbb{1}/v^n}(\mathcal{C})$ for $\ell \geqslant m + n$. This reduces to $v^n\colon \mathcal{I}^{\otimes n}/v^n \to \mathbb{1}/v^n$ being zero in $\operatorname{LMod}_{\mathbb{1}/v^n}(\mathcal{C})$. Since $\mathbb{1}/v^n \otimes -\colon \mathcal{C} \to \operatorname{LMod}_{\mathbb{1}/v^n}(\mathcal{C})$ is left adjoint to the forgetful functor, this is equivalent to $v^n\colon \mathcal{I}^{\otimes n} \to \mathbb{1}/v^n$ being zero in $\mathcal{C}$, which is true by construction. □

Proof of Lemma 2.16. Recall [Bur22, Constructions 4.7 and 4.8]: Let $\widetilde{\mathcal{C}} \coloneqq \operatorname{Def}(\mathcal{C}, \mathcal{Q})$ be Burklund's deformation of $\mathcal{C}$, which stably $\mathbb{E}_2$-monoidal and comes with $\mathbb{E}_2$-monoidal functors $\nu\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ and $(-)^{\tau=1}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$. Let furthermore $\widetilde{\mathbb{1}} \coloneqq \nu(\mathbb{1})$ denote the tensor unit of $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{I}} \coloneqq \Sigma^{-1}\nu(\Sigma\mathcal{I})$, so that there's a map $\widetilde{v}\colon \widetilde{\mathcal{I}} \to \widetilde{\mathbb{1}}$ deforming v. Note that $\widetilde{\mathcal{I}}$ is still $\otimes$-invertible, so it'll be enough to check the assertions for $\widetilde{v}\colon \widetilde{\mathcal{I}} \to \widetilde{\mathbb{1}}$.

Burklund constructs his $\mathbb{E}_1$-structure on $\widetilde{\mathbb{1}}/\widetilde{v}^m$ using the obstruction theory from [Bur22, Proposition 2.4] in $\widetilde{\mathcal{C}}$ (with its underlying $\mathbb{E}_1$-monoidal structure). Hence the $\mathbb{E}_1$-structure on $\widetilde{\mathbb{1}}/\widetilde{v}^n \otimes \widetilde{\mathbb{1}}/\widetilde{v}^m$ is obtained via Burklund's obstruction theory in the $\mathbb{E}_1$-monoidal ∞-category $\operatorname{LMod}_{\widetilde{\mathbb{1}}/\widetilde{v}^n}(\widetilde{\mathcal{C}})$. Now we claim that for all $k \geqslant 2$ and all $\ell \geqslant m \geqslant 2n$,

$$(\boxtimes)\qquad \pi_0 \operatorname{Hom}_{\operatorname{LMod}_{\widetilde{\mathbb{1}}/\widetilde{v}^n}(\widetilde{\mathcal{C}})}\left(\Sigma^{-2}\big(\Sigma^2(\widetilde{\mathcal{I}}/\widetilde{v}^n)^{\otimes \ell}\big)^{\otimes k}, \widetilde{\mathbb{1}}/\widetilde{v}^n \otimes \widetilde{\mathbb{1}}/\widetilde{v}^m\right) = 0\,.$$

Believing ($\boxtimes$) for the moment, (*b*) as well as the first part of (*a*) immediately follow. Indeed, in the case $\ell = m$, ($\boxtimes$) combined with [Bur22, Remark 2.5] shows that the $\mathbb{E}_1$-structure on $\widetilde{\mathbb{1}}/\widetilde{v}^n \otimes \widetilde{\mathbb{1}}/\widetilde{v}^m$ is unique, so it has to be the trivial square zero structure. For general $\ell \geqslant m$, the same argument shows that the $\mathbb{E}_1$-map $\widetilde{\mathbb{1}}/\widetilde{v}^n \otimes \widetilde{\mathbb{1}}/\widetilde{v}^\ell \to \widetilde{\mathbb{1}}/\widetilde{v}^n \otimes \widetilde{\mathbb{1}}/\widetilde{v}^m$ is unique, proving (*b*). It remains to show the second part of (*a*). This follows again from the same kind of arguments, as we'll see below.

To show ($\boxtimes$), we use that $\widetilde{\mathbb{1}}/\widetilde{v}^n \otimes -\colon \widetilde{\mathcal{C}} \to \operatorname{LMod}_{\widetilde{\mathbb{1}}/\widetilde{v}^n}(\widetilde{\mathcal{C}})$ is left adjoint to the forgetful functor, that $\widetilde{\mathbb{1}}/\widetilde{v}^n \otimes \widetilde{\mathbb{1}}/\widetilde{v}^m \simeq \widetilde{\mathbb{1}}/\widetilde{v}^n \oplus \Sigma(\widetilde{\mathcal{I}}^{\otimes m}/\widetilde{v}^n)$, and that $\widetilde{\mathcal{I}}$ is $\otimes$-invertible. The left-hand side can then be rewritten as follows:

$$\begin{aligned} &\pi_0 \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{2k-2}\widetilde{\mathcal{I}}^{\otimes \ell k}, \widetilde{\mathbb{1}}/\widetilde{v}^n \oplus \Sigma(\widetilde{\mathcal{I}}^{\otimes m}/\widetilde{v}^n)\right)\\ &\qquad\cong \pi_0 \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{2k-2}\widetilde{\mathcal{I}}^{\otimes \ell k}, \widetilde{\mathbb{1}}/\widetilde{v}^n\right) \oplus \pi_0 \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{2k-1}\widetilde{\mathcal{I}}^{\otimes \ell k - m}, \widetilde{\mathbb{1}}/\widetilde{v}^n\right)\\ &\qquad\cong \pi_0 \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{-\ell k+2k-2}\nu(X), \widetilde{\mathbb{1}}/\widetilde{v}^n\right) \oplus \pi_0 \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{-\ell k+m+2k-1}\nu(Y), \widetilde{\mathbb{1}}/\widetilde{v}^n\right), \end{aligned}$$

where $X \simeq (\Sigma\mathcal{I})^{\otimes \ell k}$ and $Y \simeq (\Sigma\mathcal{I})^{\otimes(\ell k - m)}$. According to [Bur22, Lemma 4.8], both summands on the right-hand side vanish as soon as $\ell k - m - 2k + 1 \geqslant n$. This is true under our assumptions $\ell \geqslant m \geqslant 2n$, $k \geqslant 2$, and $n \geqslant 3$ and so we've proved ($\boxtimes$). Vanishing of the first summand also shows that $\widetilde{\mathbb{1}}/\widetilde{v}^n \otimes \widetilde{\mathbb{1}}/\widetilde{v}^m \to \widetilde{\mathbb{1}}/\widetilde{v}^n$ is unique, and so it has to be the augmentation map. □

Let's finish this subsection with two nice applications. In the following, we'll always equip $\mathbb{S}/p^\alpha$ with a Burklund-style $\mathbb{E}_1$-structure.

2.20. Corollary. — *Let $p \geqslant 3$ be a prime and $\alpha \geqslant 3$. Then $\mathbb{S}/p^{3\alpha} \otimes \mathbb{S}/p^\alpha \to \mathbb{S}/p^{2\alpha} \otimes \mathbb{S}/p^\alpha$ factors through $\mathbb{S}/p^\alpha$ as $\mathbb{E}_1$-algebras. The same conclusion holds for $p = 2$ if α is even and $\geqslant 6$.*

Proof. This follows immediately from Corollary 2.19. □

2.21. Corollary. — *Let $p \geqslant 3$ be prime and $\alpha \geqslant 3$. Then the base change functor*

$$- \otimes_{\mathbb{S}/p^{3\alpha}} \mathbb{S}/p^\alpha \colon \operatorname{RMod}_{\mathbb{S}/p^{3\alpha}}(\mathrm{Sp}) \longrightarrow \operatorname{RMod}_{\mathbb{S}/p^\alpha}(\mathrm{Sp})$$

is trace-class in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$. The same conclusion holds for $p = 2$ if α is even and $\geqslant 6$.

Proof. As we've seen in the proof of Lemma 4.5, it's enough to show that $\mathbb{S}/p^\alpha$ is compact in $\mathrm{RMod}_{\mathbb{S}/p^\alpha}(\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/p^{3\alpha}}(\mathrm{Sp}^\omega)))$. We'll show that the multiplication $\mathbb{S}/p^{2\alpha} \otimes \mathbb{S}/p^\alpha \to \mathbb{S}/p^\alpha$ exhibits $\mathbb{S}/p^\alpha$ as a retract of the compact object $\mathbb{S}/p^{2\alpha} \otimes \mathbb{S}/p^\alpha$. This map has a section, given by the unit map $\mathbb{S}/p^\alpha \to \mathbb{S}/p^{2\alpha} \otimes \mathbb{S}/p^\alpha$, so we must equip the latter with the structure of a map of $\mathbb{S}/p^{3\alpha}$-$\mathbb{S}/p^\alpha$-bimodules. But we know from Corollary 2.20 that the $\mathbb{E}_1$-algebra map $\mathbb{S}/p^{3\alpha} \otimes \mathbb{S}/p^\alpha \to \mathbb{S}/p^{2\alpha} \otimes \mathbb{S}/p^\alpha$ factors through $\mathbb{S}/p^\alpha \to \mathbb{S}/p^\alpha \otimes \mathbb{S}/p^\alpha$, which provides the desired structure of a bimodule map.

This shows that $\mathbb{S}/p^\alpha$ is a retract of $\mathbb{S}/p^{2\alpha} \otimes \mathbb{S}/p^\alpha$ in $\mathrm{RMod}_{\mathbb{S}/p^\alpha}(\mathrm{LMod}_{\mathbb{S}/p^{3\alpha}}(\mathrm{Sp}))$. Since all objects in sight are compact on underlying spectra, we may pass to the full sub-∞-category $\mathrm{RMod}_{\mathbb{S}/p^\alpha}(\mathrm{LMod}_{\mathbb{S}/p^{3\alpha}}(\mathrm{Sp}^\omega))$ and then also to $\mathrm{RMod}_{\mathbb{S}/p^\alpha}(\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/p^{3\alpha}}(\mathrm{Sp}^\omega)))$. $\square$

§3. Derived q-Hodge complexes

We've explained in §1.1 why the q-Hodge filtration on a q-de Rham complex in coordinates can probably not be made functorial; at the very least, not in a way as to preserve the rich q-de Rham–Witt structure. In this section, we'll give an ad-hoc construction of a functorial q-Hodge filtration on certain derived q-de Rham complexes (Construction 3.5). At first, this construction will seem hopelessly naive—and indeed, it often gives nonsensical results. However, as we'll see in Theorem 3.10, in many more cases the filtration actually behaves as desired! In such cases, we can define a *derived q-Hodge complex* and show that it satisfies all expected properties.

This section is organised as follows: In §3.1, we'll construct the desired q-Hodge filtration in a p-complete setting. In §3.2 we show that it is often a q-deformation of the usual Hodge filtration. In §3.3, we derived some formal properties. Finally, in §3.4, we explain how to construct a q-Hodge filtration and derived q-Hodge complexes in the global setting.

§3.1. The q-Hodge filtration on q-de Rham cohomology

Throughout §§3.1–3.3 we fix a prime p and work in a p-complete setting. In particular, all (q-)de Rham or cotangent complexes will be implicitly p-completed. Let us also fix a p-complete δ-ring A which is *p-completely perfectly covered* in the sense defined below.

3.1. Rings of interest. — A δ-ring A is called *p-completely perfectly covered* if the map $A \to A_\infty$ into its p-completed colimit perfection is p-completely faithfully flat. By [Wag24, Remark 2.46], an equivalent condition is for the Frobenius $\phi\colon A \to A$ to be p-completely flat. Since perfect δ-rings are p-torsion free, it follows that A must be p-torsion free too.

Throughout, we will consider *quasiregular quotients over A*: These are p-complete rings R for which the cotangent complex $\mathrm{L}_{R/A}$ (which we always take to be implicitly p-completed) has p-complete Tor-amplitude over R concentrated in degree 1. Additionally, we'll usually assume that for $R_\infty \coloneqq R \mathbin{\widehat{\otimes}}_A A_\infty$ the quotient $\overline{R}_\infty \coloneqq R_\infty/p$ is *semiperfect*, meaning that the Frobenius $(-)^p\colon \overline{R}_\infty \to \overline{R}_\infty$ is surjective. An important special case are A-algebras of *perfect-regular presentation*: These are the quotients $R \cong B/J$, where B is a p-complete relatively perfect δ-A-algebra (by which we mean that the relative Frobenius $\phi_{B/A}\colon B \mathbin{\widehat{\otimes}}_{A,\phi} A \to B$ is an isomorphism) and $J \subseteq B$ is an ideal generated by a Koszul-regular sequence. We'll sometimes refer to B/J as a *perfect-regular presentation of R.*

The reason for restricting to rings R as above is the following lemma.

3.2. Lemma. — *Let R be a p-torsion free quasiregular quotient over A.*

(*a*) *The de Rham complex $\mathrm{dR}_{R/A}$, its Hodge-completion $\widehat{\mathrm{dR}}_{R/A}$, every degree in the completed Hodge filtration $\mathrm{Fil}^*_{\mathrm{Hdg}}\widehat{\mathrm{dR}}_{R/A}$, and the q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$ are all static and p-torsion free.*

(*b*) *The un-completed Hodge filtration $\mathrm{Fil}^*_{\mathrm{Hdg}}\mathrm{dR}_{R/A}$ is static in every degree if and only if $\overline{R}_\infty$ is semiperfect.*

Proof. To show that every degree in the completed Hodge filtration is static and p-torsion free, just observe that the same is true for the associated graded $\mathrm{gr}^*_{\mathrm{Hdg}}\widehat{\mathrm{dR}}_{R/A} \simeq \Sigma^{-*}\bigwedge^* \mathrm{L}_{R/A}$, because our assumption on R guarantees that $\Sigma^{-1}\mathrm{L}_{R/A}$ is a p-completely flat module over the p-torsion free ring R. To show that the $(q-1)$-complete object $q\text{-}\mathrm{dR}_{R/A}$ is static and p-torsion free, it will be enough to show the same for $q\text{-}\mathrm{dR}_{R/A}/(q-1) \simeq \mathrm{dR}_{R/A}$. Now all assertions about $\mathrm{dR}_{R/A}$ and its Hodge filtration can be reduced to the corresponding assertions about $\mathrm{dR}_{R_\infty/A_\infty}$ via base change along the p-completely faithfully flat map $A \to A_\infty$. Furthermore, since A_∞ is a perfect δ-ring, $\mathrm{L}_{A_\infty/\mathbb{Z}_p} \simeq 0$, and so we may as well consider $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$.

To see that $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ is static and p-torsion free, it suffices to check that its modulo p reduction $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}/p \simeq \mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is static. The latter admits an ascending exhaustive filtration, the *conjugate filtration*, whose associated graded $\Sigma^{-*}\bigwedge^* \mathrm{L}_{\overline{R}_\infty/\mathbb{F}_p} \simeq \Sigma^{-*}\bigwedge^* \mathrm{L}_{R_\infty/\mathbb{Z}_p}/p$ is static in every degree since $\Sigma^{-1}\mathrm{L}_{R_\infty/\mathbb{Z}_p}$ is p-completely flat over the p-torsion free ring R_∞. This shows that $\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is indeed static and we've finished the proof of (a).

For (b), we've already seen that $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ and the associated graded of the Hodge filtration are static and p-torsion free in every degree. Hence $\mathrm{Fil}^*_{\mathrm{Hdg}}\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ is degree-wise static if and only if it consists of sub-modules of $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$, which must be p-torsion free to. Thus $\mathrm{Fil}^*_{\mathrm{Hdg}}\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ is degree-wise static if and only if the same is true for $\mathrm{Fil}^*_{\mathrm{Hdg}}\mathrm{dR}_{R_\infty/\mathbb{Z}_p}/p \simeq \mathrm{Fil}^*_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$. In the case where $\overline{R}_\infty$ is semiperfect, this holds by [BMS19, Proposition 8.14]. Conversely, assume $\mathrm{Fil}^*_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is degree-wise static. If $\mathrm{Fil}^*_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}$ denotes the Nygaard filtration on the derived de Rham–Witt complex, then

$$\mathrm{Fil}^n_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}/p\,\mathrm{Fil}^{n-1}_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p} \simeq \mathrm{Fil}^n_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$$

holds by deriving [BMS19, Lemma 8.3], so inductively it follows that $\mathrm{WdR}_{R_\infty/\mathbb{Z}_p}$ and each step in its Nygaard filtration must be static too. By definition, $\mathrm{Fil}^n_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}$ is the fibre of

$$\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p} \xrightarrow{\phi} \mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p} \longrightarrow \mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}/p^n\,,$$

so each of these maps must be surjective. Then $\phi\colon \mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p} \to \mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}$ must be surjective as well. Since $\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}/p \simeq \mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p} \to \overline{R}_\infty$ is surjective by our assumption that $\mathrm{Fil}^1_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is static, we conclude that the Frobenius on R_∞/p must be surjective too. ☐

3.3. Remark. — In the case where $R \cong B/J$ is of perfect-regular presentation over A, everything can be made explicit: $\mathrm{dR}_{R/A} \simeq D_B(J)$ is the (p-completed) PD-envelope of J, the Hodge filtration is just the PD-filtration, and the q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$ is the corresponding q-PD-envelope in the sense of [BS19, Lemma 16.10].

3.4. Remark. — For $p \geqslant 3$, the $\mathbb{F}_p$-algebra constructed in [Gul21] can be lifted in a straightforward way to a p-complete $\mathbb{Z}_p$-algebra R. This gives an example of a p-torsion free quasiregular quotient over $\mathbb{Z}_p$, whose reduction modulo p is not semiperfect.

3.5. Construction. — Suppose R is a p-torsion free quasiregular quotient over A such that R_∞/p is semiperfect. By Lemma A.3, after rationalisation, $\mathrm{dR}_{R/A}$ and $q\text{-}\mathrm{dR}_{R/A}$ are related via a functorial equivalence

$$q\text{-}\mathrm{dR}_{R/A}\,\widehat{\otimes}_{\mathbb{Z}[\![q-1]\!]}\,\mathbb{Q}[\![q-1]\!] \simeq \big(\mathrm{dR}_{R/A}\otimes_{\mathbb{Z}}\mathbb{Q}\big)[\![q-1]\!]\,.$$

Observe that $(\mathrm{dR}_{R/A}\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$ carries two multiplicative filtrations: the Hodge filtration on $\mathrm{dR}_{R/A}$ and the $(q-1)$-adic filtration. These can be combined into one multiplicative filtration, which we'll call the *combined Hodge- and $(q-1)$-adic filtration*. Formally, it is is given by taking the tensor product of the filtered objects $\mathrm{Fil}^*_{\mathrm{Hdg}}\mathrm{dR}_{R/A}$ and $(q-1)^*\mathbb{Q}[\![q-1]\!]$ and then passing to degreewise $(q-1)$-completions.

Using the combined Hodge- and $(q-1)$-adic filtration, we can now give the main construction of this section:

(*a*) The *q-Hodge filtration* $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{R/A}$ is the preimage (in the underived sense) of the combined Hodge and $(q-1)$-adic filtration under $q\text{-dR}_{R/A} \to (\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$.

From the q-Hodge filtration, the following related objects can be constructed:

(*b*) The *q-Hodge-completed derived q-de Rham complex* $q\text{-}\widehat{\mathrm{dR}}_{R/A}$ is the completion of $q\text{-dR}_{R/A}$ at the q-Hodge filtration.

(*c*) The *derived q-Hodge complex of R over A* is the ring

$$q\text{-Hdg}_{R/A} \coloneqq \operatorname{colim}\Bigl(\operatorname{Fil}^0_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/A} \xrightarrow{(q-1)} \operatorname{Fil}^1_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/A} \xrightarrow{(q-1)} \cdots\Bigr)^\wedge_{(p,q-1)}.$$

3.6. Remark. — For any filtered object $X = \operatorname{Fil}^0 X \supseteq \operatorname{Fil}^1 X \supseteq \cdots$ with completion $\widehat{X}$, any filtration step $\operatorname{Fil}^n X$ is the preimage of $\operatorname{Fil}^n \widehat{X}$ under $X \to \widehat{X}$. Thus, in Construction 3.5(*a*), we could have taken the combined Hodge- and $(q-1)$-adic filtration on $(\widehat{\mathrm{dR}}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$ as well, or even its completion with respect to that filtration.

In the case where $R \cong B/J$ is of perfect-regular presentation, after completion at the Hodge filtration, $D_B(J) \otimes_{\mathbb{Z}} \mathbb{Q}$ becomes simply $(B \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_J$. Thus, the q-Hodge filtration on $q\text{-dR}_{R/A}$ is equivalently the preimage of the $(J, q-1)$-adic filtration on the ring $(B \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_J[\![q-1]\!]$.

3.7. Remark. — To define the derived q-Hodge complex we could have used $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{R/A}$ as well. Indeed, by design, elements in $\operatorname{Fil}^n q\text{-dR}_{R/A}$ become divisible by $(q-1)^n$ in $q\text{-Hdg}_{R/A}$, so after passing to the $(q-1)$-adic completion the filtration converges.

Our ultimate goal is to compare the q-Hodge filtration to the usual Hodge filtration. The starting point is the following easy lemma.

3.8. Lemma. — *If R is a p-torsion free quasi-regular quotient over A and $\overline{R}_\infty$ is semiperfect, there exists a canonical injection the canonical projection $q\text{-dR}_{R/A} \to \mathrm{dR}_{R/A}$ induces a degreewise injective morphism of filtered objects*

$$\bigl(\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{R/A}\bigr)/(q-1) \longrightarrow \operatorname{Fil}^*_{\mathrm{Hdg}} \mathrm{dR}_{R/A}\,,$$

where the quotient is taken in filtered objects, with $(q-1)$ sitting in filtration degree 1. In other words, we get injections $\operatorname{Fil}^n_{q\text{-Hdg}} q\text{-dR}_{R/A}/(q-1)\operatorname{Fil}^{n-1}_{q\text{-Hdg}} q\text{-dR}_{R/A} \to \operatorname{Fil}^n_{\mathrm{Hdg}} \mathrm{dR}_{R/A}$ for all n.

Proof. The assumptions guarantee that the Hodge filtration on $\mathrm{dR}_{R/A}$ is the preimage of the induced filtration on $\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q}$. Indeed, by Remark 3.6 it's enough to check this after completion, and then we simply observe that the map on associated gradeds is injective: It is the rationalisation map

$$\Sigma^{-n} \bigwedge^n \mathrm{L}_{R/A} \to \Sigma^{-n} \bigwedge^n \mathrm{L}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q}\,,$$

and each $\Sigma^{-n} \bigwedge^n \mathrm{L}_{R/A}$ is a p-completely flat module over the p-torsion free ring R. It follows that the composition $\operatorname{Fil}^n_{q\text{-Hdg}} q\text{-dR}_{R/A} \to q\text{-dR}_{R/A} \to \mathrm{dR}_{R/A}$ factors through $\operatorname{Fil}^n_{\mathrm{Hdg}} \mathrm{dR}_{R/A}$ for all n, which yields the desired morphism of filtered objects. To show injectivity, we need to check

$$(q-1)\operatorname{Fil}^{n-1}_{q\text{-Hdg}} q\text{-dR}_{R/A} = \operatorname{Fil}^n_{q\text{-Hdg}} q\text{-dR}_{R/A} \cap (q-1)\, q\text{-dR}_{R/A}\,.$$

This immediately reduces to the analogous assertion for the combined Hodge- and $(q-1)$-adic filtration on $(\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$, which is straightforward to check. □

3.9. Definition. — We'll say *the q-Hodge filtration* $\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-dR}_{R/A}$ *is a q-deformation of the Hodge filtration* $\mathrm{Fil}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}$ if the morphism of filtered objects in Lemma 3.8 is an isomorphism

$$\big(\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-dR}_{R/A}\big)/(q-1) \xrightarrow{\simeq} \mathrm{Fil}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}\,.$$

In general, the q-Hodge filtration is *not* a q-deformation of the Hodge filtration. We'll see an explicit counterexample in Example 3.14 below, but there's also a meta-mathematical objection: If the q-Hodge filtration were well-behaved in general, we could use quasi-syntomic descent to define it for all A-algebras. This would provide a way of making the filtration from 1.2 coordinate-independent, but this is very likely impossible, as we've discussed in §1.1.

With this in mind, the following theorem shows that the q-Hodge filtration *is* a q-deformation of the Hodge filtration in surprisingly many cases.

3.10. Theorem. — *Let R be a p-torsion free quasiregular quotient over A such that $\overline{R}_\infty$ is semiperfect. Suppose that one of the following two assumptions is satisfied:*

(a) *There exists a perfect-regular presentation $R \cong B/J$, where the ideal $J \subseteq B$ is generated by a Koszul-regular sequence of higher powers, that is, a Koszul-regular sequence $(x_1^{\alpha_1}, \dots, x_r^{\alpha_r})$ with $\alpha_i \geqslant 2$ for all i.*

(b) *The ring R_∞ admits a lift to a p-complete connective $\mathbb{E}_1$-ring spectrum $\mathbb{S}_{R_\infty}$ satisfying $R_\infty \simeq \mathbb{S}_{R_\infty} \widehat{\otimes}_{\mathbb{S}_p} \mathbb{Z}_p$.*

Then $\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-dR}_{R/A}$ *is a q-deformation of* $\mathrm{Fil}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}$.

Theorem 3.10 will be proved in §3.2. We'll now make three comments about the existence of $\mathbb{E}_1$-lifts as in Theorem 3.10(b) and then give two examples for the theorem.

3.11. On $\mathbb{E}_1$-lifts I. — Since A_∞ is a perfect δ-ring, $A_\infty \cong \mathrm{W}(A_\infty^\flat)$. Thus, [L-Ell$_{\mathrm{II}}$, Example 5.2.7] shows that A_∞ admits a unique lift to a p-complete connective $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_{A_\infty}$ satisfying $A_\infty \simeq \mathbb{S}_{A_\infty} \widehat{\otimes}_{\mathbb{S}_p} \mathbb{Z}_p$. Explicitly, $\mathbb{S}_{A_\infty}$ is given by Lurie's *spherical Witt vectors* and satisfies $\pi_*(\mathbb{S}_{A_\infty}) \cong \pi_*(\mathbb{S}) \otimes_{\mathbb{Z}} A_\infty$ as graded-commutative rings. If B is relatively perfect over A, then $B_\infty \coloneqq B \widehat{\otimes}_A A_\infty$ is a perfect δ-ring as well, hence it also admits a p-complete $\mathbb{E}_\infty$-lift $\mathbb{S}_{B_\infty}$.

It can be shown (see Lemma 2.2) that any $\mathbb{E}_1$-lift $\mathbb{S}_{R_\infty}$ as in Theorem 3.10(b) automatically refines to an $\mathbb{E}_1$-algebra in $\mathbb{S}_{A_\infty}$-modules, or, if $R \cong B/J$ is of perfect-regular presentation, even to an $\mathbb{E}_1$-algebra in $\mathbb{S}_{B_\infty}$-modules.

3.12. On $\mathbb{E}_1$-lifts II. — For $p \geqslant 3$, Burklund's breakthrough [Bur22] shows that the condition from Theorem 3.10(a) implies the existence of such an $\mathbb{E}_1$-lift $\mathbb{S}_{R_\infty}$, and so Theorem 3.10(b) implies (a). The same conclusion holds in the case $p = 2$ if all α_i are even and $\geqslant 4$.

Indeed, if $p \geqslant 3$, each $\mathbb{S}_{B_\infty}/x_i$ admits a right-unital multiplication, because the relevant obstruction $Q_1(x_i)$ is 2-torsion and hence vanishes after p-completion. Then [Bur22, Theorem 1.4] shows that $\mathbb{S}_{B_\infty}/x_i^{\alpha_i}$ admits an $\mathbb{E}_1$-algebra structure in $\mathbb{S}_{B_\infty}$-modules for all $\alpha_i \geqslant 2$. Hence

$$\mathbb{S}_{R_\infty} \coloneqq \mathbb{S}_{B_\infty}/x_1^{\alpha_1} \,\widehat{\otimes}_{\mathbb{S}_{B_\infty}} \cdots \widehat{\otimes}_{\mathbb{S}_{B_\infty}} \mathbb{S}_{B_\infty}/x_r^{\alpha_r}$$

is a lift as desired. If $p = 2$, then $\mathbb{S}_{B_\infty}/x_i^2$ admits a right-unital multiplication by [Bur22, Remark 5.5] and we can perform the same construction, provided all α_i are even and $\geqslant 4$.

Somewhat surprisingly though, Theorem 3.10(a) is true without such a modification at $p = 2$. This cannot simply be explained by the existence of $\mathbb{E}_1$-lifts that are not covered by

Burklund's theorem: If $B \coloneqq \mathbb{Z}_2\{x\}_2^\wedge$ is the free 2-complete δ-ring on a generator x, then $\mathbb{S}_{B_\infty}/x^2$ can't be equipped with an $\mathbb{E}_1$-structure, because there exists an $\mathbb{E}_\infty$-map $\mathbb{S}_{B_\infty} \to \mathbb{S}_2$ sending $x \mapsto 2$, so any $\mathbb{E}_1$-structure would base change to an $\mathbb{E}_1$-structure on $\mathbb{S}/4$.

3.13. On $\mathbb{E}_1$-lifts III. — In light of Theorem 3.22 below, it is tempting to replace the existence of an $\mathbb{E}_1$-lift $\mathbb{S}_{R_\infty}$ to the sphere spectrum in Theorem 3.10(b) with the existence of a lift ku_{R_∞} to the connective complex K-theory spectrum. This can't work! By [HW18], such an $\mathbb{E}_1$-lift *always* exists, but in Example 3.14, we'll see an instance where the q-Hodge filtration is not a q-deformation of the Hodge filtration.

It seems possible that an $\mathbb{E}_2$-lift of R_∞ to ku is sufficient to guarantee that the q-Hodge filtration is a q-deformation of the Hodge filtration, as in this case it is expected (but so far conjectural) that we have an S^1-equivariant version of Pstrągowski's perfect even filtration [Pst23] available on $\mathrm{TC}^-(\mathrm{ku}_{R_\infty}/\mathrm{ku})_p^\wedge$

3.14. Example. — Let us give an example of Theorem 3.10(a) to get a feeling for why passing to higher powers results in the q-Hodge filtration being better behaved. We'll see a much refined version of this analysis later in Lemma 5.23.

Let $A \coloneqq \mathbb{Z}_p\{x\}_p^\wedge$ be the free p-complete δ-ring on a generator x and let $R \coloneqq \mathbb{Z}_p\{x\}_p^\wedge/x^\alpha$ for some $\alpha \geqslant 1$. Then $q\text{-}\mathrm{dR}_{R/A}$ is the q-PD envelope

$$q\text{-}D_\alpha \coloneqq \mathbb{Z}_p\{x\}[\![q-1]\!]\left\{\frac{\phi(x^\alpha)}{[p]_q}\right\}^\wedge_{(p,q-1)}\,.$$

If the q-Hodge filtration were to be a q-deformation of the Hodge filtration, then $\mathrm{Fil}^p_{q\text{-Hdg}}\, q\text{-}D_\alpha$ would need to contain a lift $\widetilde{\gamma}_q(x^\alpha)$ of the divided power $\gamma(x^\alpha) \coloneqq x^{\alpha p}/p$. Certainly, $q\text{-}D_\alpha$ itself contains such a lift; namely, the q-divided power

$$\gamma_q(x^\alpha) \coloneqq \frac{\phi(x^\alpha)}{[p]_q} - \delta(x^\alpha)\,.$$

The problem is that $\gamma_q(x^\alpha)$ is usually not contained in $\mathrm{Fil}^p_{q\text{-Hdg}}\, q\text{-}D_\alpha$. By Remark 3.6, we can express $\mathrm{Fil}^p_{q\text{-Hdg}}\, q\text{-}D_\alpha$ as the preimage of the ideal $(x^\alpha, q-1)^p$ in the completed rationalisation $\mathbb{Q}_p\langle\delta(x), \delta^2(x), \dots\rangle[\![x, q-1]\!]$ of $q\text{-}D_\alpha$. So our task is to modify $\gamma_q(x^\alpha)$ by elements from the ideal $(q-1)\, q\text{-}D_\alpha$ such that the result is contained in $(x^\alpha, q-1)^p$ after completed rationalisation.

Write $[p]_q = pu + (q-1)^{p-1}$, where $u \equiv 1 \mod q-1$. In particular, u is a unit in $q\text{-}D_\alpha$. After completed rationalisation, we can rewrite $\gamma_q(x^\alpha)$ as

$$\frac{x^{\alpha p}}{[p]_q} + \left(\frac{p}{[p]_q} - 1\right)\delta(x^\alpha) = \frac{x^{\alpha p}}{[p]_q} + \left((u^{-1}-1) - u^{-2}\frac{(q-1)^{p-1}}{p} + \mathrm{O}\big((q-1)^p\big)\right)\delta(x^\alpha)\,.$$

Here $\mathrm{O}((q-1)^p)$ denotes "error terms" which are divisible by $(q-1)^p$. Observe that these error terms are contained in $(x^\alpha, q-1)^p$, so we can safely ignore them. Also $x^{\alpha p}/[p]_q$ is clearly contained in $(x^\alpha, q-1)^p$. The term $(u^{-1}-1)\delta(x^\alpha)$ is contained in $(q-1)\, q\text{-}D_\alpha$, so we can just kill it. This leaves the term $u^{-2}(q-1)^{p-1}\delta(x^\alpha)/p$.

If $\alpha = 1$, there's nothing we can do: No modification by elements from $(q-1)\, q\text{-}D_\alpha$ will ever get rid of a non-integral multiple of the polynomial variable $\delta(x)$. This shows that for $\alpha = 1$, the q-Hodge filtration on $q\text{-}D_\alpha$ is *not* a q-deformation of the Hodge filtration. For $\alpha = 2$, however, we have $\delta(x^2) = 2x^p\delta(x) + p\delta(x)^2$. Now the term $2x^p\delta(x)u^{-2}(q-1)^{p-1}/p$ is contained in $(x^2, q-1)^p$ and so

$$\widetilde{\gamma}_q(x^2) \coloneqq \gamma_q(x^2) - (u^{-1}-1)\delta(x^2) + u^{-2}(q-1)^{p-1}\delta(x)^2$$

is contained in $\mathrm{Fil}_{q\text{-Hdg}}^p\, q\text{-}D_\alpha$ and satisfies $\widetilde{\gamma}_q(x^2) \equiv x^{2p}/p \mod q-1$, as desired. For $\alpha \geqslant 3$, we can similarly decompose $\delta(x^\alpha)$ into a multiple of $x^{p(\alpha-1)}$ and a multiple of p.

3.15. Example. — An example for Theorem 3.10(b) that is not covered by Theorem 3.10(a) is the case $A \cong B \cong \mathbb{Z}_p[x]_p^\wedge$, with δ-structure defined by $\delta(x) = 0$, and $R \cong \mathbb{Z}_p$, with $B \to R$ sending $x \mapsto 1$. Then B lifts to an $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_B$ given by p-completing the flat polynomial ring $\mathbb{S}[x]$, and $B \to R$ lifts to an $\mathbb{E}_\infty$-map $\mathbb{S}_B \to \mathbb{S}_p$. Base changing along $\mathbb{S}_B \to \mathbb{S}_{B_\infty}$ yields a lift of R_∞, even as an $\mathbb{E}_\infty$-ring. In this case, $q\text{-}\mathrm{dR}_{R/A}$ is the q-PD envelope

$$q\text{-}D \coloneqq \mathbb{Z}_p[x]\llbracket q-1\rrbracket\left\{\frac{x^p-1}{[p]_q}\right\}^\wedge_{(p,q-1)}.$$

It follows for example from [Pri19, Lemma 1.3] that this ring contains elements of the form $(x-1)(x-q)\cdots(x-q^{n-1})/[n]_q!$ for all $n \geqslant 1$. After completed rationalisation, these elements are visibly contained in the ideal $(x-1, q-1)^n$. Hence they belong to $\mathrm{Fil}_{q\text{-Hdg}}^n\, q\text{-}\mathrm{dR}_{R/A}$ and lift the usual divided powers.

