On q-de Rham Cohomology

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Abstract. — This thesis consists of two parts. In the first part we study q-de Rham cohomology. We give a detailed argument how the results on the p-completed q-de Rham complex from [BS19, §16] can be used to show the existence of a well-behaved and functorial q-de Rham complex for smooth algebras over $\mathbb{Z}$, and we prove a q-crystalline analogue of a theorem of Berthelot–Ogus.

In the second part of this thesis, we study a variant of the q-de Rham complex, which we call the *q-Hodge complex*. It is given as an explicit complex $q\text{-}\mathrm{Hdg}^*_{R,\square}$ for every smooth $\mathbb{Z}$-algebra R equipped with a choice of étale coordinates $\square$. We show that the cohomology $H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ for all $m \in \mathbb{N}$ is independent of the choice of étale coordinates $\square$ and functorial in R. To this end, we introduce q-versions of (both big and p-typical) Witt vectors as well as a q-version of the de Rham–Witt pro-complex and show that the latter coincides with $(H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)))_{m\in\mathbb{N}}$. However, we also show that the complex $q\text{-}\mathrm{Hdg}^*_{R,\square}$ itself can not satisfy the same pleasant functoriality as the q-de Rham complex.

Contents

§1. Introduction

1.1. The q-de Rham Complex. — Let q be a formal variable. Classically, the *q-derivative* (or *Jackson derivative* after [Jac10]) of a function $f(t)$ is defined as

$$\nabla_q f(t) \coloneqq \frac{f(qt) - f(t)}{qt - t}\,,$$

provided this expression makes sense in the respective context. For instance, if $f(t) = t^m$ for some integer $m \geqslant 0$, then $\nabla_q f(t) = [m]_q t^{m-1}$ could be regarded as an element of the polynomial ring $\mathbb{Z}[q,t]$, where $[m]_q = 1 + q + \dots + q^{m-1}$ denotes Gauß's q-analogue of m. Using the q-derivative, it's possible to define q-analogues of the de Rham complex, as was first done by Aomoto in [Aom90]. For example, if P is a polynomial ring over $\mathbb{Z}$, one can take its base changed de Rham complex $\Omega_P^* \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ and replace its differentials by the q-derivatives as above to obtain a complex q-Ω_P^* which is a q-deformation of the usual de Rham complex Ω_P^* in the sense that q-$\Omega_P^*/(q-1) \cong \Omega_P^*$.

These q-deformations have recently come into the focus of arithmetic interest. In [Sch17], Scholze observes that, after $(q-1)$-completion, a complex as above can not only for polynomial rings, but for arbitrary smooth $\mathbb{Z}$-algebras equipped with a choice of étale coordinates; furthermore, he explains various connections to p-adic Hodge theory and singular cohomology. Scholze's construction goes as follows: A *framed smooth $\mathbb{Z}$-algebra* is a pair $(R, \square)$, where R is smooth over $\mathbb{Z}$ and $\square\colon \mathbb{Z}[T_1, \dots, T_d] \rightarrow R$ is an étale map from a polynomial ring; we'll often call $\square$ an *étale framing*. Let furthermore $\gamma_i\colon \mathbb{Z}[T_1, \dots, T_d]\llbracket q-1\rrbracket \rightarrow \mathbb{Z}[T_1, \dots, T_d]\llbracket q-1\rrbracket$ be given by $T_i \mapsto qT_i$ and $T_j \mapsto T_j$ for $j \neq i$. Observe that γ_i is the identity modulo $q-1$, that $\square$ induces a $(q-1)$-completely étale map $\mathbb{Z}[T_1, \dots, T_d]\llbracket q-1\rrbracket \rightarrow R\llbracket q-1\rrbracket$ (see 1.10 for the terminology), and that $R\llbracket q-1\rrbracket \rightarrow R$ is a $(q-1)$-complete pro-infinitesimal thickening. Hence there exists a unique dotted lift in the solid diagram

$$\begin{tikzcd}
\mathbb{Z}[T_1, \dots, T_d]\llbracket q-1\rrbracket \arrow[r, "\gamma_i"] \arrow[d, "\square"'] & R\llbracket q-1\rrbracket \arrow[d] \\
R\llbracket q-1\rrbracket \arrow[r] \arrow[ur, dotted, "\exists!"'] & R
\end{tikzcd}$$

This lift will also be denoted γ_i. Using a similar unique lifting argument and the analogous assertion in the polynomial ring case, we also see that γ_i is not only congruent to the identity modulo $q-1$, but also modulo $(q-1)T_i$. This allow us define algebraic versions $\nabla_{q,i}\colon R\llbracket q-1\rrbracket \rightarrow R\llbracket q-1\rrbracket$ of Jackson's q-derivatives using the formula

$$\nabla_{q,i}(x) \coloneqq \frac{\gamma_i(x) - x}{qT_i - T_i}$$

for $i = 1, \dots, d$. Upon taking the Koszul complex of the commuting $\mathbb{Z}\llbracket q-1\rrbracket$-module endomorphisms $\nabla_{q,1}, \dots, \nabla_{q,d}$, we obtain the *$q$-de Rham complex of the framed smooth $\mathbb{Z}$-algebra* $(R, \square)$

$$q\text{-}\Omega^*_{R,\square} \coloneqq \left(R\llbracket q-1\rrbracket \xrightarrow{\nabla_q} \Omega^1_R\llbracket q-1\rrbracket \xrightarrow{\nabla_q} \dots \xrightarrow{\nabla_q} \Omega^d_R\llbracket q-1\rrbracket\right)\,.$$

We remark that q-$\Omega^*_{R,\square}/(q-1) \cong \Omega^*_R$ is a q-deformation of the de Rham complex of R, but in general q-$\Omega^*_{R,\square}$ is no base change of Ω^*_R and thus it contains strictly more information.

What makes matters complicated is that $q\text{-}\Omega^*_{R,\square}$ is neither functorial in R as a cochain complex, nor is it usually independent of the choice of the framing $\square$ up to isomorphism. Instead, functoriality and independence of $\square$ are only satisfied in the derived category; more precisely, the following theorem is the best we can say:

1.2. Theorem (Theorem 2.1)**.** — *There exists a functor*

$$q\text{-}\Omega_{(-)}\colon \mathrm{Sm}(\mathbb{Z}) \longrightarrow \widehat{\mathrm{Alg}}_{\mathbb{E}_\infty}\left(\mathbb{Z}\llbracket q-1\rrbracket\right)$$

from the category of smooth $\mathbb{Z}$-algebras into the ∞-category of $(q-1)$-complete $\mathbb{E}_\infty$-algebras over $\mathbb{Z}\llbracket q-1\rrbracket$, such that for every étale framing $\square\colon \mathbb{Z}[T_1,\dots,T_d]\to R$ of a smooth $\mathbb{Z}$-algebra R, the underlying object of $q\text{-}\Omega_R$ in the derived ∞-category of $\mathbb{Z}\llbracket q-1\rrbracket$ can be computed as

$$q\text{-}\Omega_R \simeq q\text{-}\Omega^*_{R,\square}\,.$$

Theorem 1.2 was conjectured in [Sch17, Conjecture 3.1]. After a partial result by Pridham [Pri19], who proved that the q-de Rham complex (as an $\mathbb{E}_\infty$-algebra) is a natural invariant of Λ-rings, the p-completed version of Theorem 1.2 for any prime p was proved by Bhatt and Scholze by introducing a q-crystalline site in [BS19]. This essentially also proves Theorem 1.2 via a more or less formal reduction; we'll give a detailed argument in §2.

1.3. Can there be Smaller Bases than $\mathbb{Z}\llbracket q-1\rrbracket$? — It is an interesting question to what extent q-de Rham cohomology only exists after $(q-1)$-completion. For example, one could ask the following question:

$(*)$ Can $q\text{-}\Omega_R$, or some version of it, be written as the base change of another object along the map $\mathcal{H}\to\mathbb{Z}\llbracket q-1\rrbracket$? Here

$$\mathcal{H} \coloneqq \lim_{m\geqslant 1}\lim_{n\geqslant 1}\mathbb{Z}[q]/(q^m-1)^n$$

denotes the *Habiro ring*.

The Habiro ring was first considered in [Hab04] and has since been suggested by Manin [Man10] to be related to analytic functions over the mythical field with one element $\mathbb{F}_1$. This connection to $\mathbb{F}_1$ makes $(*)$ a natural question to ask.

We won't discuss $(*)$ in this thesis, let alone answer it. Still, $(*)$ played an important motivational role in studying the *q-Hodge complex*. We'll come back to this after Theorem 1.7, but let's first introduce the main object of interest in this thesis.

1.4. The q-Hodge Complex. — The primary goal of this thesis is to study a variant of the q-de Rham complex, which we've termed *q-Hodge complex*. Here's the construction: For a framed smooth $\mathbb{Z}$-algebra $(R,\square)$ as above, we let $q\text{-}\mathrm{Hdg}^*_{R,\square}$ be the complex obtained from $q\text{-}\Omega^*_{R,\square}$ by multiplying each differential by $q-1$. That is,

$$q\text{-}\mathrm{Hdg}^*_{R,\square} \coloneqq \left(R\llbracket q-1\rrbracket \xrightarrow{(q-1)\nabla_q} \Omega^1_R\llbracket q-1\rrbracket \xrightarrow{(q-1)\nabla_q} \dots \xrightarrow{(q-1)\nabla_q} \Omega^d_R\llbracket q-1\rrbracket\right)\,.$$

Often we'll also consider a completed version. If p is a prime number, a *framed p-completely smooth $\mathbb{Z}_p$-algebra* is a pair $(R,\square)$ as above, except that now R is p-completely smooth over $\mathbb{Z}_p$ and the framing $\square\colon \mathbb{Z}_p[T_1,\dots,T_d]\to R$ is p-completely étale (see 1.10 for the terminology).