§3.2. The q-Hodge vs. the Hodge filtration

It's possible, but not at all trivial, to continue the analysis of Example 3.14 for higher divided powers and to give an elementary proof of Theorem 3.10(a). The argument will be explained in §5.3. In this subsection, we'll give a completely different proof of Theorem 3.10(b), which implies Theorem 3.10(a) in the cases covered by Burklund's result (see 3.12). So far, the elementary proof is the only argument that also covers the remaining cases for $p = 2$. It also gives us finer control over the q-Hodge filtration, which will be crucial in our proof of Theorem 1.19 in §5.4. On the other hand, through the more abstract proof that we're going to present now, the relation between q-Hodge filtrations and $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{ku})$ will become apparent.

We keep the notation from §3.1. The first step in the proof of Theorem 3.10(b) is a formal reduction to the case where A is a perfect δ-ring.

3.16. Lemma. — *Let $A \to A'$ be a p-completely flat morphism of δ-rings, where A' is also p-completely perfectly covered. Let R be a p-torsion free quasiregular quotient over A such that $\overline{R}_\infty$ is semiperfect and let $R' \coloneqq R \mathbin{\widehat{\otimes}}_A A'$. Then the canonical map*

$$\mathrm{Fil}_{q\text{-Hdg}}^\ast\, q\text{-}\mathrm{dR}_{R/A} \mathbin{\widehat{\otimes}}_{A\llbracket q-1\rrbracket} A'\llbracket q-1\rrbracket \xrightarrow{\ \simeq\ } \mathrm{Fil}_{q\text{-Hdg}}^\ast\, q\text{-}\mathrm{dR}_{R'/A'}$$

is an equivalence.

Proof. This is not completely automatic since we have to be careful with completions. Fix n. Since $A \to A'$ is p-completely flat and $\mathrm{Fil}_{q\text{-Hdg}}^n\, q\text{-}\mathrm{dR}_{R/A}$ is p-torsion free, being a submodule of $q\text{-}\mathrm{dR}_{R/A}$, we see that $-\mathbin{\widehat{\otimes}}_{A\llbracket q-1\rrbracket} A'\llbracket q-1\rrbracket$ can be replaced by $-\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{A\llbracket q-1\rrbracket} A'\llbracket q-1\rrbracket$. By Remark A.6, the canonical map $q\text{-}\mathrm{dR}_{R/A} \to (\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})\llbracket q-1\rrbracket/(q-1)^n$ already factors through $p^{-N}\mathrm{dR}_{R/A}\llbracket q-1\rrbracket/(q-1)^n$ for sufficiently large N. Since $\mathrm{Fil}_{q\text{-Hdg}}^n\, q\text{-}\mathrm{dR}_{R/A}$ contains $(q-1)^n\, q\text{-}\mathrm{dR}_{R/A}$, we can also express it as a pullback of $A\llbracket q-1\rrbracket$-modules

$$\begin{array}{ccc} \mathrm{Fil}_{q\text{-Hdg}}^n\, q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & q\text{-}\mathrm{dR}_{R/A} \\ \big\downarrow & \lrcorner & \big\downarrow \\ p^{-N}\bigoplus_{i=0}^{n-1} \mathrm{Fil}_{\mathrm{Hdg}}^i\, \mathrm{dR}_{R/A}(q-1)^{n-i} & \longrightarrow & p^{-N}\mathrm{dR}_{R/A}\llbracket q-1\rrbracket/(q-1)^n \end{array}$$

It will be enough to show that the pullback is preserved under the $(p, q-1)$-completed derived tensor product $-\widehat{\otimes}^{\mathrm{L}}_{A[\![q-1]\!]} A'[\![q-1]\!]$. To this end, let P denote the derived pullback (that is, the pullback taken in the derived ∞-category $\mathcal{D}(A[\![q-1]\!])$) and recall that derived tensor products preserve derived pullbacks. It is then enough to check that $\mathrm{H}_{-1}(P)\,\widehat{\otimes}^{\mathrm{L}}_{A[\![q-1]\!]} A'[\![q-1]\!]$ is static. We claim that $\mathrm{H}_{-1}(P)$ is p^m-torsion for sufficiently large m. Believing this for the moment, p-complete flatness of $A \to A'$ guarantees that $\mathrm{H}_{-1}(P) \otimes^{\mathrm{L}}_A A'$ is static. Since it is also p^m- and $(q-1)^n$-torsion, the completion doesn't change anything and we're done.

To prove the claim, observe that the cokernel of $q\text{-}\mathrm{dR}_{R/A} \to p^{-N}\mathrm{dR}_{R/A}$ must clearly be p^N-torsion. Hence the cokernel of the right vertical map

$$q\text{-}\mathrm{dR}_{R/A} \longrightarrow p^{-N}\mathrm{dR}_{R/A}[\![q-1]\!]/(q-1)^n$$

is p^{nN}-torsion. Since $\pi_{-1}(P)$ is a quotient of that cokernel (explicitly the quotient by the bottom left corner of the pullback diagram), we conclude that $\mathrm{H}_{-1}(P)$ is p^{nN}-torsion too, as desired. $\square$

3.17. Replacing A by A_∞. — Whether $\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}/(q-1) \to \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A}$ is an equivalence can be checked after p-completed basechange along the p-completely faithfully flat map $A \to A_\infty$ (and similarly for the completed filtrations). By Lemma 3.16 this amounts to replacing A and R by A_∞ and R_∞. Thus, from now on we assume A to be a perfect δ-ring and R is a p-torsion free quasiregular quotient over A such that $\overline{R} \coloneqq R/p$ is semiperfect. Since A is perfect, we can (and will, whenever it is convenient) replace $q\text{-}\mathrm{dR}_{R/A}$ and $\mathrm{dR}_{R/A}$ by their absolute variants $q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ and $\mathrm{dR}_{R/\mathbb{Z}_p}$.

The crucial input in the proof of Theorem 3.10(*b*) is the following result that we learned from Arpon Raksit, who in turn learned it from Thomas Nikolaus.

3.18. Theorem. — *If we equip* THH *with its natural S^1 action and $\mathrm{ku}^{\mathrm{t}C_p}$ with its residual $S^1 \simeq S^1/C_p$-action, then there is an S^1-equivariant equivalence of $\mathbb{E}_1$-ring spectra*[(3.1)]

$$\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!]\big)^\wedge_p \xrightarrow{\ \simeq\ } \tau_{\geqslant 0}\big(\mathrm{ku}^{\mathrm{t}C_p}\big)\,,$$

where ζ_p denotes a primitive p^{th} root of unity and $q \mapsto \zeta_p$. Modulo $q-1 = \beta t$, this equivalence recovers the underlying S^1-equivariant $\mathbb{E}_1$-equivalence of the S^1-equivariant $\mathbb{E}_\infty$-equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}^{\mathrm{t}C_p})$ *from* [*TC18*, *Corollary* IV.4.13].

Here $\beta \in \pi_2(\mathrm{ku})$ denotes the Bott element and $t \in \pi_{-2}(\mathrm{ku}^{\mathrm{h}S^1})$ is a complex orientation class, which can be chosen in such a way that $1 + \beta t \in \pi_0(\mathrm{ku}^{\mathrm{h}S^1})$ corresponds to the standard representation of S^1 on $\mathbb{C}$. If we denote this class by q, then the $\mathbb{E}_1$-equivalence from Theorem 3.18 sends $q \mapsto q$.

Proof of Theorem 3.18. We learned the following argument from Sanath Devalapurkar. Let us first construct an S^1-equivariant $\mathbb{E}_\infty$-map $\mathbb{S}[\![q-1]\!] \to \mathrm{ku}^{\mathrm{t}C_p}$, where the left-hand side receives the trivial S^1 action and the right-hand side the residual $S^1 \simeq S^1/C_p$-action. It's enough to construct an S^1-equivariant $\mathbb{E}_\infty$-map $\mathbb{S}[\![q-1]\!] \to \mathrm{ku}^{\mathrm{h}C_p}$, or equivalently, an $\mathbb{E}_\infty$-map

(3.1) Conjecturally, there should even be an $\mathbb{E}_\infty$-equivalence. The second author has been informed that Devalapurkar and Raksit have some ideas on how to make the $\mathbb{E}_1$-equivalence from Theorem 3.18 into an $\mathbb{E}_\infty$-equivalence, using their forthcoming work [DR] on an $\mathbb{E}_\infty$-equivalence of cyclotomic spectra $\mathrm{THH}(\mathbb{Z}_p) \simeq \tau_{\geqslant 0}(j^{\mathrm{t}C_p})$, where $j \coloneqq \tau_{\geqslant 0}(\mathrm{L}_{\mathrm{K}(1)}\mathbb{S})$.

$\mathbb{S}[\![q-1]\!] \to (\mathrm{ku}^{\mathrm{h}C_p})^{\mathrm{h}(S^1/C_p)} \simeq \mathrm{ku}^{\mathrm{h}S^1}$. Since $q \in \pi_0(\mathrm{ku}^{\mathrm{h}S^1})$ is a strict element, there exists an $\mathbb{E}_\infty$-map $\mathbb{S}[q] \to \mathrm{ku}^{\mathrm{h}S^1}$, which factors over the $(q-1)$-completion $\mathbb{S}[q] \to \mathbb{S}[\![q-1]\!]$ and so we obtain the desired map.

Now let us construct an $\mathbb{E}_2$-$\mathbb{S}[\![q-1]\!]$-algebra map $\mathbb{Z}_p[\zeta_p] \to \mathrm{ku}^{\mathrm{t}C_p}$. To this end, observe that $\mathbb{Z}_p[\zeta_p]$ is the free $(q-1)$-complete $\mathbb{E}_2$-$\mathbb{S}[\![q-1]\!]$-algebra satisfying $[p]_q = 0$. Indeed, since $[p]_q = 0$ holds in $\mathbb{Z}_p[\zeta_p]$, it certainly receives an $\mathbb{E}_2$-$\mathbb{S}[\![q-1]\!]$-map from the free guy. Whether this map is an equivalence can be checked modulo $(q-1)$, where it reduces to the classical fact that $\mathbb{F}_p$ is the free $\mathbb{E}_2$-algebra satisfying $p = 0$. Since $\pi_*(\mathrm{ku}^{\mathrm{t}C_p}) \cong \pi_*(\mathrm{ku}^{\mathrm{t}S^1})/[p]_q$, we get our desired $\mathbb{E}_2$-$\mathbb{S}[\![q-1]\!]$-algebra map $\mathbb{Z}_p[\zeta_p] \to \mathrm{ku}^{\mathrm{t}C_p}$. It induces S^1-equivariant $\mathbb{E}_1$-$\mathbb{S}[\![q-1]\!]$-algebra map

$$\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!]\big)_p^\wedge \longrightarrow \mathrm{THH}\big(\mathrm{ku}^{\mathrm{t}C_p}/\mathbb{S}[\![q-1]\!]\big)_p^\wedge \longrightarrow \mathrm{ku}^{\mathrm{t}C_p}\,,$$

where the arrow on the right comes from the universal property of $\mathrm{THH}(-/\mathbb{S}[\![q-1]\!])$ on $\mathbb{E}_\infty$-$\mathbb{S}[\![q-1]\!]$-algebras.[(3.2)] is the usual augmentation (which is an S^1-equivariant $\mathbb{E}_\infty$-$\mathbb{S}[\![q-1]\!]$-algebra map). Since the left-hand side is connective, the above composition factors through an S^1-equivariant $\mathbb{E}_1$-$\mathbb{S}[\![q-1]\!]$-algebra map $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!])_p^\wedge \to \tau_{\geqslant 0}(\mathrm{ku}^{\mathrm{t}C_p})$.

We wish to show that this map is an equivalence. This can be checked modulo $(q-1)$, so it will be enough to prove that modulo $(q-1)$ we obtain the usual equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}^{\mathrm{t}C_p})$. To this end, observe that by the universal property of $\mathbb{Z}_p[\zeta_p]$ and $\mathbb{F}_p$ as free $\mathbb{E}_2$-algebras, the $\mathbb{E}_\infty$-map $\mathrm{ku}^{\mathrm{t}C_p} \to \mathbb{Z}^{\mathrm{t}C_p}$ fits into a commutative diagram of $\mathbb{E}_2$-algebras

$$\begin{array}{ccc} \mathbb{Z}_p[\zeta_p] & \longrightarrow & \mathrm{ku}^{\mathrm{t}C_p} \\ \big\downarrow & & \big\downarrow \\ \mathbb{F}_p & \longrightarrow & \mathbb{Z}^{\mathrm{t}C_p} \end{array}$$

which on the level of underlying spectra exhibits the bottom row as the mod-$(q-1)$-reduction of the top row. Using the same recipe as above, the bottom row induces an S^1-equivariant map of $\mathbb{E}_1$-algebras $\mathrm{THH}(\mathbb{F}_p) \to \tau_{\geqslant 0}(\mathbb{Z}_p)$. This map necessarily agrees with the underlying $\mathbb{E}_1$-map of the S^1-equivariant $\mathbb{E}_\infty$-equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}^{\mathrm{t}C_p})$. Indeed, the latter is uniquely determined by a non-equivariant $\mathbb{E}_\infty$-map $\mathbb{F}_p \to \mathbb{Z}^{\mathrm{t}C_p}$, so we only need to check that this agrees, as an $\mathbb{E}_2$-map with the one above. But $\mathbb{F}_p$ is the free $\mathbb{E}_2$-algebra with $p = 0$, so there can be only one such $\mathbb{E}_2$-map. □

Using Theorem 3.18, we'll relate the q-Hodge-completed q-de Rham complex to $\mathrm{TC}^-(-/\mathrm{ku})$.

3.19. The ku-comparison I. — With notation as in 3.17, let $\mathrm{ku}_R \coloneqq (\mathrm{ku} \otimes \mathbb{S}_R)_p^\wedge$. Consider the following diagram of S^1-equivariant right modules over $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!])_p^\wedge$:

$$\begin{array}{ccccc} \mathrm{THH}(\mathbb{S}_R) \otimes \mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!]\big)_p^\wedge & \xrightarrow{\phi \otimes (3.18)} & \mathrm{THH}(\mathbb{S}_R)^{\mathrm{t}C_p} \otimes \mathrm{ku}^{\mathrm{t}C_p} & \longrightarrow & \big(\mathrm{THH}(\mathbb{S}_R) \otimes \mathrm{ku}\big)^{\mathrm{t}C_p} \\ \big\downarrow & & & & \big\downarrow \simeq \\ \mathrm{THH}\big(R[\zeta_p]/\mathbb{S}[\![q-1]\!]\big)_p^\wedge & & \dashrightarrow & & \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})^{\mathrm{t}C_p} \end{array}$$

(3.2) In particular, this map $\mathrm{THH}(\mathrm{ku}^{\mathrm{t}C_p}/\mathbb{S}[\![q-1]\!])_p^\wedge \to \mathrm{ku}^{\mathrm{t}C_p}$ is *not* the usual augmentation, as the augmentation would only be S^1-equivariant for the trivial S^1-action on $\mathrm{ku}^{\mathrm{t}C_p}$.

Here the top-left map is induced by the cyclotomic Frobenius $\phi\colon \mathrm{THH}(\mathbb{S}_R) \to \mathrm{THH}(\mathbb{S}_R)^{\mathrm{t}C_p}$ and the map from Theorem 3.18. The left vertical map is an equivalence after p-completion, so the bottom dashed arrow exists uniquely. After taking S^1-fixed points and using the Tate fixed point lemma [TC18, Lemma II.4.2], the bottom dashed map induces a right $\mathrm{TC}^-(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!])_p^\wedge$-module map

$$\psi_R^{(1)}\colon \mathrm{TC}^-\bigl(R[\zeta_p]/\mathbb{S}[\![q-1]\!]\bigr)_p^\wedge \longrightarrow \mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge\,.$$

This looks promising because the left-hand side is almost of the form used in [BMS19, §11.2] to construct Frobenius-twisted Breuil–Kisin cohomology. The only difference is that we're working relative to $\mathbb{S}[\![q-1]\!]$, with $q \mapsto \zeta_p$, rather than relative to $\mathbb{S}[z]$, with $z \mapsto \zeta_p - 1$. But this difference is inconsequential, as the following lemma shows.

3.20. Lemma. — *We have an isomorphism of graded $\mathbb{Z}_p[\![q-1]\!]$-algebras*

$$\pi_*\Bigl(\mathrm{TC}^-\bigl(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!]\bigr)_p^\wedge\Bigr) \cong \mathbb{Z}_p[\![q-1]\!][u,t]/\bigl(ut - [p]_q\bigr)$$

where t is our usual complex orientation class in degree -2 and u is a generator in degree 2. More generally, the spectrum $\mathrm{TC}^-(R[\zeta_p]/\mathbb{S}[\![q-1]\!])_p^\wedge$ is concentrated in even degrees and

$$\pi_{2*}\Bigl(\mathrm{TC}^-\bigl(R[\zeta_p]/\mathbb{S}[\![q-1]\!]\bigr)_p^\wedge\Bigr) \cong \mathrm{Fil}_{\mathcal{N}}^*\,\widehat{\mathbb{\Delta}}^{(1)}_{R[\zeta_p]/\mathbb{Z}_p[\![q-1]\!]}$$

is the completion of the Nygaard filtration on Frobenius-twisted prismatic cohomology of $R[\zeta_p]$ relative to the q-de Rham prism $(\mathbb{Z}_p[\![q-1]\!], [p]_q)$. Furthermore, after localisation at u, we get the actual q-de Rham cohomology of R on π_0:

$$\pi_0\Bigl(\mathrm{TC}^-\bigl(R[\zeta_p]/\mathbb{S}[\![q-1]\!]\bigr)\bigl[\tfrac{1}{u}\bigr]_{(p,q-1)}^\wedge\Bigr) \cong q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}\,.$$

Proof sketch. Observe that the arguments from [BMS19; BS19] go through verbatim if we just replace base change along $\mathbb{S}[z] \to \mathbb{S}[z^{1/p^\infty}]$ with base change along $\mathbb{S}[\![q-1]\!] \to \mathbb{S}[\![q^{1/p^\infty}-1]\!]$. Then the first isomorphism is the analogue of [BMS19, Proposition 11.10], the second is the analogue of [BS19, §15.2] and the final one is the analogue of the Frobenius descent from [BMS19, §11.3]. □

3.21. The ku-comparison II. — Using 3.19 and Lemma 3.20, we see that $\psi_R^{(1)}$ induces a map from the Nygaard-completed Frobenius-twisted q-de Rham cohomology of R into $\pi_0(\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge)$. To promote this to a map from the actual q-de Rham cohomology, we need to show that the generator u from Lemma 3.20 is sent to a unit. This can be checked modulo $(q-1)$. Using the compatibility with $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}^{\mathrm{t}C_p})$ from Theorem 3.18, we're reduced to checking that the cyclotomic Frobenius $\phi^{\mathrm{h}S^1}\colon \mathrm{TC}^-(\mathbb{F}_p) \to \mathrm{TP}(\mathbb{F}_p)$ sends the eponymous element $u \in \pi_2\,\mathrm{TC}^-(\mathbb{F}_p)$ to a unit. But in fact, u is sent to $t^{-1} \in \pi_2\,\mathrm{TP}(\mathbb{F}_p)$.

Therefore, $\psi_R^{(1)}$ factors through a map of right $\mathrm{TC}^-(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!])_p^\wedge$-modules

$$\psi_R\colon \mathrm{TC}^-\bigl(R[\zeta_p]/\mathbb{S}[\![q-1]\!]\bigr)\bigl[\tfrac{1}{u}\bigr]_{(p,q-1)}^\wedge \longrightarrow \mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge\,.$$

3.22. Theorem. — *Let ku_R be as above and let $\mathrm{KU}_R \coloneqq (\mathrm{KU} \otimes \mathbb{S}_R)_p^\wedge$. Then the spectra $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$ and $\mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU})_p^\wedge$ are concentrated in even degrees and $\pi_0(\psi_R)$ induces graded $\mathbb{Z}_p[\![q-1]\!]$-linear isomorphisms*

$$\begin{aligned}\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} &\xrightarrow{\ \cong\ } \pi_{2*}\bigl(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge\bigr)\,,\\ q\text{-}\mathrm{Hdg}_{R/\mathbb{Z}_p}[\beta^{\pm 1}] &\xrightarrow{\ \cong\ } \pi_{2*}\bigl(\mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU})_p^\wedge\bigr)\,.\end{aligned}$$

3.23. Remark. — A priori $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$ is only an $\mathbb{E}_0$-ring spectrum, so it doesn't make sense to ask for multiplicativity in Theorem 3.22. However, if $\mathbb{S}_R$ refines to an $\mathbb{E}_2$-ring spectrum, then the map ψ_R respects $\mathbb{E}_1$-$\mathbb{S}[\![q-1]\!]$-algebra structures and so the isomorphism from Theorem 3.22 will be one of graded rings.

To prove Theorem 3.22 we analyse the filtration on $\pi_0(\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge)$ coming from the Tate spectral sequence (which we'll call the *Tate filtration*) and relate it to the q-Hodge filtration. This is the content of Lemmas 3.27 and 3.28. Then we compute $\pi_0(\psi_R)$ modulo $(q-1)$ in Lemma 3.29. Once we have this, Theorem 3.22 will be an easy consequence.

3.24. Completion issues. — To analyse the filtration on $\pi_0(\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge)$, we would like to compare filtrations along the rationalisation map $\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku}) \to \mathrm{TP}(\mathrm{ku}_R\otimes\mathbb{Q}/\mathrm{ku}\otimes\mathbb{Q})$. The problem with this is the p-completion: It would kill the right-hand side, but without p-completion, the left-hand side is a rather pathological object. There are at least two ways to overcome this problem:

(*a*) If $R\cong B/J$ is of perfect-regular presentation, we can consider $\mathrm{ku}_B \coloneqq (\mathrm{ku}\otimes\mathbb{S}_B)_p^\wedge$ and promote ku_R to an $\mathbb{E}_1$-algebra in ku_B-modules by Lemma 2.2. Then the same argument as in [BMS19, Proposition 11.7] shows $\mathrm{THH}(\mathrm{ku}_B/\mathrm{ku})\to\mathrm{ku}_B$ is an equivalence after p-completion, hence $\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge \simeq \mathrm{TP}(\mathrm{ku}_R/\mathrm{ku}_B)_p^\wedge$. But $\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku}_B)$ is already p-complete, so we can simply work relative to ku_B.

(*b*) We can use solid condensed mathematics.

We'll follow strategy (*b*), since it allows us to prove Theorem 3.22 in the most generality (and condensed mathematics will show up anyway in our proof of Theorem 1.19).

3.25. Solid condensed recollections. — Let $\mathrm{Cond}(\mathrm{Sp})$ denote the ∞-category of (*light*) *condensed spectra*, that is, hypersheaves of spectra on the site of light profinite sets as defined by Clausen and Scholze [CS24]. The evaluation at the point $(-)(*)\colon\mathrm{Cond}(\mathrm{Sp})\to\mathrm{Sp}$ admits a fully faithful symmetric monoidal left adjoint $\underline{(-)}\colon\mathrm{Sp}\to\mathrm{Cond}(\mathrm{Sp})$, sending a spectrum X to the *discrete* solid condensed spectrum $\underline{X}$.

If X is a p-complete spectrum, then $\underline{X}$ is usually *not* p-complete in $\mathrm{Cond}(\mathrm{Sp})$ because $\underline{(-)}$ doesn't commute with limits. After passing to p-completions, we still get an adjunction on p-complete objects $\underline{(-)}_p^\wedge\colon\mathrm{Sp}_p^\wedge \rightleftarrows \mathrm{Cond}(\mathrm{Sp})_p^\wedge :(-)(*)$ and the left adjoint is still fully faithful because the unit is still an equivalence. For readability we'll make the following abusive convention: If X is a p-complete spectrum, we'll identify X with $\underline{X}_p^\wedge$, otherwise we identify X with the discrete condensed spectrum $\underline{X}$. In particular, we'll regard ku as a discrete condensed spectrum, but ku_R as a p-complete one.

One can develop a theory of *solid condensed spectra* along the lines of [CS24, Lectures 5–6]. Consider the condensed spectrum $P_{\mathbb{S}}\coloneqq\mathrm{cofib}(\mathbb{S}[\{\infty\}]\to\mathbb{S}[\mathbb{N}\cup\{\infty\}])$. Let $\sigma\colon P_{\mathbb{S}}\to P_{\mathbb{S}}$ be the endomorphism induced by the shift map $(-)+1\colon\mathbb{N}\cup\{\infty\}\to\mathbb{N}\cup\{\infty\}$. Recall that a condensed spectrum M is called *solid* if

$$1-\sigma^*\colon \underline{\mathrm{Hom}}_{\mathbb{S}}(P_{\mathbb{S}},M)\xrightarrow{\simeq}\underline{\mathrm{Hom}}_{\mathbb{S}}(P_{\mathbb{S}},M)$$

is an equivalence, where $\underline{\mathrm{Hom}}_{\mathbb{S}}$ denotes the internal Hom in $\mathrm{Cond}(\mathrm{Sp})$. We let $\mathrm{Sp}_{\blacksquare}\subseteq\mathrm{Cond}(\mathrm{Sp})$ denote the full sub-∞-category of solid condensed spectra. Then $\mathrm{Sp}_{\blacksquare}$ is closed under all limits and colimits, hence the inclusion admits a left adjoint $(-)^{\blacksquare}\colon\mathrm{Cond}(\mathrm{Sp})\to\mathrm{Sp}_{\blacksquare}$. It satisfies $(M\otimes N)^{\blacksquare}\simeq(M^{\blacksquare}\otimes N)^{\blacksquare}$, which allows us to endow $\mathrm{Sp}_{\blacksquare}$ with a symmetric monoidal structure, called the *solid tensor product*, via $M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}$. The solid tensor product has the magical property that if M and N are p-complete and bounded below, then $M \otimes^{\blacksquare} N$ is again p-complete; see [CS24, Lecture 6] or [Bos23, Proposition A.3].

For every $\mathbb{E}_\infty$-algebra k in $\mathrm{Sp}_{\blacksquare}$, we let $- \otimes_k^{\blacksquare} -$ denote the tensor product on $\mathrm{Mod}_k(\mathrm{Sp}_{\blacksquare})$. We can then consider topological Hochschild homology inside the symmetric monoidal ∞-category $\mathrm{Mod}_k(\mathrm{Sp}_{\blacksquare})$. This yields a functor

$$\mathrm{THH}(-/k_{\blacksquare})\colon \mathrm{Alg}\big(\mathrm{Mod}_k(\mathrm{Sp}_{\blacksquare})\big) \longrightarrow \mathrm{Mod}_k(\mathrm{Sp}_{\blacksquare})^{\mathrm{B}S^1}\,.$$

We also let $\mathrm{TC}^-(-/k_{\blacksquare}) := \mathrm{THH}(-/k_{\blacksquare})^{\mathrm{h}S^1}$ and $\mathrm{TP}(-/k_{\blacksquare}) := \mathrm{THH}(-/k_{\blacksquare})^{\mathrm{t}S^1}$. Since every discrete spectrum is solid, we can regard ku as an $\mathbb{E}_\infty$-algebra in $\mathrm{Sp}_{\blacksquare}$. Since $\mathrm{ku}_R \simeq \lim_{\alpha \geqslant 0} \mathrm{ku}_R/p^\alpha$ is a limit of discrete condensed spectra, it must be solid as well, hence it defines an object in $\mathrm{Alg}(\mathrm{Mod}_{\mathrm{ku}}(\mathrm{Sp}_{\blacksquare}))$.

3.26. Lemma. — *Then condensed spectrum* $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_{\blacksquare})$ *is the p-completion of the discrete condensed spectrum* $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})$.

Proof. By the magical property of the solid tensor product,

$$\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_{\blacksquare}) \simeq \mathrm{ku}_R \otimes^{\blacksquare}_{\mathrm{ku}_R^{\mathrm{op}} \otimes^{\blacksquare}_{\mathrm{ku}} \mathrm{ku}_R} \mathrm{ku}_R$$

is again p-complete. Hence we get a map $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge \rightarrow \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_{\blacksquare})$. Whether this map is an equivalence can be checked modulo p^3. Since ku/p admits a right unital multiplication (in fact an $\mathbb{E}_1$-ku-algebra structure by [HW18]), Burklund's result [Bur22, Theorem 1.5] shows that ku/p^3 admits an $\mathbb{E}_2$-ku-algebra structure, and so we may regard $\mathrm{ku}_R/p^3 \simeq \mathrm{ku}_R \otimes^{\blacksquare}_{\mathrm{ku}} \mathrm{ku}/p^3$ as an $\mathbb{E}_1$-algebra in the $\mathbb{E}_1$-monoidal ∞-category $\mathrm{RMod}_{\mathrm{ku}/p^3}(\mathrm{Sp}_{\blacksquare})$. Since ku_R/p^3 is discrete and the inclusion of discrete objects into all condensed spectra preserves tensor products, we obtain

$$\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge/p^3 \simeq (\mathrm{ku}_R/p^3) \otimes_{(\mathrm{ku}_R/p^3)^{\mathrm{op}} \otimes_{\mathrm{ku}/p^3} (\mathrm{ku}_R/p^3)} (\mathrm{ku}_R/p^3) \simeq \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_{\blacksquare})/p^3\,. \qquad \square$$

3.27. Lemma. — *The Tate filtration on* $\pi_0\,\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku}_{\blacksquare})$ *is the preimage of the Tate filtration on* $\pi_0\,\mathrm{TP}((\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q})/(\mathrm{ku} \otimes^{\blacksquare} \mathbb{Q})_{\blacksquare})$ *under the rationalisation map*

$$\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku}_{\blacksquare}) \longrightarrow \mathrm{TP}\big((\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q})/(\mathrm{ku} \otimes^{\blacksquare} \mathbb{Q})_{\blacksquare}\big)\,.$$

Proof. As both filtrations are complete, it's enough to check that the map on associated gradeds is injective. By $(q-1)$-completeness, this can, in turn, be checked modulo $q - 1 = \beta t$. Since t is invertible, reducing modulo $(q-1)$ amounts to base change along $\mathrm{ku} \rightarrow \mathbb{Z}$. So we reduce to the same question about $\mathrm{HP}(R/\mathbb{Z}_{\blacksquare}) \rightarrow \mathrm{HP}(R \otimes^{\blacksquare}_{\mathbb{Z}} \mathbb{Q}/\mathbb{Q}_{\blacksquare})$ (according to our convention in 3.25, $\mathbb{Z}$ and $\mathbb{Q}$ are regarded as discrete condensed rings, whereas R is a p-complete condensed ring). The same argument as in Lemma 3.26 shows that $\mathrm{HH}(R/\mathbb{Z}_{\blacksquare}) \simeq \mathrm{HH}(R/\mathbb{Z})_p^\wedge$. Hence $\pi_0\,\mathrm{HP}(R/\mathbb{Z}_{\blacksquare}) \cong \widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$ and this isomorphism identifies the Tate filtration with the Hodge filtration. Similarly $\mathrm{HH}(R \otimes^{\blacksquare} \mathbb{Q}/\mathbb{Q}_{\blacksquare}) \simeq \mathrm{HH}(R/\mathbb{Z}_{\blacksquare}) \otimes^{\blacksquare}_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{HH}(R/\mathbb{Z})_p^\wedge \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus we obtain that $\pi_0\,\mathrm{HH}(R \otimes^{\blacksquare} \mathbb{Q}/\mathbb{Q}_{\blacksquare}) \cong (\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{\mathrm{Hdg}}$ is the completion of $\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q}$ at its Hodge filtration, and the Tate filtration gets again identified with the Hodge filtration.

So it'll be enough to check that the Hodge filtration on $\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$ is the preimage of the Hodge filtration on $\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q}$. This follows from the argument in the proof of Lemma 3.8. $\square$

3.28. Lemma. — *There exists an isomorphism*

$$\pi_0 \operatorname{TP}\bigl((\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q})/(\mathrm{ku} \otimes^{\blacksquare} \mathbb{Q})_{\blacksquare}\bigr) \cong (\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{\mathrm{Hdg}}[\![q-1]\!]$$

in such a way that the Tate filtration becomes identified with the combined Hodge- and $(q-1)$-adic filtration from Construction 3.5. Furthermore, $\pi_0(\psi_R)$ becomes identified with the canonical map $q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p} \to (\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{\mathrm{Hdg}}[\![q-1]\!]$

Proof. Note that the polynomial ring $\mathbb{Q}[\beta]$ is the free $\mathbb{E}_\infty$-$\mathbb{Q}$-algebra on a generator β in degree 2. Hence we get an $\mathbb{E}_\infty$-map $\mathbb{Q}[\beta] \to \mathrm{ku} \otimes^{\blacksquare} \mathbb{Q}$ and an $\mathbb{E}_1$-map $R \otimes^{\blacksquare}_{\mathbb{Z}} \mathbb{Q}[\beta] \simeq \mathbb{S}_R \otimes^{\blacksquare} \mathbb{Q}[\beta] \to \mathrm{ku}_R$. Both maps are equivalences, as one immediately checks on homotopy groups. It follows that

$$\operatorname{THH}\bigl((\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q})/(\mathrm{ku} \otimes^{\blacksquare} \mathbb{Q})_{\blacksquare}\bigr) \simeq \operatorname{HH}(R/\mathbb{Z}_{\blacksquare}) \otimes^{\blacksquare}_{\mathbb{Z}} \mathbb{Q}[\beta] \simeq \operatorname{HH}(R/\mathbb{Z})^{\wedge}_p \otimes^{\blacksquare}_{\mathbb{Z}} \mathbb{Q}[\beta]\,.$$

By lax monoidality of $(-)^{\mathrm{t}S^1}$, we get a map

$$\operatorname{TP}(R/\mathbb{Z})^{\wedge}_p \otimes^{\blacksquare}_{\mathbb{Z}(\!(t)\!)} \mathbb{Q}[\beta](\!(t)\!) \longrightarrow \operatorname{TP}\bigl((\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q})/(\mathrm{ku} \otimes^{\blacksquare} \mathbb{Q})_{\blacksquare}\bigr)\,.$$

The THH-computation above shows that on π_0 this map is an equivalence up to completion at the Tate filtration. Hence indeed $\pi_0 \operatorname{TP}\bigl((\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q})/(\mathrm{ku} \otimes^{\blacksquare} \mathbb{Q})_{\blacksquare}\bigr) \cong (\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{\mathrm{Hdg}}[\![q-1]\!]$, where $q-1 = \beta t$, and the Tate filtration becomes the combined Hodge- and $(q-1)$-adic filtration.

It remains to show that $\pi_0(\psi_R)$ agrees with the canonical map. To this end, recall our assumption that $\overline{R} \coloneqq R/p$ is semiperfect. This implies that $\mathbb{A}_{\mathrm{inf}} \coloneqq \mathrm{W}(R^\flat) \to R$ is surjective. If $J \coloneqq \ker(\mathbb{A}_{\mathrm{inf}} \to R)$ and $\mathbb{A}_{\mathrm{crys}} \coloneqq D_{\mathbb{A}_{\mathrm{inf}}}(J)$ denotes the p-completed PD-envelope of J, then it's well-known that $\mathrm{dR}_{R/\mathbb{Z}_p} \simeq \mathrm{dR}_{R/\mathbb{A}_{\mathrm{inf}}} \simeq \mathbb{A}_{\mathrm{crys}}$.(3.3) So $\pi_0(\psi_R)$ appears as a map $q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p} \to (\mathbb{A}_{\mathrm{crys}} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{\mathrm{Hdg}}[\![q-1]\!]$. Since the right-hand side is rational and $(q-1)$-complete, $\pi_0(\psi_R)$ factors through $(q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \simeq (\mathbb{A}_{\mathrm{crys}} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$. In total, we're given a map

$$(\mathbb{A}_{\mathrm{crys}} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!] \longrightarrow (\mathbb{A}_{\mathrm{crys}} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{\mathrm{Hdg}}[\![q-1]\!]$$

and we must show that it is the obvious one. To see this, observe that the un-p-completed PD-envelope $\mathbb{A}_{\mathrm{crys}}^{\circ}$ of $J \subseteq \mathbb{A}_{\mathrm{inf}}$ is contained in $\mathbb{A}_{\mathrm{inf}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, the obvious map is uniquely characterised by the following two properties:

(*a*) *It is a map of $\mathbb{A}_{\mathrm{inf}}[\![q-1]\!]$-modules.*

(*b*) *It is continuous with respect to the $(p, q-1)$-adic topology on either side.*

It's clear from the construction that $\pi_0(\psi_R)$ also satisfies (*b*). To see (*a*), since $\mathbb{A}_{\mathrm{inf}}$ is a perfect δ-ring, it lifts uniquely to a p-complete connective $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}}$. Then $\operatorname{THH}(\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}}) \to \mathbb{S}_{\mathbb{A}_{\mathrm{inf}}}$ is an equivalence after p-completion by the same argument as in [BMS19, Proposition 11.7] and so $\operatorname{THH}(\mathbb{S}_R)^{\wedge}_p \simeq \operatorname{THH}(\mathbb{S}_R/\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}})^{\wedge}_p$. Thus, we can redo the constructions from 3.19 and 3.21 and work relative to $\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}}$ everywhere. This yields the desired $\mathbb{A}_{\mathrm{inf}}[\![q-1]\!]$-linearity.

Here we've implicitly used two facts: First that the $\mathbb{E}_1$-algebra structure on $\mathbb{S}_R$ refines to an $\mathbb{E}_1$-algebra in $\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}}$-modules; this is shown in Lemma 2.2. Second, we've used that $\operatorname{THH}(-/\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}})$ admits a cyclotomic structure; this is shown in Lemma 2.1. We should also

(3.3) Indeed, the first equivalence follows from $\mathbb{A}_{\mathrm{inf}}$ being a perfect δ-ring. For the second, note that $\mathrm{dR}_{R/\mathbb{A}_{\mathrm{inf}}}$ is p-torsion free and contains divided powers for all $x \in J$, as can be seen from $\mathrm{dR}_{\mathbb{Z}/\mathbb{Z}[x]} \to \mathrm{dR}_{R/\mathbb{A}_{\mathrm{inf}}}$. Hence there's a map $\mathbb{A}_{\mathrm{crys}} \to \mathrm{dR}_{R/\mathbb{A}_{\mathrm{inf}}}$, and this map is an equivalence modulo p by [BMS19, Proposition 8.12].

remark that the cyclotomic Frobenius in 3.19 introduces a Frobenius-twist, so that $\pi_0(\psi_R^{(1)})$ is not $\mathbb{A}_{\mathrm{inf}}[\![q-1]\!]$-linear, but only semi-linear with respect to the Frobenius on $\mathbb{A}_{\mathrm{inf}}$. However, this twist gets untwisted in 3.21[(3.4)] so $\pi_0(\psi_R)$ really is $\mathbb{A}_{\mathrm{inf}}[\![q-1]\!]$-linear. □

This finishes our analysis of $\pi_0(\psi_R)$ after rationalisation. Let us now compute it after reduction modulo $(q-1)$.

3.29. Lemma. — *Let $\mathrm{WdR}_{-/\mathbb{F}_p}$ denote the p-typical derived de Rham–Witt complex of $\mathbb{F}_p$-algebras and let $\overline{R} \coloneqq R/p$. Then modulo $(q-1)$, the map $\pi_0(\psi_R)$ can be canonically identified with*

$$\mathrm{WdR}_{\overline{R}/\mathbb{F}_p} \simeq \mathrm{dR}_{R/\mathbb{Z}_p} \longrightarrow \widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}\,.$$

Proof. If we reduce the maps $\psi_R^{(1)}$ and ψ_R from 3.19 and 3.21 modulo $(q-1) = \beta t$, we obtain maps $\overline{\psi}_R^{(1)}$ and $\overline{\psi}_R$ sitting inside a commutative diagram

$$\begin{array}{ccc} \mathrm{TC}^-(\overline{R})_p^\wedge & \xrightarrow{\overline{\psi}_R^{(1)}} & \mathrm{HP}(R/\mathbb{Z})_p^\wedge \\ \big\downarrow & \overset{\overline{\psi}_R}{\nearrow} & \\ \mathrm{TC}^-(\overline{R})\big[\tfrac{1}{u}\big]_p^\wedge & & \end{array}$$

where ϕ is the cyclotomic Frobenius. We know that $\pi_0(\mathrm{HP}(R/\mathbb{Z})_p^\wedge) \cong \widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$. According to [BMS19, Theorem 8.17], $\pi_0(\mathrm{TC}^-(\overline{R})_p^\wedge) \cong \widehat{\mathrm{WdR}}_{\overline{R}/\mathbb{F}_p}$ is the Nygaard-completed de Rham–Witt complex of $\overline{R}$. Furthermore, the Frobenius-descent from [BMS19, §11.3] (except that we're working absolutely, not relative to $\mathbb{S}[z]$ or $\mathbb{S}[\![q-1]\!]$) shows that the left vertical map can be identified with the Frobenius $\phi\colon \widehat{\mathrm{WdR}}_{\overline{R}/\mathbb{F}_p} \to \mathrm{WdR}_{\overline{R}/\mathbb{F}_p}$ on π_0. So $\pi_0(\overline{\psi}_R)$ really appears as a map $\mathrm{WdR}_{\overline{R}/\mathbb{F}_p} \simeq \mathrm{dR}_{R/\mathbb{Z}_p} \to \widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$. To show that it is the indeed the obvious such map, we can use the same argument as in the proof of Lemma 3.28: The obvious map is uniquely determined by $\mathbb{A}_{\mathrm{inf}}$-linearity and continuity; the former property is obviously satisfied for $\pi_0(\overline{\psi}_R)$, whereas the latter follows since we can again work relative to $\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}}$. □

Before we continue with our proof of Theorem 3.22, let us mention the following curious observation.

3.30. Remark. — Recall the usual equivalence $\mathrm{dR}_{R/\mathbb{Z}_p} \simeq \mathrm{WdR}_{\overline{R}/\mathbb{F}_p}$. For $p \geqslant 3$, it's straightforward to check that the Hodge filtration is contained in the Nygaard filtration, and so we get a map $\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \to \widehat{\mathrm{WdR}}_{\overline{R}/\mathbb{F}_p}$. This map is an equivalence because it reduces to an equivalence modulo p; see [BMS19, Theorem 8.17].

However, for $p = 2$ this argument breaks down and the author suspects that in the case $R = \mathbb{Z}_2\{x\}_\infty/x$ (where $\mathbb{Z}_2\{x\}_\infty$ denotes the free 2-complete perfect δ-ring on a generator x) the Hodge and Nygaard filtrations are incommensurable. But in the presence of an $\mathbb{E}_1$-lift $\mathbb{S}_R$, such a pathology cannot occur. Even better: We get an equivalence on the level of spectra!

Indeed, recall from [TC18, Corollary IV.4.13] there exists a map $\mathbb{Z} \to \mathrm{THH}(\mathbb{F}_p)$ of S^1-equivariant $\mathbb{E}_\infty$-ring spectra. It is an equivalence after $(-)^{\mathrm{t}C_p}$ and induces an S^1-equivariant map $(\mathrm{THH}(\mathbb{S}_R) \otimes \mathbb{Z})^{\mathrm{t}C_p} \to (\mathrm{THH}(\mathbb{S}_R) \otimes \mathrm{THH}(\mathbb{F}_p))^{\mathrm{t}C_p}$ of spectra. After taking S^1-fixed points, this yields a map

$$c\colon \mathrm{HP}(R/\mathbb{Z})_p^\wedge \longrightarrow \mathrm{TP}(\overline{R})_p^\wedge\,.$$

(3.4) This probably not clear from the construction alone, but it becomes apparent from [BMS19, §11.3].

On π_0, we obtain a map $\pi_0(c)\colon \widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}\to\widehat{\mathrm{WdR}}_{\overline{R}/\mathbb{F}_p}$. The same $\mathbb{S}_{\mathbb{A}_{\mathrm{inf}}}$-trick as in the proofs of Lemmas 3.28 and 3.29 shows that the map $\pi_0(c)$ must be induced by the usual equivalence $\mathrm{dR}_{R/\mathbb{Z}_p}\simeq\mathrm{WdR}_{\overline{R}/\mathbb{F}_p}$ after passing to completions on both sides (so in particular, the completions must be compatible). The same argument as above then shows that $\pi_0(c)$ must be an equivalence. Thus c must be an equivalence as well, as both sides are even periodic and the periodicity generators are already detected in $\mathbb{Z}^{\mathrm{t}C_p}\simeq\mathrm{THH}(\mathbb{F}_p)^{\mathrm{t}C_p}$.

Proof of Theorem 3.22. Consider $\pi_0(\psi_R)\colon q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}\to\pi_0(\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge)$. By Lemmas 3.27 and 3.28, the q-Hodge filtration on $q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ is the preimage of the Tate filtration under $\pi_0(\psi_R)$. This immediately implies that $\pi_0(\psi_R)$ factors through an injective map

$$q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}\longrightarrow\pi_0\bigl(\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge\bigr)\,.$$

To show surjectivity, we only need to check that the image of this map is dense. This can be checked modulo $(q-1)$, where it follows from Lemma 3.29.

By the usual comparison between the homotopy fixed point and Tate spectral sequences, we see that $\pi_{2n}(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge)$ agrees for all $n\geqslant 0$ with the n^{th} step in the Tate filtration on $\pi_0(\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge)\simeq q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$, so we can upgrade this equivalence to the desired filtered equivalence

$$\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}\simeq\pi_{2*}\bigl(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge\bigr)\,.$$

Using the base change formula $\mathrm{THH}(-\otimes_{\mathrm{ku}}\mathrm{KU}/\mathrm{KU})\simeq\mathrm{THH}(-/\mathrm{ku})\otimes_{\mathrm{ku}}\mathrm{KU}$ we obtain

$$\mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU})_p^\wedge\simeq\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge\Bigl[\tfrac{1}{\beta}\Bigr]^\wedge_{(p,t)}\,.$$

Since $q-1=\beta t$, we may as well complete at $(p,q-1)$. Via the above computation of $\pi_{2*}(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge)$, we see that $\pi_0(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge[1/\beta])$ arises from $q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$ by adjoining elements $(q-1)^{-n}\omega_n$ for all $\omega_n\in\mathrm{Fil}^n_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$. After completion at $(p,q-1)$, this precisely yields the q-Hodge complex $q\text{-}\mathrm{Hdg}_{R/\mathbb{Z}_p}$ from Construction 3.5(*c*). To finish the proof, it remains to check that in this case $(p,q-1)$-completion commutes with passing to homotopy groups and to establish the evenness assertions. This will be done in Lemma 3.31 below. □

3.31. Lemma. — *The spectra* $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$ *and* $\mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU})_p^\wedge$ *are concentrated in even degrees. Furthermore,* $(q-1,p)$ *is a regular sequence in* $\pi_{2*}(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge[1/\beta])$*, so* $(p,q-1)$*-completion commutes with passing to homotopy groups.*

Proof. The $\mathrm{ku}^{\mathrm{h}S^1}$-module spectrum $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$ is t-complete, so it's enough to check that $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})/t\simeq\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$ is even. This is a connective ku-module, hence β-complete. So we can further reduce to $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge/\beta\simeq\mathrm{HH}(R/\mathbb{Z})_p^\wedge$, which is even with π_{2n} given by $\Sigma^{-n}\bigwedge^n\mathrm{L}_{R/\mathbb{Z}_p}$; see [BMS19, Lemma 5.14] and observe that the proof goes through in our situation as well even though R need not be quasiregular semiperfectoid.

That $\pi_{2*}(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge[1/\beta])$ is $(q-1)$-torsion free, or equivalently, t-torsion free, follows since $(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge[1/\beta])/t\simeq\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge[1/\beta]$ is even, as we've just checked that $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$ is even. Furthermore, to see that the homotopy groups of $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge[1/\beta]$ are p-torsion free, it's enough to check the same for $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge$. Since this is β-complete, we can again reduce to $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku})_p^\wedge/\beta\simeq\mathrm{HH}(R/\mathbb{Z})_p^\wedge$, and then we win since $\Sigma^{-n}\bigwedge^n\mathrm{L}_{R/\mathbb{Z}_p}$ are p-completely flat modules over the p-torsion free ring R. This shows that $(q-1,p)$ is a regular sequence, as desired. Then $\mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU})_p^\wedge\simeq\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})_p^\wedge[1/\beta]^\wedge_{(p,q-1)}$ must again be even, since $(p,q-1)$-completion commutes with passing to homotopy groups in this case. □

Proof of Theorem 3.10(b). By Lemma 3.16, we can reduce to the case where $A \cong A_\infty$ is a perfect δ-ring. Then $R \cong R_\infty$. Furthermore, $\operatorname{Fil}^*_{\mathrm{Hdg}} \mathrm{dR}_{-/A} \simeq \operatorname{Fil}^*_{\mathrm{Hdg}} \mathrm{dR}_{-/\mathbb{Z}_p}$ and then the same follows for the derived q-de Rham complexes with their q-Hodge filtration. So we may as well work relative to $\mathbb{Z}_p$. Since $q-1 = \beta t$, we see that under the equivalence from Theorem 3.22 we get an identification

$$\operatorname{Fil}^n_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}/(q-1)\operatorname{Fil}^{n-1}_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p} \cong \pi_{2n}\big(\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})^\wedge_p/\beta\big)$$

But on homotopy groups, $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku})^\wedge_p/\beta \simeq \mathrm{HC}^-(R/\mathbb{Z})^\wedge_p$ recovers the Hodge filtration on $\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$. This shows that $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$ is a q-deformation of $\operatorname{Fil}^*_{\mathrm{Hdg}} \widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$.

To show that the same holds true before completing the filtrations, thanks to Lemma 3.8 we only need to prove that $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p} \to \operatorname{Fil}^*_{\mathrm{Hdg}} \mathrm{dR}_{R/\mathbb{Z}_p}$ is surjective in every degree. So let $\overline{\omega} \in \operatorname{Fil}^n_{\mathrm{Hdg}} \mathrm{dR}_{R/\mathbb{Z}_p}$. We know that $\operatorname{Fil}^n_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$ contains a lift $\widehat{\omega}$ of $\overline{\omega}$. We also know that $q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ contains a lift ω of $\overline{\omega}$. Then $\widehat{\omega} = \omega + (q-1)\widehat{\eta}$ for some $\widehat{\eta} \in q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}_p}$. Choose an approximation $\eta \in q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ of η such that $\eta - \widehat{\eta}$ lies in the n^{th} step of the completed q-Hodge filtration. Then $\omega + (q-1)\eta \in \operatorname{Fil}^n_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/\mathbb{Z}_p}$ is a lift of $\overline{\omega}$, as desired. □

§3.3. Some formal properties of derived q-Hodge complexes

In this subsection we show a few easy results about $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A}$ and $q\text{-}\mathrm{Hdg}_{R/A}$. We start with a result which is very convenient to streamline arguments.

3.32. Lemma. — *Let $\operatorname{Fil}^* M \in \mathrm{Mod}_{(q-1)^*\mathbb{Z}[\![q-1]\!]}(\operatorname{Fil}\mathcal{D}(\mathbb{Z}))$ be a filtered module over the $(q-1)$-adically filtered ring $(q-1)^*\mathbb{Z}[\![q-1]\!]$ such that for all $n<0$, $\operatorname{Fil}^n M \simeq \operatorname{Fil}^0 M$, and for all $n \geqslant 0$, $\operatorname{Fil}^n M$ is $(q-1)$-complete and the multiplication $(q-1)^n\colon \operatorname{Fil}^0 M \to \operatorname{Fil}^0 M$ factors through $\operatorname{Fil}^n M$ as $\mathbb{Z}[\![q-1]\!]$-modules. Then M is also $(q-1)$-complete in the filtered sense, with $(q-1)$ sitting in filtration degree 1.*

In particular, this applies to $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A}$, as $(q-1)^n q\text{-}\mathrm{dR}_{R/A} \subseteq \operatorname{Fil}^n_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A}$.

Proof of Lemma 3.32. For all n, we obtain

$$\operatorname{Fil}^n M \simeq \lim_{i\geqslant 0} \operatorname{Fil}^n M/(q-1)^i \operatorname{Fil}^n M \simeq \lim_{i \geqslant 0} \operatorname{Fil}^n M/(q-1)^i \operatorname{Fil}^{n-i} M\,.$$

The first equivalence is the assumption that $\operatorname{Fil}^n M$ is $(q-1)$-complete, the second equivalence follows from the factorisation assumption. In combination, this precisely says that $\operatorname{Fil}^* M$ is $(q-1)$-complete in the filtered sese. □

Next, we'll show base change and a Künneth formula. We've already seen in Lemma 3.16 that the q-Hodge filtration satisfies base change along p-completely flat maps of δ-rings. We'll now show that the flatness hypothesis can be removed as long as the q-Hodge filtration is a q-deformation of the Hodge filtration.

3.33. Lemma. — *Let $A \to A'$ be a morphism of p-complete δ-rings such that A' is also p-completely perfectly covered. Let R be a p-torsion free quasiregular quotient over A such that $\overline{R}_\infty$ is semiperfect. Suppose that the derived base change $R' \coloneqq R \widehat{\otimes}^{\mathrm{L}}_A A'$ is static and p-torsion free.*

(a) *R' is again a quasiregular quotient over A' and $\overline{R}'_\infty$ is semiperfect (so we can define $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R'/A'}$ and $q\text{-}\mathrm{Hdg}_{R'/A'}$).*

Furthermore, suppose that the q-Hodge filtration on $q\text{-}\mathrm{dR}_{R/A}$ is a q-deformation of the Hodge filtration. Then:

(b) *The q-Hodge filtration on $q\text{-}\mathrm{dR}_{R'/A'}$ is again a q-deformation of the Hodge filtration and the canonical base change morphism*

$$\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{A[\![q-1]\!]} A'[\![q-1]\!] \xrightarrow{\simeq} \mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R'/A'}\,.$$

is an equivalence. On the left-hand side the tensor product is degreewise $(p, q-1)$-completed.

(c) *The canonical base change morphism for derived q-Hodge complexes is also an equivalence*

$$q\text{-}\mathrm{Hdg}_{R/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{A[\![q-1]\!]} A'[\![q-1]\!] \xrightarrow{\simeq} q\text{-}\mathrm{Hdg}_{R'/A'}\,.$$

Proof. Part (a) is almost trivial: Our assumptions imply that $R'/p \simeq R/p \otimes^{\mathrm{L}}_{A/p} A'/p$ and that $\Sigma^{-1}\mathrm{L}_{R/A}/p$ is a flat R/p-module. Then $\Sigma^{-1}\mathrm{L}_{R'/A'}/p \simeq \Sigma^{-1}\mathrm{L}_{R/A}/p \otimes^{\mathrm{L}}_{A/p} A'/p$ is a flat module over R'/p, proving that R' is a quasiregular quotient over A. Furthermore, it's clear that $\overline{R}'_\infty \cong \overline{R}_\infty \otimes_{A_\infty/p} A'_\infty/p$ is semiperfect again.

Next we show (b). To show that the q-Hodge filtration on $q\text{-}\mathrm{dR}_{R'/A'}$ is a q-deformation of the Hodge filtration, we only need to check that $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R'/A'}/(q-1) \to \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R'/A'}$ is degreewise surjective. Since the usual Hodge filtration satisfies derived base change, we have $\mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R'/A'} \simeq \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A A'$; the right-hand side must be static by (a). The desired surjectivity now follows since already

$$\Big(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{A[\![q-1]\!]} A'[\![q-1]\!]\Big)/(q-1) \longrightarrow \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A A'$$

is degreewise surjective by our assumption on R. To show the base change equivalence, observe that both sides are $(q-1)$-complete as filtered objects by Lemma 3.32. So we may reduce both sides modulo $(q-1)$ in the filtered sense, and then we obtain the base change morphism $\mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A A' \to \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R'/A'}$ for the ordinary Hodge filtrations, which we know to be an equivalence. This proves (b).

To show (c), recall from Remark 3.7 that we may as well use the un-completed q-Hodge filtration in the definition of the derived q-Hodge complex. Since we know the latter to satisfy base change by (b), we're done. □

3.34. Lemma. — *Let R_1 and R_2 be p-torsion free quasiregular quotients over A such that $\overline{R}_{1,\infty}$ and $\overline{R}_{2,\infty}$ are semiperfect. Suppose that $R \coloneqq R_1 \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A R_2$ is static and p-torsion free.*

(a) *R is again a quasiregular quotient over A and $\overline{R}_\infty$ is semiperfect (so we can define $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ and $q\text{-}\mathrm{Hdg}_{R/A}$).*

Furthermore, suppose that the q-Hodge filtrations on $q\text{-}\mathrm{dR}_{R_1/A}$ and $q\text{-}\mathrm{dR}_{R_2/A}$ are q-deformations of the respective Hodge filtrations. Then:

(b) *The q-Hodge filtration on $q\text{-}\mathrm{dR}_{R/A}$ is again a q-deformation of the Hodge filtration and we have a canonical Künneth equivalence*

$$\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R_1/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{(q-1)^*A[\![q-1]\!]} \mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R_2/A} \xrightarrow{\simeq} \mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}\,.$$

Here the tensor product is taken in filtered objects and degree-wise $(p, q-1)$-completed; $(q-1)^*A[\![q-1]\!]$ denotes the $(q-1)$-adic filtration on $A[\![q-1]\!]$ (which agrees with the q-Hodge filtration $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{A/A}$).

(c) *Likewise, for derived q-Hodge complexes there is a canonical Künneth equivalence*

$$q\text{-}\mathrm{Hdg}_{R_1/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{A[\![q-1]\!]} q\text{-}\mathrm{Hdg}_{R_2/A} \xrightarrow{\simeq} q\text{-}\mathrm{Hdg}_{R/A}\,.$$

Proof. Again, (a) is almost trivial: Since $R_1/p \otimes^{\mathrm{L}}_{A/p} R_2/p$ is static and $\Sigma^{-1}\mathrm{L}_{R_1/A}/p$ is flat over R_1/p, we see that $\Sigma^{-1}\mathrm{L}_{R_1/A}/p \otimes^{\mathrm{L}}_{A/p} R_2/p$ must be static again. The same conclusion holds if we reverse the roles of R_1 and R_2. Hence both summands in

$$\mathrm{L}_{R/A} \simeq \left(\mathrm{L}_{R_1/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A R_2\right) \oplus \left(R_1 \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A \mathrm{L}_{R_2/A}\right)$$

have p-complete Tor-amplitude concentrated in degree 1, so R is a quasiregular quotient over A again. Also $\overline{R}_\infty \cong \overline{R}_{1,\infty} \otimes_{A_\infty/p} \overline{R}_{2,\infty}$is clearly semiperfect again.

To show that the q-Hodge filtration on $q\text{-}\mathrm{dR}_{R'/A'}$ is a q-deformation of the Hodge filtration, we only need to check that $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}/(q-1) \to \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A}$ is degreewise surjective. But $\mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A} \simeq \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R_1/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_A \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R_2/A}$ and so surjectivity for R follows from surjectivity for R_1 and R_2. By Lemma 3.32, the Künneth equivalence from (b) can be checked after reducing both sides modulo $(q-1)$ as filtered objects. Then we get the usual Künneth equivalence for the Hodge filtration. This shows (b). As in the proof of Lemma 3.16, thanks to Remark 3.7 we can deduce (c) as a formal consequence of (b). $\square$

Finally, we'll introduce yet another filtration.

3.35. The conjugate filtration. — Let R be a p-torsion free quasiregular quotient over A such that $\overline{R}_\infty$ is semiperfect. We will construct an ascending filtration $\mathrm{Fil}^{\mathrm{conj}}_*(q\text{-}\mathrm{Hdg}_{R/A}/(q-1))$ on $q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$, which we'll call the *conjugate filtration*, whose associated graded is $\mathrm{gr}^{\mathrm{conj}}_n(q\text{-}\mathrm{Hdg}_{R/A}/(q-1)) \simeq \mathrm{dR}^n_{R/A} \simeq \Sigma^{-n}\bigwedge^n \mathrm{L}_{R/A}$. This gives some justification to our proposal that $q\text{-}\mathrm{Hdg}_{R/A}$ should be a q-deformation of Hodge cohomology.

To construct the conjugate filtration, let's consider the p-completed localisation of the filtered object $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ at the element $(q-1)$, sitting in filtration degree 1, as always:

$$\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}\left[\tfrac{1}{q-1}\right]^\wedge_p \simeq \operatorname{colim}\left(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \mathrm{Fil}^{*+1}_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \dots\right)^\wedge_p.$$

Let $q\text{-}\mathrm{Hdg}^\circ_{R/A}$ denote the degree-0-part. It's completely formal to see that the filtered object above is the $(q-1)$-adic filtration on $q\text{-}\mathrm{Hdg}^\circ_{R/A}[1/(q-1)]^\wedge_p$. In particular, if we complete the filtration (or in other words, take the localisation in the category of *complete* filtered objects), then we obtain the $(q-1)$-adic filtration on $q\text{-}\mathrm{Hdg}_{R/A}[1/(q-1)]^\wedge_p$. By passing to the associated graded, it follows that

$$q\text{-}\mathrm{Hdg}_{R/A}/(q-1) \simeq \operatorname{colim}\left(\mathrm{gr}^0_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \mathrm{gr}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \dots\right)^\wedge_p.$$

This representation as a colimit yields an exhaustive ascending filtration on $q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$ via $\mathrm{Fil}^{\mathrm{conj}}_n(q\text{-}\mathrm{Hdg}_{R/A}/(q-1)) \coloneqq \mathrm{gr}^n_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$. It remains to check:

3.36. Lemma. — *If the q-Hodge filtration on $q\text{-}\mathrm{dR}_{R/A}$ is a q-deformation of the Hodge filtration, then the associated graded of the conjugate filtration on $q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$ is indeed given by*

$$\mathrm{gr}^{\mathrm{conj}}_n\left(q\text{-}\mathrm{Hdg}_{R/A}/(q-1)\right) \simeq \mathrm{dR}^n_{R/A} \simeq \Sigma^{-n}\bigwedge^n \mathrm{L}_{R/A}\,.$$

Proof. The filtered localisation $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{R/A}[1/(q-1)]^\wedge_p$ can similarly be promoted to a doubly filtered object, with one ascending ("horizontal") and one descending ("vertical") direction. The associated graded in the vertical direction is $\operatorname{Fil}_*^{\mathrm{conj}}(q\text{-Hdg}_{R/A}/(q-1))$ by construction. For any doubly filtered object, it doesn't matter in which order one passes to the associated graded. Passing to the associated graded in the horizontal direction first, we get $(\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{R/A})/(q-1) \simeq \operatorname{Fil}^*_{\mathrm{Hdg}} \mathrm{dR}_{R/A}$. In the vertical direction we then obtain $\operatorname{gr}^n_{\mathrm{Hdg}} \mathrm{dR}_{R/A} \simeq \Sigma^{-n} \bigwedge^n \mathrm{L}_{R/A} \simeq \mathrm{dR}^n_{R/A}$, as desired. $\square$

§3.4. Global q-Hodge complexes

In the same way as an integral q-de Rham complex can be defined for arbitrary animated rings (see Appendix A), we can also globalise the constructions from Construction 3.5.

3.37. Construction. — Let A be a Λ-ring which is p-torsion free for all primes p and such that all Adams operations $\psi^p\colon A \to A$ are p-completely faithfully flat. Let R be an A-algebra such that for all primes p, R is p-torsion free, the p-completion $\widehat{R}_p$ is a quasiregular quotient over $\widehat{A}_p$, and its base change $\widehat{R}_{p,\infty} \coloneqq \widehat{R}_p \mathbin{\widehat{\otimes}}_{\widehat{A}_p} \widehat{A}_{p,\infty}$ satisfies that $\widehat{R}_{p,\infty}/p$ is semiperfect.

(a) We define the *q-Hodge filtration on* $q\text{-dR}_{R/A}$ to be the pullback

$$\begin{array}{ccc} \operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{R/A} & \longrightarrow & \prod_p \operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{\widehat{R}_p/\widehat{A}_p} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \operatorname{Fil}^*_{(\mathrm{Hdg},q-1)}\Big(\big(\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q}\big)[\![q-1]\!]\Big) & \longrightarrow & \operatorname{Fil}^*_{(\mathrm{Hdg},q-1)}\bigg(\Big(\prod_p \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \otimes_{\mathbb{Z}} \mathbb{Q}\Big)[\![q-1]\!]\bigg) \end{array}$$

taken in the ∞-category $\operatorname{CAlg}(\operatorname{Fil}\mathcal{D}(A[\![q-1]\!]))$ of filtered $\mathbb{E}_\infty$-algebras in the derived ∞-category $\mathcal{D}(A[\![q-1]\!])$. Here $\operatorname{Fil}^*_{(\mathrm{Hdg},q-1)}$ denotes the combined Hodge- and $(q-1)$-adic filtration as in Construction 3.5.

The existence of the right vertical map in the diagram above follows directly from the definition of the q-Hodge filtration on $q\text{-dR}_{\widehat{S}_p/\widehat{A}_p}$. Via the pullback square from Construction A.11, we can regard $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-dR}_{R/A}$ as a filtration on the global q-de Rham complex $q\text{-dR}_{R/A}$, as the notation suggests.

As in the p-complete case, the global q-Hodge filtration can be used to construct two more objects of interest:

(b) The *q-Hodge-completed derived q-de Rham complex* $q\text{-}\widehat{\mathrm{dR}}_{R/A}$ is the completion of $q\text{-dR}_{R/A}$ at the q-Hodge filtration.

(c) The *derived q-Hodge complex of R over A* is the $\mathbb{E}_\infty$-$A[\![q-1]\!]$-algebra

$$q\text{-Hdg}_{R/A} \coloneqq \operatorname{colim}\Big(\operatorname{Fil}^0_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/A} \xrightarrow{(q-1)} \operatorname{Fil}^1_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{R/A} \xrightarrow{(q-1)} \cdots\Big)^\wedge_{(q-1)}$$

3.38. Remark. — As in Remark 3.7, it doesn't matter whether we use $q\text{-dR}_{R/A}$ or its q-Hodge completion $q\text{-}\widehat{\mathrm{dR}}_{R/A}$ in Construction 3.37(c).

3.39. Lemma. — *Let A and R be as in Construction 3.37. If, for all primes p, the q-Hodge filtration on $q\text{-dR}_{\widehat{R}_p/\widehat{A}_p}$ is a q-deformation of the Hodge filtration on $\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}$ (for example,* *this is true if each $\widehat{R}_p$ satisfies one of the two conditions of Theorem 3.10), then the q-Hodge filtration on $q\text{-}\mathrm{dR}_{R/A}$ is a q-deformation of the Hodge filtration on $\mathrm{dR}_{R/A}$.*

Proof. Upon reducing modulo $(q-1)$ in $\mathrm{Fil}(\mathcal{D}(A[\![q-1]\!]))$ (or in $\mathrm{Fil}\,\mathrm{DAlg}_{A[\![q-1]\!]}$), the pullback square from Construction 3.37(*a*) becomes the usual arithmetic fracture square for the Hodge filtration on the derived de Rham complex $\mathrm{dR}_{R/A}$. □

3.40. Remark. — It's straightforward to show that the global q-Hodge filtration satisfies the obvious analogues of Lemmas 3.33 and 3.34. Furthermore, we can construct a conjugate filtration on $q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$ in the same way as in 3.35 (except that the localisation won't be p-completed), and then the same argument as in Lemma 3.36 shows

$$\mathrm{gr}_n^{\mathrm{conj}}\big(q\text{-}\mathrm{Hdg}_{R/A}/(q-1)\big) \simeq \mathrm{dR}_{R/A}^n \simeq \Sigma^{-n} \bigwedge^n \mathrm{L}_{R/A}\,,$$

provided the q-Hodge filtration is a q-deformation of the Hodge filtration.

3.41. Upgrade to derived commutative $A[\![q-1]\!]$-algebras. — It's straightforward to lift $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ functorially to a $(q-1)$-complete object in the ∞-*category of filtered derived commutative $A[\![q-1]\!]$-algebras* $\mathrm{Fil}\,\mathrm{DAlg}_{A[\![q-1]\!]}$, as defined in [Rak21, Construction 4.3.4]. Here $(q-1)$ sits in filtration degree 1, as usual.

Indeed, as in A.12, it'll be enough to lift the three components of the pullback from Construction 3.37(*a*) to $\mathrm{Fil}\,\mathrm{DAlg}_{A[\![q-1]\!]}$. For all p, $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}$ is just an ordinary ring together with a descending filtration by ideals, so there's a unique way to promote it to an object of $\mathrm{Fil}\,\mathrm{DAlg}_{A[\![q-1]\!]}$. The same reasoning applies for each $\mathrm{Fil}^*_{(\mathrm{Hdg},q-1)}(\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$. Finally, to equip $\mathrm{Fil}^*_{(\mathrm{Hdg},q-1)}(\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$ with such a structure, we can first reduce to the case of smooth A-algebras S (and then pass to animations). In the smooth case, we can use a cosimplicial argument as in A.12 to reduce once again to the case of ordinary rings equipped with a descending filtration by ideals.

This argument also provides a functorial lift of $q\text{-}\mathrm{Hdg}_{R/A}$ to a $(q-1)$-complete object in derived commutative $A[\![q-1]\!]$-algebras.

§4. Refined TC^- of $\mathbb{Q}$

In this section, we'll compute $\pi_* \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\pi_* \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$. In §4.1, we explain what one has to do to compute $\mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)$ for any $\mathbb{E}_\infty$-ring spectrum k. In §4.2 we'll show how to do these computations for $k=\mathrm{ku}$ or $k=\mathrm{KU}$. The computation will make heavy use of the q-Hodge filtration on derived q-de Rham complexes, even though it is no longer present in the final result.

Throughout §4 and §5, we'll often use the notion of *killing a pro-idempotent algebra object*. This is a variant of the usual notion of killing idempotent algebras and we'll review it in §2.3.

§4.1. A recipe for computing $\mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)$

Fix an $\mathbb{E}_\infty$-ring spectrum k. Let $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}\subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathbb{S}}$ denote the ∞-category of presentable stable ∞-categories and left adjoint functors as well as its non-full sub-∞-category spanned by compactly generated stable ∞-categories and compact objects-preserving functors. Let $\mathrm{Cat}^{\mathrm{perf}}_{\mathbb{S}}$ be the ∞-category of small idempotent-complete stable ∞-categories. It's well-known that passing to ind-∞-categories and restricting to compact objects provides inverse equivalences

$$\mathrm{Ind}\colon \mathrm{Cat}^{\mathrm{perf}}_{\mathbb{S}} \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega} :(-)^\omega$$

and we'll freely pass back and forth between them. Let's also fix the following:

4.1. Compatible $\mathbb{E}_1$-structures on $\mathbb{S}/m$. — According to [Bur22, Theorem 1.5], for every prime $p\geqslant 3$ there exists a tower of $\mathbb{E}_1$-algebras $(\cdots\to\mathbb{S}/p^4\to\mathbb{S}/p^3\to\mathbb{S}/p^2)$. For $p=2$ there's a similar tower $(\cdots\to\mathbb{S}/4^4\to\mathbb{S}/4^3\to\mathbb{S}/4^2)$. Let's fix such a tower for every prime. For convenience, let's call a positive integer m *high-powered* if in its prime factorisation $m=\prod_p p^{\alpha_p}$ we have either $\alpha_p=0$ or $\alpha_p\geqslant 2$ for all odd primes and either $\alpha_2=0$ or α_2 is even and $\geqslant 4$. We let $\mathbb{N}^{\sharp}$ denote the set of high-powered positive integers, partially ordered by divisibility. Through our choice of tower for every prime, we obtain a preferred $\mathbb{E}_1$-algebra structure on

$$\mathbb{S}/m\simeq\prod_p \mathbb{S}/p^{\alpha_p}$$

for every $m\in\mathbb{N}^{\sharp}$. These assemble into a functor $\mathbb{S}/-:(\mathbb{N}^{\sharp})^{\mathrm{op}}\to\mathrm{Alg}(\mathrm{Sp})$.