Given a framed p-completely smooth $\mathbb{Z}_p$-algebra $(R, \square)$, we can define a p-completed q-Hodge complex

$$q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square} \coloneqq \left(R\llbracket q-1\rrbracket \xrightarrow{(q-1)\nabla_q} \widehat{\Omega}^1_R\llbracket q-1\rrbracket \xrightarrow{(q-1)\nabla_q} \dots \xrightarrow{(q-1)\nabla_q} \widehat{\Omega}^d_R\llbracket q-1\rrbracket \right) ,$$

where $\widehat{\Omega}^i_R$ denotes the p-completion of the Kähler differential module Ω^i_R (so that we again obtain a complex of degree-wise finite free $R\llbracket q-1\rrbracket$-modules).

To the authors knowledge, the q-Hodge complex first appeared in Pridham's work [Pri19], who defined $\mathbb{E}_\infty$-$A\llbracket q-1\rrbracket$-algebras $q\widehat{\mathrm{DR}}_P(B/A)$ for any set P of primes and any flat morphism $A \to B$ of Λ_P-rings. If P is the set of all primes and $(R, \square)$ is a framed smooth $\mathbb{Z}$-algebra such that the constant Λ-ring structure on the polynomial ring $\mathbb{Z}[T_1, \dots, T_d]$ extends along $\square\colon \mathbb{Z}[T_1, \dots, T_d] \to R$, then [Pri19, Theorem 2.8] shows that the underlying complex of $q\widehat{\mathrm{DR}}_P(R/\mathbb{Z})$ is quasi-isomorphic to $q\text{-}\mathrm{Hdg}^*_{R,\square}$. If $P = \{p\}$, then for any framed p-completely smooth $\mathbb{Z}_p$-algebra $(R, \square)$ the framing $\square\colon \mathbb{Z}_p[T_1, \dots, T_d] \to R$ induces a δ-structure on the right-hand side (see [BS19, Lemma 2.18] for example), and in this case Pridham's result shows that $q\widehat{\mathrm{DR}}_{\{p\}}(R/\mathbb{Z}_p)$ is quasi-isomorphic to $q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}$. Despite these identifications, we've decided to deviate from Pridham's notation, to distinguish his homotopical constructions from our explicit complexes, but also to emphasise that $q\text{-}\mathrm{Hdg}^*_{R,\square}$ really is a q-deformation of Hodge cohomology.

Our main goal in this thesis is to study to what extent $q\text{-}\mathrm{Hdg}^*_{R,\square}$ is functorial in R and independent of $\square$ in the derived category. More precisely, we'll investigate the following analogue of Theorem 1.2.[1]

1.5. Conjecture. — *There exists a functor*

$$q\text{-}\mathrm{Hdg}_{(-)}\colon \mathrm{Sm}(\mathbb{Z}) \longrightarrow \widehat{\mathrm{Alg}}_{\mathbb{E}_\infty}\big(\mathbb{Z}\llbracket q-1\rrbracket\big)$$

from the category of smooth $\mathbb{Z}$-algebras into the ∞-category of $(q-1)$-complete $\mathbb{E}_\infty$-algebras over $\mathbb{Z}\llbracket q-1\rrbracket$, such that for every étale framing $\square\colon \mathbb{Z}[T_1, \dots, T_d] \to R$ of a smooth $\mathbb{Z}$-algebra R, the underlying object of $q\text{-}\mathrm{Hdg}_R$ in the derived ∞-category of $\mathbb{Z}\llbracket q-1\rrbracket$ can be computed as

$$q\text{-}\mathrm{Hdg}_R \simeq q\text{-}\mathrm{Hdg}^*_{R,\square} .$$

Unfortunately, it turned out during this project that Conjecture 1.5 is *most likely wrong*! However, the reason why we think it fails rests upon a closely related assertion, which is actually true and already rather interesting on its own. So even though it's usually pointless to gather evidence for a wrong conjecture, it will be worthwhile to explain our motivation behind Conjecture 1.5 before we tell the reader what goes wrong.

First of all, observe $\eta_{q-1}(q\text{-}\mathrm{Hdg}^*_{R,\square}) \cong q\text{-}\Omega^*_{R,\square}$, where η_{q-1} denotes the Berthelot–Ogus décalage operator (see [BO78] or [Stacks, Tag 0F7N]). Hence $q\text{-}\mathrm{Hdg}^*_{R,\square}$ already contains all information about $q\text{-}\Omega^*_{R,\square}$. Furthermore, Pridham's results [Pri19] about functoriality of the q-de Rham complex as a functor on Λ-rings are all deduced from corresponding assertions about the q-Hodge complex in this way. This already suggests that $q\text{-}\mathrm{Hdg}^*_{R,\square}$ might be a more fundamental object than $q\text{-}\Omega^*_{R,\square}$. But perhaps the most compelling evidence for Conjecture 1.5, and the reason why the question 1.3($*$) has lead to studying q-Hodge complexes, comes from the fact that the cohomology rings $H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ for all $m \in \mathbb{N}$ admit an incredibly nice structure, which actually is fully functorial in R and independent of the choice of framing $\square$.

[1] Conjecture 1.5 was suggested by Peter Scholze.

1.6. q-Witt Vector Structures. — It turns out that $H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ for $m \geqslant 1$ form a system of commutative differential-graded algebras which very much resembles Illusie's de Rham–Witt pro-complex of an $\mathbb{F}_p$-algebra from [Ill79, Définition I.1.4].

To set the stage, we first define certain q-versions of truncated big Witt vectors. Recall that Hesselholt [Hes15, Section 1] defined a ring of truncated big Witt vectors $\mathbb{W}_\Sigma(R)$ for every subset $\Sigma \subseteq \mathbb{N}$ closed under divisors. If $\Sigma = \Sigma_m$ is the set of positive divisors of some $m \in \mathbb{N}$, we put $\mathbb{W}_m(R) \coloneqq \mathbb{W}_{\Sigma_m}(R)$. Now let $(q\text{-}\mathbb{W}_m(R))_{m\in\mathbb{N}}$ be the universal system of rings satisfying the following two conditions:

(a) $q\text{-}\mathbb{W}_m(R)$ is a $\mathbb{W}_m(R)[\![q-1]\!]/(q^m-1)$-algebra for all $m \in \mathbb{N}$.

(b) For all divisors $d \mid m$, the Frobenius and Verschiebung maps on the ordinary big Witt vectors of R extend to $\mathbb{Z}[\![q-1]\!]$-linear maps $F_{m/d}\colon q\text{-}\mathbb{W}_m(R) \to q\text{-}\mathbb{W}_d(R)$ and $V_{m/d}\colon q\text{-}\mathbb{W}_d(R) \to q\text{-}\mathbb{W}_m(R)$ satisfying the relations

$$F_{m/d} \circ V_{m/d} = m/d \quad \text{and} \quad V_{m/d} \circ F_{m/d} = \frac{q^m-1}{q^d-1}\,.$$

We call $q\text{-}\mathbb{W}_m(R)$ the ring of *m-truncated big q-Witt vectors*[2] *over R*. One can construct $q\text{-}\mathbb{W}_m(R)$ explicitly as a certain quotient $\mathbb{W}_m(R)[\![q-1]\!]/\mathbb{I}_m$ (which is actually how we'll define them in Definition 5.3; the universal property will be proved afterwards in Lemma 5.5).

The idea behind $q\text{-}\mathbb{W}_m(R)$ is pretty natural. While the Frobenius and Verschiebung on ordinary big Witt vectors satisfy $F_{m/d} \circ V_{m/d} = m/d$, there's usually no nice description of $V_{m/d} \circ F_{m/d}$, unless the ring in question has characteristic p and $m = p^n$ is a power of p, so that $\mathbb{W}_m(-)$ coincides with the functor of truncated p-typical Witt vectors $W_{n+1}(-)$. Also it wouldn't make much sense to artificially enforce $F_{m/d}$ and $V_{m/d}$ to commute. So the next best thing one could try is to enforce that $F_{m/d}$ and $V_{m/d}$ commute up to q-twist. And indeed, condition (b) above precisely makes sure that

$$V_{m/d} \circ F_{m/d} = 1 + q^d + (q^d)^2 + \dots + (q^d)^{m/d-1} \eqqcolon [m/d]_q^d$$

is a q-deformation of m/d.

We'll prove in Proposition 5.7 that $H^0(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ is isomorphic to $q\text{-}\mathbb{W}_m(R)$ for all $m \in \mathbb{N}$. Already the fact that there is a map $\mathbb{W}_m(R) \to R[\![q-1]\!]/(q^m-1)$ (which depends on the choice of framing $\square$) seems rather unexpected. But even more is true. In Definition 5.11 and Proposition 5.16 we'll construct a system of commutative differential-graded algebras $(q\text{-}\mathbb{W}_m\Omega^*_R)_{m\geqslant 1}$, which we call the with a universal property that is very much reminiscent of the de Rham–Witt pro-complex. Since these universal properties involve quite a lot of conditions, we won't recall them here but refer to §4.3 and §5.2 instead; but let it be mentioned that we again obtain Frobenius and Verschiebung maps $F_{m/d}\colon q\text{-}\mathbb{W}_m\Omega^*_R \to q\text{-}\mathbb{W}_d\Omega^*_R$ and $V_{m/d}\colon q\text{-}\mathbb{W}_d\Omega^*_R \to q\text{-}\mathbb{W}_m\Omega^*_R$ for all divisors $d \mid m$ as part of the structure. Our main result will then be the following theorem:

1.7. Theorem (Theorem 5.18)**.** — *Let $(R,\square)$ be a framed smooth $\mathbb{Z}$-algebra. For all $m \in \mathbb{N}$, there are isomorphisms*

$$q\text{-}\mathbb{W}_m\Omega^*_R \xrightarrow{\sim} H^*\big(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)\big)\,.$$

[2] The terminology we choose in this thesis is rather pretentious. It may well be that a future resolution of question 1.3($*$) features objects which are more deserving of the titles (*big*) *q-Witt vectors* and (*big*) *q-de Rham–Witt complex* and which should then be called thusly.