4.2. Lemma. — *The pro-system* “$\lim$”$_{m\in(\mathbb{N}^{\sharp})^{\mathrm{op}}}\mathbb{S}/m$ *is not only pro-idempotent* Sp, *but also in* $\mathrm{Alg}(\mathrm{Sp})$. *In particular,* “$\lim$”$_{m\in(\mathbb{N}^{\sharp})^{\mathrm{op}}}\mathrm{THH}((k\otimes\mathbb{S}/m)/k)$ *is pro-idempotent in* $\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$.

Proof. Let $A\coloneqq$ “$\lim$”$_{m\in(\mathbb{N}^{\sharp})^{\mathrm{op}}}\mathbb{S}/m$. We have a unit map $A\simeq\mathbb{S}\otimes A\to A\otimes A$. Since, again by Burklund's result, for all $m\in\mathbb{N}^{\sharp}$ the $\mathbb{E}_1$-structures on $\mathbb{S}/m^2$ refine to $\mathbb{E}_2$-structures in a compatible way, we also get a multiplication map $A\otimes A\to A$ in $\mathrm{Pro}(\mathrm{Alg}(\mathrm{Sp}))$. The composition $A\to A\otimes A\to A$ is clearly the identity. To see that $A\otimes A\to A\to A\otimes A$ is the identity as well, we use Corollary 2.20 to see that $\mathbb{S}/m^6\otimes\mathbb{S}/m^2\to\mathbb{S}/m^4\otimes\mathbb{S}/m^2$ factors through $\mathbb{S}/m^2\to\mathbb{S}/m^4\otimes\mathbb{S}/m^2$ for every $m\in\mathbb{N}^{\sharp}$. □

Thanks to Lemma 4.2, we can now formulate an explicit description of $\mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)$.

4.3. Theorem. — $\mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)$ *is an idempotent* $\mathbb{E}_\infty$*-algebra in* $\mathrm{Nuc}(\mathrm{Ind}\,\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1})$. *Its underlying* $\mathbb{E}_\infty$*-algebra in* $\mathrm{Ind}(\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1})$ *is obtained by killing the pro-idempotent algebra*

"$\lim$"$_{m\in(\mathbb{N}^{\natural})^{\mathrm{op}}}\mathrm{THH}((k\otimes\mathbb{S}/m)/k)$. *In particular, there exists a cofibre sequence of nuclear S^1-equivariant k-module spectra*

$$\text{“}\underset{m\in\mathbb{N}^{\natural}}{\mathrm{colim}}\text{”}\,\mathrm{THH}\big((k\otimes\mathbb{S}/m)/k\big)^{\vee}\longrightarrow k\longrightarrow \mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)\,,$$

where $(-)^{\vee}\coloneqq \mathrm{Hom}_{\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}}(-,k)$ *denotes the dual in S^1-equivariant k-module spectra.*

Efimov has explained several computations of refined invariants in various talks; see for example [Efi24, Talk 6]. For instance, to compute $\mathrm{HC}^{-,\mathrm{ref}}(\mathbb{Q}[x^{\pm1}]/\mathbb{Q}[x])$, one has to "cut away" the ∞-category of x^{∞}-torsion objects in $\mathcal{D}(\mathbb{Q}[x])$. The x^{∞}-torsion objects can be resolved by the system "$\lim$"$_{n\geqslant 1}\mathcal{D}^b_{\mathrm{coh}}(\mathbb{Q}[x]/x^n)$, which leads to a similar cofibre sequence as we claim in Theorem 4.3. One crucial input for this approach to work is that the transition functors $\mathcal{D}^b_{\mathrm{coh}}(\mathbb{Q}[x]/x^m)\rightarrow\mathcal{D}^b_{\mathrm{coh}}(\mathbb{Q}[x]/x^n)$ become trace-class for $m\geqslant 2n$.

Our strategy to prove Theorem 4.3 is entirely analogous: Let $\mathrm{Sp}_{\mathbb{Q}}$ denote the ∞-category of rational spectra and let $\mathrm{Sp}^{\mathrm{tors}}\subseteq\mathrm{Sp}$ be the kernel of $\mathrm{Sp}\rightarrow\mathrm{Sp}_{\mathbb{Q}}$. Applying $\mathrm{THH}^{\mathrm{ref}}(k\otimes-/k)$ to the cofibre sequence $\mathrm{Sp}^{\mathrm{tors}}\rightarrow\mathrm{Sp}\rightarrow\mathrm{Sp}_{\mathbb{Q}}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$ yields a cofibre sequence

$$\mathrm{THH}^{\mathrm{ref}}(\mathrm{Sp}^{\mathrm{tors}})\otimes k\longrightarrow k\longrightarrow\mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)$$

in $\mathrm{Nuc}(\mathrm{Ind}\,\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1})$. The goal is then to identify the left term in this cofibre sequence with the corresponding term in Theorem 4.3. To do that, we'll resolve $\mathrm{Sp}^{\mathrm{tors}}$ by "$\lim$"$_{m\in\mathbb{N}^{\natural}}\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega})$, analogous to what Efimov does with "$\lim$"$_{n\geqslant1}\mathcal{D}^b_{\mathrm{coh}}(\mathbb{Q}[x]/x^n)$. Again, there will be a trace-class property to check.

It's clear that for all $m\in\mathbb{N}^{\natural}$ the forgetful functor $\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega})\rightarrow\mathrm{Sp}^{\omega}$ lands in $\mathrm{Sp}^{\omega}\cap\mathrm{Sp}^{\mathrm{tors}}$. Hence it extends to a functor $L_m\colon\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega}))\rightarrow\mathrm{Sp}^{\mathrm{tors}}$.

4.4. Lemma. — *The induced functor*

$$L\colon\underset{m\in\mathbb{N}^{\natural}}{\mathrm{colim}}\,\mathrm{Ind}\big(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega})\big)\xrightarrow{\ \simeq\ }\mathrm{Sp}^{\mathrm{tors}}$$

is an equivalence $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S}}$. Here the colimit on the left-hand side is taken along the forgetful functors $\mathrm{LMod}_{\mathbb{S}/d}(\mathrm{Sp}^{\omega})\rightarrow\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega})$ for all $d,m\in\mathbb{N}^{\natural}$ such that d divides m.

Proof. The presentable ∞-category $\mathrm{Sp}^{\mathrm{tors}}$ is compactly generated, with a generating set given by $\{\Sigma^n\mathbb{S}/m\}_{m\in\mathbb{N}^{\natural},n\in\mathbb{Z}}$. Clearly, each $\Sigma^n\mathbb{S}/m$ is in the image of $\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega})\rightarrow\mathrm{Sp}^{\mathrm{tors}}$, hence L is essentially surjective.

To show that L is fully faithful, let R denote its right adjoint. We'll verify that the unit $u\colon\mathrm{id}\Rightarrow R\circ L$ is an equivalence. By construction, $\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega}))\rightarrow\mathrm{Sp}^{\mathrm{tors}}$ and $\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/d}(\mathrm{Sp}^{\omega}))\rightarrow\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega}))$ preserve compact objects. Since the inclusion $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}\subseteq\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S}}$ preserves colimits, we conclude that L preserves compact objects too. Hence R preserves filtered colimits. Therefore it's enough to check that $u_M\colon M\rightarrow RL(M)$ is an equivalence for M ranging through a set of compact generators of $\mathrm{colim}_{m\in\mathbb{N}^{\natural}}\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega}))$. We may thus restrict to the case $M\in\mathrm{LMod}_{\mathbb{S}/m_0}(\mathrm{Sp}^{\omega})$ for some $m_0\in\mathbb{N}^{\natural}$. For all $m\in\mathbb{N}^{\natural}$, the functor $L_m\colon\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^{\omega}))\rightarrow\mathrm{Sp}^{\mathrm{tors}}$ has a right adjoint R_m. It follows formally that

$$RL(M)\simeq\underset{m\in\mathbb{N}^{\natural},\,m_0|m}{\mathrm{colim}}R_mL_m(M)\,.$$

We compute that $R_m L_m(M) \simeq \mathrm{Hom}_{\mathbb{S}}(\mathbb{S}/m, M) \simeq \mathrm{Hom}_{\mathbb{S}/m}((\mathbb{S}/m)/m, M) \simeq \Sigma^{-1}M/m$ holds in $\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega))$. Since M is a left $\mathbb{S}/m_0$-module the multiplication $m_0\colon M \to M$ is zero. It follows formally that $M \simeq \operatorname{colim}_{m \in \mathbb{N}^{\natural},\, m_0 | m} \Sigma^{-1}M/m$ holds in $\operatorname{colim}_{m\in\mathbb{N}^{\natural}} \mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega))$ and so $u_M\colon M \to RL(M)$ is indeed an equivalence. □

4.5. Lemma. — *For every fixed $m_0 \in \mathbb{N}^{\natural}$, the base change functor*

$$- \otimes_{\mathbb{S}/m} \mathbb{S}/m_0 \colon \mathrm{RMod}_{\mathbb{S}/m}(\mathrm{Sp}) \longrightarrow \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp})$$

is trace-class in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$ for all sufficiently large $m \in \mathbb{N}^{\natural}$.

Proof. Recall from [L-HA, Remark 4.8.4.8] that $\mathrm{RMod}_{\mathbb{S}/m}(\mathrm{Sp})$ is dualisable in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S}}$ with dual $\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp})$. Therefore, the base change functor is always trace-class in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S}}$. The witnessing functor $\mathrm{Sp} \to \mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}) \otimes \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp}) \simeq \mathrm{RMod}_{\mathbb{S}/m^{\mathrm{op}} \otimes \mathbb{S}/m_0}(\mathrm{Sp})$ is the classifier of $\mathbb{S}/m_0$ as a right module over $\mathbb{S}/m^{\mathrm{op}} \otimes \mathbb{S}/m_0$, or equivalently, a $\mathbb{S}/m$-$\mathbb{S}/m_0$-bimodule. If we work in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$, then $\mathrm{RMod}_{\mathbb{S}/m}(\mathrm{Sp})$ will no longer be dualisable, but we can still form

$$\underline{\mathrm{Hom}}_{\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}}\big(\mathrm{RMod}_{\mathbb{S}/m}(\mathrm{Sp}), \mathrm{Sp}\big) \simeq \mathrm{Ind}\big(\mathrm{Fun}_{\mathbb{S}}\big(\mathrm{RMod}_{\mathbb{S}/m}(\mathrm{Sp})^\omega, \mathrm{Sp}^\omega\big)\big) \simeq \mathrm{Ind}\big(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega)\big)\,,$$

where we've used [L-HA, Theorem 4.8.4.1]. Using [L-HA, Theorem 4.8.4.6], we still have a functor in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S}}$,

$$\mathrm{Sp} \longrightarrow \mathrm{Ind}\big(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega)\big) \otimes \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp}) \simeq \mathrm{RMod}_{\mathbb{S}/m_0}\big(\mathrm{Ind}\big(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega)\big)\big)\,,$$

that classifies $\mathbb{S}/m_0$ has a right $\mathbb{S}/m_0$-module in $\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}))$. For the desired trace-class property to hold, this functor needs to be contained in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$. That is, we need $\mathbb{S}/m_0$ to be a compact object in $\mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega)))$. To this end, it will be enough that $\mathbb{S}/m_0$ is a retract of $\mathbb{S}/m_0 \otimes \mathbb{S}/m_0$. Indeed, the object $\mathbb{S}/m_0 \in \mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega))$ is compact[(4.1)] and so $\mathbb{S}/m_0 \otimes \mathbb{S}/m_0$ must be compact in $\mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega)))$.

Consider the multiplications $\mu_1\colon \mathbb{S}/m_0 \otimes \mathbb{S}/m_0 \to \mathbb{S}/m_0$ and $\mu_2\colon \mathbb{S}/m_0^2 \otimes \mathbb{S}/m_0 \to \mathbb{S}/m_0$. It will be enough to show that the canonical map $\mathrm{fib}(\mu_2) \to \mathrm{fib}(\mu_1)$ vanishes as a map of right $\mathbb{S}/m^{\mathrm{op}} \otimes \mathbb{S}/m_0$-modules for sufficiently large m, because then $\mathbb{S}/m_0^2 \otimes \mathbb{S}/m_0 \to \mathbb{S}/m_0 \otimes \mathbb{S}/m_0$ factors through μ_2, which exhibits $\mathbb{S}/m_0$ as the desired retract. As right modules over $\mathbb{S}/m_0$ we have $\mathrm{fib}(\mu_1) \simeq \Sigma^{-1}\mathbb{S}/m_0 \simeq \mathrm{fib}(\mu_2)$ and the canonical map $\mathrm{fib}(\mu_1) \to \mathrm{fib}(\mu_2)$ is multiplication by m_0. Hence it vanishes in $\mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp}) \simeq \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp}^{\mathrm{tors}}) \simeq \mathrm{Sp}^{\mathrm{tors}} \otimes \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp})$. Using Lemma 4.4 and the fact that the Lurie tensor product commutes with colimits in either factor by [L-HA, Remark 4.8.1.24], we have

$$\mathrm{Sp}^{\mathrm{tors}} \otimes \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp}) \simeq \operatorname*{colim}_{m \in \mathbb{N}^{\natural}} \Big(\mathrm{Ind}\big(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega)\big) \otimes \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp})\Big)\,.$$

Hence $\mathrm{fib}(\mu_2) \to \mathrm{fib}(\mu_1)$ must already vanish in $\mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp}^\omega)) \otimes \mathrm{RMod}_{\mathbb{S}/m_0}(\mathrm{Sp})$ for sufficiently large m, which is what we wanted to show. □

4.6. Remark. — The implicit bounds in Lemma 4.5 can be made explicit; see Corollary 2.21. This crucially uses that the $\mathbb{E}_1$-structures on $\mathbb{S}/m$ are constructed Burklund-style, as recalled in 4.1. By contrast, the proofs of Lemmas 4.4 and 4.5 work for any choice of compatible $\mathbb{E}_1$-structures on $\mathbb{S}/m$, $m \in \mathbb{N}^{\natural}$, and they can be carried over to other bases than $\mathbb{S}$, where

(4.1) By contrast, $\mathbb{S}/m_0$ is usually *not* compact in $\mathrm{LMod}_{\mathbb{S}/m}(\mathrm{Sp})$.

such compatible systems might be easier to construct. For example, over MU (or ku, KU, ...) there's another construction due to Jeremy Hahn of a compatible system of $\mathbb{E}_1$-MU-algebras MU/m, $m \in \mathbb{N}$.

Proof of Theorem 4.3. Fix a coinitial subcategory $\{m_0 \leftarrow m_1 \leftarrow \cdots\} \subseteq \mathbb{N}^{\natural}$ such that the base change functors $-\otimes_{\mathbb{S}/m_{i+1}} \mathbb{S}/m_i \colon \mathrm{RMod}_{\mathbb{S}/m_{i+1}}(\mathrm{Sp}) \to \mathrm{RMod}_{\mathbb{S}/m_i}(\mathrm{Sp})$ are trace-class in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$ for all $i \geqslant 0$. Such a subcategory exists by Lemma 4.5. Combining Lemmas 4.4 and 2.7(*b*), we see that $\mathrm{Sp}^{\mathrm{tors}} \simeq \operatorname{colim}_{i\geqslant 0} \mathrm{Ind}(\mathrm{LMod}_{\mathbb{S}/m_i}(\mathrm{Sp}^\omega))$ is a resolution in $\mathrm{Pr}^{\mathrm{L}}_{\mathbb{S},\omega}$ with trace-class transition maps. It follows that the underlying object of $\mathrm{Sp}_\mathbb{Q}$ in $\mathrm{Mot}^{\mathrm{loc}}_{\mathbb{S}} \simeq \mathrm{Nuc}(\mathrm{Mot}^{\mathrm{loc}}_{\mathbb{S}})$ is the idempotent nuclear $\mathbb{E}_\infty$-algebra obtained via Lemma 2.15(*b*) by killing the pro-idempotent algebra "$\lim_i$" $\mathrm{RMod}_{\mathbb{S}/m_i}(\mathrm{Sp})$. Since idempotents admit a unique $\mathbb{E}_\infty$-algebra structure, this must also be true as $\mathbb{E}_\infty$-algebras.

By Lemma 2.15(*c*), the symmetric monoidal functor

$$\mathrm{THH}^{\mathrm{ref}}\big((k\otimes -)/k\big)\colon \mathrm{Mot}^{\mathrm{loc}}_{\mathbb{S}} \to \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$$

preserves killing pro-idempotents with trace-class transition maps. This shows the desired description of $\mathrm{THH}^{\mathrm{ref}}((k\otimes\mathbb{Q})/k)$. ☐

§4.2. The cases $k=\mathrm{ku}$ and $k=\mathrm{KU}$

If k is a complex orientable $\mathbb{E}_\infty$-ring spectrum, then computing $\mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)$ together with its S^1-action is equivalent to computing $\mathrm{TC}^{-,\mathrm{ref}}(k\otimes\mathbb{Q}/k)$. This is due to the following lemma.

4.7. Lemma. — *If k is a complex orientable $\mathbb{E}_\infty$-ring spectrum, equipped with trivial S^1-action, then taking S^1-fixed points defines a symmetric monoidal equivalence*

$$(-)^{\mathrm{h}S^1}\colon \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1} \xrightarrow{\simeq} \mathrm{Mod}_{k^{\mathrm{h}S^1}}(\mathrm{Sp})^\wedge_t\,.$$

Here $t\in\pi_{-2}(k^{\mathrm{h}S^1})$ denotes a chosen complex orientation and $\mathrm{Mod}_{k^{\mathrm{h}S^1}}(\mathrm{Sp})^\wedge_t$ denotes the symmetric monoidal ∞-category of t-complete $k^{\mathrm{h}S^1}$-module spectra together with the t-completed tensor product $-\mathbin{\widehat{\otimes}}_{k^{\mathrm{h}S^1}}-$.

Proof. By construction $(-)^{\mathrm{h}S^1}$ is lax symmetric monoidal. To see that it is strictly symmetric monoidal, we must check whether $M^{\mathrm{h}S^1}\mathbin{\widehat{\otimes}}_{k^{\mathrm{h}S^1}} N^{\mathrm{h}S^1} \to (M\otimes_k N)^{\mathrm{h}S^1}$ is an equivalence. As both sides are t-complete, this can be checked modulo t, where it follows from [HRW22, Lemma 2.2.10] for example.

For formal reasons, $(-)^{\mathrm{h}S^1}$ admits a left adjoint L which can be described as follows: We first have a symmetric monoidal functor $\mathrm{const}\colon \mathrm{Sp}\to\mathrm{Sp}^{\mathrm{B}S^1}$, sending a spectrum X to itself equipped with the trivial S^1-action. It induces a functor $\mathrm{Mod}_{k^{\mathrm{h}S^1}}(\mathrm{Sp}) \to \mathrm{Mod}_{k^{\mathrm{h}S^1}}(\mathrm{Sp}^{\mathrm{B}S^1})$. Then L is given as the composition of this functor with the base change

$$-\otimes_{k^{\mathrm{h}S^1}} k\colon \mathrm{Mod}_{k^{\mathrm{h}S^1}}\big(\mathrm{Sp}^{\mathrm{B}S^1}\big) \longrightarrow \mathrm{Mod}_k\big(\mathrm{Sp}^{\mathrm{B}S^1}\big)\simeq \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}\,.$$

In particular, on underlying k-modules, L is simply given by $(-)/t$. Since $(-)/t$ is conservative on t-complete $k^{\mathrm{h}S^1}$-modules, it follows that L must be conservative too. Furthermore, the counit $c\colon L((-)^{\mathrm{h}S^1})\Rightarrow \mathrm{id}$ is an equivalence, as follows from [HRW22, Lemma 2.2.10]. Thus $(-)^{\mathrm{h}S^1}$ must be fully faithful. We conclude using the standard fact that an adjunction in which the right adjoint is fully faithful and the left adjoint is conservative must be a pair of inverse equivalences. ☐

We'll now explain the computation of $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$.

4.8. Battle plan. — For ease of notation, in the following we'll write $\mathrm{ku}/m \coloneqq \mathrm{ku}\otimes\mathbb{S}/m$ and $\mathrm{KU}/m \coloneqq \mathrm{KU}\otimes\mathbb{S}/m$, where it is understood that the $\mathbb{E}_1$-structure is always base changed from $\mathbb{S}/m$. By Theorem 4.3 and Lemma 4.7, there exists a cofibre sequence

$$\underset{m\in\mathbb{N}^\natural}{\text{“colim”}} \operatorname{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)^\vee \longrightarrow \mathrm{ku}^{\mathrm{h}S^1} \longrightarrow \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$$

(where now $(-)^\vee \coloneqq \operatorname{Hom}_{\mathrm{ku}^{\mathrm{h}S^1}}(-,\mathrm{ku}^{\mathrm{h}S^1})$ denotes the dual in $\mathrm{ku}^{\mathrm{h}S^1}$-modules) and a similar one for KU. We will thus proceed in three steps:

(a) We compute the homotopy groups of $\operatorname{TC}^-((\mathrm{ku}\otimes\mathbb{S}/m)/\mathrm{ku})$ and $\operatorname{TC}^-((\mathrm{KU}\otimes\mathbb{S}/m)/\mathrm{KU})$ using Theorem 3.22. This will be achieved in Corollary 4.14.

(b) We compute the homotopy groups of the dual modules $\operatorname{TC}^-((\mathrm{ku}\otimes\mathbb{S}/m)/\mathrm{ku})^\vee$ and $\operatorname{TC}^-((\mathrm{KU}\otimes\mathbb{S}/m)/\mathrm{KU})^\vee$. This leads to a preliminary description of the homotopy rings $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$ in Theorem 4.20.

(c) We derive the simpler description of Theorem 1.19 via a careful analysis of q-Hodge filtrations. This will be the content of §5.

4.9. Reduction to the p-complete case. — Decomposing $m = \prod_p p^{\alpha_p}$ into prime powers, we have

$$\operatorname{TC}^-\big((\mathrm{ku}\otimes\mathbb{S}/m)/\mathrm{ku}\big) \simeq \prod_p \operatorname{TC}^-\big((\mathrm{ku}\otimes\mathbb{S}/p^{\alpha_p})/\mathrm{ku}\big)\,,$$

so we may reduce to the case where $m = p^\alpha$ is a high-powered prime power. Let us also remark that $\operatorname{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ is automatically p-complete. Indeed, it is (β,t)-complete and $\operatorname{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})/(\beta,t) \simeq \operatorname{HH}((\mathbb{Z}/p^\alpha)/\mathbb{Z})$ is p^α-torsion, hence p-complete.

4.10. Reduction to the p-torsion free case. — Now let $\mathbb{Z}_p\{x\}_\infty$ be the free p-complete perfect δ-ring on a generator x and let $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}$ be its unique lift to a p-complete $\mathbb{E}_\infty$-ring spectrum (see 3.11). By [Bur22, Theorem 1.5], there exists a tower of $\mathbb{E}_1$-algebras in $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}$-modules

$$\Big(\cdots \longrightarrow \mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^4 \longrightarrow \mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^3 \longrightarrow \mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^2\Big)$$

for $p \geqslant 3$; the case $p=2$ needs powers of x^2 instead.

The unique map of perfect δ-rings $\mathbb{Z}_p\{x\}_\infty \to \mathbb{Z}_p$ sending $x \mapsto p$ lifts uniquely to an $\mathbb{E}_\infty$-map $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty} \to \mathbb{S}_p$ and we're free to choose our tower of $\mathbb{E}_1$-algebras $(\mathbb{S}/p^\alpha)_{\alpha\geqslant 2}$ in such a way that it arises via base change from the tower $(\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^\alpha)_{\alpha\geqslant 2}$ above.[(4.2)] Putting $\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty} \coloneqq (\mathrm{ku}\otimes\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty})_p^\wedge$, we can now compute $\operatorname{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ via base change. Taking into account that $\operatorname{TC}^-(-/\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty})_p^\wedge \simeq \operatorname{TC}^-(-/\mathrm{ku})_p^\wedge$ (see the argument before Lemma 3.27), we obtain

$$\operatorname{TC}^-\big((\mathrm{ku}/p^\alpha)/\mathrm{ku}\big) \simeq \operatorname{TC}^-\big((\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty}/x^\alpha)/\mathrm{ku}\big)\,\widehat{\otimes}_{\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty}} \mathrm{ku}_p^\wedge\,,$$

where the tensor product is (p,t)-completed.

(4.2) In fact, if we use Burklund's construction, this will be automatically satisfied. Indeed, with notation as in Burklund's paper, there is a *unique* morphism $\nu\mathbb{1}_\mathcal{C}/\widetilde{v}^m \to \nu\mathbb{1}_\mathcal{C}/\widetilde{v}^n$ of $\mathbb{E}_1$-algebras in $\operatorname{Def}(\mathcal{C},\mathcal{Q})$ for all $m > n \geqslant 2$. $v\colon \mathcal{I} \to \mathbb{1}_\mathcal{C}$. This is because [Bur22, Lemma 4.8] guarantees that $\pi_* \operatorname{Hom}_{\operatorname{Def}(\mathcal{C},\mathcal{Q})}(\nu((\Sigma\mathcal{I})^{\otimes mk}),\nu\mathbb{1}_\mathcal{C}/\widetilde{v}^n)$ vanishes not only in degree $* = (2-m)k-2$, which guarantees vanishing of the relevant obstructions, but for $m > n$ the homotopy groups also vanish in degree $* = (2-m)k-1$, which guarantees uniqueness of all nullhomotopies.

4.11. Construction. — The considerations in 4.10 suggest to define a q-Hodge filtration on the q-de Rham complex of $\mathbb{Z}/p^\alpha$ via base change from $\mathbb{Z}_p\{x\}_\infty/x^\alpha$. By Lemma 3.16, we may as well base change the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}}$. Concretely, this leads to the following definition:

(*a*) The *q-Hodge filtration on* $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is

$$\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \coloneqq \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}_p\{x\}[\![q-1]\!]} \mathbb{Z}_p[\![q-1]\!]\,.$$

We remark that the filtered object $\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is degreewise static with injective transition maps. Indeed, this can be checked modulo $(q-1)$ (in the filtered sense, with $(q-1)$ sitting in filtration degree 1) and we have $(\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p})/(q-1) \simeq \mathrm{Fil}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ by construction.

As in the p-torsion free case, we will also need to consider the following two related constructions:

(*b*) The *q-Hodge-completed derived q-de Rham complex* $q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is the completion of $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ at the q-Hodge filtration.

(*c*) The *derived q-Hodge complex* $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is defined as

$$\operatorname{colim}\Bigl(\mathrm{Fil}^0_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \xrightarrow{(q-1)} \mathrm{Fil}^1_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \xrightarrow{(q-1)} \cdots\Bigr)^\wedge_{(p,q-1)}$$

Observe that the un-completed colimit contains an element $p^\alpha/(q-1)$, hence it would have been enough to just complete at $(q-1)$.

4.12. Remark. — A priori, Construction 4.11 depends on the choice of writing $\mathbb{Z}/p^\alpha$ as the base change $\mathbb{Z}_p\{x\}/x^\alpha \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}_p\{x\}} \mathbb{Z}_p$, but it turns out that the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is, at least in a suitable sense, canonical:

4.13. Lemma. — *If $A \to \mathbb{Z}_p$ is any map from a p-completely perfectly covered δ-ring and R is a p-torsion free A-algebra which admits a perfect-regular presentation $R \cong B/J$, where the ideal $J = (x_1^{\alpha_1}, \dots, x_r^{\alpha_r})$ is generated by a Koszul-regular sequence of higher powers, then for any map of A-algebras $R \to \mathbb{Z}/p^\alpha$, the induced map*

$$q\text{-}\mathrm{dR}_{R/A} \longrightarrow q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$$

is compatible with q-Hodge filtrations. Moreover, the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is the smallest mutliplicative filtration with this property.

Proof. We can first use Lemma 3.16(*b*) to reduce to the universal case $A = \mathbb{Z}_p\{x_1, \dots, x_r\}$, $R = \mathbb{Z}_p\{x_1, \dots, x_r\}/(x_1^{\alpha_1}, \dots, x_r^{\alpha_r})$ and then Lemma 3.34(*b*) to reduce to $r = 1$. Suppose $\mathbb{Z}_p\{x_1\} \to \mathbb{Z}_p$ sends $x_1 \mapsto ap^\beta$, where $(a, p) = 1$. In order to have a map $\mathbb{Z}_p\{x_1\}/x_1^{\alpha_1} \to \mathbb{Z}/p^\alpha$, we must have $\alpha_1\beta \geqslant \alpha$. Then the map $\mathbb{Z}_p\{x_1\} \to \mathbb{Z}_p\{x\}$ sending $x_1 \mapsto ax^\beta$ induces a map $\mathbb{Z}_p\{x_1\}/x_1^{\alpha_1} \to \mathbb{Z}_p\{x\}/x^\alpha$ and so the desired compatibility of q-Hodge filtrations follows from functoriality in the p-torsion free case.

This shows that for $(A, R) \to (\mathbb{Z}_p, \mathbb{Z}/p^\alpha)$ as above, $q\text{-}\mathrm{dR}_{R/A} \to q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ preserves q-Hodge filtrations. Since the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is basechanged from such a map, it must automatically be minimal with this property. $\square$

It would be desirable to get compatibility also for A-algebras R that satisfy the condition from Theorem 3.10(*b*), but the author doesn't know how to do that without additionally assuming existence of an $\mathbb{E}_1$-algebra map $\mathbb{S}_{R_\infty} \to \mathbb{S}/p^\alpha$. It would also be nice to generalise the argument above to more rings than just $\mathbb{Z}/p^\alpha$, ideally removing the p-torsion freeness assumption from Theorem 3.10(*a*), but again, the author doesn't know how to do that in general. The argument above crucially needs that we have strict control over the nilpotent elements in $\mathbb{Z}/p^\alpha$.

The upshot of 4.9, 4.10, and Construction 4.11 is the following.

4.14. Corollary. — *For $p \geqslant 3$ and $\alpha \geqslant 2$, or $p = 2$ and α is even and $\geqslant 4$, the spectra* $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ *and* $\mathrm{TC}^-((\mathrm{KU}/p^\alpha)/\mathrm{KU})$ *are concentrated in even degrees and we have*

$$\pi_{2*}\,\mathrm{TC}^-\big((\mathrm{ku}/p^\alpha)/\mathrm{ku}\big) \cong \mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\,,$$
$$\pi_{2*}\,\mathrm{TC}^-\big((\mathrm{KU}/p^\alpha)/\mathrm{KU}\big) \cong q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}[\beta^{\pm 1}]\,.$$

Proof. It's enough to check evenness modulo t, so we may pass from TC^- to THH. Since $\mathrm{THH}((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ is connective, we may further pass to $\mathrm{THH}((\mathrm{ku}/p^\alpha)/\mathrm{ku})/\beta \simeq \mathrm{HH}((\mathbb{Z}/p^\alpha)/\mathbb{Z})$, which is indeed even. This shows evenness for $\mathrm{THH}((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ and then the same follows for $\mathrm{THH}((\mathrm{ku}/p^\alpha)/\mathrm{ku})[1/\beta] \simeq \mathrm{THH}((\mathrm{KU}/p^\alpha)/\mathrm{KU})$.

Thanks to 4.10, we get a map $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \to \pi_{2*}\,\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$. Whether this is an equivalence can be checked after reducing $(q-1)$ in the filtered sense, and then we recover the well-known fact that the even homotopy groups of $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})/\beta \simeq \mathrm{HC}^-((\mathbb{Z}/p^\alpha)/\mathbb{Z})$ are $\mathrm{Fil}^*_{\mathrm{Hdg}}\, \widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. The claim that the even homotopy groups of

$$\mathrm{TC}^-\big((\mathrm{KU}/p^\alpha)/\mathrm{KU}\big) \simeq \mathrm{TC}^-\big((\mathrm{ku}/p^\alpha)/\mathrm{ku}\big)\Big[\tfrac{1}{\beta}\Big]^\wedge_t$$

are given by $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}[\beta^{\pm 1}]$ follows formally the description of $\pi_{2*}\,\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ combined with our remark at the end of Construction 4.11(*c*) that completing at $(q-1)$ is already enough. ☐

This finishes step (*a*) of our battle plan 4.8 and we move onwards to step (*b*). We start with a general fact (which is usually formulated as a spectral sequence).

4.15. Lemma. — *Let k be an even $\mathbb{E}_1$-ring spectrum and let M, N be even left-k-modules. Then the mapping spectrum $\mathrm{Hom}_k(M, N)$ admits a complete exhaustive descending filtration with graded pieces*

$$\mathrm{gr}^*\,\mathrm{Hom}_k(M,N) \simeq \Sigma^{2*}\,\mathrm{R}\underline{\mathrm{Hom}}_{\pi_{2*}(k)}\big(\pi_{2*}(M), \pi_{2*}(N)\big)\,.$$

Here $\Sigma^{2*}\colon \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$ is the "double shearing" functor and $\mathrm{R}\underline{\mathrm{Hom}}_{\pi_{2*}(k)}$ denotes the derived internal Hom in graded $\pi_{2*}(k)$-modules.

Proof. In the usual adjunction $\mathrm{colim}\colon \mathrm{Fil}(\mathrm{Sp}) \rightleftarrows \mathrm{Sp} :\!\mathrm{const}$, the left adjoint is symmetric monoidal and the right adjoint is lax symmetric monoidal. Furthermore, $\mathrm{colim}\,\tau_{\geqslant 2*}(k) \simeq k$. It follows formally that $\mathrm{colim}\colon \mathrm{LMod}_{\tau_{\geqslant 2*}(k)}(\mathrm{Fil}(\mathrm{Sp})) \rightleftarrows \mathrm{LMod}_k(\mathrm{Sp}) :\!\mathrm{const}$ is an adjunction as well and so $\mathrm{Hom}_k(M,N) \simeq \mathrm{Hom}_{\tau_{\geqslant 2*}(k)}(\tau_{\geqslant 2*}(M), \mathrm{const}\, N)$. Hence we may define the desired filration via

$$\mathrm{Fil}^n\,\mathrm{Hom}_k(M,N) \coloneqq \mathrm{Hom}_{\tau_{\geqslant 2*}(k)}\big(\tau_{\geqslant 2*}(M), \tau_{\geqslant 2(*+n)}(N)\big)\,.$$

This filtration is clearly complete since we may pull $0 \simeq \lim_{n\to\infty} \tau_{\geqslant 2(*+n)}(N)$ out of the Hom. To show that the filtration is exhaustive, we need to check that $\operatorname{const} N \simeq \operatorname{colim}_{n\to-\infty} \tau_{\geqslant 2(*+n)}(N)$ can similarly be pulled out of the Hom. To this end, recall that $\mathrm{Fil}(\mathrm{Sp})$ can be equipped with the *double Postnikov t-structure* in which objects in the image of $\tau_{\geqslant 2*}(-)$ are connective and connective objects are closed under tensor products (see [Rak21, Construction 3.3.6] for example and double everything). Then $\mathrm{Mod}_{\tau_{\geqslant 2*}(k)}(\mathrm{Fil}(\mathrm{Sp}))$ inherits a t-structure in which $\tau_{\geqslant 2*}(M)$ is connective and the cofibres of $\tau_{\geqslant 2(*+n)}(N) \to \operatorname{const} N$ get more and more coconnective as $n \to -\infty$. This shows that the colimit can be pulled out.

It remains to determine the associated graded. By construction, the n^{th} graded piece is given by $\mathrm{gr}^n \mathrm{Hom}_k(M,N) \simeq \mathrm{Hom}_{\tau_{\geqslant 2*}(k)}(\tau_{\geqslant 2*}(M), \Sigma^{2(*+n)}\pi_{2(*+n)}(N))$. To simplify this further, let $\mathbb{S}_{\mathrm{Gr}}$ and $\mathbb{S}_{\mathrm{Fil}}$ denote the tensor units in graded and filtered spectra, respectively. By abuse of notation, we identify $\mathbb{S}_{\mathrm{Fil}}$ with its underlying graded spectrum. Then [L-Rot, Proposition 3.1.6] shows $\mathrm{Fil}(\mathrm{Sp}) \simeq \mathrm{Mod}_{\mathbb{S}_{\mathrm{Fil}}}(\mathrm{Gr}(\mathrm{Sp}))$; this identifies passing to the associated graded with the base change functor $-\otimes_{\mathbb{S}_{\mathrm{Fil}}} \mathbb{S}_{\mathrm{Gr}}$. Since the $\mathbb{S}_{\mathrm{Fil}}$-module structure on $\Sigma^{2(*+n)}\pi_{2(*+n)}(N)$ already factors through $\mathbb{S}_{\mathrm{Fil}} \to \mathbb{S}_{\mathrm{Gr}}$, we obtain

$$\begin{aligned}\mathrm{Hom}_{\tau_{\geqslant 2*}(k)}\bigl(\tau_{\geqslant 2*}(M), \Sigma^{2(*+n)}\pi_{2(*+n)}(N)\bigr) &\simeq \mathrm{Hom}_{\Sigma^{2*}\pi_{2*}(k)}\bigl(\Sigma^{2*}\pi_{2*}(M), \Sigma^{2(*+n)}\pi_{2(*+n)}(N)\bigr)\\ &\simeq \Sigma^{2n}\,\mathrm{Hom}_{\pi_{2*}(k)}\bigl(\pi_{2*}(M), \pi_{2*}(N)(-n)\bigr)\,.\end{aligned}$$

The first step is the usual base change equivalence for $\tau_{\geqslant 2*}(k) \to \tau_{\geqslant 2*}(k) \otimes_{\mathbb{S}_{\mathrm{Fil}}} \mathbb{S}_{\mathrm{Gr}} \simeq \Sigma^{2*}\pi_{2*}(k)$, the second step uses that the shearing functor $\Sigma^{2*}\colon \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$ is an $\mathbb{E}_1$-monoidal equivalence (even $\mathbb{E}_2$-monoidal, see [DHL+23, Proposition 3.10], but we don't need that). Now the right-hand side is precisely the n^{th} graded piece of $\mathrm{R}\underline{\mathrm{Hom}}_{\pi_{2*}(k)}(\pi_{2*}(M), \pi_{2*}(N))$ and so we're done. ∎

We'll apply this now in the case $k \simeq \mathrm{ku}^{\mathrm{h}S^1}$, so that $\pi_{2*}(k) \cong \mathbb{Z}[\beta][\![t]\!]$. We also let $\underline{\mathrm{Ext}}^i_{\mathbb{Z}[\beta][\![t]\!]}$ denote the graded $\mathbb{Z}[\beta][\![t]\!]$-module $\mathrm{H}_{-i}\,\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[\beta][\![t]\!]}$ for all $i \geqslant 0$.

4.16. Corollary. — *The spectra* $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})^\vee$ *and* $\mathrm{TC}^-((\mathrm{KU}/p^\alpha)/\mathrm{KU})^\vee$ *are concentrated in odd degrees and we have*

$$\begin{aligned}\pi_{-(2*+1)}\,\mathrm{TC}^-\bigl((\mathrm{ku}/p^\alpha)/\mathrm{ku}\bigr)^\vee &\cong \underline{\mathrm{Ext}}^1_{\mathbb{Z}[\beta][\![t]\!]}\Bigl(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}, \mathbb{Z}[\beta][\![t]\!]\Bigr)\\ \pi_{-(2*+1)}\,\mathrm{TC}^-\bigl((\mathrm{KU}/p^\alpha)/\mathrm{KU}\bigr)^\vee &\cong \mathrm{Ext}^1_{\mathbb{Z}[\![q-1]\!]}\Bigl(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}, \mathbb{Z}[\![q-1]\!]\Bigr)[\beta^{\pm 1}]\,.\end{aligned}$$

Here $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ *is regarded as a graded module over* $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{\mathbb{Z}/\mathbb{Z}} \simeq \mathbb{Z}[\beta][\![t]\!]$.

Proof. According to Corollary 4.14 and Lemma 4.15, the spectrum $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})^\vee$ admits a complete exhaustive filtration with associated graded $\Sigma^{2*}(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p})^\vee$, where now the dual is taken in graded $\mathbb{Z}[\beta][\![t]\!]$. It'll be enough to show that this dual is concentrated in homological degree -1 (which precisely accounts for the $\underline{\mathrm{Ext}}^1_{\mathbb{Z}[\![q-1]\!][\beta^{\pm 1}]}$-terms). Since $\mathbb{Z}[\beta][\![t]\!]$ is (β,t)-complete as a graded object, the same is true for any dual in graded $\mathbb{Z}[\beta][\![t]\!]$-modules, and so it'll be enough that

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[\beta][\![t]\!]}\Bigl(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}, \mathbb{Z}[\beta][\![t]\!]\Bigr)/(\beta,t) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}\Bigl(\mathrm{gr}^*_{\mathrm{Hdg}}\, \widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}, \mathbb{Z}\Bigr)$$

is concentraded in homological degree -1. Since $\mathrm{gr}^n_{\mathrm{Hdg}}\, \widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \simeq \Sigma^{-n} \bigwedge^n \mathrm{L}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \simeq \mathbb{Z}/p^\alpha$, the n^{th} graded piece of the right-hand side is precisely $\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}/p^\alpha, \mathbb{Z})$, which is indeed concentrated in homological degree -1. This finishes the proof for $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$.

The proof for $\mathrm{TC}^-((\mathrm{KU}/p^\alpha)/\mathrm{KU})^\vee$ is analogous, except that we need a different argument to show that the dual $(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p})^\vee$ in $\mathbb{Z}[\![q-1]\!]$-modules is concentrated in homological degree -1. By $(q-1)$-completeness, it'll be enough to check the same for $\mathrm{RHom}_\mathbb{Z}(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/(q-1),\mathbb{Z})$. By 3.35 and base change we see that $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/(q-1)$ admits an exhaustive ascending filtration with associated graded given by $\mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. It follows that $\mathrm{RHom}_\mathbb{Z}(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/(q-1),\mathbb{Z})$ admits a descending filtration with associated graded $\mathrm{R}\underline{\mathrm{Hom}}_\mathbb{Z}(\mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p},\mathbb{Z})$. This is indeed concentrated in homological degree -1 as we've seen above, so we're done. □

We have now all ingredients together to finish the computation. To formulate the result, we'll use another ad-hoc construction to define a q-Hodge filtration and a derived q-Hodge complex for $\mathbb{Z}/m$.

4.17. Construction. — For any high-powered integer $m \in \mathbb{N}^\sharp$ with prime factorisation $m = \prod_p p^{\alpha_p}$, we define the *q-Hodge filtration on* $q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$ to be the product

$$\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}} \coloneqq \prod_p \mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^{\alpha_p})/\mathbb{Z}_p}\,.$$

In the same way, we define the *q-Hodge completed derived q-de Rham complex* $q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and the *derived q-Hodge complex* $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$. It's clear from 4.9 that the conclusions of Corollaries 4.14 and 4.16 remain true if we replace p^α by an arbitrary $m \in \mathbb{N}^\sharp$.

We also need to check the following two lemmas, but we'll postpone their proofs until after the statement of the main result.

4.18. Lemma. — *"$\lim$"$_{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and "$\lim$"$_{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ are pro-idempotent algebras, respectively, in the derived ∞-categories of t-complete graded $\mathbb{Z}[\beta][\![t]\!]$-modules and of $(q-1)$-complete $\mathbb{Z}[\![q-1]\!]$-modules.*

4.19. Lemma. — *"$\lim$"$_{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and "$\lim$"$_{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ are equivalent to pro-objects with trace-class transition maps.*

Here's the result of our computation.

4.20. Theorem. — *$\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes\mathbb{Q})/\mathrm{KU})$ are concentrated in even degrees. Furthermore, their even homotopy groups are given as follows:*

(a) $\pi_{2*}\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku}) \cong \mathrm{A}^*_{\mathrm{ku}}$, *where $\mathrm{A}^*_{\mathrm{ku}}$ is obtained by killing the pro-idempotent graded $\mathbb{Z}[\beta][\![t]\!]$-algebra "$\lim$"$_{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. In particular, there's a short exact sequence*

$$0 \longrightarrow \mathbb{Z}[\beta][\![t]\!] \longrightarrow \mathrm{A}^*_{\mathrm{ku}} \longrightarrow \underset{m\in\mathbb{N}^\sharp}{\text{“colim”}}\,\underline{\mathrm{Ext}}^1_{\mathbb{Z}[\beta][\![t]\!]}\Bigl(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\beta][\![t]\!]\Bigr) \longrightarrow 0\,,$$

and $\mathrm{A}^*_{\mathrm{ku}}$ is an idempotent nuclear graded $\mathbb{Z}[\beta][\![t]\!]$-algebra.

(b) $\pi_{2*}\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}\otimes\mathbb{Q})/\mathrm{KU}) \cong \mathrm{A}_{\mathrm{KU}}[\beta^{\pm1}]$, *where A_{KU} is obtained by killing the pro-idempotent $\mathbb{Z}[\![q-1]\!]$-algebra "$\lim$"$_{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}_p}$. In particular, there's a short exact sequence*

$$0 \longrightarrow \mathbb{Z}[\![q-1]\!] \longrightarrow \mathrm{A}_{\mathrm{KU}} \longrightarrow \underset{m\in\mathbb{N}^\sharp}{\text{“colim”}}\,\mathrm{Ext}^1_{\mathbb{Z}[\![q-1]\!]}\Bigl(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}_p},\mathbb{Z}[\![q-1]\!]\Bigr) \longrightarrow 0\,,$$

and A_{KU} is an idempotent nuclear $\mathbb{Z}[\![q-1]\!]$-algebra.

Proof. We use the cofibre sequence of 4.8. To compute $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})$, we must study the cofibres of $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee \to \mathrm{ku}^{\mathrm{h}S^1}$ for high-powered integers $m\in\mathbb{N}^{\natural}$. Put

$$\begin{aligned}\mathrm{Fil}^*\, q\text{-}\overline{\mathrm{dR}}_m &\coloneqq \operatorname{cofib}\Big(\mathbb{Z}[\beta]\llbracket t\rrbracket \to \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}_p}\Big)\,,\\ \overline{\mathrm{TC}}{}^-_m &\coloneqq \operatorname{cofib}\Big(\mathrm{ku}^{\mathrm{h}S^1} \to \mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)\Big)\,.\end{aligned}$$

Since $\mathrm{ku}^{\mathrm{h}S^1}$ and $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$ are even spectra, the sequence of double speed Postnikov filtrations $\tau_{\geqslant 2*}(\mathrm{ku}^{\mathrm{h}S^1}) \to \tau_{\geqslant 2*}\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku}) \to \tau_{\geqslant 2*}\overline{\mathrm{TC}}{}^-_m$ is still a cofibre sequence in filtered spectra. Applying the construction from the proof of Lemma 4.15, we get complete exhaustive filtrations on the duals of $\mathrm{ku}^{\mathrm{h}S^1}$, $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$, and $\overline{\mathrm{TC}}{}^-_m$ in such a way that they fit into a cofibre sequence $\mathrm{Fil}^*(\overline{\mathrm{TC}}{}^-_m)^\vee \to \mathrm{Fil}^*\,\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee \to \mathrm{Fil}^*(\mathrm{ku}^{\mathrm{h}S^1})^\vee$. After passing to associated gradeds, we get a cofibre sequence of graded $\Sigma^{2*}\mathbb{Z}[\beta]\llbracket t\rrbracket$-modules

$$\mathrm{gr}^*(\overline{\mathrm{TC}}{}^-_\alpha)^\vee \longrightarrow \Sigma^{2*}\big(\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\big)^\vee \longrightarrow \Sigma^{2*}\mathbb{Z}[\beta]\llbracket t\rrbracket^\vee\,,$$

where $\Sigma^{2*}\colon \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$ denotes the "double shearing" functor. It's clear from the construction that the morphism on the right must really be given by $\Sigma^{2*}(-)^\vee$ applied to the unit map $\mathbb{Z}[\beta]\llbracket t\rrbracket \to \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}_p}$. It follows that $\mathrm{gr}^*(\overline{\mathrm{TC}}{}^-_m)^\vee \simeq \Sigma^{2*}(\mathrm{Fil}^*\, q\text{-}\overline{\mathrm{dR}}_m)^\vee$. Observe that $(\mathrm{Fil}^*\, q\text{-}\overline{\mathrm{dR}}{}^*_m)^\vee$ sits in homological degree -1. Indeed, this can be checked modulo (β,t). Then $\mathrm{Fil}^*\, q\text{-}\overline{\mathrm{dR}}_m/(\beta,t) \simeq \operatorname{cofib}(\mathbb{Z}\to \mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}})$ is given by $\Sigma\mathbb{Z}$ in graded degree 0 and $\mathbb{Z}/m$ in every other graded degree, so it's straightforward to see that its graded dual over $\mathbb{Z}$ sits indeed in homological degree -1.

Thus, $\mathrm{Fil}^*(\overline{\mathrm{TC}}{}^-_m)^\vee$ must be the double speed Postnikov filtration, $(\overline{\mathrm{TC}}{}^-_m)^\vee$ is concentrated in odd degrees, and $\pi_{2*-1}((\overline{\mathrm{TC}}{}^-_m)^\vee) \cong \mathrm{H}_{-1}(\mathrm{Fil}^*(q\text{-}\overline{\mathrm{dR}}_m)^\vee)$ as a graded $\mathbb{Z}[\beta]\llbracket t\rrbracket$-module. Combining this with what Corollary 4.16 tells us about $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee$, we see that the long exact homotopy sequence of the rotated cofibre sequence $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee \to (\mathrm{ku}^{\mathrm{h}S^1})^\vee \to \Sigma(\overline{\mathrm{TC}}{}^-_m)^\vee$ breaks up into a short exact sequence of graded $\mathbb{Z}[\beta]\llbracket t\rrbracket$-modules

$$0 \longrightarrow \mathbb{Z}[\beta]\llbracket t\rrbracket \longrightarrow \mathrm{H}_{-1}\big(\mathrm{Fil}^*(q\text{-}\overline{\mathrm{dR}}_m)^\vee\big) \longrightarrow \underline{\mathrm{Ext}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\Big(\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}, \mathbb{Z}[\beta]\llbracket t\rrbracket\Big) \longrightarrow 0\,.$$

Since $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku}) \simeq \text{“}\mathrm{colim}\text{”}_{m\in\mathbb{N}^{\natural}}\,\Sigma(\overline{\mathrm{TC}}{}^-_m)$ by the cofibre sequence from 4.8, it follows at once that $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})$ is concentrated in even degrees and that $\mathrm{A}^*_{\mathrm{ku}}$ fits into the desired short exact sequence. Furthermore, it's clear from our considerations above that

$$\big(\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}\big)^\vee \simeq \Sigma^{-1}\,\underline{\mathrm{Ext}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\Big(\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}, \mathbb{Z}[\beta]\llbracket t\rrbracket\Big) \longrightarrow \mathbb{Z}[\beta]\llbracket t\rrbracket\,,$$

induced by the short exact sequence, is given by dualising the canonical unit morphism $\mathbb{Z}[\beta]\llbracket t\rrbracket \to \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. Then the underlying graded ind-$\mathbb{Z}[\beta]\llbracket t\rrbracket$-module of $\mathrm{A}^*_{\mathrm{ku}}$ must really be given by killing the pro-idempotent “$\lim$”$_{m\in(\mathbb{N}^{\natural})^{\mathrm{op}}}\,\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. Idempotence and nuclearity of $\mathrm{A}^*_{\mathrm{ku}}$ follow from Lemmas 2.15 and 4.19. Since idempotents admit a unique $\mathbb{E}_\infty$-algebra structure, it follows that the desired description of $\mathrm{A}^*_{\mathrm{ku}}$ also holds as an ind-$\mathbb{Z}[\beta]\llbracket t\rrbracket$-algebra. This finishes the proof of (*a*), up to the two technical lemmas that we have postponed.

The proof of (*b*) is analogous; the only difference is that we need a different argument why $\operatorname{cofib}(\mathbb{Z}\llbracket q-1\rrbracket \to q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}})^\vee$ is concentrated in homological degree -1. This can be checked modulo $(q-1)$. In the case of a prime power p^α, $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ carries an ascending filtration, given by base changing the conjugate filtration from 3.35, whose graded pieces are copies of $\mathbb{Z}/p^\alpha$. Moreover, $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/(q-1)$ is a $\mathbb{Z}/p^\alpha$-algebra, since $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ contains an element $p^\alpha/(q-1)$. Thus, abstractly, $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/(q-1) \simeq \bigoplus_{\mathbb{N}} \mathbb{Z}/p^\alpha$. It's clear from Construction 4.17 that then also $q\text{-Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1) \simeq \bigoplus_{\mathbb{N}} \mathbb{Z}/m$, and the morphism $\mathbb{Z} \to q\text{-Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1)$ surjects onto one of the summands. The rest is an elementary computation. $\square$

It remains to show Lemmas 4.18 and 4.19. Neither one is automatic from the corresponding properties of "$\lim_{m \in (\mathbb{N}^\nmid)^{\mathrm{op}}}$" $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$, since $\pi_*(-)$—or really passing to the associated graded of the Postnikov filtration—is not a symmetric monoidal functor. To fix this, we'll work with the double speed Postnikov filtration instead.

Our starting point is the followng general fact, which is quite similar to Lemma 4.15 (and is also usually formulated as a spectral sequence).

4.21. Lemma. — *Let k be an even $\mathbb{E}_\infty$-ring spectrum, let $t \in \pi_{2*}(k)$ be a homogeneous element, and let M, N be even k-modules. Then the t-completed tensor product $M \widehat{\otimes}_k N$ admits a complete exhaustive descending filtration with graded pieces*

$$\mathrm{gr}^*(M \widehat{\otimes}_k N) \simeq \Sigma^{2*}\left(\pi_{2*}(M) \widehat{\otimes}^{\mathrm{L}}_{\pi_{2*}(k)} \pi_{2*}(N)\right).$$

Here $- \widehat{\otimes}^{\mathrm{L}}_{\pi_{2*}(k)} -$ denotes the graded t-completed derived tensor product over $\pi_{2*}(k)$.

Proof. The filtered spectrum $\tau_{\geqslant 2*}(M) \otimes_{\tau_{\geqslant 2*}(k)} \tau_{\geqslant 2*}(N)$ defines a filtration on $M \otimes_k N$. This filtration is exhaustive, since $\mathrm{colim}\colon \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$ is symmetric monoidal, and complete, since $\tau_{\geqslant 2*}(M) \otimes_{\tau_{\geqslant 2*}(k)} \tau_{\geqslant 2*}(N)$ is a connective object in the double Postnikov t-structure (see the proof of Lemma 4.15).

To incorporate the t-completion, we consider $\tau_{\geqslant 2*}(M) \widehat{\otimes}_{\tau_{\geqslant 2*}(k)} \tau_{\geqslant 2*}(N)$, where the completion is the t-adic completion in $\mathrm{Fil}(\mathrm{Sp})$, with t sitting in its assigned degree. This now defines a filtration on $M \widehat{\otimes}_k N$, which is clearly still complete. It is also still exhaustive. Indeed, for all n, the cofibre of $(\tau_{\geqslant 2*}(M) \otimes_{\tau_{\geqslant 2*}(k)} \tau_{\geqslant 2*}(N))_{-n} \to M \otimes N$ is $(2n+1)$-coconnective. Upon t-adic completion, the coconnectivity can go down by at most 1, and so we see that the cofibre of $(\tau_{\geqslant 2*}(M) \widehat{\otimes}_{\tau_{\geqslant 2*}(k)} \tau_{\geqslant 2*}(N))_{-n} \to M \widehat{\otimes} N$ will still be $2n$-coconnective. This ensures exhaustiveness.

Passing to the associated graded is symmetric and commutes with t-adic completion (in the filtered and graded setting, respectively). Moreover, the shearing functor Σ^{2*} is $\mathbb{E}_1$-monoidal (even $\mathbb{E}_2$, but we won't need that). Hence

$$\mathrm{gr}^*(M \widehat{\otimes}_k N) \simeq \Sigma^{2*}\pi_{2*}(M) \widehat{\otimes}_{\Sigma^{2*}\pi_{2*}(k)} \Sigma^{2*}\pi_{2*}(N) \simeq \Sigma^{2*}\left(\pi_{2*}(M) \widehat{\otimes}^{\mathrm{L}}_{\pi_{2*}(k)} \pi_{2*}(N)\right). \qquad \square$$

Proof of Lemma 4.18. It'll be enough to show pro-idempotence of the sub-systems indexed by p^α for a fixed prime p. In the following, we put $\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^\alpha} \coloneqq \mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ for short. Also let $A \coloneqq$ "$\lim_{\alpha \geqslant 2}$" $\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^\alpha}$ (the case $p = 2$ needs α even and $\geqslant 4$ instead). Since each $\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^\alpha}$ is a graded $\mathbb{Z}[\beta][\![t]\!]$-algebra, we get a unit map $\mathbb{Z}[\beta][\![t]\!] \to A$ and a multiplication $A \widehat{\otimes}^{\mathrm{L}}_{\mathbb{Z}[\beta][\![t]\!]} A \to A$ such that the composition

$$A \simeq \mathbb{Z}[\beta][\![t]\!] \widehat{\otimes}^{\mathrm{L}}_{\mathbb{Z}[\beta][\![t]\!]} A \longrightarrow A \widehat{\otimes}^{\mathrm{L}}_{\mathbb{Z}[\beta][\![t]\!]} A \longrightarrow A$$

is the identity. For the other composition, choose exponents α_1, α_2 and consider the t-completed tensor product

$$\mathrm{TC}^-\big((\mathrm{ku}/p^{\alpha_1}\otimes_{\mathrm{ku}}\mathrm{ku}/p^{\alpha_2})/\mathrm{ku}\big)\simeq \mathrm{TC}^-\big((\mathrm{ku}/p^{\alpha_1})/\mathrm{ku}\big)\mathbin{\widehat{\otimes}}_{\mathrm{ku}^{\mathrm{h}S^1}}\mathrm{TC}^-\big((\mathrm{ku}/p^{\alpha_2})/\mathrm{ku}\big)\,.$$

By Lemma 4.21, this has a complete exhaustive filtration with graded pieces given by $\Sigma^{2*}(\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{\alpha_1}}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{\alpha_2}})$. Observe that this graded completed tensor product is concentrated in homological degrees $[0,1]$. Indeed, this can be checked modulo (β,t), and then $\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{\alpha_i}}/(\beta,t)\simeq \mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/p^{\alpha_i})/\mathbb{Z}_p}$ is given by $\mathbb{Z}/p^{\alpha_i}$ in every graded degree for $i=1,2$. It remains to observe that $\mathbb{Z}/p^{\alpha_1}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Z}/p^{\alpha_2}$ is concentrated in homological degrees $[0,1]$. It follows that the filtration on $\mathrm{TC}^-((\mathrm{ku}/p^{\alpha_1}\otimes_{\mathrm{ku}}\mathrm{ku}/p^{\alpha_2})/\mathrm{ku})$ must be the double speed Postnikov filtration.

By Corollary 2.20, $\mathrm{TC}^-((\mathrm{ku}/p^{3\alpha}\otimes_{\mathrm{ku}}\mathrm{ku}/p^{\alpha})/\mathrm{ku})\to\mathrm{TC}^-((\mathrm{ku}/p^{2\alpha}\otimes_{\mathrm{ku}}\mathrm{ku}/p^{\alpha})/\mathrm{ku})$ factors through the even spectrum $\mathrm{TC}^-((\mathrm{ku}/p^{\alpha})/\mathrm{ku})$ for all $\alpha\geqslant 3$ (the case $p=2$ needs α even and $\geqslant 6$). By passing to the associated graded of the double speed Postnikov filtration, we see that

$$\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{3\alpha}}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{\alpha}}\longrightarrow \mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{2\alpha}}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{\alpha}}\,.$$

factors through $\mathrm{Fil}^*\, q\text{-}\widehat{\mathrm{dR}}_{p^{\alpha}}$. This finishes the proof that $A=\text{“}\lim\text{”}_{\alpha\geqslant 2}\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}$ is pro-idempotent.

The argument for “$\lim$”$_{\alpha\geqslant 2}\, q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}$ is analogous, except that we work with KU instead of ku, and to show that $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha_1})/\mathbb{Z}_p}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}\llbracket q-1\rrbracket}q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha_1})/\mathbb{Z}_p}$ is concentrated in homological degrees $[0,1]$, we need a slightly different argument: First, we can reduce modulo $(q-1)$. Then we use that for $i=1,2$, $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha_i})/\mathbb{Z}_p}$ carries an ascending filtration, given by base changing the conjugate filtration from 3.35, whose graded pieces are copies of $\mathbb{Z}/p^{\alpha_i}$. Moreover, $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}/(q-1)$ is an $\mathbb{Z}/p^{\alpha_i}$-algebra, since $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha_i})/\mathbb{Z}_p}$ contains an element $p^{\alpha_i}/(q-1)$. Thus, abstractly, $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha_i})/\mathbb{Z}_p}/(q-1)\simeq\bigoplus_{\mathbb{N}}\mathbb{Z}/p^{\alpha_i}$. So we're done since $\mathbb{Z}/p^{\alpha_1}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Z}/p^{\alpha_2}$ is concentrated in homological degrees $[0,1]$. ☐

Proof of Lemma 4.19. As in the proof of Lemma 4.18, we can reduce to the pro-systems indexed by high-powered prime powers p^{α}. For $\alpha\geqslant 3$ (the case $p=2$ needs α even and $\geqslant 6$), it follows from Corollary 2.21 that $\mathrm{TC}^-((\mathrm{ku}/p^{3\alpha})/\mathrm{ku})\to\mathrm{TC}^-((\mathrm{ku}/p^{\alpha})/\mathrm{ku})$ is trace-class in t-complete $\mathrm{ku}^{\mathrm{h}S^1}$-modules, hence it must be induced by a map

$$\eta\colon \mathrm{ku}^{\mathrm{h}S^1}\longrightarrow \mathrm{TC}^-\big((\mathrm{ku}/p^{3\alpha})/\mathrm{ku}\big)^\vee\mathbin{\widehat{\otimes}}_{\mathrm{ku}^{\mathrm{h}S^1}}\mathrm{TC}^-\big((\mathrm{ku}/p^{\alpha})/\mathrm{ku}\big)$$

By Lemma 4.21 (applied to the shift $\Sigma\,\mathrm{TC}^-((\mathrm{ku}/p^{3\alpha})/\mathrm{ku})^\vee$ to get an even spectrum, then we shift back afterwards), the right-hand side has a complete exhaustive filtration with graded pieces $(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^{3\alpha})/\mathbb{Z}_p})^\vee\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}$. As in the proof of Lemma 4.18, one easily checks that this graded completed tensor product is concentrated in homological degrees $[-1,0]$. It follows that the filtration must be given by $\tau_{\geqslant 2*-1}(-)$. Thus, by considering $\tau_{\geqslant 2*-1}(\eta)$ and then passing to associated gradeds, we obtain a morphism

$$\mathbb{Z}[\beta]\llbracket t\rrbracket\longrightarrow\big(\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^{3\alpha})/\mathbb{Z}_p}\big)^\vee\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}\,.$$

which witnesses that the morphism $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^{3\alpha})/\mathbb{Z}_p}\to\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}$ is indeed trace-class, as desired.

The argument for $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{3\alpha})/\mathbb{Z}_p} \to q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}$ being trace-class is analogous, except that we use KU instead of ku. Moreover, we need a different argument to show that $(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{3\alpha})/\mathbb{Z}_p})^\vee \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\![q-1]\!]} q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}$ is concentrated in homological degrees $[-1, 0]$: First, we can reduce modulo $(q-1)$. As we've seen in the proof of Lemma 4.18, on underlying abelian groups we get an equivalence $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{\alpha})/\mathbb{Z}_p}/(q-1) \simeq \bigoplus_{\mathbb{N}} \mathbb{Z}/p^{\alpha}$. An analogous conclusion holds for $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^{3\alpha})/\mathbb{Z}_p}/(q-1)$. Thus, the tensor product modulo $(q-1)$ becomes $\Sigma^{-1} \prod_{\mathbb{N}} \mathbb{Z}/p^{3\alpha} \otimes^{\mathrm{L}}_{\mathbb{Z}} \bigoplus_{\mathbb{N}} \mathbb{Z}/p^{\alpha}$, which is clearly concentrated in homological degrees $[-1, 0]$. □

§5. Algebras of overconvergent functions

In this section we prove Theorems 1.19 and 1.20. The arguments from §4.2 can be immediately adapted to show that $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}_p^\wedge\otimes\mathbb{Q})/\mathrm{ku}_p^\wedge)$ and $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}_p^\wedge\otimes\mathbb{Q})/\mathrm{KU}_p^\wedge)$ are concentrated in even degrees. Moreover,

$$\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}\big((\mathrm{ku}_p^\wedge\otimes\mathbb{Q})/\mathrm{ku}_p^\wedge\big)\cong \mathrm{A}^*_{\mathrm{ku},p}\,,\quad \pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}\big((\mathrm{KU}_p^\wedge\otimes\mathbb{Q})/\mathrm{KU}_p^\wedge\big)\cong \mathrm{A}_{\mathrm{KU},p}[\beta^{\pm1}]\,,$$

where $\mathrm{A}^*_{\mathrm{ku},p}$ is obtained by killing the pro-idempotent "$\lim_{\alpha\geqslant 2}$" $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ in graded (p,t)-complete $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$-modules and similarly $\mathrm{A}_{\mathrm{KU},p}$ is obtained by killing the pro-idempotent "$\lim_{\alpha\geqslant 2}$" $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ in $(p,q-1)$-complete $\mathbb{Z}_p\llbracket q-1\rrbracket$-modules.[(5.1)]

To make $\mathrm{A}^*_{\mathrm{ku},p}$ and $\mathrm{A}_{\mathrm{KU},p}$ more explicit, we must understand the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. This will be the content of §5.3. Our methods also yield an elementary proof of Theorem 3.10(a). The proof of Theorems 1.19 and 1.20 will then be finished in §5.4. But before we do all that, in §§5.1–5.2 we'll review Clausen's and Scholze's approach to adic spaces via solid analytic rings [CS24, Lecture 10] and study algebras of overconvergent functions and gradings in this setup.

§5.1. Adic spaces as analytic stacks

In the following, we'll use the formalism of analytic stacks from [CS24]. Recall the notion of *solid condensed spectra* from 3.25. We let $\mathcal{D}(\mathbb{Z}_\blacksquare)\simeq \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}_\blacksquare)$ denote the derived ∞-category of solid abelian groups. Let also $P\coloneqq P_{\mathbb{S}}\otimes\mathbb{Z}\simeq \mathrm{cofib}(\mathbb{Z}[\{\infty\}]\rightarrow\mathbb{Z}[\mathbb{N}\cup\{\infty\}])$ denote the free condensed abelian group on a nullsequence and let $\sigma\colon P\rightarrow P$ denote the shift endomorphism.

5.1. Huber pairs à la Clausen–Scholze. — Recall that to any Huber pair (R,R^+) one can associate an analytic ring $(R,R^+)_\blacksquare$ in the sense of [CS24, Lecture 1] as follows: First consider R as a condensed ring via its given topology. For $f\in R(*)$ and $M\in\mathrm{Mod}_R(\mathcal{D}(\mathbb{Z}_\blacksquare))$ we say that M is f-*solid* if

$$1-f\sigma^*\colon \mathrm{R}\underline{\mathrm{Hom}}_R(P,M)\xrightarrow{\simeq}\mathrm{R}\underline{\mathrm{Hom}}_R(P,M)$$

is an equivalence. The inclusion of the full sub-∞-category of f-solid R-modules admits a left adjoint $(-)^{f\text{-}\blacksquare}$, called f-*solidification*. The underlying animated condensed ring of $(R,R^+)_\blacksquare$ is then defined as

$$(R,R^+)^\triangleright_\blacksquare\coloneqq\operatorname*{colim}_{\{f_1,\dots,f_r\}\subseteq R^+}R^{f_1\text{-}\blacksquare,\dots,f_r\text{-}\blacksquare}\,,$$

where the colimit is taken over all finite subsets of R^+, and $\mathcal{D}((R,R^+)_\blacksquare)\subseteq\mathrm{Mod}_R(\mathcal{D}(\mathbb{Z}_\blacksquare))$ is the full sub-∞-category of solid condensed R-modules that are x-solid for all $x\in R^+\subseteq R(*)$. In the following, we'll always work with Huber pairs for which $(R,R^+)^\triangleright_\blacksquare$ is just R itself.

The classical notion of *affinoid open subsets* fits naturally into this formalism. Suppose we're given $f_1,\dots,f_r\in R(*)$ generating an open ideal as well as another element $g\in R(*)$, so that $U\coloneqq\big\{x\in\mathrm{Spa}(R,R^+)\ \big|\ |f_1|_x,\dots,|f_r|_x\leqslant|g|_x\neq0\big\}$ defines a rational open subset. We can define an analytic ring $\mathcal{O}(U_\blacksquare)$ as follows: The underlying animated condensed ring is the solidification

$$\mathcal{O}(U)\coloneqq R\Big[\tfrac{1}{g}\Big]^{(f_1/g)\text{-}\blacksquare,\dots,(f_r/g)\text{-}\blacksquare}$$

(5.1) In the case $p=2$, the pro-systems need to be indexed by α even and $\geqslant 4$, but we'll ignore this since it makes no difference and we let $\mathcal{D}(U_\blacksquare) \coloneqq \mathcal{D}(\mathcal{O}(U_\blacksquare)) \subseteq \operatorname{Mod}_{R[1/g]}(\mathcal{D}((R,R^+)_\blacksquare))$ be the full sub-∞-category spanned by those $R[1/g]$-modules in $\mathcal{D}((R,R^+)_\blacksquare)$ that are also (f_i/g)-solid for $i=1,\dots,r$. If $\mathcal{O}(U)$ is static and quasi-separated, it agrees with the Huber ring from the classical theory of adic spaces. In practice, this will almost always be the case.

5.2. Adic spaces à la Clausen–Scholze. — Clausen and Scholze associate to any Tate[(5.2)] adic space X an analytic stack $X_\blacksquare$ over $\mathbb{Z}_\blacksquare \to \operatorname{AnSpec}\mathbb{Z}_\blacksquare$. If $X=\operatorname{Spa}(R,R^+)$ is Tate affinoid, we simply put $X_\blacksquare \coloneqq \operatorname{AnSpec}(R,R^+)_\blacksquare$. If $U\subseteq \operatorname{Spa}(R,R^+)$ is an open subset of a Tate affinoid adic space, choose a cover $V\coloneqq \coprod_{i\in I} V_i \to U$ by rational open subsets and form the Čech nerve $V_\bullet \coloneqq \check{\mathrm{C}}_\bullet(V\to X)$. Every V_n is a disjoint union of affinoid adic spaces, hence $V_{n,\blacksquare}$ is already defined. Then we can put $U_\blacksquare \coloneqq \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} V_{n,\blacksquare}$. Finally, if X is an arbitrary Tate adic space, choose a cover $W\coloneqq \coprod_{j\in J} W_j\to X$ by affinoids and form the Čech nerve $W_\bullet \coloneqq \check{\mathrm{C}}_\bullet(W\to X)$. Each W_m is a disjoint union of open subsets of Tate affinoid adic spaces, so $W_{m,\blacksquare}$ is already defined, and we put $X_\blacksquare \coloneqq \operatorname{colim}_{[m]\in\Delta^{\mathrm{op}}} W_{m,\blacksquare}$.

It can be shown that these constructions are well-defined and independent of the choices involved. We'll omit the verification, but let us at least mention the crucial input.