Under these isomorphisms, the Frobenius map $F_{m/d}\colon q\text{-}\mathbb{W}_m\Omega^*_R \to q\text{-}\mathbb{W}_d\Omega^*_R$ *for all divisors* $d \mid m$ *corresponds to the map induced by the projection*

$$q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1) \longrightarrow q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^d-1)\,.$$

Similarly, the Verschiebung $V_{m/d}\colon q\text{-}\mathbb{W}_d\Omega^*_R \to q\text{-}\mathbb{W}_m\Omega^*_R$ *corresponds to the scalar multiplication map*

$$[m/d]_{q^d}\colon q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^d-1) \longrightarrow q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)\,.$$

The hope for such a big q-de Rham–Witt structure is one of the considerations that lead to considering $q\text{-}\mathrm{Hdg}^*_{R,\square}$ rather than $q\text{-}\Omega^*_{R,\square}$. To make this a bit clearer, recall from [Sch17, Proposition 3.4] that

$$H^*\bigl(q\text{-}\Omega^*_{R,\square}/[p]_q\bigr) \cong \Omega^*_R\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathbb{Z}_p[\zeta_p]$$

is a base change of the Hodge cohomology of R. Now one can show (which we won't do, but it should be tractable by the same methods as in §4) that the cohomology rings $H^*(q\text{-}\Omega^*_{R,\square}/[p^n]_q)$ for $n \geqslant 2$ extend the structure from $n=1$ to a similar kind of q-de Rham–Witt structure as in Theorem 1.7, in such a way that $H^*(q\text{-}\Omega^*_{R,\square}/[p^n]_q)$ plays the role of W_n, the ring of truncated p-typical Witt vectors of length n. However, there's no way to extend this structure to a big q-de Rham–Witt structure in which, say, $H^*(q\text{-}\Omega^*_{R,\square}/[m]_q)$ plays the role of $\mathbb{W}_m$, simply because $H^*(q\text{-}\Omega^*_{R,\square}/[p^n]_q)$ doesn't correspond to $\mathbb{W}_{p^n}$, but to $\mathbb{W}_{p^{n-1}} = W_n$. So the q-de Rham–Witt structures on $(H^*(q\text{-}\Omega^*_{R,\square}/[p^n]_q))_{n\geqslant 1}$ for all primes p are "inconveniently shifted", preventing us from unifying them into one big structure. To fix this issue, a natural approach is to look for a complex C^* for which Hodge cohomology already occurs as $H^*(C^*/(q-1))$ rather than as $H^*(C^*/[p]_q)$, which in turn leads to $q\text{-}\mathrm{Hdg}^*_{R,\square}$ as a likely candidate. And indeed, the candidate $q\text{-}\mathrm{Hdg}^*_{R,\square}$ lives up to our expectations, as Theorem 1.7 shows.

Theorem 1.7 also looks encouraging in view of question 1.3($*$). Indeed, it's not at all necessary to take the $(q-1)$-completion in the definition of $q\text{-}\mathbb{W}_m(R)$. We could as well define an "uncompleted" version $q\text{-}\mathbb{W}^\circ_m(R)$ if we replace 1.6(a) by the condition that $q\text{-}\mathbb{W}^\circ_m(R)$ is a $\mathbb{W}_m(R)[q]/(q^m-1)$-algebra, and merely require that Frobenius and Verschiebung are $\mathbb{Z}[q]$-linear rather than $\mathbb{Z}[\![q-1]\!]$-linear in 1.6(b). On that same note, we could also define "uncompleted" q-de Rham–Witt complexes $q\text{-}\mathbb{W}^\circ_m\Omega^*_R$. If $q\text{-}\mathrm{Hdg}^*_{R,\square}$ were functorial, this would suggest that $q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)$ might be the $(q-1)$-completion of a complex defined over $\mathbb{Z}[q]/(q^m-1)$—which would certainly look like a step towards a theory defined over the Habiro ring $\mathcal{H}$, if only $q\text{-}\mathrm{Hdg}^*_{R,\square}$ had better functoriality properties.

1.8. Ok, But What Goes Wrong? — A precise formulation of "Conjecture 1.5 is most likely wrong" is that there exists no functor $q\text{-}\mathrm{Hdg}_{(-)}$ together with functorial isomorphisms

$$q\text{-}\mathbb{W}_m\Omega^*_R \xrightarrow{\sim} H^*\bigl(q\text{-}\mathrm{Hdg}_R/^L(q^m-1)\bigr)$$

(where we use derived quotient notation as defined in 1.10) which identify the Frobenius maps $F_{m/d}\colon q\text{-}\mathbb{W}_m\Omega^*_R \to q\text{-}\mathbb{W}_d\Omega^*_R$ on the left-hand side with the map induced by the projection $q\text{-}\mathrm{Hdg}_R/^L(q^m-1) \to q\text{-}\mathrm{Hdg}_R/^L(q^d-1)$ for all divisors $d \mid m$. Of course, this doesn't rule that a functor $q\text{-}\mathrm{Hdg}_{(-)}$ might still exist somehow, but in light of Theorem 1.7 that would be pretty weird, and it would be questionable how useful such a functor could be at all.

If a functor $q\text{-}\mathrm{Hdg}_{(-)}$ exists, we could define its non-abelian left-derived functor, or in more modern terms, its left Kan extension to the ∞-category of animated rings. This left

Kan extension will be denoted $Lq\text{-}\mathrm{Hdg}_{(-)}$. If a functorial isomorphism as above exists, then $Lq\text{-}\mathrm{Hdg}_{(-)}/^L(q^m-1)$ would acquire an ascending filtration, with filtration quotients given by the non-abelian left-derived functors of $R \mapsto q\text{-}\mathbb{W}_m\Omega^i_R[-i]$, where $[-i]$ denotes the shift by $-i$ in the derived category. However, by analysing these filtrations, one can show that the p-completion of $Lq\text{-}\mathrm{Hdg}_{(-)}$ is both zero and nonzero when evaluated at a p-torsion free perfectoid ring, which is clearly a contradiction. We'll give a detailed version of these arguments in §6.

1.9. Leitfaden of this Thesis. — This thesis consists of two mostly separate parts: §2 and §3 are concerned with the q-de Rham complex, whereas in §§4–6 we study the q-Hodge complex.

In §2, we give a detailed argument how the existence of a q-de Rham complex over $\mathbb{Z}$ with all expected functoriality properties can be reduced to p-complete case from [BS19, §16], thus proving Theorem 1.2. In §3, we study the Frobenius on the p-completed q-de Rham complex and prove a q-crystalline analogue of a result of Berthelot–Ogus. This will be used later to determine, out of curiosity, the cohomology of the (p-typical) q-de Rham–Witt complexes.

§4 is devoted to computing the cohomology $H^*(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1))$ for all primes p, all $n \geqslant 0$, and all framed p-completely smooth $\mathbb{Z}_p$-algebras $(R,\square)$. To this end, we introduce p-typical q-Witt vectors and study their basic properties. Once we know enough about these rings, we can define our q-de Rham–Witt complexes and prove a p-completed version of Theorem 1.7. In §5, we globalise these results using Beauville–Laszlo type arguments and prove Theorem 1.7. Finally, in §6 we give a detailed explanation of why we think Conjecture 1.5 is wrong.

1.10. Notations and Conventions. — Throughout, a *framed smooth $\mathbb{Z}$-algebra* is defined to be a pair $(R,\square)$, where R is a smooth $\mathbb{Z}$-algebra and $\square\colon \mathbb{Z}[T_1,\dots,T_d] \to R$ is an étale map from a polynomial ring. Similarly, a *framed p-completely smooth $\mathbb{Z}_p$-algebra* is a pair $(R,\square)$ in which R is p-completely smooth over $\mathbb{Z}_p$ (see below for the definition) and $\square\colon \mathbb{Z}_p[T_1,\dots,T_d] \to R$ is p-completely étale.

As usual, we'll write $[m]_q = 1+q+\dots+q^{m-1}$ for Gauß's q-analogue of an integer $m \geqslant 0$. More generally, if d is any positive divisor of m, we'll use the notation

$$[m/d]_{q^d} \coloneqq 1+q^d+(q^d)^2\dots+(q^d)^{m/d-1} = \frac{q^m-1}{q^d-1}\,.$$

In particular, if $m=p^n$ is a prime power and $d=p^{n-1}$, then $[p]_{q^{p^{n-1}}}$ is the $(p^n)^{\mathrm{th}}$ cyclotomic polynomial, and we'll always use this notation in favour of $\Phi_{p^n}(q)$.