5.3. Lemma. — *Let (R,R^+) be a Huber pair and let $X_\blacksquare \coloneqq \operatorname{AnSpec}(R,R^+)_\blacksquare$ be the associated affine analytic stack.*

(a) *If $U,U'\subseteq \operatorname{Spa}(R,R^+)$ are rational open subsets, then*

$$\operatorname{AnSpec}\mathcal{O}(U_\blacksquare)\times_{\operatorname{AnSpec}(R,R^+)_\blacksquare}\operatorname{AnSpec}\mathcal{O}(U'_\blacksquare)\simeq \operatorname{AnSpec}\mathcal{O}\big((U\cap U')_\blacksquare\big)\,.$$

(b) *If R is Tate and $U\subseteq\operatorname{Spa}(R,R^+)$ is a rational open subset, then $j\colon U_\blacksquare\to X_\blacksquare$ is an open immersion of affine analytic stacks in the sense of [CS24, Lecture 16]. That is, j^* admits a fully faithful left adjoint $j_!$ satisfying the projection formula.*

(c) *If R is Tate and $\coprod_{i=1}^n U_i\to \operatorname{Spa}(R,R^+)$ is a cover by rational open subsets, then $\coprod_{i=1}^n U_{i,\blacksquare}\to X_\blacksquare$ is a !-cover of affine analytic stacks.*

5.4. Remark. — The Tate condition in Lemma 5.3(b) and (c) is crucial and it is the reason why we restrict to the Tate case when we describe adic spaces in terms of analytic stacks. Without this assumption, (b) will be wrong. For example, if R is a discrete ring, any Zariski-open also determines a rational open of $\operatorname{Spa}(R,R)$, but in this case j^* almost never preserves limits, so it can't have a left adjoint $j_!$.

Proof sketch of Lemma 5.3. Suppose U and U' are given by $|f_1|,\dots,|f_r|\leqslant |g|\neq 0$ and $|f_1'|,\dots,|f_s'|\leqslant|g'|\neq 0$, respectively. Using the description of pushouts from [CS24, Lecture 11], it's clear that $\mathcal{O}(U_\blacksquare)\otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare}\mathcal{O}(U'_\blacksquare)$ is the solidification of $R[1/(gg')]$ at the elements f_i/g and f_j'/g' for $i=1,\dots,r$, $j=1,\dots,s$. But that's precisely $\mathcal{O}((U\cap U')_\blacksquare)$, proving ($a$).

For (b), assume U is given by $|f_1|,\dots,|f_r|\leqslant|g|\neq 0$. Since R is assumed to be Tate, the open ideal generated by $f_1,\dots,f_r$ must be all of R. Hence g will aready be invertible in $R[T_1,\dots,T_r]/(gT_i-f_i\mid i=1,\dots,r)$ and this quotient is automatically a derived quotient as well. It follows that the functor $j^*\colon\mathcal{D}(X_\blacksquare)\to\mathcal{D}(U_\blacksquare)$ can also be written as

$$(-)[T_1,\dots,T_r]^{T_1\text{-}\blacksquare,\dots,T_r\text{-}\blacksquare}/(gT_i-f_i\mid i=1,\dots,r)\,.$$

(5.2) To avoid confusion with analytic stacks, we'll call an adic space *Tate* rather than *analytic* if, locally, there exists a topologically nilpotent unit.

By [CS24, Lecture 7], the functor $(-)[T]^{T\text{-}\blacksquare}$ of adjoining a variable and then solidifying can be explicitly described as $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}(\!(T^{-1})\!)/\mathbb{Z}[T], -)$ and so $j^*(-) \simeq \mathrm{R}\underline{\mathrm{Hom}}_R(Q, -)$, where

$$Q \coloneqq \left(\bigotimes_{i=1}^{r} \Sigma^{-1}\mathbb{Z}(\!(T_i^{-1})\!)/\mathbb{Z}[T_i] \otimes_{\mathbb{Z}}^{\mathrm{L}\blacksquare} R\right)/(gT_i - f_i \mid i = 1, \dots, r)\,.$$

It follows immediately that j^* admits a left adjoint $j_!(-) \simeq Q \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} -$. It remains to check the projection formula

$$j_!(M) \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} N \simeq j_!\big(M \otimes^{\mathrm{L}}_{\mathcal{O}(U_\blacksquare)} j^*(N)\big)\,.$$

By the same argument as above, Q is already an $R[1/g]$-module and the functor j^* is insensitive to inverting g. Therefore, it's enough to check the projection in the case where N is an $R[1/g]$-module. When restricting to $R[1/g]$-modules, j^* is just given by successively killing the idempotent algebras $\mathbb{Z}(\!(T_i^{-1})\!) \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}[T_i], T_i \mapsto f_i/g} R[1/g]$ for $i = 1, \dots, r$. Now for killing an idempotent it's completely formal to see that the left adjoint indeed satisfies the projection formula. This finishes the proof of (*b*).

To show (*c*), since we already know that each $j_i \colon U_{i,\blacksquare} \to X_\blacksquare$ is an open immersion, we can use the criterion from [CS24, Lecture 18] to verify that $\coprod_{i=1}^n U_{i,\blacksquare} \to X_\blacksquare$ is indeed a !-cover. That is, if $A_i \coloneqq \operatorname{cofib}((j_i)_!\mathcal{O}(U_i) \to R)$, we need to show $A_1 \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} \cdots \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} A_n \simeq 0$. Using [Hub94, Lemma 2.6] and an inductive argument as in [CS19, Lemma 10.3], this can be reduced to the special case where $n = 2$ and $U_1 = \{x \in X \mid 1 \leqslant |f|_x\}$, $U_2 = \{x \in X \mid |f|_x \leqslant 1\}$.for some $f \in R$. This is now a straightforward calculation. □

5.5. Remark. — Let $U \subseteq X$ be an open inclusion of Tate adic spaces and let $j \colon U_\blacksquare \to X_\blacksquare$ be the corresponding map of analytic stacks. In the following, if its clear that we're working in $\mathcal{D}(X_\blacksquare)$, we often abuse notation and write $\mathcal{O}_U$ instead of $j_*\mathcal{O}_{U_\blacksquare}$ for the pushforward of the structure sheaf of $U_\blacksquare$. We also use $- \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare}$ to denote the functor $j_*j^* \colon \mathcal{D}(X_\blacksquare) \to \mathcal{D}(X_\blacksquare)$.

Let us point out that $- \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare}$ is *not* just the tensor product with $\mathcal{O}_U$ in the symmetric monoidal ∞-category $\mathcal{D}(X_\blacksquare)$. We can already see the difference if $X = \operatorname{Spa}(R, R^+)$ and $U \subseteq X$ is a rational open given by $|f_1|, \dots, |f_r| \leqslant |g| \neq 0$: In this case,

$$- \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare} \simeq \big(- \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_U\big)^{(f_1/g)\text{-}\blacksquare, \dots, (f_r/g)\text{-}\blacksquare}\,.$$

In particular, even though $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare} \simeq \mathcal{O}_U$ (see Lemma 5.3(*a*) and Lemma 5.10(*b*) below), it's rarely true that $\mathcal{O}_U$ is idempotent in $\mathcal{D}(X_\blacksquare)$.

Thus, there's a priori no reason to expect that sheaves of overconvergent functions $\mathcal{O}_{Z^\dagger}$ would be idempotent. In the following, we'll investigate why idempotence is satisfied in the situation of Theorems 1.19 and 1.20. Let's start by introducing a notion of open immersions for analytic stacks that need not be affine.

5.6. Open immersions of analytic stacks. — We call a map of analytic stacks $j \colon U \to X$ a *naive open immersion* if j is a !-able monomorphism and $j^* \simeq j^!$. Since j is a monomorphism, $U \times_X U \simeq U$. Combining this with proper base change, we get $j^*j_! \simeq \mathrm{id}_{\mathcal{D}(U)}$ and so $j_!$ is fully faithful. Then the right adjoint j_* of j^* must be fully faithful as well.

Using the projection formula and $j^*j_! \simeq \mathrm{id}_{\mathcal{D}(U)}$, we see that $j_!\mathcal{O}_U \to \mathcal{O}_X$ exhibits $j_!\mathcal{O}_U$ as an idempotent coalgebra in $\mathcal{D}(X)$. Then $\operatorname{cofib}(j_!\mathcal{O}_U \to \mathcal{O}_X)$ must be an idempotent algebra. In this way, we can associate to any naive open immersion an idempotent algebra in $\mathcal{D}(X)$, which we call the *complementary idempotent determined by* U and denote $\mathcal{O}_{X \smallsetminus U}$. It's straightforward to check that the forgetful functor $i_* \colon \operatorname{Mod}_{\mathcal{O}_{X\smallsetminus U}}(\mathcal{D}(X)) \to \mathcal{D}(X)$, which is fully faithful by idempotence, fits into a recollement

$$\operatorname{Mod}_{\mathcal{O}_{X\smallsetminus U}}\big(\mathcal{D}(X)\big) \xrightarrow{\;i_*\;} \mathcal{D}(X) \xrightarrow{\;j^*\;} \mathcal{D}(U), \qquad i^*, i^! \colon \mathcal{D}(X) \to \operatorname{Mod}_{\mathcal{O}_{X\smallsetminus U}}\big(\mathcal{D}(X)\big), \quad j_!, j_* \colon \mathcal{D}(U) \to \mathcal{D}(X)$$

and so $j_*\mathcal{O}_U$ is obtained from $\mathcal{O}_X$ by killing the idempotent algebra $\mathcal{O}_{X\smallsetminus U}$. As long as it's clear that we're working in $\mathcal{D}(X)$, we often abuse notation and just write $\mathcal{O}_U$ instead of $j_*\mathcal{O}_X$.

5.7. Remark. — Every open immersion of affine analytic stacks in the sense of [CS24, Lecture 16] is also a naive open immersion.

5.8. Remark. — If $A \in \mathcal{D}(X)$ is an idempotent algebra, we can define an analytic substack $U_A \subseteq X$ by declaring that a map $f \colon Y \to X$ factors through U_A if and only if $f^* \colon \mathcal{D}(X) \to \mathcal{D}(Y)$ factors through the localisation $\mathcal{D}(X)/\operatorname{Mod}_A(\mathcal{D}(X))$, or equivalently, if and only if $f^*(A) \simeq 0$. However, it's *not* true that the constructions $U \mapsto \mathcal{O}_{X\smallsetminus U}$ and $A \mapsto U_A$ are inverses; it's not even clear why $\mathcal{D}(U_A)$ would coincide with $\mathcal{D}(X)/\operatorname{Mod}_A(\mathcal{D}(X))$.

It's not obvious what conditions should be put on U and A to make these constructions mutually inverse (moreover, whatever the condition, it should be satisfied for open immersions of affine analytic stacks). This explains why we call the notion from 5.6 *naive*: An *honest* open immersion of analytic stacks should be a naive open immersion for which the idempotent algebra $\mathcal{O}_{X\smallsetminus U}$ meets the putative condition. In the following, we'll work with the naive notion, since it is enough for our purposes.

5.9. Lemma. — *Let $U' \to U \to X$ be naive open immersions of analytic stacks. Suppose that U contains the closure of U' in the sense that there exists another naive open immersion $j \colon V \to X$ such that $U' \times_X V \simeq \emptyset$ and $\mathcal{O}_{X\smallsetminus V} \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{X\smallsetminus U} \simeq 0$. Then $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq \mathcal{O}_{U'}$. Moreover, the map $\mathcal{O}_U \to \mathcal{O}_{U'}$ is trace-class in $\mathcal{D}(X)$ and factors through $\mathcal{O}_{X\smallsetminus V}$.*

Proof. The condition $U' \times_X V \simeq \emptyset$ implies that $\mathcal{O}_{U'}$ is in the kernel of the pullback functor $j^* \colon \mathcal{D}(X) \to \mathcal{D}(V)$ and so $\mathcal{O}_{U'}$ is an algebra over the idempotent $A \coloneqq \mathcal{O}_{X\smallsetminus V}$ by 5.6. We also know that $\mathcal{O}_U$ is obtained from $\mathcal{O}_X$ by killing the idempotent $B \coloneqq \mathcal{O}_{X\smallsetminus U}$. Hence $\mathcal{O}_U \simeq \operatorname{cofib}(B^\vee \to \mathcal{O}_X)$. Since $B^\vee$ is a B-module, $\mathcal{O}_{U'}$ is an A-module, and $A \otimes B \simeq 0$, we get $B^\vee \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq 0$, hence indeed $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq \mathcal{O}_{U'}$.

Since the double dual $B^{\vee\vee}$ is still a B-module, the same argument shows $\mathcal{O}_U^\vee \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq \mathcal{O}_{U'}$. Hence $\mathcal{O}_U \to \mathcal{O}_{U'}$ is trace-class, with classifier given by the unit $\mathcal{O}_X \to \mathcal{O}_{U'}$. We've already seen that $\mathcal{O}_{U'}$ is an A-algebra. The condition $A \otimes B \simeq 0$ also implies $\mathrm{R}\underline{\operatorname{Hom}}_X(B, A) \simeq 0$, since $\mathrm{R}\underline{\operatorname{Hom}}_X(B, A)$ is both an A-module and a B-module. It follows that A is contained in the image of $j_* \colon \mathcal{D}(U) \to \mathcal{D}(X)$ and hence A is an $\mathcal{O}_U$-algebra. This shows that $\mathcal{O}_U \to \mathcal{O}_{U'}$ factors through A. □

5.10. Lemma. — *Let X be a Tate adic space with associated analytic stack $X_\blacksquare \to \operatorname{AnSpec} \mathbb{Z}_\blacksquare$, and let $U, U' \subseteq X$ be open subsets.*

(*a*) *The map $j \colon U_\blacksquare \to X_\blacksquare$ is a naive open immersion of analytic stacks. Moreover, an arbitary map $f \colon Y \to X_\blacksquare$ of analytic stacks factors through $U_\blacksquare$ if and only if $f^*(\mathcal{O}_{X\smallsetminus U}) \simeq 0$.*

(*b*) *We have* $U_\blacksquare \times_{X_\blacksquare} U'_\blacksquare \simeq (U \cap U')_\blacksquare$*. In particular,* $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U'_\blacksquare} \simeq \mathcal{O}_{U\cap U'}$ *and vice versa if* U *and* U' *are exchanged.*

(*c*) *If* $\overline{U}' \subseteq U$*, then* $U_\blacksquare$ *contains the closure of* $U'_\blacksquare$ *in the sense of Lemma 5.9.*

Proof sketch. Let's start with (*b*). In the case where U and U' are affinoid, $U_\blacksquare \times_{X_\blacksquare} U'_\blacksquare \simeq (U\cap U')_\blacksquare$ follows essentially by the construction of $X_\blacksquare$ in 5.2, because we can choose both U and U' to be part of an affinoid cover of X (and to prove that said construction is independent of the choice of cover, we need Lemma 5.3(*a*)). To show the general case, just cover U and U' by affinoid open subsets.

Let's show (*a*) next. Let's first consider the case where $X = \operatorname{Spa}(R, R^+)$ is affinoid and $U \subseteq X$ is a rational open. We've already seen in Lemma 5.3(*b*) that $j\colon U_\blacksquare \to X_\blacksquare$ is a naive open immersion. Suppose $f\colon Y \to X_\blacksquare$ is a map of analytic stacks such that $f^*(\mathcal{O}_{X\smallsetminus U}) \simeq 0$. If $Y \simeq \operatorname{AnSpec} S$ is affine, then the map of analytic rings $(R, R^+)_\blacksquare \to S$ factors through $\mathcal{O}(U_\blacksquare)$ if and only if $f^*\colon \mathcal{D}((R, R^+)_\blacksquare) \to \mathcal{D}(S)$ factors through $\mathcal{D}(U_\blacksquare)$. Since $f^*(\mathcal{O}_{X\smallsetminus U}) \simeq 0$, this is satisfied in our case. This proves the claim in the case where $Y \simeq \operatorname{AnSpec} S$ is affine. In particular, $U_\blacksquare \times_{X_\blacksquare} \operatorname{AnSpec} S \simeq \operatorname{AnSpec} S$. For the general case, write Y as a colimit of affines to see $U_\blacksquare \times_{X_\blacksquare} Y \simeq Y$. Then $f\colon Y \to X_\blacksquare$ clearly factors through $U_\blacksquare$.

Now let U and X be arbitrary. Proving that $j\colon U_\blacksquare \to X_\blacksquare$ is a naive open immersion formally reduces to the special case considered above; we omit the argument. Now let $f\colon Y \to X_\blacksquare$ be a map of analytic stacks such that $f^*(\mathcal{O}_{X\smallsetminus U}) \simeq 0$. Whether f factors through $U_\blacksquare$ can be checked locally on $X_\blacksquare$. By (*b*), if $\operatorname{Spa}(R, R^+) \to X$ is an affinoid open supset, then $U_\blacksquare \times_{X_\blacksquare} \operatorname{AnSpec}(R, R^+)_\blacksquare \simeq (U \cap \operatorname{Spa}(R, R^+))_\blacksquare$, so we can reduce to the case where X is affinoid. As above, we may also assume that $Y \simeq \operatorname{AnSpec} S$ is affine. Let $\coprod_{i\in I} U_i \to U$ be a cover by rational open subsets. Then

$$\mathcal{O}_{X\smallsetminus U} \simeq \operatorname*{colim}_{\{i_1,\dots,i_n\}\subseteq I} \left(\mathcal{O}_{X\smallsetminus U_{i_1}} \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \cdots \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{X\smallsetminus U_{i_n}}\right),$$

where the colimit is taken over all finite subsets of I. Since the colimit is filtered and $f^*(\mathcal{O}_{X\smallsetminus U})$ is detected by the single condition $1 = 0$, there exists a finite subset $\{i_1, \dots, i_n\} \subseteq I$ such that already $f^*(\mathcal{O}_{X\smallsetminus U_{i_1}}) \otimes^{\mathrm{L}}_S \cdots \otimes^{\mathrm{L}}_S f^*(\mathcal{O}_{X\smallsetminus U_{i_n}}) \simeq 0$ in $\mathcal{D}(S)$. By the criterion from [CS24, Lecture 18], it follows that $\coprod_{j=1}^n U_{i_j,\blacksquare} \times_{X_\blacksquare} \operatorname{AnSpec} S \to \operatorname{AnSpec} S$ is a !-cover. We may therefore replace S by the constituents of this cover, and for each of them it's clear that they factor through $U_\blacksquare$. This finishes the proof of (*a*).

Part (*c*) is a formal consequence: If $V \coloneqq X \smallsetminus \overline{U}'$, then $V_\blacksquare \to X_\blacksquare$ is a naive open immersion by (*a*), $U_\blacksquare \times_{X_\blacksquare} V_\blacksquare \simeq \emptyset$ follows from (*b*), and if $A \coloneqq \mathcal{O}_{X\smallsetminus U} \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{X\smallsetminus V}$, then it's formal to see that $\operatorname{Mod}_A(\mathcal{D}(X_\blacksquare))$ is the kernel of the pullback functor $\mathcal{D}(X_\blacksquare) \to \mathcal{D}(U_\blacksquare) \times_{\mathcal{D}((U\cap V)_\blacksquare)} \mathcal{D}(V_\blacksquare)$. But this functor is an equivalence as $U \cup V = X$, and so $A \simeq 0$. □

We can finally show the desired criterion for idempotence.

5.11. Definition. — If X is a Tate adic space and $Z \subseteq X$ is a closed subset, the *overconvergent neighbourhood of* Z is the analytic stack

$$Z^\dagger \coloneqq \lim_{U\supseteq Z} U_\blacksquare,$$

where the limit is taken over all open neighbourhoods of Z. If it's clear that we're working in $\mathcal{D}(X_\blacksquare)$, we often abuse notation and denote by $\mathcal{O}_{Z^\dagger} \coloneqq \operatorname{colim}_{U\supseteq Z} \mathcal{O}_U \in \mathcal{D}(X_\blacksquare)$ the *sheaf of overconvergent functions on* Z. This is in favorable situations, but not always, the pushforward of the structure sheaf of $Z^\dagger$; see Theorem 5.12(*b*) below.

5.12. Theorem. — *Let X be a quasi-compact quasi-separated Tate adic space and let $Z \subseteq X$ be a closed subset such that for all points $z \in Z$ and all generalisations $z' \rightsquigarrow z$ also $z' \in Z$.*

(*a*) *The ind-object*

$$\underset{U \supseteq Z}{\text{“colim”}}\, \mathcal{O}_U \in \operatorname{Ind}\mathcal{D}(X_\blacksquare)$$

is idempotent, nuclear, and obtained by killing the pro-idempotent “$\lim$”$_{Z\cap \overline{W}=\emptyset}\, \mathcal{O}_W$, where the limit is taken over all open subsets $W \subseteq X$ such that $Z \cap \overline{W} = \emptyset$. In particular, $\mathcal{O}_{Z^\dagger} \in \mathcal{D}(X_\blacksquare)$ is idempotent and nuclear.

(*b*) *If for every affinoid open $j\colon \operatorname{Spa}(R, R^+) \to X$ the pullback $j^*(\mathcal{O}_{Z^\dagger}) \in \mathcal{D}((R, R^+)_\blacksquare)$ is connective, then pushforward along $Z^\dagger \to X_\blacksquare$ induces a symmetric monoidal equivalence $\mathcal{D}(Z^\dagger) \simeq \operatorname{Mod}_{\mathcal{O}_{Z^\dagger}}(\mathcal{D}(X_\blacksquare))$. In particular, in this case $\mathcal{O}_{Z^\dagger}$ is really the pushforward of the structure sheaf of $Z^\dagger$.*

5.13. Remark. — Following discussions with Ben Antieau and Peter Scholze, we believe that connectivity in Theorem 5.12(*b*) can be replaced by the much weaker condition that $\operatorname{Mod}_{j^*(\mathcal{O}_{Z^\dagger})}(\mathcal{D}(R))$ is left-complete should already be enough, using an adaptation of [MM24, Proposition 2.16].

To prove Theorem 5.12, we send a lemma in advance.

5.14. Lemma. — *Let X be a spectral space and let $Y, Z \subseteq X$ be closed subsets such that for $z \in Z$ and $y \in Y$ there never exists a common generalisation $z \leftsquigarrow x \rightsquigarrow y$ (in particular $Z \cap Y = \emptyset$). Then there exist open neighbourhoods $U \supseteq Z$ and $V \supseteq Y$ such that $U \cap V = \emptyset$.*

Proof. Fix $z \in Z$. By [Stacks, Tag 0906], $y \in Y$ there exist open neighbourhoods $U_y \ni z$ and $V_y \ni y$ such that $U_y \cap V_y = \emptyset$. By compactness of Y, there exist finitely many $y_1, \dots, y_n \in Y$ such that $Y \subseteq V_z \coloneqq V_{y_1} \cup \dots \cup V_{y_n}$. Let also $U_z \coloneqq U_{y_1} \cap \cdots U_{y_n}$, so that $U_z \cap V_z = \emptyset$. By compactness of Z, there exist finitely many $z_1, \dots, z_m \in Z$ such that $Z \subseteq U \coloneqq U_{z_1} \cup \dots \cup U_{z_m}$. Putting $V \coloneqq V_{z_1} \cap \dots \cap V_{z_m}$, we have constructed U and V with the required properties. □

Proof of Theorem 5.12. First observe that Lemma 5.14 can be applied to any closed subset $Y \subseteq X$ such that $Z \cap Y = \emptyset$. Indeed, for any common generalisation $z \leftsquigarrow x \rightsquigarrow y$, we would have $x \in Z$, as Z is closed under generalisations, but then $y \in Z$, as Z is also closed under specialisations.

It follows that in the ind-object “colim”$_{U \supseteq Z}\, \mathcal{O}_U$ we can restrict to open neighbouhoods of the form $U = X \smallsetminus \overline{W}$ for some open subset W such that $Z \cap \overline{W} = \emptyset$. Indeed, for arbitrary U, apply Lemma 5.14 to Z and $X \smallsetminus U$ to get an open neighbourhood $W \supseteq (X \smallsetminus U)$ such that $Z \cap \overline{W} = \emptyset$. Then $(X \smallsetminus \overline{W}) \subseteq U$, as desired.

Let $\mathcal{O}_{\overline{W}} \coloneqq \mathcal{O}_{X \smallsetminus (X \smallsetminus \overline{W})} \in \mathcal{D}(X_\blacksquare)$ be the complementary idempotent determined by the open subset $X \smallsetminus \overline{W}$. Since each $\mathcal{O}_U$ is obtained by killing the idempotent $\mathcal{O}_{X \smallsetminus U}$, our observation implies that “colim”$_{U \supseteq Z}\, \mathcal{O}_U$ is obtained by killing the pro-idempotent “lim”$_{Z \cap \overline{W} = \emptyset}\, \mathcal{O}_{\overline{W}}$. For all such W, applying Lemma 5.14 to Z and $\overline{W}$ provides another open neighbourhood $W' \supseteq \overline{W}$ such that still $Z \cap \overline{W'} = \emptyset$. By Lemma 5.9 and Lemma 5.10(*c*), $\mathcal{O}_{W'} \to \mathcal{O}_W$ is trace-class and factors through $\mathcal{O}_{\overline{W}}$. It follows that “lim”$_{Z \cap \overline{W} = \emptyset}\, \mathcal{O}_W \simeq$ “lim”$_{Z \cap \overline{W} = \emptyset}\, \mathcal{O}_{\overline{W}}$ and that the condition of Lemma 2.15 is satisfied, so that “colim”$_{U \supseteq Z}\, \mathcal{O}_U$ is indeed idempotent and nuclear in $\operatorname{Ind}\mathcal{D}(X_\blacksquare)$. Since $\operatorname{colim}\colon \operatorname{Ind}\mathcal{D}(X_\blacksquare) \to \mathcal{D}(X_\blacksquare)$ preserves idempotents and nuclear objects, it follows that $\mathcal{O}_{Z^\dagger} \in \mathcal{D}(X_\blacksquare)$ is idempotent and nuclear as well. This finishes the proof of (*a*).

For (b), note that $Z^\dagger$ is clearly compatible with base change and so is $\mathcal{O}_{Z^\dagger}$ by (a) and Lemma 2.15(c). We may therefore assume that $X = \operatorname{Spa}(R, R^+)$ is affinoid and $\mathcal{O}_{Z^\dagger}$ is connective. Then $\mathcal{O}_{Z^\dagger}$ can be turned into an analytic ring using the induced analytic ring structure from $(R, R^+)_{\blacksquare}$. It follows that a map $f\colon \operatorname{AnSpec} S \to \operatorname{AnSpec}(R, R^+)_{\blacksquare}$ factors through $\mathcal{O}_{Z^\dagger}$ if and only if $S \simeq f^*(\mathcal{O}_{Z^\dagger})$. By Lemma 2.15($b$), we have $\mathcal{O}_{Z^\dagger} \otimes^{\mathrm{L}}_{(R,R^+)_{\blacksquare}} \mathcal{O}_W \simeq 0$ for all open W such that $Z \cap \overline{W} = \emptyset$. Thus $S \simeq f^*(\mathcal{O}_{Z^\dagger})$ implies $f^*(\mathcal{O}_W) \simeq 0$ for all such W. By sandwiching open and closed subsets, we get $f^*(\mathcal{O}_{X \smallsetminus U}) \simeq 0$ for all open neighbourhoods $U \supseteq Z$. By Lemma 5.10(a), this implies that f factors through $Z^\dagger \simeq \lim_{U \supseteq Z} U_{\blacksquare}$.

Conversely, if f factors through $Z^\dagger$, then $f^*(\mathcal{O}_{X \smallsetminus U}) \simeq 0$ for all U and thus $f^*(\mathcal{O}_W) \simeq 0$ for all W as above, using the same sandwiching argument. It follows that S is a module over the nuclear idempotent ind-algebra obtained by killing "$\lim$"$_{Z \cap \overline{W} = \emptyset} f^*(\mathcal{O}_W)$ in $\mathcal{D}(S)$. By Lemma 2.15(c), this is "colim"$_{U \supseteq Z} f^*(\mathcal{O}_U)$. Then S is also a module over the honest colimit $\operatorname{colim}_{U \supseteq Z} f^*(\mathcal{O}_U) \simeq f^*(\mathcal{O}_{Z^\dagger})$, proving $S \simeq f^*(\mathcal{O}_{Z^\dagger})$.

In conclusion, this argument shows that $Z^\dagger \simeq \operatorname{AnSpec} \mathcal{O}_{Z^\dagger}$ is an affine analytic stack and so $\mathcal{D}(Z^\dagger) \simeq \operatorname{Mod}_{\mathcal{O}_{Z^\dagger}}(\mathcal{D}((R, R^+)_{\blacksquare}))$ follows by construction, as we've put the induced analytic ring structure on $\mathcal{O}_{Z^\dagger}$. □

This implies idempotence and nuclearity in the situation of Theorem 1.19.

5.15. Corollary. — *Let* $X \coloneqq \operatorname{Spa} \mathbb{Z}_p[\![q-1]\!] \smallsetminus \{p = 0, q = 1\}$ *and let* $Z \subseteq X$ *be the union of the closed subsets* $\operatorname{Spa}(\mathbb{F}_p(\!(q-1)\!), \mathbb{F}_p[\![q-1]\!])$ *and* $\operatorname{Spa}(\mathbb{Q}_p(\zeta_{p^n}), \mathbb{Z}_p[\zeta_{p^n}])$ *for all* $n \geqslant 0$.

(a) *Z is closed and closed under generalisations.*

(b) *For $n, r, s \geqslant 1$ such that $(p-1)p^n > s$, let $W_{n,r,s} \subseteq X$ be the rational open subset determined by $|p^r| \leqslant |q^{p^n} - 1| \neq 0$, $|(q-1)^s| \leqslant |p| \neq 0$. Then $\mathcal{O}_{Z^\dagger}$ is the colimit of the idempotent nuclear ind-algebra obtained by killing the pro-idempotent "$\lim$"$_{n,r,s} \mathcal{O}_{W_{n,r,s}}$.*

Proof. Let $x \in X \smallsetminus Z$. Then $|p|_x \neq 0$, hence $|(q-1)^s|_x \leqslant |p|_x$ for $s \gg 0$. Choose such an s. Moreover, $|q^{p^n} - 1|_x \neq 0$ holds for all $n \geqslant 0$. Choose n such that $(p-1)p^n > s$ and choose $r \gg 0$ such that $|p^r|_x \leqslant |q^{p^n} - 1|_x$. Then $x \in W_{n,r,s}$. If we can show $Z \cap \overline{W}_{n,r,s} = \emptyset$, both ($a$) and ($b$) will follow. Indeed, this will imply that $X \smallsetminus Z$ is open and closed under specialisations, proving (a). Moreover, $X \smallsetminus Z = \bigcup_{n,r,s} W_{n,r,s}$ and so for any open subset W such that $Z \cap \overline{W} = \emptyset$ we must have $W_{n,r,s} \supseteq \overline{W}$ for sufficiently large n, r, and s by quasi-compactness of $\overline{W}$. Hence (b) follows from Theorem 5.12(a).

To show $Z \cap \overline{W}_{n,r,s} = \emptyset$, let $w \in W_{n,r,s}$. Since $(p-1)p^n > s$, we get $|(q-1)^{(p-1)p^{i-1}}|_w < |p|_x$ for all $i > n$ and so $|\Phi_{p^i}(q)|_w = |p|_w$, where $\Phi_{p^i}(q)$ denotes the $(p^i)^{\mathrm{th}}$ cyclotomic polynomial. Thus $0 < |p^{r+i-n}|_w \leqslant |q^{p^i} - 1|_w$ for $i > n$. In particular, $w \notin Z$. Even better: If U_i denotes the rational open subset determined by $|q^{p^i} - 1| \leqslant |p^{r+i-n+1}| \neq 0$ and V denotes the rational open subset determined by $|p| \leqslant |(q-1)^{s+1}| \neq 0$, then the open set $\bigcup_{i \geqslant n} U_i \cup V$ contains Z and doesn't intersect $W_{n,r,s}$, so indeed $Z \cap \overline{W}_{n,r,s} = \emptyset$. □

§5.2. Graded adic spaces

To deduce idempotence and nuclearity in the situation of Theorem 1.20, let us describe how to encode gradings in terms of actions of the analytic stack

$$\mathbb{T} \coloneqq \operatorname{AnSpec} \mathbb{Z}[u^{\pm 1}]_{\blacksquare}\,,$$

where $\mathbb{Z}[u^{\pm 1}]_{\blacksquare}$ is obtained from $\mathbb{Z}[u^{\pm 1}]$ by solidifying both u and u^{-1}. Equivalently, $\mathbb{Z}[u^{\pm 1}]_{\blacksquare}$ is the analytic ring associated to the discrete Huber pair $(\mathbb{Z}[u^{\pm 1}], \mathbb{Z}[u^{\pm 1}])$.

5.16. Graded adic spaces via actions of $\mathbb{T}$. — Classically, the grading on $\mathbb{Z}[\beta,t]$ in which β and t receive degree 2 and -2, respectively, is encoded by an action of $\mathbb{G}_m \coloneqq \operatorname{Spec}\mathbb{Z}[u^{\pm 1}]$ on $\operatorname{Spec}\mathbb{Z}[\beta,t]$. The action map $\operatorname{Spec}\mathbb{Z}[\beta,t]\times\mathbb{G}_m\rightarrow\operatorname{Spec}\mathbb{Z}[\beta,t]$ corresponds to the ring map $\Delta\colon \mathbb{Z}[\beta,t]\rightarrow\mathbb{Z}[\beta,t]\otimes_{\mathbb{Z}}\mathbb{Z}[u^{\pm1}]$ given by $\Delta(\beta)\coloneqq u^2\beta$, $\Delta(t)\coloneqq u^{-2}t$.

In our situation, if we want to give geometric meaning to our computation of $\mathrm{A}^*_{\mathrm{ku},p}$, we're forced to work with the adic spectrum $\overline{X}^*\coloneqq\operatorname{Spa}\mathbb{Z}[\beta,t]^\wedge_{(p,t)}$ instead. But in the map Δ we can't just replace $\mathbb{Z}[\beta,t]$ by its (p,t)-completion, since the tensor product $\mathbb{Z}[\beta,t]^\wedge_{(p,t)}\otimes_{\mathbb{Z}}\mathbb{Z}[u^{\pm1}]$ won't be (p,t)-complete anymore.

To fix this, consider $\pi\colon\mathbb{T}\rightarrow\operatorname{AnSpec}\mathbb{Z}_\blacksquare$ and let $-\otimes^{\mathrm{L}}_{\mathbb{Z}_\blacksquare}\mathbb{Z}[u^{\pm1}]_\blacksquare$ denote the pullback $\pi^*\colon\mathcal{D}(\mathbb{Z}_\blacksquare)\rightarrow\mathcal{D}(\mathbb{T})$. By [CS24, Lecture 7], the process of adjoining a variable and then solidifying it preserves limits, and so

$$\mathbb{Z}[\beta,t]^\wedge_{(p,t)}\otimes^{\mathrm{L}}_{\mathbb{Z}_\blacksquare}\mathbb{Z}[u^{\pm1}]_\blacksquare\simeq\mathbb{Z}[\beta,t,u^{\pm1}]^\wedge_{(p,t)}\,.$$

Thus, we do get an action $\overline{X}^*_\blacksquare\times\mathbb{T}\rightarrow\overline{X}^*_\blacksquare$ simply by (p,t)-completing the map Δ above. Here and in the following, all products are taken in the ∞-category $\mathrm{AnStk}_{\mathbb{Z}_\blacksquare}$ of analytic stacks over $\mathbb{Z}_\blacksquare$. We let $\mathbb{T}^\bullet\colon\Delta^{\mathrm{op}}\rightarrow\mathrm{AnStk}_{\mathbb{Z}_\blacksquare}$ denote the simplicial analytic stack corresponding to the underlying $\mathbb{E}_1$-structure of the $\mathbb{E}_\infty$-group object $\mathbb{T}$, and we let $\overline{X}^*_\blacksquare\times\mathbb{T}^\bullet\colon\Delta^{\mathrm{op}}\rightarrow\mathrm{AnStk}_{\mathbb{Z}_\blacksquare}$ denote the simplicial analytic stack corresponding to the $\mathbb{T}$-action on $\overline{X}^*_\blacksquare$. Finally, let

$$\mathrm{B}\mathbb{T}\coloneqq\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}}\mathbb{T}^n\quad\text{and}\quad\overline{X}^*_\blacksquare/\mathbb{T}\coloneqq\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}}\overline{X}^*_\blacksquare\times\mathbb{T}^n\,.$$

5.17. Lemma. — *Let $\mathcal{O}_{\overline{X}^*/\mathbb{T}}\in\mathcal{D}(\mathrm{B}\mathbb{T})$ denote the pushforward of the structure sheaf of $\overline{X}^*_\blacksquare/\mathbb{T}$. Then pushforward along $\overline{X}^*_\blacksquare/\mathbb{T}\rightarrow\mathrm{B}\mathbb{T}$ induces a symmetric monoidal equivalence of ∞-categories*

$$\mathcal{D}(\overline{X}^*_\blacksquare/\mathbb{T})\simeq\mathrm{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

Proof. The same argument as in 5.16 shows $\overline{X}^*_\blacksquare\times\mathbb{T}^n\simeq\operatorname{AnSpec}(\mathbb{Z}[\beta,t,u_1^{\pm1},\dots,u_n^{\pm1}]^\wedge_{(p,t)})_\blacksquare$. By definition, $\mathcal{D}(\mathrm{B}\mathbb{T})\simeq\lim_{[n]\in\Delta}\mathcal{D}(\mathbb{T}^n)$ and $\mathcal{D}(\overline{X}^*_\blacksquare/\mathbb{T})\simeq\lim_{[n]\in\Delta}\mathcal{D}(\overline{X}^*_\blacksquare\times\mathbb{T}^n)$, where the cosimplicial limits are taken along the pullback functors. Observe that the pushforward functors $\pi_*\colon\mathcal{D}(\overline{X}^*_\blacksquare\times\mathbb{T}^n)\rightarrow\mathcal{D}(\mathbb{T}^n)$ commute with these pullbacks. Indeed, if we would take the limit along the !-pullbacks, this would follow from proper base change (by passing to right adjoints). Since $\mathbb{Z}\rightarrow\mathbb{Z}[u^{\pm1}]$ is smooth of relative dimension 1 and $\Omega^1_{\mathbb{Z}[u^{\pm1}]/\mathbb{Z}}\cong\mathbb{Z}[u^{\pm1}]\,\mathrm{d}u$ is a free module of rank 1, [CS19, Theorem 11.6] shows $\pi^!\simeq\Sigma^{-1}\pi^*$, and so we get commutativity for the $*$-pullbacks as well.