We have to use some ∞-categoric language. If A is a ring, the derived ∞-category of A will be denoted $\mathcal{D}(A)$, whereas $D(A)$ denotes the ordinary derived category. Occasionally, we use ∞-categoric language even if we don't have to. A complex $M \in D(A)$ is called *discrete* if it is concentrated in degree 0. We'll also call a sequence $K \to L \to M$ in $\mathcal{D}(A)$ a *fibre/cofibre sequence* instead of writing that $K \to L \to M \to K[1]$ is a distinguished triangle in $D(A)$. Furthermore, we often use the derived quotient notation: If $f \in A$ and $M \in D(A)$, we let

$$M/^L f \coloneqq \operatorname{cofib}\left(f\colon M \to M\right)$$

denote the cofibre taken in $\mathcal{D}(A)$, or equivalently the cone in $D(A)$, of the multiplication map $f\colon M \to M$. For multiple elements $f_1,\dots,f_r \in A$, we let

$$M/^L(f_1,\dots,f_r) \coloneqq \bigl(\dots(M/^L f_1)/^L\dots\bigr)/^L f_r\,.$$

Finally, the notion of *derived I-completeness* for $I \subseteq A$ a finitely generated ideal will be ubiquitous throughout the text. We give an overview of all necessary facts in the appendix, §A.1. We'll denote by $\mathcal{D}_{I\text{-comp}}(A)$ and $D_{I\text{-comp}}(A)$ the full sub(-∞-)categories spanned by the derived I-complete objects. Usually, the ideal I is clear from the context and we just write $\mathcal{D}_{\mathrm{comp}}(A)$ and $D_{\mathrm{comp}}(A)$. A complex $M \in D(A)$ is called *I-completely flat* if $M \otimes_A^L A/I$ is discrete and flat over A/I. A ring morphism $A \to B$ is called *I-completely smooth* if B is derived I-complete, I-completely flat, and $B \otimes_A^L A/I$ is smooth over A/I. In the same way, the terms *I-completely étale* and *I-completely ind-smooth/étale* are defined. By Elkik's algebraisation results [Elk73], B is I-completely smooth/étale over A iff it is the derived I-completion of a smooth/étale A-algebra.

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§2. q-de Rham Cohomology over $\mathbb{Z}$

In this section we'll prove the following theorem.

2.1. Theorem ([Sch17, Conjecture 3.1]). — *There exists a functor*

$$q\text{-}\Omega_{(-)}\colon \mathrm{Sm}(\mathbb{Z}) \longrightarrow \widehat{\mathrm{Alg}}_{\mathbb{E}_\infty}\big(\mathbb{Z}[\![q-1]\!]\big)$$

from the category of smooth $\mathbb{Z}$-algebras into the ∞-category of $(q-1)$-complete $\mathbb{E}_\infty$-algebras over $\mathbb{Z}[\![q-1]\!]$, such that for every étale framing $\square\colon \mathbb{Z}[T_1,\dots,T_d] \to R$ of a smooth $\mathbb{Z}$-algebra R, the underlying object of $q\text{-}\Omega_R$ in $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}[\![q-1]\!])$ can be computed as

$$q\text{-}\Omega_R \simeq q\text{-}\Omega^*_{R,\square}\,,$$

where $q\text{-}\Omega^*_{R,\square}$ denotes the q-de Rham complex

$$q\text{-}\Omega^*_{R,\square} = \Big(R[\![q-1]\!] \xrightarrow{\nabla_q} \Omega^1_R[\![q-1]\!] \xrightarrow{\nabla_q} \dots \xrightarrow{\nabla_q} \Omega^d_R[\![q-1]\!]\Big)\,.$$

The p-completed analogue of [Sch17, Conjecture 3.1] was resolved by the construction of the *q-crystalline site* in [BS19, §16]. We'll explain how the p-complete result can be used to prove the global case, i.e. Theorem 2.1. The argument we'll present is largely formal and is undoubtedly already known to the experts in some form or another. Still we believe it doesn't hurt to work out the argument in some detail.

A different approach to constructing the global version of $q\text{-}\Omega_{(-)}$ was given by Kedlaya in the course notes [Ked21, Section 29].

2.2. Outline of the Strategy. — Roughly, we'll construct $q\text{-}\Omega_{(-)}$ from its p-completions for all primes p and its rationalisation, using a Beauville–Laszlo type argument. Note that for a framed smooth $\mathbb{Z}$-algebra $(R,\square)$ we already know the following:

(*a*) The rationalisation $q\text{-}\Omega^*_{R,\square}\,\widehat{\otimes}_{\mathbb{Z}[\![q-1]\!]}\,\mathbb{Q}[\![q-1]\!]$ is just a base change of the ordinary de Rham complex. We'll recall the precise result in Lemma 2.6 below.

(*b*) If $\widehat{R}_p$ denotes the p-completion of R, which is a p-completely smooth $\mathbb{Z}_p$-algebra, then the derived p-completion of $q\text{-}\Omega^*_{R/\mathbb{Z},\square}$ satisfies

$$\big(q\text{-}\Omega^*_{R,\square}\big)^\wedge_p \simeq q\text{-}\widehat{\Omega}^*_{\widehat{R}_p,\square} \simeq R\Gamma_{q\text{-crys}}\big(\widehat{R}_p/\mathbb{Z}_p[\![q-1]\!]\big)\,,$$

where the complexes in the middle is the p-complete q-de Rham complex and the complex on right-hand side denotes q-crystalline cohomology; these constructions will be recalled in 2.4 and 2.5 below.

In §2.1 we'll study the rationalisation of q-de Rham cohomology further. In particular, we'll show a coordinate-free analogue of (*a*); namely, that the rationalisation of q-crystalline cohomology is canonically a base change of crystalline cohomology. Once this is done, we can construct $q\text{-}\Omega_{(-)}$ and prove Theorem 2.1 in §2.2.

2.3. Recollections from [BS19], Part I: q-Divided Power Algebras. — From here on until §2.2, we fix a prime p. Let's first recall the notion of q-divided powers. Equip $\mathbb{Z}_p[\![q-1]\!]$ with the δ-structure given by $\delta(q) = 0$. Let D be a δ-ring over $\mathbb{Z}_p[\![q-1]\!]$. We assume additionally that D is $[p]_q$-torsion free and derived $(p, q-1)$-complete (or equivalently, derived $(p, [p]_q)$-complete). If $x \in D$ is an element such that $\phi(x) \in [p]_q D$, we write

$$\gamma_q(x) \coloneqq \frac{\phi(x)}{[p]_q} - \delta(x) \in D\,.$$

This operation γ_q should be thought of as a q-analogue of the divided power operation $\gamma(x) \coloneqq x^p/p$ on $\mathbb{Z}_{(p)}$-algebras.

We say that a $(p, q-1)$-complete ideal $I \subseteq D$ *has q-divided powers* if γ_q is defined on I and preserves I, that is, if $\phi(I) \subseteq [p]_q D$ and $\gamma_q(I) \subseteq I$. Moreover, [BS19, Definition 16.2] defines a *q-PD pair* to be a pair (D, I) as above, which also satisfies the following two technical conditions:

(a) $D/[p]_q$ has bounded p^∞-torsion. Together with the above assumptions on D, this implies that $(D, [p]_q D)$ is a bounded prism over $(\mathbb{Z}_p[\![q-1]\!], [p]_q\mathbb{Z}_p[\![q-1]\!])$.

(b) The ring $D/(q-1)$ is p-torsion free with finite $(p, [p]_q)$-complete Tor amplitude over D.

In such a situation, D is sometimes called a *q-PD thickening of D/I*.

We also remark that there is a notion of q-PD envelope for sufficiently well-behaved rings and ideals. We won't recall the precise formulation, but only the special case relevant to us. Let R be a p-completely smooth $\mathbb{Z}_p$-algebra and P a p-completely ind-smooth $\mathbb{Z}_p$-algebra, equipped with a surjection $P \twoheadrightarrow R$ with kernel J. Then by [BS19, Lemma 16.10] (or rather its upgrade to our situation, by using Zariski descent and taking filtered colimits), there exists a universal map

$$\big(P[\![q-1]\!], J[\![q-1]\!]\big) \longrightarrow \big(D_{J[\![q-1]\!],q}(P[\![q-1]\!]), K\big)$$

into a q-PD pair. The ring $D_{J[\![q-1]\!],q}(P[\![q-1]\!])$ is called the *q-PD envelope of the pair* $(P[\![q-1]\!], J[\![q-1]\!])$. It is $(p, q-1)$-completely flat over $\mathbb{Z}_p[\![q-1]\!]$ (and thus flat on the nose by Lemma A.7), and $D_{J[\![q-1]\!],q}(P[\![q-1]\!])/^L(q-1)$ coincides with the p-completion of the ordinary divided power envelope of (P, J).

2.4. Recollections from [BS19], Part II: The q-Crystalline Site. — For simplicity, we'll work over $(\mathbb{Z}_p[\![q-1]\!], (q-1))$, whereas [BS19, Definition 16.12] works relative to any q-PD pair (D, I). Let R be a p-completely smooth $\mathbb{Z}_p$-algebra. The *q-crystalline site of R relative to $\mathbb{Z}_p[\![q-1]\!]$*, denoted $(R/\mathbb{Z}_p[\![q-1]\!])_{q\text{-crys}}$, is the category of q-PD thickenings of R, that is, the category of triples (D, I, η), where (D, I) is a q-PD pair and $\eta\colon D/I \xrightarrow{\sim} R$ is an isomorphism. We equip $(R/\mathbb{Z}_p[\![q-1]\!])_{q\text{-crys}}$ with the indiscrete Grothendieck topology, so that every presheaf is a sheaf.

The assignment $(D, I) \mapsto D$ then defines a sheaf $\mathcal{O}_{q\text{-crys}}$ of δ-$\mathbb{Z}_p[\![q-1]\!]$-algebras on the q-crystalline site $(R/\mathbb{Z}_p[\![q-1]\!])_{q\text{-crys}}$, and we call

$$R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big) \coloneqq R\Gamma\big((R/\mathbb{Z}_p[\![q-1]\!])_{q\text{-crys}}, \mathcal{O}_{q\text{-crys}}\big)_p$$

the *q-crystalline cohomology of R*. Here the author decided to deviate from the original notation $q\Omega_{R/\mathbb{Z}_p[\![q-1]\!]}$, since otherwise he would get utterly confused by the statement of Corollary 2.13 below.