It follows that $\mathcal{O}_{\overline{X}^*/\mathbb{T}}\in\mathcal{D}(\mathrm{B}\mathbb{T})$ is given by the degree-wise pushforwards of the structure sheaves $\mathcal{O}_{\overline{X}^*_\blacksquare\times\mathbb{T}^n}$, that is, by $\mathbb{Z}[\beta,t,u_1^{\pm1},\dots,u_n^{\pm1}]^\wedge_{(p,t)}\in\mathcal{D}(\mathbb{T}^n)$ for all $[n]\in\Delta$. In every degree, the pushforward induces an equivalence

$$\mathcal{D}(\overline{X}^*_\blacksquare\times\mathbb{T}^n)\xrightarrow{\simeq}\mathrm{Mod}_{\mathbb{Z}[\beta,t,u_1^{\pm1},\dots,u_n^{\pm1}]^\wedge_{(p,t)}}\big(\mathcal{D}(\mathbb{T}^n)\big)\,.$$

Using this observation, $\mathcal{D}(\overline{X}^*_\blacksquare/\mathbb{T})\simeq\mathrm{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$ is completely formal. □

5.18. Graded objects and sheaves on $\mathrm{B}\mathbb{T}$. — Let $\mathbb{G}_{m,\mathbb{Z}_\blacksquare}\coloneqq\mathbb{G}_m\times\operatorname{AnSpec}\mathbb{Z}_\blacksquare$. By adapting the usual proof, it's not hard to show that

$$\mathcal{D}(\mathrm{B}\mathbb{G}_{m,\mathbb{Z}_\blacksquare})\simeq\mathrm{Gr}\,\mathcal{D}(\mathbb{Z}_\blacksquare)\,.$$

is the ∞-category of graded solid condensed abelian groups. Since we have a map of analytic stacks $c\colon \mathrm{B}\mathbb{T}\rightarrow \mathrm{B}\mathbb{G}_{m,\mathbb{Z}_\blacksquare}$, we get a pullback functor $c^*\colon \operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)\rightarrow \mathcal{D}(\mathrm{B}\mathbb{T})$. In this way, we can associate to any graded solid condensed $\mathbb{Z}$-module a quasi-coherent sheaf on $\mathrm{B}\mathbb{T}$.

We don't know if c^* is fully faithful (it probably isn't), but at least it's fully faithful when restricted to the full sub-∞-category $\operatorname{Gr}\mathcal{D}(\mathbb{Z})\subseteq \operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)$ spanned by the discrete graded $\mathbb{Z}$-modules. Indeed, for discrete objects, solidification doesn't do anything, and so for all $[n]\in\Delta$ the functor $\mathcal{D}(\mathbb{G}_{m,\mathbb{Z}_\blacksquare}^n)\rightarrow\mathcal{D}(\mathbb{T}^n)$, given by solidifying $u_i^{\pm1}$ for $i=1,\dots,n$, is fully faithful when restricted to discrete objects.

The following lemma takes this one step further and allows us to regard the graded $\mathbb{Z}_p[\beta][\![t]\!]$-modules $\operatorname{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as sheaves on $\mathcal{D}(\mathrm{B}\mathbb{T})$ without loss of information.

5.19. Lemma. — *Let $\mathbb{Z}_p[\beta][\![t]\!]\in\operatorname{Gr}\mathcal{D}(\mathbb{Z})$ denote the graded (p,t)-completion of the discrete graded ring $\mathbb{Z}[\beta,t]$ and equip $\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}))^\wedge_{(p,t)}$ with the (p,t)-completed graded tensor product. Then after (p,t)-completion, c^* induces a fully faithful symmetric monoidal functor*

$$\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}\big(\operatorname{Gr}\mathcal{D}(\mathbb{Z})\big)^\wedge_{(p,t)}\longrightarrow \operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

Proof. To construct the desired functor, we compose with (p,t)-completion to obtain functors

$$\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}\big(\operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)\big)\xrightarrow{c^*}\operatorname{Mod}_{c^*(\mathbb{Z}_p[\beta][\![t]\!])}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\xrightarrow{(-)^\wedge_{(p,t)}}\operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

Note that this composition is symmetric monoidal. Indeed, c^* is symmetric monoidal; to see the same for $(-)^\wedge_{(p,t)}$, we need to check that the tensor product in the symmetric monoidal ∞-category $\operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$ preserves (p,t)-complete objects.[5.3] But the pullback functors, along which the limit $\operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))\simeq\lim_{[n]\in\Delta}\mathcal{D}(\overline{X}^*_\blacksquare\times\mathbb{T}^n)$ is taken, all preserve (p,t)-complete objects, because they preserve limits (see the argument in 5.16). Moreover, by the same argument as for the solid tensor product, the tensor product in $\mathcal{D}(\overline{X}^*_\blacksquare\times\mathbb{T})$ also preserves (p,t)-complete objects. Hence we get the same for $\operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$.

Clearly $(-)^\wedge_{(p,t)}\circ c^*$ factors through $\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare))^\wedge_{(p,t)}$. By restricting to the full sub-∞-category $\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}))^\wedge_{(p,t)}$, we get the desired functor

$$\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}\big(\operatorname{Gr}\mathcal{D}(\mathbb{Z})\big)^\wedge_{(p,t)}\longrightarrow \operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

We've already seen that this functor is symmetric monoidal. Fully faithfulness can be checked modulo (p,t), so it'll be enough to check that $\operatorname{Mod}_{\mathbb{F}_p[\beta]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}))\rightarrow\operatorname{Mod}_{c^*(\mathbb{F}_p[\beta])}(\mathcal{D}(\mathrm{B}\mathbb{T}))$ is fully faithful. This follows from the fact that $c^*\colon\operatorname{Gr}\mathcal{D}(\mathbb{Z})\rightarrow\mathcal{D}(\mathrm{B}\mathbb{T})$ is fully faithful, as we've seen in 5.18. ☐

5.20. Lemma. — *Let $X^*\subseteq\overline{X}^*$ be the subset $\operatorname{Spa}\mathbb{Z}[\beta,t]^\wedge_{(p,t)}\smallsetminus\{p=0,\beta t=0\}$. Then X^* is a Tate adic space and its associated analytic stack $X^*_\blacksquare$ can be written as the following pushout*

[5.3] By contrast, even though the solid tensor product preserves p-complete objects, it's not true that the graded solid tensor product on $\operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)$ preserves graded p-complete objects, because being p-complete is not preserved under infinite direct sums.

in analytic stacks:

$$\begin{array}{ccc} \operatorname{AnSpec}\Big(\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big[\tfrac{1}{p\beta t}\Big],\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big)_{\blacksquare} & \longrightarrow & \operatorname{AnSpec}\Big(\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big[\tfrac{1}{\beta t}\Big],\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big)_{\blacksquare} \\ \downarrow & \ulcorner. & \downarrow \\ \operatorname{AnSpec}\Big(\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big[\tfrac{1}{p}\Big],\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big)_{\blacksquare} & \longrightarrow & X^*_{\blacksquare} \end{array}$$

Moreover, the $\mathbb{T}$-action on $\overline{X}^*_{\blacksquare}$ restricts to an action on $X^*_{\blacksquare}$, and if $\mathcal{O}_{X^*/\mathbb{T}} \in \mathcal{D}(\mathrm{B}\mathbb{T})$ denotes the pushforward of the structure sheaf of $X^*_{\blacksquare}/\mathbb{T}$, then pushforward along $X^*_{\blacksquare}/\mathbb{T} \to \mathrm{B}\mathbb{T}$ induces a symmetric monoidal equivalence

$$\mathcal{D}(X^*_{\blacksquare}/\mathbb{T}) \simeq \operatorname{Mod}_{\mathcal{O}_{X^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

Proof. By 5.2, $X^*_{\blacksquare}$ is glued together from rational open subsets of $\overline{X}^*$. For example, one can take $U_1 = \{x \in \overline{X}^* \mid |\beta t|_x \leqslant |p|_x \neq 0\}$ and $U_2 = \{x \in \overline{X}^* \mid |p|_x \leqslant |\beta t|_x \neq 0\}$ and then

$$X^*_{\blacksquare} \simeq U_{1,\blacksquare} \sqcup_{(U_1\cap U_2)_{\blacksquare}} U_{2,\blacksquare}\,.$$

To show the desired pushout, it's enough that $Y_{1,\blacksquare} \coloneqq \operatorname{AnSpec}(\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}[1/p],\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)})_{\blacksquare}$ and $Y_{2,\blacksquare} \coloneqq \operatorname{AnSpec}(\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}[1/(\beta t)],\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)})_{\blacksquare}$ form a !-cover after pullback to $U_{1,\blacksquare}$ and $U_{2,\blacksquare}$. This is clear, as $Y_{1,\blacksquare} \times_{\overline{X}^*_{\blacksquare}} U_{1,\blacksquare} \simeq U_{1,\blacksquare}$ and similarly $Y_{2,\blacksquare} \times_{\overline{X}^*_{\blacksquare}} U_{2,\blacksquare} \simeq U_{2,\blacksquare}$.

To see that the $\mathbb{T}$-action on $\overline{X}^*_{\blacksquare}$ restricts to an action on $X^*_{\blacksquare}$, just observe that p and βt are homogeneous elements. The pushout above implies that the pushforward $\mathcal{O}_{X^*} \in \mathcal{D}(\mathbb{Z}_{\blacksquare})$ of the structure sheaf of $X^*_{\blacksquare}$ is given by

$$\mathcal{O}_{X^*} \simeq \mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big[\tfrac{1}{p}\Big] \times_{\mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\left[\frac{1}{p\beta t}\right]} \mathbb{Z}[\beta,t]^{\wedge}_{(p,t)}\Big[\tfrac{1}{\beta t}\Big]\,,$$

the pullback being taken in the derived sense. Now $\mathcal{D}(X^*_{\blacksquare} \times \mathbb{T}^n) \simeq \operatorname{Mod}_{\mathcal{O}_{X^*\times\mathbb{T}^n}}(\mathcal{D}(\mathbb{T}^n))$ holds for all $[n] \in \Delta$, since the same is true for $Y_{1,\blacksquare}$, $Y_{2,\blacksquare}$, and $Y_{1,\blacksquare} \times_{X^*_{\blacksquare}} Y_{2,\blacksquare}$. This finally implies $\mathcal{D}(X^*_{\blacksquare}/\mathbb{T}) \simeq \operatorname{Mod}_{\mathcal{O}_{X^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$, as desired. □

We can finally show idempotence and nuclearity in the situation of Theorem 1.20. To this end, let $Z^* \subseteq X^*$ be union of the closed subsets $\{p = 0\}$ and $\{[p^n]_{\mathrm{ku}}(t) = 0\}$ for all $n \geqslant 0$, where $[p^n]_{\mathrm{ku}}(t) \coloneqq ((1+\beta t)^{p^n} - 1)/\beta$ denotes the p^n-series of the formal group law of ku. For $n, r, s \geqslant 1$ such that $(p-1)p^n > s$, we also let $W^*_{n,r,s} \subseteq X^*$ be the rational open subset determined by $|p^r| \leqslant |[p^n]_{\mathrm{ku}}(t)| \neq 0$, $|(\beta t)^s| \leqslant |p| \neq 0$. Observe that $W^*_{n,r,s}$ is $\mathbb{T}$-equivariant, since it is defined by homogeneous elements.

5.21. Corollary. — *The $\mathbb{T}$-action on $X^*_{\blacksquare}$ restricts to an action on the overconvergent neighbourhood $Z^{*,\dagger}$ of Z^*. Moreover, $\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}} \in \mathcal{D}(X^*_{\blacksquare}/\mathbb{T})$ is idempotent, nuclear, and the colimit of the ind-algebra obtained by killing the pro-idempotent "$\lim$"$_{n,r,s}\, \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$.*

Proof. The proof of Corollary 5.15 can be carried over to show that $Z^* \cap \overline{W}^*_{n,r,s} = \emptyset$ and $X^* \smallsetminus Z^* = \bigcup_{n,r,s} W^*_{n,r,s}$. In particular, the $\mathbb{T}$-equivariant open subsets $X^* \smallsetminus \overline{W}^*_{n,r,s}$ are coinitial among all open neighbourhoods of Z^*, because for an arbitrary $U \supseteq Z^*$, the complement $X^* \smallsetminus U$ is quasi-compact and thus contained in some $W^*_{n,r,s}$. This shows that $Z^{*,\dagger}$ acquires a $\mathbb{T}$-action.

Moreover, Theorem 5.12 shows that $\mathcal{O}_{Z^{*,\dagger}}$ is the colimit of the idempotent nuclear ind-algebra obtained by killing the pro-idempotent "$\lim_{n,r,s}$" $\mathcal{O}_{W^*_{n,r,s}}$. Since $Z^{*,\dagger}\times\mathbb{T}^n\simeq\lim_{U^*\supseteq Z^*}(U^*_\blacksquare\times\mathbb{T})$, where the limit is taken over all $\mathbb{T}$-equivariant open neighbourhoods, and since killing pro-idempotents is compatible with base change in the nuclear case by Lemma 2.15(*c*), we get that $\mathcal{O}_{Z^{*,\dagger}\times\mathbb{T}^n}$ is similarly given by killing "$\lim_{n,r,s}$" $\mathcal{O}_{W^*_{n,r,s}\times\mathbb{T}^n}$ in $\mathcal{D}(X^*_\blacksquare\times\mathbb{T}^n)$. Now let $A\in\mathcal{D}(X^*_\blacksquare/\mathbb{T})$ be the colimit of the ind-algebra given by killing "$\lim_{n,r,s}$" $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$. Then Lemma 5.9 shows that all sufficiently large transition maps in this pro-object are trace-class again. Hence A is idempotent, nuclear, and the base change result from Lemma 2.15(*c*) shows that the pullbacks of A to $X^*_\blacksquare\times\mathbb{T}^n$ agree with $\mathcal{O}_{Z^{*,\dagger}\times\mathbb{T}^n}$ for all $[n]\in\Delta$. This implies $\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}\simeq A$, as both of the maps

$$\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}\longrightarrow\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}\otimes^{\mathrm{L}}_{\mathcal{O}_{X^*_\blacksquare/\mathbb{T}}}A\longleftarrow A$$

become equivalences after pullback to $X^*_\blacksquare\times\mathbb{T}^n$ for all $[n]\in\Delta$. □

§5.3. Understanding the q-Hodge filtration for $\mathbb{Z}/p^\alpha$

Recall from Construction 4.11 that the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is defined via base change from the q-Hodge filtration on the derived q-de Rham complex

$$q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}}\simeq\mathbb{Z}_p\{x\}[\![q-1]\!]\left\{\frac{\phi(x^\alpha)}{[p]_q}\right\}^\wedge_{(p,q-1)}.$$

Let us denote this ring by $q\text{-}D_\alpha$ for short. In the following, we'll also write $\Phi_p(q)$ instead of $[p]_q$, since higher cyclotomic polynomials will appear as well.

5.22. The q-Hodge filtration and lifts of divided powers. — Suppose the q-Hodge filtration on $q\text{-}D_\alpha$ is a q-deformation of the Hodge filtration on $D_\alpha\coloneqq\mathrm{dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}}$. We already know this for $p\geqslant 3$ and all $\alpha\geqslant 2$ as well as for $p=2$ and all even $\alpha\geqslant 4$. We know that D_α is the p-completed PD-envelope of $(x^\alpha)\subseteq\mathbb{Z}_p\{x\}$ and the Hodge filtration is precisely the PD-filtration. Thus, if $\gamma(-)\coloneqq(-)^p/p$ denotes the divided power operation, then $\mathrm{Fil}^*_{\mathrm{Hdg}}D_\alpha$ is the p-complete filtered D_α-algebra generated by the iterated divided powers $\gamma^{(n)}(x^\alpha)$ in filtration degree p^n for all $n\geqslant 0$.

Since $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}D_\alpha/(q-1)\cong\mathrm{Fil}^*_{\mathrm{Hdg}}D_\alpha$, it follows that $\mathrm{Fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}D_\alpha$ must be generated as a $(p,q-1)$-complete filtered $q\text{-}D_\alpha$-algebra by the element $(q-1)$ in filtration degree 1 and lifts $\tilde{\gamma}_q^{(n)}(x^\alpha)$ of $\gamma^{(n)}(x^\alpha)$ in filtration degree p^n for all $n\geqslant 0$. Thus, to describe the q-Hodge filtration, it will be enough to give a description of these lifts. By construction, the q-PD envelope $q\text{-}D_\alpha$ contains lifts of divided powers, but it's not clear at all that these can be chosen to lie in the required degrees of the q-Hodge filtration.

The following technical lemma shows existence of these lifts along with some structural information about them, and we'll even see an explicit recursive construction in the proof. Moreover, all of this works for $\alpha\geqslant 2$ without any restrictions in the case $p=2$.

5.23. Lemma. — *For $\alpha\geqslant 2$ and all primes p, there are polynomials $\Gamma_n\in\mathbb{Z}_p\{x\}[q]$ with the following properties:*

(*a*) $\Gamma_n\equiv x^{p^n}\mod (q-1)^{p-1}$ *and* $\Gamma_n\in(x^p,(q-1)^{p-1})^{p^{n-1}}$.

(*b*) $\Gamma_n\in((\phi^i(x),\Phi_{p^i}(q))^p,\Phi_{p^i}(q)^{p-1})^{p^{n-1-i}}$ *for all* $1\leqslant i\leqslant n-1$.

(*c*) $\Gamma_n \in (\phi^n(x), \Phi_{p^n}(q))$.

(*d*) $(\Gamma_n)^\alpha \in \prod_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha$

In particular, $(\Gamma_n)^\alpha$ *is contained in the ideal* $(x^\alpha, q-1)^{p^n}$ *and*

$$\widetilde{\gamma}_q^{(n)}(x^\alpha) \coloneqq \frac{(\Gamma_n)^\alpha}{\prod_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}}} \in \mathrm{Fil}^{p^n}_{q\text{-Hdg}}\, q\text{-}D_\alpha$$

is a lift of the n*-fold iterated divided power* $\gamma^{(n)}(x^\alpha)$ *and contained in the* $(p^n)^{th}$ *step of the* q*-Hodge filtration.*

Proof. We'll do a proof by induction. For the base case of the induction, $n=0$, let $\Gamma_0 \coloneqq x$. All of the statements are trivial in this case.

For the induction step, we first want to construct the element Γ_n. For this, let P_n, Q_n be some polynomials in $\mathbb{Z}[q]$ such that $p = P_n(q)(q-1)^{(p-1)p^{n-1}} + Q_n(q)\Phi_{p^n}(q)$. Note that such polynomials always exist, since $\Phi_{p^n}(1) = p$ and $\Phi_{p^n}(q) \equiv (q-1)^{(p-1)p^{n-1}} \mod p$, so

$$\frac{\Phi_{p^n}(q) - (q-1)^{(p-1)p^{n-1}}}{p}$$

is a unit modulo $(q-1)^{(p-1)p^{n-1}}$. Now define

$$\Gamma_n \coloneqq (\Gamma_{n-1})^p + P_n(q)(q-1)^{(p-1)p^{n-1}}\delta(\Gamma_{n-1}) = \phi(\Gamma_{n-1}) - Q_n(q)\Phi_{p^n}(q)\delta(\Gamma_{n-1}).$$

Statement (*a*) follows trivially. For (*b*) and (*c*), by Lemma 5.24 below it's enough to check that $p \cdot \Gamma_n$ is contained in these ideals. We have

$$\begin{aligned} p \cdot \Gamma_n &= p \cdot (\Gamma_{n-1})^p + P_n(q)(q-1)^{(p-1)p^{n-1}}\big(\phi(\Gamma_{n-1}) - (\Gamma_{n-1})^p\big) \\ &= p \cdot \phi(\Gamma_{n-1}) - Q_n(q)\Phi_{p^n}(q)\big(\phi(\Gamma_{n-1}) - (\Gamma_{n-1})^p\big). \end{aligned}$$

Now $(\Gamma_{n-1})^p$ and $\phi(\Gamma_{n-1})$ are contained in each one of the ideals from (*b*). Indeed, for $(\Gamma_{n-1})^p$, this follows from statements (*b*) and (*c*) of the induction hypothesis, and for $\phi(\Gamma_{n-1})$ this follows similarly from (*a*) and (*b*). Therefore, the first of the two equations above shows that $p \cdot \Gamma_n$ is contained in each of the ideals from (*b*). Similarly, using statement (*c*) of the induction hypothesis, we get $\phi(\Gamma_{n-1}) \in (\phi^n(x), \Phi_{p^n}(q))$ and so the second of the equations above shows that $p \cdot \Gamma_n$ is contained in this ideal as well. This finishes the induction step for (*b*) and (*c*).

It remains to show statement (*d*). By [BS19, Lemma 16.10], $q\text{-}D_\alpha$ is $(p, q-1)$-completely flat over $\mathbb{Z}_p[\![q-1]\!]$ and thus flat on the nose over $\mathbb{Z}[q]$. Therefore

$$\prod_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha = \bigcap_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha\,.$$

To show that $(\Gamma_n)^\alpha \in \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha$ for $1 \leqslant i \leqslant n-1$, by the already proven statement (*b*), it's enough to show the same for any element in the ideal $((\phi^i(x).\Phi_{p^i}(q))^p, \Phi_{p^i}(q)^{p-1})^{\alpha p^{n-1-i}}$. So consider a monomial of the form

$$\big(\phi^i(x)^j\Phi_{p^i}(q)^k\big)^\ell\Phi_{p^i}(q)^{(p-1)m}\,,$$

where $j+k=p$ and $\ell+m=\alpha p^{n-1-i}$. By construction, $\phi(x)^\alpha$ becomes divisible by $\Phi_p(q)$ in q-D_α and so $\phi^i(x)^\alpha\in\Phi_{p^i}(q)\cdot q$-$D_\alpha$. Hence $\phi^i(x)^{j\ell}$ is divisible by $\Phi_{p^i}(q)^{\lfloor j\ell/\alpha\rfloor}$. It will therefore be enough to show

$$\left\lfloor\frac{j\ell}{\alpha}\right\rfloor+k\ell+(p-1)m\geqslant p^{n-i}\,.$$

This is straightforward: For $\ell=0$, the inequality follows from $\alpha(p-1)\geqslant p$ as $\alpha\geqslant 2$. In general, if we replace (j,k) by $(j-1,k+1)$, the left-hand side changes by at least $\ell-\lfloor\ell/\alpha\rfloor-1$; for $\ell\geqslant 1$ and $\alpha\geqslant 2$ this term is always nonnegative. Therefore we may assume $j=p$, $k=0$, and we must show $\lfloor p\ell/\alpha\rfloor+(p-1)m\geqslant p^{n-i}$. If $p=2$ and $\alpha=2$, this becomes the equality $\ell+m=2^{n-i}$ and so the inequality is sharp in this case. If $p\geqslant 3$ or $\alpha\geqslant 3$, we have $(p-1)-\lfloor p/\alpha\rfloor-1\geqslant 0$ and so by the same argument as before we may assume $\ell=\alpha p^{n-1-i}$, $m=0$. The the desired inequality follows from $\alpha(p-1)\geqslant p$ again.

A similar but easier argument shows that every element in $(\phi^n(x),\Phi_{p^n}(q))^\alpha$ becomes divisible by $\Phi_{p^n}(q)$ in q-D_α and we have an inclusion of ideals $(x^p,(q-1)^{p-1})^{\alpha p^{n-1}}\subseteq(x^\alpha,q-1)^{p^n}$ in $\mathbb{Z}_p\{x\}[q]$. This finishes the proof of (*d*) and shows $(\Gamma_n)^\alpha\in(x^\alpha,q-1)^{p^n}$. Hence $\widetilde{\gamma}_q^{(n)}(x^\alpha)$ is really contained in the $(p^n)^{\text{th}}$ step of the q-Hodge filtration and it lifts $\gamma^{(n)}(x^\alpha)$ by (*a*). □

5.24. Lemma. — *If $J\subseteq\mathbb{Z}_p\{x\}[q]$ is any of the ideals in Lemma 5.23(b) or (c), then $\mathbb{Z}_p\{x\}[q]/J$ is p-torsion free.*

Proof. Consider the map $\psi_i\colon\mathbb{Z}_p\{x\}[q]\to\mathbb{Z}_p\{x\}[q]$ given by the i-fold iterated Frobenius $\phi^i\colon\mathbb{Z}_p\{x\}\to\mathbb{Z}_p\{x\}$ and $q\mapsto\Phi_{p^i}(q)$. If we replace $\phi^i(x)$ and $\Phi_{p^i}(q)$ in the definition of J by x and q, respectively, we obtain an ideal $J_0\subseteq\mathbb{Z}_p\{x\}[q]$ such that

$$\mathbb{Z}_p\{x\}/J\cong\mathbb{Z}_p\{x\}/J_0\otimes_{\mathbb{Z}_p\{x\}[q],\psi_i}\mathbb{Z}_p\{x\}[q]\,.$$

Now ϕ^i is flat by [BS19, Lemma 2.11] and $q\mapsto\Phi_{p^i}(q)$ is finite free, as the polynomial $\Phi_{p^i}(q)$ is monic. So ψ_i is flat and it suffices to show that $\mathbb{Z}_p\{x\}[q]/J_0$ is p-torsion free. But $\mathbb{Z}_p\{x\}[q]$ is a free module over $\mathbb{Z}_p$ with basis given by monomials in $x,\delta(x),\delta^2(x),\dots$ and q. By construction, J_0 is a free submodule on a subset of that basis. It follows that $\mathbb{Z}_p\{x\}[q]/J_0$ is free over $\mathbb{Z}_p$, hence p-torsion free. □

Finally, as a simple corollary of Lemma 5.23, we get an elementary proof of Theorem 3.10(*a*).

Proof of Theorem 3.10(a). We already know from Lemma 3.8 that

$$\big(\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\big)/(q-1)\longrightarrow\mathrm{Fil}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}$$

is degree-wise injective, so it suffices to show surjectivity. It'll be enough to show that for each of the generators of $J=(x_1^{\alpha_1},\dots,x_r^{\alpha_r})$ and all $n\geqslant 0$, the n-fold iterated divided power $\gamma^{(n)}(x_i^{\alpha_i})$ admits a lift which lies in the $(p^n)^{\text{th}}$ step of the q-Hodge filtration. Thus, it's enough to treat the case $A=\mathbb{Z}_p\{x\}$, $R=\mathbb{Z}_p\{x\}/x^\alpha$, where $\alpha\geqslant 2$. Then Lemma 5.23 finishes the proof. □

§5.4. Proof of Theorems 1.19 and 1.20

Recall from 2.11 that Efimov [Efi-Lim] constructs a fully faithful strongly continuous functor

$$\mathrm{Nuc}\big(\mathcal{D}(\mathbb{Z}_p[\![q-1]\!]_\blacksquare)\big)\longrightarrow\mathrm{Nuc}\big(\mathbb{Z}_p[\![q-1]\!]\big)\,.$$

Efimov shows that this functor is an equivalence on bounded objects. Since $\mathrm{A}_{\mathrm{KU},p}$ is bounded, it's therefore contained in the essential image of $\mathrm{Nuc}(\mathcal{D}(\mathbb{Z}_p[\![q-1]\!]_\blacksquare))$. Its preimage (and in fact, the right adjoint to Efimov's functor) can be explicitly described: $\mathrm{A}_{\mathrm{KU},p}$ is obtained by killing the pro-idempotent "$\lim_{\alpha\geqslant 2}$" $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. We can regard each $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as a $(p,q-1)$-complete solid condensed $\mathbb{Z}_p[\![q-1]\!]$-module by $(p,q-1)$-completing the associated discrete condensed abelian group. By killing the pro-idempotent "$\lim_{\alpha\geqslant 2}$" $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ in $\mathcal{D}(\mathbb{Z}_p[\![q-1]\!]_\blacksquare)$, we get an idempotent nuclear algebra in $\mathrm{Ind}\,\mathcal{D}(\mathbb{Z}_p[\![q-1]\!]_\blacksquare)$. Its colimit is the preimage of $\mathrm{A}_{\mathrm{KU},p}$.

In the following, we'll regard $\mathrm{A}_{\mathrm{KU},p}$ as a solid condensed $\mathbb{Z}_p[\![q-1]\!]$-module. In the same way, using the formalism developed in §5.2, we'll regard $\mathrm{A}^*_{\mathrm{ku},p}$ as a quasi-coherent sheaf on the analytic stack $\overline{X}^*_\blacksquare/\mathbb{T}$, where $\overline{X}^* \coloneqq \mathrm{Spa}\,\mathbb{Z}[\beta,t]^\wedge_{(p,t)}$. As in §§5.1–5.2, we'll also denote $X^* \coloneqq \overline{X}^* \smallsetminus \{p=0,\beta t=0\}$ and $X \coloneqq \mathrm{Spa}\,\mathbb{Z}_p[\![q-1]\!] \smallsetminus \{p=0,q=1\}$.

5.25. Lemma. — *$\mathrm{A}^*_{\mathrm{ku},p}$ vanishes after (p,β)-completion and after (p,t)-completion. $\mathrm{A}_{\mathrm{KU},p}$ vanishes after $(p,q-1)$-completion. In particular, $\mathrm{A}^*_{\mathrm{ku},p}$ and $\mathrm{A}_{\mathrm{KU},p}$ are already contained in the full sub-∞-categories $\mathcal{D}(X^*_\blacksquare/\mathbb{T}) \simeq \mathrm{Mod}_{\mathcal{O}_{X^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$ and $\mathcal{D}(X_\blacksquare) \simeq \mathrm{Mod}_{\mathcal{O}_X}(\mathcal{D}(\mathbb{Z}_\blacksquare))$.*

Proof. By Nakayama's lemma it's enough to show $\mathrm{A}^*_{\mathrm{ku},p}/(p,\beta)\simeq 0$ and $\mathrm{A}^*_{\mathrm{ku},p}/(p,t)\simeq 0$. Since $\mathrm{A}_{\mathrm{KU},p}[\beta^{\pm 1}]$ is a $\mathrm{A}^*_{\mathrm{ku},p}$-algebra, this will also show $\mathrm{A}_{\mathrm{KU},p}/(p,q-1)\simeq 0$. Since $\mathrm{A}^*_{\mathrm{ku},p}/t$ is concentrated in nonnegative graded degrees, it is automatically β-complete, so it's already enough to show $\mathrm{A}^*_{\mathrm{ku},p}/(p,\beta)\simeq 0$. Now $\mathrm{ku}\to\mathrm{ku}/(p,\beta)\simeq\mathbb{F}_p$ is a map of $\mathbb{E}_\infty$-ring spectra, so we can invoke base change to see $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{ku}\otimes\mathbb{Q})/\mathrm{ku})/(p,\beta)\simeq\mathrm{TC}^{-,\mathrm{ref}}((\mathbb{F}_p\otimes\mathbb{Q})/\mathbb{F}_p)\simeq 0$.

It follows that $(\mathrm{A}^*_{\mathrm{ku},p})^\wedge_{(p,\beta t)}\simeq 0$. Using the pullback square from Lemma 5.20 and a version of the Beauville–Laszlo theorem (see [Wag24, Lemma 2.4] for example), we get

$$\mathrm{A}^*_{\mathrm{ku},p}\simeq\mathrm{A}^*_{\mathrm{ku},p}\otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}}\mathcal{O}_{X^*/\mathbb{T}}$$

and so $\mathrm{A}^*_{\mathrm{ku},p}$ is indeed a $\mathcal{O}_{X^*/\mathbb{T}}$-module. The argument for $\mathrm{A}_{\mathrm{KU},p}$ is analogous. □

To finish the proof, we analyse the pro-systems "$\lim_{n,r,s}$" $\mathcal{O}_{W_{n,r,s}}$ and "$\lim_{n,r,s}$" $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$ from Corollaries 5.15 and 5.21.

5.26. Lemma. — *For every fixed $\alpha\geqslant 2$ and all sufficiently large n, r, s, there exist maps*

$$\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}\longrightarrow\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}}\mathcal{O}_{X^*/\mathbb{T}}\,,$$
$$\mathcal{O}_{W_{n,r,s}}\longrightarrow q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\otimes^{\mathrm{L}}_{\mathbb{Z}_p[\![q-1]\!]_\blacksquare}\mathcal{O}_X$$

in $\mathcal{D}(X^*_\blacksquare/\mathbb{T})$ and $\mathcal{D}(X_\blacksquare)$, respectively.

Proof. By construction, the q-de Rham complex $q\text{-dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ contains elements of the form $\phi^i(\phi(p^\alpha)/\Phi_p(q))=p^\alpha/\Phi_{p^{i+1}}(q)$ for all $i\geqslant 0$, and $p^\alpha\in\mathrm{Fil}^1_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. When we regard $\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as a graded $\mathbb{Z}_p[\beta][\![t]\!]$-module, this precisely means that p^α is divisible by t. Hence we have elements of the form

$$\frac{p^{(n+1)\alpha}}{[p^n]_{\mathrm{ku}}(t)}=\frac{p^\alpha}{t}\cdot\frac{\phi(p^\alpha)}{\Phi_p(q)}\cdots\frac{\phi^n(p^\alpha)}{\Phi_{p^n}(q)}\in\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$$

for all $n\geqslant 0$. Similarly, there exist elements of the form $(\beta t)^N/p$ in $\mathrm{Fil}^*_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ for sufficiently large N. Indeed, the ring $q\text{-dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is $(p,\Phi_p(q))$-complete and contains an element of the form $p^\alpha/\Phi_p(q)$. Applying the nilpotence criterion from [BCM20, Proposition 2.5], we see that $\Phi_p(q)$ is nilpotent in $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/p$. Then $(q-1)^{p-1}$ must be nilpotent as well, and so $(q-1)^N$ must be divisible by p in $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ for $N \gg 0$.

In particular, as soon as we invert βt in $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/p$, we see that p will be invertible as well, and so

$$\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}} \simeq \operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\left[\tfrac{1}{p}\right].$$

Moreover, as soon as p is invertible, $[p^n]_{\mathrm{ku}}(t)$ will be invertible for all $n \geqslant 0$. Choosing $s > N$, we see that $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ contains an element of the form $(\beta t)^s/p$ which is topologically nilpotent, hence automatically solid. Moreover, for $(p-1)p^n > s$ and $r > (n+1)\alpha$, we get an element of the form $p^r/[p^n]_{\mathrm{ku}}(t)$, which is again topologically nilpotent and thus solid. Thus, for such n, r, and s, a map $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}} \to \operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}[1/p]$ exists. The argument in the q-Hodge case is analogous. □

5.27. Remark. — As a consequence of the derived version of Theorem 1.3(*a*) that we'll show in [W-Hab], $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/(q^{p^n}-1)$ is an algebra over the p-typical Witt vectors $\mathbb{W}_{p^n}(\mathbb{Z}/p^\alpha)$. Since this ring is $p^{\alpha+n}$-torsion, we already have elements of the form $p^{\alpha+n}/(q^{p^n}-1)$ in $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ for all $n \geqslant 0$.

5.28. Lemma. — *For all fixed n, r, s such that $(p-1)p^n > s$ and all sufficiently large $\alpha \geqslant 2$, there exist canonical maps*

$$\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}} \longrightarrow \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}\,,$$
$$q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathbb{Z}_p[\![q-1]\!]_\blacksquare} \mathcal{O}_X \longrightarrow \mathcal{O}_{W_{n,r,s}}$$

in $\mathcal{D}(X^*_\blacksquare/\mathbb{T})$ and $\mathcal{D}(X_\blacksquare)$, respectively.

Proof. Let $q\text{-}D_\alpha \coloneqq q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}}$ as in §5.3 and let $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{D}_\alpha$ denote its completed q-Hodge filtration. It follows from 5.22 that $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{D}_\alpha$ is generated as a (p,t)-complete graded $\mathbb{Z}_p[\beta][\![t]\!]$-algebra by lifts of the iterated divided powers $\gamma^{(d)}(x^\alpha)$ sitting in filtration degree $2p^d$. Thanks to Lemma 5.23, we know that these lifts can be chosen to be of the form

$$\frac{(\Gamma_d)^\alpha}{t^{p^d}\prod_{i=1}^d \Phi_{p^i}(q)^{p^{d-i}}}$$

for $\Gamma_d \in (x^p, (q-1)^{p-1})^{p^{d-1}}$. The extra t^{p^d} in the denominator accomodates for the fact that this element must sit in degree $2p^N$. Note that the denominators all become invertible in $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$, but that's not enough to obtain the desired map: We must send the generators to *solid* elements, to ensure that the map extends over the (p,t)-completion.

By construction, $(q-1)^s/p$ and $p^r/[p^n]_{\mathrm{ku}}(t)$ are solid. In particular, $p^r/(t\Phi_{p^i}(q))$ is solid for all $i = 1, \dots, n$. For $i > n$, we have $(p-1)p^{i-1} > s$ by assumption. Hence $(q-1)^{(p-1)p^{i-1}}/p$ is topologically nilpotent in $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$. It follows that $\Phi_{p^i}(q) = p(1+w)$, where w is topologically nilpotent, and so $p^r/\Phi_{p^i}(q)$ is solid in $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$ for $i > n$. Therefore the elements $p^{2r}/(t\Phi_{p^i}(q))$ are solid for all $i \geqslant 1$.

By choosing α large enough, we can ensure that for every monomial $x^{pi}(q-1)^{(p-1)j}$ in the ideal $(x^p, (q-1)^{p-1})^{\alpha p^{d-1}}$ we have $pi \geqslant 2rp^d$ or $(p-1)j \geqslant sp^d$. Now $(\Gamma_d)^\alpha$ is a $\mathbb{Z}_p\{x\}[q]$-linear combination of such terms. It follows that the δ-ring map $\mathbb{Z}_p\{x\} \to \mathbb{Z}_p$ sending $x \mapsto p$ can really be extended to a map $\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{D}_\alpha \to \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$ of graded solid condensed $\mathbb{Z}_p[\beta][\![t]\!]$-algebras. Via (p,t)-completed base change along $\mathbb{Z}_p\{x\} \to \mathbb{Z}_p$ and extension of scalars to $\mathcal{O}_{X^*/\mathbb{T}}$, this yields the desired map

$$\operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}} \longrightarrow \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$$

The argument in the q-Hodge case is analogous. □

Proof of Theorems 1.19 and 1.20. By Lemma 5.25 and Lemma 2.15(c), we see that $\mathrm{A}^*_{\mathrm{ku},p}$ is the colimit of the idempotent nuclear ind-algebra given by killing the pro-idempotent

$$\text{“}\lim_{\alpha\geqslant 2}\text{”} \operatorname{Fil}^*_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}}$$

in $\mathcal{D}(X^*_\blacksquare/\mathbb{T})$. By Lemmas 5.26 and 5.28, we see that this pro-system is equivalent to “$\lim$”$_{n,r,s} \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$, which proves $\mathrm{A}^*_{\mathrm{ku},p} \simeq \mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}$. The argument for $\mathrm{A}_{\mathrm{KU},p} \simeq \mathcal{O}_{Z^\dagger}$ is completely analogous. □

5.29. Remark. — An obvious adaptation of Theorem 4.20 shows that $\mathrm{A}_{\mathrm{KU},p}$ and $\mathrm{A}^*_{\mathrm{ku},p}$ are connective. Therefore the condition from Theorem 5.12(b) is satisfied and so $\mathcal{O}_{Z^\dagger}$ and $\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}$ are really the pushforwards of the respective structure sheaves.

Appendix A. The global q-de Rham complex

Let p be a prime. In [BS19, §16], Bhatt and Scholze construct a functorial $(p, q-1)$-complete q-de Rham complex relative to any q-PD pair (D, I). This verifies Scholze's conjecture [Sch17, Conjecture 3.1] after p-completion, but leaves open the global case. There are (at least) two strategies to tackle the global case:

(a) One can glue the global q-de Rham complex from its p-completions and its rationalisation using an arithmetic fracture square.

(b) Following Kedlaya [Ked21, §29], one can construct the global q-de Rham complex as the cohomology of a global q-crystalline site.

Strategy (a) is what Bhatt and Scholze originally had in mind, but they never published the argument. It is essentially straightforward, but not entirely trivial. Since all the global constructions in this paper and the follow-up [W-Hab] proceed similarly by gluing p-completions and rationalisations, with no site-theoretic interpretation like strategy (b) in sight, it will be worthwhile to fill in the missing details of strategy (a). Our goal is to show the following theorem.

A.1. Theorem. — *If A is a Λ-ring and p-torsion free for all primes p, there exists a functor*

$$q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_A \longrightarrow \mathrm{CAlg}\Big(\widehat{\mathcal{D}}_{(q-1)}\big(A[\![q-1]\!]\big)\Big)$$

from the ∞-category of smooth A-algebras into the ∞-category of $(q-1)$-complete $\mathbb{E}_\infty$-algebras over $A[\![q-1]\!]$, satisfying the following properties:

(a) *$q\text{-}\Omega_{-/A}/(q-1) \simeq \Omega_{-/A}$ agrees with the usual de Rham complex functor. Moreover, if $A \rightarrow A'$ is a map of $\mathbb{Z}$-torsion free Λ-rings, there's a base change equivalence*

$$q\text{-}\Omega_{-/A} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{A[\![q-1]\!]} A'[\![q-1]\!] \xrightarrow{\ \simeq\ } q\text{-}\Omega_{(-\otimes_A A')/A'}\,.$$

Modulo $(q-1)$ this reduces to the usual base change equivalence of the de Rham complex.

(b) *For every framed smooth A-algebra $(S, \square)$, the underlying object of $q\text{-}\Omega_{S/A}$ in the derived ∞-category of $A[\![q-1]\!]$ can be represented as*

$$q\text{-}\Omega_{S/A} \simeq q\text{-}\Omega^*_{S/A,\square}\,,$$

where the coordinate-dependent q-de Rham complex $q\text{-}\Omega^*_{S/A,\square}$ is as in [Sch17, §3].

A.2. Remark. — It will be apparent from our proof (and we'll give a precise argument in A.12) that the q-de Rham complex functor lifts canonically to a functor

$$q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_A \longrightarrow \big(\mathrm{DAlg}_{A[\![q-1]\!]}\big)^\wedge_{(q-1)}$$

into $(q-1)$-complete objects of the *the ∞-category of derived commutative $A[\![q-1]\!]$-algebras* $\mathrm{DAlg}_{A[\![q-1]\!]}$ as defined in [Rak21, Definition 4.2.22].

§A.1. Rationalised q-crystalline cohomology

Fix a prime p. Then $(\widehat{A}_p[\![q-1]\!], (q-1))$ is a q-PD pair as in [BS19, Definition 16.1] and so we can use q-crystalline cohomology to construct a functorial $(p, q-1)$-complete q-de Rham complex $q\text{-}\Omega_{S/\widehat{A}_p}$ for every p-completely smooth $\widehat{A}_p$-algebra S. We let $q\text{-}\mathrm{dR}_{-/\widehat{A}_p}$ denote its non-abelian derived functor (or *animation*), which is now defined for all p-complete animated $\widehat{A}_p$-algebras. Observe that animation leaves the values on p-completely smooth $\widehat{A}_p$-algebras unchanged, as can be seen modulo $(p, q-1)$, where it reduces to a well-known fact about derived de Rham cohomology in characteristic p.

Our first goal is to show that after rationalisation derived q-de Rham cohomology is just a base change of derived de Rham cohomology relative to $\widehat{A}_p$. In coordinates, such an equivalence was already constructed in [Sch17, Lemma 4.1] (see A.7 for a review), but here we need a different argument: We want a coordinate-independent equivalence, so we have to work with the definition of the q-de Rham complex via q-crystalline cohomology.

A.3. Lemma. — *For all p-complete animated $\widehat{A}_p$-algebras R there is a functorial equivalence of $\mathbb{E}_\infty$-$(\widehat{A}_p \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$-algebras*

$$q\text{-}\mathrm{dR}_{R/\widehat{A}_p} \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!] \simeq \big(\mathrm{dR}_{R/\widehat{A}_p} \otimes_{\mathbb{Z}} \mathbb{Q}\big)[\![q-1]\!]\,.$$

Proof. By passing to non-abelian derived functors, it's enough to construct such a functorial equivalence for p-completely smooth $\widehat{A}_p$-algebras S. In this case, we can identify derived (q-)de Rham and (q-)crystalline cohomology:

$$q\text{-}\mathrm{dR}_{S/\widehat{A}_p} \simeq \mathrm{R\Gamma}_{q\text{-crys}}\big(S/\widehat{A}_p[\![q-1]\!]\big) \quad \text{and} \quad \mathrm{dR}_{S/\widehat{A}_p} \simeq \mathrm{R\Gamma}_{\mathrm{crys}}\big(S/\widehat{A}_p\big)\,.$$

To construct the desired identification between q-crystalline and crystalline cohomology after rationalisation, let $P \twoheadrightarrow S$ be a surjection from a p-completely ind-smooth δ-$\widehat{A}_p$-algebra. Extend the δ-structure on P to $P[\![q-1]\!]$ via $\delta(q) \coloneqq 0$. Let J be the kernel of $P \twoheadrightarrow S$ and let $D \coloneqq D_P(J)$ be its p-completed PD-envelope. Finally, let $q\text{-}D$ denote the corresponding q-PD-envelope as defined in [BS19, Lemma 16.10]. It will be enough to construct a functorial equivalence

$$q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!] \simeq (D \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]\,.$$

If D° denotes the un-p-completed PD-envelope of J, then $P \to q\text{-}D \to q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!]$ uniquely factors through $D^\circ \to q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!]$. The tricky part is to show that this map extends over the p-completion. Since D° is p-torsion free, its p-completion agrees with $D^\circ[\![t]\!]/(t-p)$. By Lemma A.5 below, for every fixed $n \geqslant 0$, every p-power series in D° converges in the p-adic topology on $(q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!])/(q-1)^n$, so we get indeed our desired extension $D \to q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!]$.

Extending further, we get a map $(D \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!] \to q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!]$ of the desired form. Whether this is an equivalence can be checked modulo $(q-1)$ by the derived Nakayama lemma. Then the base change property from [BS19, Lemma 16.10(3)] finishes the proof—up to verifying convergence for p-power series in D°. ☐

To complete the proof of Lemma A.3, we need to prove two technical lemmas about (q-)divided powers. Let's fix the following notation: According to [BS19, Lemmas 2.15 and 2.17], we may uniquely extend the δ-structure from $q\text{-}D$ to $q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!]$. We still let ϕ and δ denote the extended Frobenius and δ-map. Furthermore, we denote by

$$\gamma(x) = \frac{x^p}{p} \quad \text{and} \quad \gamma_q(x) = \frac{\phi(x)}{[p]_q} - \delta(x)$$

the maps defining a PD-structure and a q-PD structure, respectively. Note that $\gamma(x)$ and $\gamma_q(x)$ make sense for all $x \in q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!]$ since p and $[p]_q$ are invertible.

A.4. Lemma. — *With notation as above, the following is true for the self-maps δ and γ_q of $(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$:*

(*a*) *For all $n\geqslant 1$ and all $\alpha\geqslant 1$, the map δ sends $(q-1)^n\,q\text{-}D$ into itself, and $p^{-\alpha}(q-1)^n\,q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^n\,q\text{-}D$.*

(*b*) *For all $n\geqslant 1$ and all $\alpha\geqslant 1$, the map γ_q sends $(q-1)^n\,q\text{-}D$ into $(q-1)^{n+1}\,q\text{-}D$, and $p^{-\alpha}(q-1)^n\,q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^{n+1}\,q\text{-}D$.*

Proof. Let's prove (*a*) first. Let $x=p^{-\alpha}(q-1)^n y$ for some $y\in q\text{-}D$. Since $q\text{-}D$ is flat over $\mathbb{Z}_p[\![q-1]\!]$ and thus p-torsion free, we can compute

$$\delta(x)=\frac{\phi(x)-x^p}{p}=\frac{(q^p-1)^n\phi(y)}{p^{\alpha+1}}-\frac{(q-1)^{pn}y^p}{p^{p\alpha+1}}\,.$$

As q^p-1 is divisible by $q-1$, the right-hand side lies in $p^{-(p\alpha+1)}(q-1)^n\,q\text{-}D$. If $\alpha=0$, then the right-hand side must also be contained in $q\text{-}D$. But $q\text{-}D\cap p^{-1}(q-1)^n\,q\text{-}D=(q-1)^n\,q\text{-}D$ by flatness again. This proves both parts of (*a*). Now for (*b*), we first compute

$$\gamma_q(q-1)=\frac{\phi(q-1)}{[p]_q}-\delta(q-1)=-(q-1)^2\sum_{i=2}^{p-1}\frac{1}{p}\binom{p}{i}(q-1)^{i-2}\,.$$

Hence $\gamma_q(q-1)$ is divisible by $(q-1)^2$. In the following, we'll repeatedly use the relation $\gamma_q(xy)=\phi(y)\gamma_q(x)-x^p\delta(y)$ from [BS19, Remark 16.6] repeatedly. First off, it shows that

$$\gamma_q\big((q-1)^n x\big)=\phi\big((q-1)^{n-1}x\big)\gamma_q(q-1)-(q-1)^p\delta\big((q-1)^{n-1}x\big)$$

It follows from (*a*) that $\delta((q-1)^{n-1}x)$ and $\phi((q-1)^{n-1}x)$ are divisible by $(q-1)^{n-1}$. Hence $\gamma_q((q-1)^n x)$ is indeed divisible by $(q-1)^{n+1}$. Moreover, we obtain

$$\gamma_q\big(p^{-\alpha}(q-1)^n x\big)=\phi(p^{-\alpha})\gamma_q\big((q-1)^n x\big)-(q-1)^{np}x^p\delta(p^{-\alpha})\,.$$

Now $\phi(p^{-\alpha})=p^{-\alpha}$ and $\delta(p^{-\alpha})$ is contained in $p^{-(p\alpha+1)}\,q\text{-}D$, hence $\gamma_q(p^{-\alpha}(q-1)^n x)$ is contained in $p^{-(p\alpha+1)}(q-1)^n\,q\text{-}D$. This finishes the proof of (*b*). ∎

A.5. Lemma. — *Let $x\in J$. For every $n\geqslant 1$, there are elements $y_0,\dots,y_n\in q\text{-}D$ such that y_0 admits q-divided powers in $q\text{-}D$ and*

$$\gamma^{(n)}(x)=y_0+\sum_{i=1}^{n}p^{-2(p^{i-1}+\dots+p+1)}(q-1)^{(p-2)+i}y_i$$

holds in $q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q}$, where $\gamma^{(n)}=\gamma\circ\dots\circ\gamma$ denotes the n-fold iteration of γ.

Proof. We use induction on n. For $n=1$, we compute

$$\gamma(x)=\frac{x^p}{p}=\gamma_q(x)+\frac{[p]_q-p}{p}\big(\gamma_q(x)+\delta(x)\big)\,.$$

Note that x admits q-divided powers in $q\text{-}D$ since we assume $x\in J$. Then $\gamma_q(x)$ admits q-divided powers again by [BS19, Lemma 16.7]. Moreover, writing $[p]_q=pu+(q-1)^{p-1}$, we find that $([p]_q-p)/p=(u-1)+p^{-1}(q-1)^{p-1}$. Then $(u-1)(\gamma_q(x)+\delta(x))$ admits q-divided powers since $u\equiv 1\mod (q-1)$. This settles the case $n=1$. We also remark that the above equation for $\gamma(x)$ remains true without the assumption $x \in J$ as long as the expression $\gamma_q(x)$ makes sense.

Now assume $\gamma^{(n)}$ can be written as above. We put $z_i = p^{-2(p^{i-1}+\dots+p+1)}(q-1)^{(p-2)+i}y_i$ for short, so that $\gamma^n(x) = y_0 + z_1 + \dots + z_n$. Recall the relations

$$\gamma_q(a+b) = \gamma_q(a) + \gamma_q(b) + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a^i b^{p-i}\,,\ \ \delta(a+b) = \delta(a) + \delta(b) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a^i b^{p-i}\,.$$

The first relation implies that $\gamma_q(y_0 + z_1 + \dots + z_n)$ is equal to $\gamma_q(y_0) + \gamma_q(z_1) + \dots + \gamma_q(z_n)$ plus a linear combination of terms of the form $y_0^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with $0 \leqslant \alpha_i < p$ and $\alpha_0 + \dots + \alpha_n = p$. Now $\gamma_q(y_0)$ admits q-divided powers again. Moreover, Lemma A.4(*b*) makes sure that each $\gamma_q(z_i)$ is contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{(p-2)+i+1}\, q\text{-}D$. It remains to consider monomials $y_0^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Put $m \coloneqq \max\{i \mid \alpha_i \neq 0\}$. If $\alpha_0 = p-1$, then all other α_i must vanish except $\alpha_m = 1$. In this case, the monomial is contained in $p^{-2(p^{m-1}+\dots+p+1)}(q-1)^{(p-2)+m}\, q\text{-}D$. If $\alpha_0 < p-1$, then we get at least one more factor $(q-1)$ and the monomial $y_0^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is contained in $p^{-2(p^m+\dots+p+1)}(q-1)^{(p-2)+m+1}\, q\text{-}D$.

A similar analysis, using the second of the above relations as well as Lemma A.4(*a*), shows that $(u-1)\delta(y_0 + z_1 + \dots + z_n)$ and $p^{-1}(q-1)^{p-1}\delta(y_0 + z_1 + \dots + z_n)$ can be decomposed into a bunch of terms, each of which is either a multiple of $(q-1)$ in q-D, so that it admits q-divided powers, or contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{i+1}\, q\text{-}D$ for some $1 \leqslant i \leqslant n+1$. We conclude that

$$\gamma^{(n+1)}(x) = \gamma_q\big(\gamma^{(n)}(x)\big) + \frac{[p]_q - p}{p}\Big(\gamma_q\big(\gamma^{(n)}(x)\big) + \delta\big(\gamma^{(n)}(x)\big)\Big)$$

can be written in the desired form. ☐

The following remark is irrelevant for the proof of Theorem A.1, but it will be used once in the main text.

A.6. Remark. — There's also an analogue of Lemma A.5 with the roles of D and q-D reversed. For every $x \in J$ and $n \geqslant 1$, there's an infinite sequence $y_0, y_1, \dots, \in D$ such that y_0 admits divided powers and

$$\gamma_q^{(n)}(x) = y_0 + \sum_{i \geqslant 1} p^{-2(p^{i-1}+\dots+1)}(q-1)^{(p-2)+i} y_i$$

holds in $(D \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$. The proof is very similar to Lemma A.5: We write

$$\gamma_q(x) = \left(\gamma(x) + \frac{[p]_q - p}{p}\delta(x)\right)\frac{p}{[p]_q}$$

and $[p]_q = pu + (q-1)^{p-1}$. Then we use induction on $n \geqslant 1$. For the inductive step, we first check that the operations $\gamma(-)$, $(u-1)\delta(-)$ and $p^{-1}(q-1)^{p-1}\delta(-)$ all preserve expressions of the desired form. Then we observe that u is a unit in $\mathbb{Z}_p[\![q-1]\!]$ and so multiplication by $p/[p]_q = u^{-1}\sum_{i \geqslant 0} p^{-i}u^{-i}(q-1)^{(p-1)i}$ also preserves expressions of the desired form.

A.7. The equivalence on q-de Rham complexes. — Suppose we're given a p-completely smooth $\widehat{A}_p$-algebra S together with a p-completely étale framing $\square\colon \widehat{A}_p\langle T_1, \dots, T_d\rangle \to S$. In this case, the q-crystalline cohomology can be computed as a q-de Rham complex

$$R\Gamma_{q\text{-crys}}\big(S/\widehat{A}_p[\![q-1]\!]\big) \simeq q\text{-}\Omega^*_{S/\widehat{A}_p,\square}$$

by [BS19, Theorem 16.22]. Similarly, it's well-known that the crystalline cohomology is given by the ordinary de Rham complex $\Omega^*_{S/\widehat{A}_p}$ (recall that according to our convention in 1.25, all (q-)de Rham complexes of the p-complete ring S will implicitly be p-completed). In this case, an explicit isomorphism of complexes

$$q\text{-}\Omega^*_{S/\widehat{A}_p,\square}\,\widehat{\otimes}_{\mathbb{Z}[\![q-1]\!]}\,\mathbb{Q}[\![q-1]\!]\cong\left(\Omega^*_{S/\widehat{A}_p}\otimes_{\mathbb{Z}}\mathbb{Q}\right)[\![q-1]\!]$$

can be constructed as explained in [Sch17, Lemma 4.1]: One first observes that, after rationalisation, the partial q-derivatives $q\text{-}\partial_i$ can be computed in terms of the usual partial derivative ∂_i via the formula

$$q\text{-}\partial_i=\left(\frac{\log(q)}{q-1}+\sum_{n\geqslant 1}\frac{\log(q)^n}{n!(q-1)}(\partial_i T_i)^{(n-1)}\right)\partial_i\,;$$

see [BMS18, Lemma 12.4]. Here $\log(q)$ refers to the usual Taylor series for the logarithm around $q=1$. Noticing that the first factor is an invertible automorphism, one can then appeal to the general fact that for any abelian group M together with commuting endomorphisms $g_1,\dots,g_d$ and commuting automorphisms $h_1,\dots,h_d$ such that h_i commutes with g_j for $i\neq j$ one always has an isomorphism $\operatorname{Kos}^*(M,(g_1,\dots,g_d))\cong\operatorname{Kos}^*(M,(h_1g_1,\dots,h_dg_d))$ of Koszul complexes.[(A.1)]

We would like to show that this explicit isomorphism is compatible with the one constructed in Lemma A.3. To this end, let's put ourselves in a slightly more general situation: Instead of a p-completely étale framing $\square$ as above, let's assume we're given a surjection $P\twoheadrightarrow S$ from a p-completely ind-smooth $\widehat{A}_p$-algebra P, which is in turn equipped with a p-completely ind-étale framing $\square\colon \widehat{A}_p\langle\{T_i\}_{i\in I}\rangle\to P$ for some (possible infinite) set I. Then $\widehat{A}_p\langle\{T_i\}_{i\in I}\rangle$ carries a δ-$\widehat{A}_p$-algebra structure characterised by $\delta(T_i)=0$ for all $i\in I$. By [BS19, Lemma 2.18], this extends uniquely to a δ-$\widehat{A}_p$-algebra structure on P. If J denotes the kernel of $P\twoheadrightarrow S$, we can form the usual PD-envelope $D\coloneqq D_P(J)^\wedge_p$ and the q-PD-envelope $q\text{-}D$ as before. Furthermore, we let $\breve{\Omega}^*_{D/\widehat{A}_p}$ and $q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}$ denote the usual PD-de Rham complex and the q-PD-de Rham complex from [BS19, Construction 16.20], respectively (both are implicitly p-completed).

A.8. Lemma. — *With notation as above, there is again an explicit isomorphism of complexes*

$$q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\,\widehat{\otimes}_{\mathbb{Z}[\![q-1]\!]}\,\mathbb{Q}[\![q-1]\!]\cong\left(\breve{\Omega}^*_{D/\widehat{A}_p}\otimes_{\mathbb{Z}}\mathbb{Q}\right)[\![q-1]\!]\,.$$

Proof. This follows from the same recipe as in A.7, provided we can show that the formula for $q\text{-}\partial_i$ in terms of ∂_i remains true under the identification $(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}\cong(D\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$ from the proof of Lemma A.3. But for every fixed n, the images of the diagonal maps in the diagram

$$\begin{array}{ccc} & (P\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!] & \\ \swarrow & & \searrow \\ (q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})/(q-1)^n & \xrightarrow{\ \cong\ } & (D\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]/(q-1)^n \end{array}$$

are dense for the p-adic topology and for elements of $(P\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$ the formula is clear. $\square$

[(A.1)] We don't require h_i to commute with g_i (and it's not true in the case at hand).

A.9. Lemma. — *With notation as above, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{R}\Gamma_{q\text{-crys}}\big(S/\widehat{A}_p[\![q-1]\!]\big)\mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]}\mathbb{Q}[\![q-1]\!] & \xrightarrow[\text{(A.3)}]{\simeq} & \Big(\mathrm{R}\Gamma_{\mathrm{crys}}\big(S/\widehat{A}_p\big)\otimes_{\mathbb{Z}}\mathbb{Q}\Big)[\![q-1]\!] \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]}\mathbb{Q}[\![q-1]\!] & \xrightarrow[\text{(A.8)}]{\cong} & \Big(\breve{\Omega}^*_{D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}}\mathbb{Q}\Big)[\![q-1]\!] \end{array}$$

Here the left vertical arrow is the quasi-isomorphism from [BS19, Theorem 16.22] and the right vertical arrow is the usual quasi-isomorphism between crystalline cohomology and PD-de Rham complexes.

Proof. Let $P^\bullet$ be the degreewise p-completed Čech nerve of $\widehat{A}_p \to P$ and let $J^\bullet \subseteq P^\bullet$ be the kernel of the augmentation $P^\bullet \twoheadrightarrow S$. Let $D^\bullet \coloneqq D_{P^\bullet}(J^\bullet)_p^\wedge$ be the PD-envelope and let q-$D^\bullet$ be the corresponding q-PD-envelope. Finally, form the cosimplicial complexes

$$M^{\bullet,*} \coloneqq \breve{\Omega}^*_{D^\bullet/\widehat{A}_p} \quad\text{and}\quad q\text{-}M^{\bullet,*} \coloneqq q\text{-}\breve{\Omega}^*_{q\text{-}D^\bullet/\widehat{A}_p,\square}\,.$$

In the proof of [BS19, Theorem 16.22] it's shown that the totalisation $\mathrm{Tot}(q\text{-}M^{\bullet,*})$ of q-$M^{\bullet,*}$ is quasi-isomorphic to the 0^{th} column $q\text{-}M^{0,*} \cong q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}$, but also to the totalisation of the 0^{th} row $\mathrm{Tot}(q\text{-}M^{\bullet,0}) \cong \mathrm{Tot}(q\text{-}D^\bullet)$. This provides the desired quasi-isomorphism

$$q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square} \simeq \mathrm{Tot}(q\text{-}M^{\bullet,*}) \simeq \mathrm{Tot}(q\text{-}D^\bullet) \simeq \mathrm{R}\Gamma_{q\text{-crys}}\big(S/\widehat{A}[\![q-1]\!]\big)\,.$$

In the exact same way, the quasi-isomorphism $\breve{\Omega}^*_{D/\widehat{A}_p} \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(S/\widehat{A}_p)$ is constructed using the cosimplicial complex $M^{\bullet,*}$ in [Stacks, Tag 07LG]. Applying Lemma A.8 column-wise gives an isomorphism of cosimplicial complexes $q\text{-}M^{\bullet,*}\mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]}\mathbb{Q}[\![q-1]\!] \cong (M^{\bullet,*}\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$. On 0^{th} columns, this is the isomorphism from Lemma A.8, whereas on 0^{th} rows it is the isomorphism from Lemma A.3. This proves commutativity of the diagram. □

§A.2. Construction of the global q-de Rham complex

From now on, we no longer work in a p-complete setting.

A.10. Doing §A.1 for all primes at once. — Fix n and put $N_n \coloneqq \prod_{\ell\leqslant n}\ell^{2(\ell^{n-1}+\dots+\ell+1)}$, where the product is taken over all primes $\ell \leqslant n$. Now fix an arbitrary prime p and let P, D, and q-D be as in §A.1. We've verified that the map $P \to q\text{-}D \to q\text{-}D/(q-1)^n\otimes_{\mathbb{Z}}\mathbb{Q}$ admits a unique continuous extension

$$\begin{array}{ccc} P & \longrightarrow & q\text{-}D/(q-1)^n\otimes_{\mathbb{Z}}\mathbb{Q} \\ \big\downarrow & \nearrow & \\ D & & \end{array}$$

But in fact, Lemma A.5 shows that this extension already factors through $N_n^{-1}\,q\text{-}D/(q-1)^n$, no matter how our implicit prime p is chosen. This observation allows us to construct canonical maps $\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \to N_n^{-1}\,q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}/(q-1)^n$ for all animated rings R and all $n \geqslant 0$. Taking the product over all p and the limit over all n allows us to construct a map

$$\prod_p q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}\mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]}\mathbb{Q}[\![q-1]\!] \xleftarrow{\simeq} \Big(\prod_p \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}\otimes_{\mathbb{Z}}\mathbb{Q}\Big)[\![q-1]\!]\,.$$

compatible with the one from Lemma A.3. This map is an equivalence as indicated, as one immediately checks modulo $q-1$.

A.11. Construction. — For all smooth A-algebras S, we construct the *q-de Rham complex of S over A* as the pullback

$$\begin{array}{ccc} q\text{-}\Omega_{S/A} & \longrightarrow & \prod_p q\text{-}\Omega_{\widehat{S}_p/\widehat{A}_p} \\ \downarrow & \lrcorner & \downarrow \\ (\Omega_{S/A}\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!] & \longrightarrow & \left(\prod_p \Omega_{\widehat{S}_p/\widehat{A}_p}\otimes_{\mathbb{Z}}\mathbb{Q}\right)[\![q-1]\!] \end{array}$$

Here the right vertical map is the one constructed in A.10 above.

Proof of Theorem A.1. We've constructed q-$\Omega_{S/A}$ in Construction A.11. Functoriality is clear since all constituents of the pullback are functorial and so are the arrows between them. Modulo $q-1$, the pullback reduces to the usual arithmetic fracture square for $\Omega_{R/A}$, proving $q\text{-}\Omega_{-/A}/(q-1)\simeq\Omega_{-/A}$. It's clear from the construction that a base change morphism

$$q\text{-}\Omega_{-/A}\,\widehat{\otimes}^{\mathrm{L}}_{A[\![q-1]\!]}A'[\![q-1]\!]\longrightarrow q\text{-}\Omega_{(-\otimes_A A')/A'}$$

exists and that it reduces modulo $(q-1)$ to the usual base change equivalence for the de Rham complex. In particular, it must be an equivalence as well. This shows (*a*).

For (*b*), suppose S is equipped with an étale framing $\square\colon A[T_1,\dots,T_d]\to S$. The same argument as in A.7 provides an isomorphism $q\text{-}\Omega^*_{S/A,\square}\,\widehat{\otimes}_{\mathbb{Z}[\![q-1]\!]}\mathbb{Q}[\![q-1]\!]\cong(\Omega^*_{S/A}\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$. The compatibility check from Lemma A.9 now allows us to identify the pullback square for q-$\Omega_{S/A}$ with the usual arithmetic fracture square for the complex $q\text{-}\Omega^*_{S/A,\square}$, completed at $(q-1)$. This shows $q\text{-}\Omega_{S/A}\simeq q\text{-}\Omega^*_{S/A,\square}$, finishing the proof. □

A.12. Upgrade to derived commutative $A[\![q-1]\!]$-algebras. — Let us explain how to lift the q-de Rham complex to a functor

$$q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_A\longrightarrow\left(\mathrm{DAlg}_{A[\![q-1]\!]}\right)^{\wedge}_{(q-1)}$$

into the ∞-category of $(q-1)$-complete derived commutative $A[\![q-1]\!]$-algebras. The key observation is that all limits and colimits in derived commutative $A[\![q-1]\!]$-algebras can be computed on the level of underlying $\mathbb{E}_\infty$-$A[\![q-1]\!]$-algebras by [Rak21, Proposition 4.2.27]. Thus, by compatibility with pullbacks, it'll be enough to lift the three components of the pullback from Construction A.11 to derived commutative $A[\![q-1]\!]$-algebras. By compatibility with cosimplicial limits, it'll be enough to construct functorial cosimplicial realisations of $\Omega_{S/A}$, $\Omega_{\widehat{S}_p/\widehat{A}_p}$, and $q\text{-}\Omega_{\widehat{S}_p/\widehat{A}_p}$.

For the latter two, the comparison with (q-)crystalline cohomology easily provides such realisations. But the same trick works just as well for $\Omega_{S/A}$: Let $P\twoheadrightarrow S$ be any surjection from an ind-smooth-A-algebra (which can be chosen functorially; for example, take $P\coloneqq A[\{T_s\}_{s\in S}]$), form the Čech nerve $P^\bullet$ of $A\to P$ and let $J^\bullet\subseteq P^\bullet$ be the kernel of the augmentation $P^\bullet\twoheadrightarrow S$. Then $\Omega_{S/A}\simeq\operatorname{Tot}D_{P^\bullet}(J^\bullet)$ holds by a straightforward adaptation of the proof of [BS19, Theorem 16.22].

A.13. Derived global q-de Rham complexes. — We let $q\text{-dR}_{-/A}$ denote the animation of $q\text{-}\Omega_{-/A}$. For all animated A-algebras R, we call $q\text{-dR}_{R/A}$ the *derived q-de Rham complex of R over A*. By construction, it sits inside a pullback square

$$\begin{array}{ccc} q\text{-dR}_{R/A} & \longrightarrow & \prod_p q\text{-dR}_{\widehat{R}_p/\widehat{A}_p} \\ \downarrow & \lrcorner & \downarrow \\ \left(\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q}\right)[\![q-1]\!] & \longrightarrow & \left(\prod_p \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \otimes_{\mathbb{Z}} \mathbb{Q}\right)[\![q-1]\!] \end{array}$$

where the right vertical map again comes from A.10. It's still true that $q\text{-dR}_{-/A}/(q-1) \simeq \mathrm{dR}_{-/A}$ and that $q\text{-dR}_{-/A}$ lifts canonically to $(q-1)$-complete derived commutative $A[\![q-1]\!]$-algebras (this follows immediately from compatibility with colimits as explained in A.12).

However, in contrast to the p-complete situation, it's no longer true that the values on smooth A-algebras remain unchanged under animation (only the values on polynomial algebras do). In fact, this already fails for the derived de Rham complex in characteristic 0. For some R, this can be fixed by our construction of a *q-Hodge-completed q-de Rham complex* $q\text{-}\widehat{\mathrm{dR}}_{R/A}$ in Construction 3.37(*b*).

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