2.5. Recollections from [BS19], Part III: q-de Rham Complexes. — Let $(R, \square)$ be a framed p-completely smooth $\mathbb{Z}_p$-algebra. From this data we can construct the p-complete q-de Rham complex

$$q\text{-}\widehat{\Omega}^*_{R,\square} = \left(R[\![q-1]\!] \xrightarrow{\nabla_q} \widehat{\Omega}^1_R[\![q-1]\!] \xrightarrow{\nabla_q} \dots \xrightarrow{\nabla_q} \widehat{\Omega}^d_R[\![q-1]\!]\right)\,,$$

where $\widehat{\Omega}_R^i$ denotes the p-completion of Ω_R^i, or equivalently, the p-completion of $\Omega^i_{R/\mathbb{Z}_p}$. We already know that $q\text{-}\widehat{\Omega}^*_{R,\square}$ is independent of the choice of framing up to quasi-isomorphism. In fact, for every choice of $\square$ there's a quasi-isomorphism

$$q\text{-}\widehat{\Omega}^*_{R,\square} \simeq R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big)$$

into the q-crystalline cohomology of R; see [BS19, Theorem 16.21].

More generally, assume that $P \twoheadrightarrow R$ is a surjection from a p-completely ind-smooth $\mathbb{Z}_p$-algebra and that P is equipped with a p-completely ind-étale framing $\square\colon \mathbb{Z}_p[\{U_i\}_{i\in\Sigma}] \to P$. There is a δ-structure on $\mathbb{Z}_p[\{U_i\}_{i\in\Sigma}]$ given by $\delta(U_i)=0$, which extends uniquely to P. We extend this δ-structure further to the $(p,q-1)$-completely ind-smooth $\mathbb{Z}_p[\![q-1]\!]$-algebra $P[\![q-1]\!]$ via $\delta(q)=0$. Let J denote the kernel of $P\to R$. As in 2.3, we may then consider the q-divided power envelope $q\text{-}D \coloneqq D_{J[\![q-1]\!],q}(P[\![q-1]\!])$. Using this, [BS19, Construction 16.19] defines the q-divided power de Rham complex

$$q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square} \coloneqq \left(q\text{-}D \xrightarrow{\nabla_q} q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \widehat{\Omega}^1_P \xrightarrow{\nabla_q} q\text{-}D \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \widehat{\Omega}^2_P \xrightarrow{\nabla_q} \dotsb \right).$$

In the case $P=R$, there are no divided powers to be added, and we get back the complex $q\text{-}\widehat{\Omega}^*_{R,\square}$ from above. Moreover, as above there's a quasi-isomorphism

$$q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square} \simeq R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big)\,;$$

for the proof we refer to [BS19, Theorem 16.21] again.

§2.1. Rationalised q-de Rham and q-Crystalline Cohomology

We begin with the result on the rationalised q-de Rham complex that was mentioned above.

2.6. Lemma ([Sch17, Lemma 4.1])**.** — *Let $(R,\square)$ be a framed smooth $\mathbb{Z}$-algebra. Then there is an isomorphism of complexes*

$$q\text{-}\Omega^*_{R,\square} \mathbin{\widehat{\otimes}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Q}[\![q-1]\!] \cong \Omega^*_R \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \mathbb{Q}[\![q-1]\!]\,,$$

where both tensor products are degree-wise $(q-1)$-adically completed (which computes the derived $(q-1)$-completions by flatness).

Proof. As in [BMS18, Lemma 12.4], one shows that

$$\nabla_{q,i} = \left(\frac{\log(q)}{q-1} + \sum_{n\geqslant 2} \frac{\log(q)^n}{n!(q-1)}(\nabla_i T_i)^{n-1}\right)\nabla_i\,. \tag{2.6.1}$$

Here $\log(q)$ is to be understood as the corresponding Taylor expansion around 1. Note that the first factor is an invertible automorphism. Indeed, $\log(q)/(q-1)$ is invertible, and the $\log(q)^n/(n!(q-1))$ for $n\geqslant 2$ are topologically nilpotent and converge to 0 in the $(q-1)$-adic topology.

Now in general, assume that M is an abelian group together with commuting endomorphisms $g_1,\dots,g_d$ and commuting automorphisms $h_1,\dots,h_d$, such that moreover g_i commutes with h_j whenever $i\neq j$. However, we don't require g_i to commute with h_i

(and it won't be satisfied in our application). In this case, the cohomological Koszul complexes $\operatorname{Kos}_c^*(M,(g_1,\dots,g_d))$ and $\operatorname{Kos}_c^*(M,(h_1g_1,\dots,h_dg_d))$ are isomorphic, with an explicit isomorphism given by

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "(g_i)"] \arrow[d, equal] & \bigoplus_i M \arrow[r] \arrow[d, "(h_i)_i"] & \bigoplus_{i<j} M \arrow[r] \arrow[d, "(h_ih_j)_{i<j}"] & \dots \\
0 \arrow[r] & M \arrow[r, "(h_ig_i)"] & \bigoplus_i M \arrow[r] & \bigoplus_{i<j} M \arrow[r] & \dots
\end{tikzcd}$$

Applying this to $g_i=\nabla_i$ and $h_ig_i=\nabla_{q,i}$, with h_i given by the converging series in the first factor on the right-hand side of (2.6.1), shows the result. $\square$

Next, we'll aim to prove two more variants of Lemma 2.6: First a version for the q-divided power de Rham complexes from 2.5 and second a coordinate-free version involving q-crystalline and crystalline cohomology. Both versions will be deduced from Lemma 2.8 below, which connects the rationalisations of q-PD envelopes and ordinary PD-envelopes.

2.7. Notation. — Until the end of the subsection, R will denote a p-completely smooth $\mathbb{Z}_p$-algebra. Let $P\twoheadrightarrow R$ be a surjection from a p-completely ind-smooth δ-$\mathbb{Z}_p$-algebra; for now the δ-structure on P need not be given as in 2.5. We extend it again to a δ-structure on $P[\![q-1]\!]$ by putting $\delta(q)=0$. Finally, let J denote the kernel of $P\to R$, let $q\text{-}D\coloneqq D_{J[\![q-1]\!],q}(P[\![q-1]\!])$ denote its q-PD envelope and let $D\coloneqq D_J(P)_p^\wedge$ denote the ordinary p-completed PD envelope of the kernel of $P\to R$.

2.8. Lemma. — *With notation as in Notation 2.7, there is a canonical isomorphism*

$$D\left[\tfrac{1}{p}\right][\![q-1]\!]\cong q\text{-}D\left[\tfrac{1}{p}\right]_{q-1}^\wedge\,,$$

where we take the $(q-1)$-adically completed localisation on the right-hand side (since q-D is $(p,q-1)$-completely flat over $\mathbb{Z}_p[\![q-1]\!]$, and thus flat on the nose by Lemma A.7, it doesn't matter whether we take the derived or underived completion) .

Observe that if $D^\circ\coloneqq D_J(P)$ denotes the uncompleted PD-envelope, then the composition $P\to P[\![q-1]\!]\to q\text{-}D\left[\frac{1}{p}\right]_{q-1}^\wedge$ automatically extends to a map

$$D^\circ\longrightarrow q\text{-}D\left[\tfrac{1}{p}\right]_{q-1}^\wedge\,,$$

since the right-hand side is a $\mathbb{Q}_p$-algebra and thus has all divided powers. However, it's not clear at all whether this map extends to the p-completion $D=(D^\circ)_p^\wedge$. Showing that this is indeed the case will be the main difficulty in the proof of Lemma 2.8.

2.9. Notation. — According to [BS19, Lemmas 2.15 and 2.17], we may uniquely extend the δ-structure from q-D to $q\text{-}D\left[\frac{1}{p}\right]_{q-1}^\wedge$. We still let ϕ and δ denote the extended Frobenius and δ-map. Furthermore, we denote by

$$\gamma(x)=\frac{x^p}{p}\quad\text{and}\quad\gamma_q(x)=\frac{\phi(x)}{[p]_q}-\delta(x)$$

the maps defining a PD-structure and a q-PD structure, respectively. Note that $\gamma(x)$ and $\gamma_q(x)$ make sense for all $x\in q\text{-}D\left[\frac{1}{p}\right]_{q-1}^\wedge$ since p and $[p]_q$ are invertible.

To prove Lemma 2.8, we have to send two technical preparatory lemmas in advance.

2.10. Lemma. — *Let notation be as in Notations 2.7 and 2.9.*

(*a*) *For all $n \geqslant 1$ and all $\alpha \geqslant 0$, the map δ sends $(q-1)^n q\text{-}D$ into itself, and $p^{-\alpha}(q-1)^n q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^n q\text{-}D$.*

(*b*) *For all $n \geqslant 1$ and all $\alpha \geqslant 0$, the map γ_q sends $(q-1)^n q\text{-}D$ into $(q-1)^{n+1} q\text{-}D$, and $p^{-\alpha}(q-1)^n q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^{n+1} q\text{-}D$.*

Proof. Let's prove (*a*) first. Let $x = p^{-\alpha}(q-1)^n y$ for some $y \in q\text{-}D$. Since $q\text{-}D$ is flat over $\mathbb{Z}_p[\![q-1]\!]$ and thus p-torsion free, we can compute

$$\delta(x) = \frac{\phi(x) - x^p}{p} = \frac{(q^p-1)^n \phi(y)}{p^{\alpha+1}} - \frac{(q-1)^{pn} y^p}{p^{p\alpha+1}}\,.$$

As $q^p - 1$ is divisible by $q-1$, the right-hand side lies in $p^{-(p\alpha+1)}(q-1)^n q\text{-}D$. If $\alpha = 0$, then the right-hand side must also be contained in $q\text{-}D$. But $q\text{-}D \cap p^{-1}(q-1)^n q\text{-}D = (q-1)^n q\text{-}D$ by flatness again. This proves both parts of (*a*).

Now for (*b*). Let's first check that $\gamma_q(q-1)$ is divisible by $(q-1)^2$. For that we compute

$$\frac{\phi(q-1)}{[p]_q} - \delta(q-1) = \frac{q^p-1}{[p]_q} - \frac{(q^p-1)-(q-1)^p}{p} = -(q-1)^2 \sum_{i=2}^{p-1} \frac{1}{p}\binom{p}{i}(q-1)^{i-2}\,.$$

In the following, we'll use the relation $\gamma_q(xy) = \phi(y)\gamma_q(x) - x^p\delta(y)$ from [BS19, Remark 16.6] repeatedly. First off, it shows that

$$\gamma_q\big((q-1)^n x\big) = \phi\big((q-1)^{n-1}x\big)\gamma_q(q-1) - (q-1)^p \delta\big((q-1)^{n-1}x\big)\,.$$

It follows from (*a*) that $\delta((q-1)^{n-1}x)$ and $\phi((q-1)^{n-1}x)$ are divisible by $(q-1)^{n-1}$. Hence $\gamma_q((q-1)^n x)$ is indeed divisible by $(q-1)^{n+1}$. Moreover, we obtain

$$\gamma_q\big(p^{-\alpha}(q-1)^n x\big) = \phi(p^{-\alpha})\gamma_q\big((q-1)^n x\big) - (q-1)^{np} x^p \delta(p^{-\alpha})\,.$$

Now $\phi(p^{-\alpha}) = p^{-\alpha}$ and $\delta(p^{-\alpha})$ is contained in $p^{-(p\alpha+1)}q\text{-}D$, hence $\gamma_q(p^{-\alpha}(q-1)^n x)$ is contained in $p^{-(p\alpha+1)}(q-1)^n q\text{-}D$. This finishes the proof of (*b*). ∎

2.11. Lemma. — *Let notation be as in Notations 2.7 and 2.9. Let $x \in J$. For every $n \geqslant 1$, there are elements $y_0, \dots, y_n \in q\text{-}D$ such that y_0 admits q-divided powers in $q\text{-}D$ and*

$$\gamma^n(x) = y_0 + p^{-2}(q-1)y_1 + p^{-2(p+1)}(q-1)^2 y_2 + \dots + p^{-2(p^{n-1}+\dots+p+1)}(q-1)^n y_n$$

holds in $q\text{-}D\big[\frac{1}{p}\big]$, where $\gamma^n = \gamma \circ \dots \circ \gamma$ denotes the n-fold iteration of γ.

Proof. We use induction on n. For $n = 1$, we compute

$$\gamma(x) = \frac{x^p}{p} = \gamma_q(x) + \frac{[p]_q - p}{p}\big(\gamma_q(x) + \delta(x)\big)\,. \tag{2.11.1}$$

Note that x admits q-divided powers in $q\text{-}D$ since we assume $x \in J$. Then $\gamma_q(x)$ admits q-divided powers again by [BS19, Lemma 16.7]. Moreover, $([p]_q - p)/p$ is contained in $p^{-1}(q-1)$. This settles the case $n = 1$. We also remark that (2.11.1) remains true without the assumption $x \in J$ as long as the expression $\gamma_q(x)$ makes sense.

Now assume γ^n can be written as above. We put $z_i = p^{-2(p^{i-1}+\dots+p+1)}(q-1)^i y_i$ for short, so that $\gamma^n(x) = y_0 + z_1 + \dots + z_n$. Recall the relations

$$\gamma_q(a+b) = \gamma_q(a) + \gamma_q(b) + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a^i b^{p-i}\,,\ \delta(a+b) = \delta(a) + \delta(b) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a^i b^{p-i}\,.$$

The first relation implies that $\gamma(y_0+z_1+\dots+z_n)$ is equal to $\gamma(y_0)+\gamma_q(z_1)+\dots+\gamma_q(z_n)$ plus a linear combination of terms of the form $y_0^{k_0} z_1^{k_1}\cdots z_n^{k_n}$ with $0 \leqslant k_i < p$ and $k_0+\dots+k_n = p$. Now $\gamma_q(y_0)$ admits q-divided powers again. Moreover, Lemma 2.10(b) makes sure that each $\gamma_q(z_i)$ is contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{i+1} q\text{-}D$. Finally, a monomial $y_0^{i_0} z_1^{i_1}\cdots z_n^{i_n}$ with $m = \max\{i \mid k_i \neq 0\}$ is contained in $p^{-2(p^m+\dots+p+1)}(q-1)^{m+1} q\text{-}D$ by inspection.

A similar analysis, using the second of the above relations as well as Lemma 2.10(a), shows that $p^{-1}(q-1)\delta(y_0+z_1+\dots+z_n)$ can be decomposed into a bunch of terms, each of which is contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{i+1} q\text{-}D$ for some $1 \leqslant i \leqslant n+1$. By (2.11.1), we conclude that

$$\gamma^{n+1}(x) = \gamma_q\bigl(\gamma^n(x)\bigr) + \frac{[p]_q - p}{p}\bigl(\gamma_q(\gamma^n(x)) + \delta(\gamma^n(x))\bigr)$$

can be written in the desired form. $\square$

Proof of Lemma 2.8. As observed below Lemma 2.8, we get a map $D^\circ \to q\text{-}D\bigl[\tfrac{1}{p}\bigr]^\wedge_{q-1}$ from the uncompleted PD-envelope $D^\circ = D_J(P)$. By our assumptions in Notation 2.7, D° can be identified with the sub-P-algebra of $P\bigl[\tfrac{1}{p}\bigr]$ generated by $\gamma^n(x)$ for all $x \in J$ and all $n \geqslant 1$. In particular, D° is p-torsion free, and so its p-completion D agrees with the "analytic p-completion" $D^\circ[\![t]\!]/(t-p)$, as can be easily checked by derived Nakayama. Thus, to construct the a map $D \to q\text{-}D\bigl[\tfrac{1}{p}\bigr]^\wedge_{q-1}$, it suffices to check that there is a map

$$D^\circ[\![t]\!] \longrightarrow q\text{-}D\bigl[\tfrac{1}{p}\bigr]^\wedge_{q-1}$$

sending $t \mapsto p$. That is, we must show that every p-power series in D° converges in $q\text{-}D\bigl[\tfrac{1}{p}\bigr]^\wedge_{q-1}$.

By Lemma 2.11, every such p-power series can be written as an infinite sum of a p-power series in $q\text{-}D$ and, for all $n \geqslant 1$, a p-power series in $p^{-2(p^{n-1}+\dots+p+1)}(q-1)^n q\text{-}D$. Each individual of these p-power series converges in $q\text{-}D\bigl[\tfrac{1}{p}\bigr]$, and their sum converges in $q\text{-}D\bigl[\tfrac{1}{p}\bigr]^\wedge_{q-1}$. Thus, we get a well-defined map as above. It extends canonically to a $\mathbb{Q}_p[\![q-1]\!]$-algebra map

$$D\bigl[\tfrac{1}{p}\bigr][\![q-1]\!] \longrightarrow q\text{-}D\bigl[\tfrac{1}{p}\bigr]^\wedge_{q-1}\,.$$

Since both sides are derived $(q-1)$-complete, whether this map is an isomorphism can be checked after derived base change along $\mathbb{Q}_p[\![q-1]\!] \to \mathbb{Q}_p$. But [BS19, Theorem 16.10(3)] ensures that the right-hand side becomes $D\bigl[\tfrac{1}{p}\bigr]$ after derived base change, as does the left-hand side. $\square$

Finally, we can prove the two promised variants of Lemma 2.6.

2.12. Corollary. — *Let notation be as in Notation 2.7, but we assume additionally that the δ-structure on P comes from a p-completely ind-étale framing $\square\colon \mathbb{Z}_p[\{U_i\}_{i\in\Sigma}] \to P$ as in 2.5. Furthermore, let $\widehat{\Omega}^*_D$ denote the p-completed divided power de Rham complex of D over $\mathbb{Z}_p$; see [Stacks, Tag 07HZ]. Then*

$$q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square} \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p[\![q-1]\!]} \mathbb{Q}_p[\![q-1]\!] \cong \widehat{\Omega}^*_D \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbb{Q}_p[\![q-1]\!]\,,$$

where both tensor products are degree-wise $(q-1)$-adically completed (which computes the derived $(q-1)$-completion by flatness).

Proof. Combine Lemma 2.8 with the argument from the proof of Lemma 2.6. The only non-obvious point is that equation (2.6.1) still holds in $D\left[\frac{1}{p}\right]\llbracket q-1\rrbracket$. But every element can be written as a certain converging infinite sum of elements from the subring $D^\circ\left[\frac{1}{p}\right]\llbracket q-1\rrbracket$. This subring coincides with $P\left[\frac{1}{p}\right]\llbracket q-1\rrbracket$, where the desired equation is clear. $\square$

2.13. Corollary. — *There is an equivalence*

$$R\Gamma_{q\text{-crys}}\bigl(-/\mathbb{Z}_p\llbracket q-1\rrbracket\bigr)\mathbin{\widehat{\otimes}}^L_{\mathbb{Z}_p\llbracket q-1\rrbracket}\mathbb{Q}_p\llbracket q-1\rrbracket\simeq R\Gamma_{\mathrm{crys}}(-/\mathbb{Z}_p)\mathbin{\widehat{\otimes}}^L_{\mathbb{Z}_p}\mathbb{Q}_p\llbracket q-1\rrbracket$$

of functors $\widehat{\mathrm{Sm}}(\mathbb{Z}_p)\to\widehat{\mathrm{Alg}}_{\mathbb{E}_\infty}(\mathbb{Z}\llbracket q-1\rrbracket)$ from the category of p-completely smooth $\mathbb{Z}_p$-algebras into the ∞-category of $(q-1)$-complete $\mathbb{E}_\infty$-$\mathbb{Z}\llbracket q-1\rrbracket$-algebras. This equivalence is compatible with the one from Corollary 2.12 in the sense that for every p-completely smooth $\mathbb{Z}_p$-algebra R and every q-divided power envelope q-D as given there, we get a commutative diagram

$$\begin{tikzcd}
q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p\llbracket q-1\rrbracket,\square}\mathbin{\widehat{\otimes}}_{\mathbb{Z}_p\llbracket q-1\rrbracket}\mathbb{Q}_p\llbracket q-1\rrbracket \arrow[r,"\simeq"] \arrow[d,"\simeq"'] & R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p\llbracket q-1\rrbracket\bigr)\mathbin{\widehat{\otimes}}^L_{\mathbb{Z}_p\llbracket q-1\rrbracket}\mathbb{Q}_p\llbracket q-1\rrbracket \arrow[d,"\simeq"] \\
\widehat{\Omega}^*_D\mathbin{\widehat{\otimes}}_{\mathbb{Z}_p}\mathbb{Q}_p\llbracket q-1\rrbracket \arrow[r,"\simeq"] & R\Gamma_{\mathrm{crys}}(R/\mathbb{Z}_p)\mathbin{\widehat{\otimes}}^L_{\mathbb{Z}_p}\mathbb{Q}_p\llbracket q-1\rrbracket
\end{tikzcd}$$

where the top row is induced by the quasi-isomorphism from [BS19, Theorem 16.21] and the bottom row is induced by the usual quasi-isomorphism $\widehat{\Omega}^*_D\simeq R\Gamma_{\mathrm{crys}}(R/\mathbb{Z}_p)$.

Proof. Let $P\twoheadrightarrow R$ be any surjection from a p-completely ind-smooth $\mathbb{Z}_p$-algebra and let $P^\bullet$ be the degree-wise p-completed Čech nerve of $\mathbb{Z}_p\to P$. Let $J^\bullet\subseteq P^\bullet$ be the kernel of the augmentation $P^n\to P\llbracket q-1\rrbracket\to R$. Consider the cosimplicial $\mathbb{Z}_p\llbracket q-1\rrbracket$-module $q\text{-}D^\bullet\coloneqq D_{J^\bullet\llbracket q-1\rrbracket,q}(P^\bullet\llbracket q-1\rrbracket)$ and the cosimplicial $\mathbb{Z}_p$-module $D^\bullet\coloneqq D_{J^\bullet}(P^\bullet)^\wedge_p$. The totalisation of $q\text{-}D^\bullet$ computes $R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p\llbracket q-1\rrbracket)$; see [BS19, Remark 16.15]. Hence

$$R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p\llbracket q-1\rrbracket\bigr)\mathbin{\widehat{\otimes}}^L_{\mathbb{Z}_p\llbracket q-1\rrbracket}\mathbb{Q}_p\llbracket q-1\rrbracket\simeq\operatorname{Tot}\left(q\text{-}D^\bullet\left[\tfrac{1}{p}\right]^\wedge_{q-1}\right).$$

But by Lemma 2.8, the right-hand side coincides with $\operatorname{Tot}(D^\bullet\left[\frac{1}{p}\right]\llbracket q-1\rrbracket)$, which computes $R\Gamma_{\mathrm{crys}}(R/\mathbb{Z}_p)\mathbin{\widehat{\otimes}}^L_{\mathbb{Z}_p}\mathbb{Q}_p\llbracket q-1\rrbracket$. It is straightforward to see that this quasi-isomorphism is independent of the choice of P. Indeed, the construction is compatible with morphisms $P\to P'$ over R, and for every two surjections $P\twoheadrightarrow R$ and $P'\twoheadrightarrow R$ as above, we get another one via $P\mathbin{\widehat{\otimes}}_{\mathbb{Z}_p}P'\twoheadrightarrow R$ together with morphisms $P\to P\mathbin{\widehat{\otimes}}_{\mathbb{Z}_p}P'$ and $P'\to P\mathbin{\widehat{\otimes}}_{\mathbb{Z}_p}P'$ over R. Moreover, if we choose P to be strictly functorial in R (for example, take P to be the free δ-$\mathbb{Z}_p$-algebra over $W(R)$), then our quasi-isomorphism upgrades to an equivalence of functors $\widehat{\mathrm{Sm}}(\mathbb{Z}_p)\to\widehat{\mathrm{Alg}}_{\mathbb{E}_\infty}(\mathbb{Z}\llbracket q-1\rrbracket)$, as desired.

To prove the claimed compatibility, we recall how the quasi-isomorphism

$$q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p\llbracket q-1\rrbracket,\square}\simeq R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p\llbracket q-1\rrbracket\bigr)$$

was constructed in the proof of [BS19, Theorem 16.21]. Namely, one considers the cosimplicial complex $q\text{-}M^{\bullet,*}\coloneqq q\text{-}\widehat{\Omega}^*_{q\text{-}D^\bullet/\mathbb{Z}_p\llbracket q-1\rrbracket,\square}$: Its rows $q\text{-}M^{\bullet,i}$ for $i>0$ are homotopy equivalent to 0, whereas its columns $q\text{-}M^{j,\bullet}$ are all quasi-isomorphic to the 0^{th} column $q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square}$. We then get the desired quasi-isomorphism between the totalisation of $q\text{-}M^{\bullet,0}$, which computes $R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])$, and the 0^{th} column $q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square}$.

Now we know from [BJ11, Proof of Theorem 2.12] that the quasi-isomorphism

$$\widehat{\Omega}^*_D \simeq R\Gamma_{\text{crys}}(R/\mathbb{Z}_p)$$

can be constructed similarly, by considering the cosimplicial complex $M^{\bullet,*} \coloneqq \widehat{\Omega}^*_{D^\bullet}$. Applying Corollary 2.12 column-wise provides an isomorphism of cosimplicial complexes

$$q\text{-}M^{\bullet,*}\,\widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]}\,\mathbb{Q}_p[\![q-1]\!] \cong M^{\bullet,*}\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathbb{Q}_p[\![q-1]\!]\,.$$

Upon inspection, this yields the desired compatibility. □

§2.2. The q-de Rham Complex over $\mathbb{Z}$

We need one more lemma before we can give the general construction and prove Theorem 2.1. The lemma roughly says that after modding out a power of $q-1$, the isomorphisms from Lemma 2.6 and Corollary 2.13 already hold away from a closed subset of $\operatorname{Spec}\mathbb{Z}$ rather than only at the generic point.

2.14. Lemma. — *Let R be a smooth $\mathbb{Z}$-algebra, p a prime and $m \leqslant p-1$ an integer.*

(a) *There is a functorial equivalence of $\mathbb{E}_\infty$-$\mathbb{Z}_p[\![q-1]\!]$-algebras*

$$R\Gamma_{q\text{-crys}}\big(\widehat{R}_p/\mathbb{Z}_p[\![q-1]\!]\big)/^L(q-1)^m \simeq R\Gamma_{\text{crys}}\big(\widehat{R}_p/\mathbb{Z}_p\big)\otimes^L_{\mathbb{Z}_p}\mathbb{Z}_p[\![q-1]\!]/(q-1)^m\,.$$

(b) *If R is equipped with an étale framing $\square\colon \mathbb{Z}[T_1,\dots,T_d]\to R$ and N is a nonzero integer divisible by $(m+1)!$, then there is an isomorphism of complexes*

$$q\text{-}\Omega^*_{R,\square}\big[\tfrac{1}{N}\big]/(q-1)^m \cong \Omega^*_R\otimes_{\mathbb{Z}}\mathbb{Z}\big[\tfrac{1}{N}\big][\![q-1]\!]/(q-1)^m\,.$$

Proof. We use the notation from Notations 2.7 and 2.9 again. Since $m\leqslant p-1$, the element $[p]_q-p$ is divisible by $p(q-1)$ in $q\text{-}D/(q-1)^m$. Hence, using (2.11.1), the expression $\gamma(x)$ makes sense in $q\text{-}D/(q-1)^m$ for all $x\in J$. Moreover, $\gamma(x)$ can then be written as a sum of $\gamma_q(x)$ and a multiple of $(q-1)$, so $\gamma(x)$ admits q-divided powers again. Thus, we can iteratively make sense of $\gamma^n(x)$ for all $n\geqslant 1$. Therefore, we get a ring map $D\to q\text{-}D/(q-1)^m$. It can be extended to a map

$$D[\![q-1]\!]/(q-1)^m \longrightarrow q\text{-}D/(q-1)^m\,.$$

This map is an isomorphism. Indeed, the case $m=1$ is clear from [BS19, Theorem 16.10(3)]. The general case follows by comparing the filtration $((q-1)^iD[\![q-1]\!]/(q-1)^m)_{0\leqslant i\leqslant m}$ on the left-hand side to the filtration $((q-1)^iq\text{-}D/(q-1)^m)_{0\leqslant i\leqslant m}$ on the right-hand side: By flatness, the filtration quotients on both sides are D in every degree. Equipped with the isomorphism $D[\![q-1]\!]/(q-1)^m\cong q\text{-}D/(q-1)^m$, the desired functorial equivalence can now be constructed as in Corollary 2.13. This proves (a).

For (b), observe that all we need to repeat the proof of Lemma 2.6 is that the Taylor series for $\log(q)$ and $\log(q)^n/(n!(q-1))$ make sense. Since we quotient out $(q-1)^m$, only the series for $n\leqslant m+1$ are nonzero, and each of them has only finitely many terms. Thus, by an inspection of denominators, all necessary Taylor series are defined over $\mathbb{Z}\big[\tfrac{1}{N}\big][\![q-1]\!]/(q-1)^m$ if N divisible by $(m+1)!$. □

2.15. Construction. — Let R be smooth over $\mathbb{Z}$. To construct $q\text{-}\Omega_R$, we'll construct the derived quotients $q\text{-}\Omega_R/^L(q-1)^m$ first. A priori, this notation is meaningless since we haven't yet defined $q\text{-}\Omega_R$, but it will follow a posteriori that $q\text{-}\Omega_R/^L(q-1)^m$ is indeed the correct quotient of $q\text{-}\Omega_R$. To construct $q\text{-}\Omega_R/^L(q-1)^m$, let N be any nonzero integer divisible by $(m+1)!$ and take the pullback

$$\begin{tikzcd}
q\text{-}\Omega_R/^L(q-1)^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_{p\mid N}\left(R\Gamma_{q\text{-crys}}\big(\widehat{R}_p/\mathbb{Z}[\![q-1]\!]\big)/^L(q-1)^m\right) \arrow[d] \\
\Omega_R^*\otimes_{\mathbb{Z}}\mathbb{Z}\big[\tfrac{1}{N}\big][\![q-1]\!]/(q-1)^m \arrow[r] & \prod_{p\mid N}\left(R\Gamma_{\mathrm{crys}}\big(\widehat{R}_p/\mathbb{Z}_p\big)\otimes^L_{\mathbb{Z}_p}\mathbb{Q}_p[\![q-1]\!]/(q-1)^m\right)
\end{tikzcd}$$

in the ∞-category $\widehat{\mathrm{Alg}}_{\mathbb{E}_\infty}(\mathbb{Z}[\![q-1]\!])$. The right vertical arrow is induced by the equivalence $R\Gamma_{q\text{-crys}}(\widehat{R}_p/\mathbb{Z}[\![q-1]\!])\widehat{\otimes}^L_{\mathbb{Z}_p[\![q-1]\!]}\mathbb{Q}_p[\![q-1]\!]\simeq R\Gamma_{\mathrm{crys}}(\widehat{R}_p/\mathbb{Z}_p)\widehat{\otimes}^L_{\mathbb{Z}_p}\mathbb{Q}_p[\![q-1]\!]$ from Corollary 2.13 and the bottom horizontal arrow is induced by $(\Omega_R^*)_p^\wedge\simeq R\Gamma_{\mathrm{crys}}(R/\mathbb{Z}_p)$. Both equivalences are functorial and thus $q\text{-}\Omega_R/^L(q-1)^m$ is functorial in R as well.

Furthermore, $q\text{-}\Omega_{(-)}/^L(q-1)^m$ doesn't depend on the choice of N. Indeed, if $N\mid N'$, then an equivalence between the version of $q\text{-}\Omega_{(-)}/^L(q-1)^m$ defined via N' and the version defined via N can be constructed using the derived Beauville–Laszlo theorem (Lemma A.8, Remark A.9) and Lemma 2.14(a).

Let $\mathbb{N}^{\mathrm{op}}$ denote the opposite category of the partially ordered set of positive integers. Then the functors $q\text{-}\Omega_{(-)}/^L(q-1)^m$ for all $m\geqslant 1$ can be arranged into a diagram

$$\mathbb{N}^{\mathrm{op}}\longrightarrow \mathrm{Fun}\left(\mathrm{Sm}(\mathbb{Z}),\widehat{\mathrm{Alg}}_{\mathbb{E}_\infty}(\mathbb{Z}[\![q-1]\!])\right).$$

Indeed, since the inclusion of the 1-skeleton of (the nerve of) $\mathbb{N}^{\mathrm{op}}$ is inner anodyne, it suffices to provide natural transformations $q\text{-}\Omega_{(-)}/^L(q-1)^{m+1}\Rightarrow q\text{-}\Omega_{(-)}/^L(q-1)^m$ for all $m\geqslant 1$. But if we use the same N to construct $q\text{-}\Omega_{(-)}/^L(q-1)^{m+1}$ and $q\text{-}\Omega_{(-)}/^L(q-1)^m$, such a transformation is obvious. Hence we get a diagram as above and can define $q\text{-}\Omega_{(-)}$ as its limit, that is,

$$q\text{-}\Omega_R\simeq R\lim_{m\geqslant 1} q\text{-}\Omega_R/^L(q-1)^m\,.$$

Proof of Theorem 2.1. We've constructed the functor $q\text{-}\Omega_{(-)}$ in Construction 2.15, so it only remains to show that its values are the correct ones. Let $(R,\square)$ be a framed smooth $\mathbb{Z}$-algebra. Let $m\geqslant 1$ and let N be a nonzero integer divisible by $(m+1)!$. By the derived Beauville–Laszlo theorem (Lemma A.8, Remark A.9), we get a pullback diagram

$$\begin{tikzcd}
q\text{-}\Omega^*_{R,\square}/(q-1)^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_{p\mid N}\left(q\text{-}\widehat{\Omega}^*_{\widehat{R}_p,\square}/(q-1)^m\right) \arrow[d] \\
q\text{-}\Omega^*_{R,\square}\big[\tfrac{1}{N}\big]/(q-1)^m \arrow[r] & \prod_{p\mid N}\left(q\text{-}\widehat{\Omega}^*_{\widehat{R}_p,\square}\big[\tfrac{1}{p}\big]/(q-1)^m\right)
\end{tikzcd}$$

in the ∞-category $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}[\![q-1]\!])$. Using Corollary 2.13 and Lemma 2.14(b), this pullback diagram can be identified with the one defining $q\text{-}\Omega_R/^L(q-1)^m$; moreover, this also identifies the transition maps on both sides. Taking $R\lim_{n\geqslant 1}$ on both sides yields the desired equivalence $q\text{-}\Omega^*_{R,\square}\simeq q\text{-}\Omega_R$ in $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}[\![q-1]\!])$. $\square$

2.16. Remark. — It follows from the constructions that there is a functorial equivalence $q\text{-}\Omega_R/^L(q-1) \simeq \Omega_R^*$, as one should expect. Furthermore, for every framed smooth $\mathbb{Z}$-algebra $(R, \square)$ this equivalence is compatible with the isomorphism of complexes $q\text{-}\Omega_{R,\square}^*/(q-1) \cong \Omega_R^*$.

§3. Frobenius Action

The goal of this short section is to prove a q-crystalline analogue of a result of Berthelot–Ogus [BO78]. It will be used in 4.42 to determine the cohomology of our q-de Rham–Witt complexes.

3.1. The Frobenius on the Level of Complexes. — Let R be a p-completely smooth $\mathbb{Z}_p$-algebra. There is a canonical "Frobenius" endomorphism

$$\phi_{q\text{-crys}}\colon R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big) \longrightarrow R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big)\,,$$

coming from the fact that the structure sheaf $\mathcal{O}_{q\text{-crys}}$ on the q-crystalline site is a sheaf of δ-$\mathbb{Z}_p[\![q-1]\!]$-algebras. However, given a p-completely étale framing $\square\colon \mathbb{Z}_p[T_1,\dots,T_d]\rightarrow R$, the Frobenius can already be constructed on the level of q-$\widehat{\Omega}_{R,\square}^*$.

Since this will be convenient in a moment, we describe the construction in a slightly more general setting. Choose a surjection $P\twoheadrightarrow R$ from a p-completely ind-smooth $\mathbb{Z}_p$-algebra with a p-completely ind-étale framing $\square\colon \mathbb{Z}_p[\{U_i\}_{i\in\Sigma}]\rightarrow P$ as in 2.5. As described there, this defines a δ-structure on P. We let $\phi_\square\colon P\rightarrow P$ denote the Frobenius on P and let q-D denote the q-divided power envelope of the kernel of $P[\![q-1]\!]\twoheadrightarrow R$. We wish to show that $\phi_\square$ extends to an endomorphism

$$\phi_\square\colon q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square}\longrightarrow q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square}$$

of the q-divided power de Rham complex. On generators in degree 1, we define it by $\phi_\square(\mathrm{d}U_i)\coloneqq [p]_qU_i^{p-1}\,\mathrm{d}U_i$, and for a general degree-m element we put

$$\phi_\square(x\,\mathrm{d}U_{i_1}\wedge\dots\wedge\mathrm{d}U_{i_m})\coloneqq\phi_\square(x)\phi_\square(\mathrm{d}U_{i_1})\wedge\dots\wedge\phi_\square(\mathrm{d}U_{i_m})\,.$$

3.2. Lemma. — *The above defines indeed an endomorphism of the cochain complex q-$\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square}$. It is compatible with the Frobenius on $R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])$; that is, the diagram*

$$\begin{tikzcd}
q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square} \arrow[r, "\simeq"] \arrow[d, "\phi_\square"'] & R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big) \arrow[d, "\phi_{q\text{-crys}}"] \\
q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square} \arrow[r, "\simeq"] & R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big)
\end{tikzcd}$$

is commutative.

Proof. By naturality of the Koszul complex construction, proving that $\phi_\square$ is a cochain complex endomorphism reduces to showing $\phi_\square\circ\nabla_{q,i}=\nabla_{q,i}\circ\phi_\square$ for all $i\in\Sigma$. So let's recall the construction of $\nabla_{q,i}$: Let $\gamma_i\colon\mathbb{Z}_p[\{U_i\}_{i\in\Sigma}]\rightarrow\mathbb{Z}_p[\{U_i\}_{i\in\Sigma}]$ be given by $X_i\mapsto qX_i$ and $X_j\mapsto X_j$ for $j\neq i$. By p-complete ind-étaleness, γ_i extends uniquely to a map $\gamma_i\colon P\rightarrow P$, which is compatible with the δ-structure on P determined by $\phi_\square$. By [BS19, Lemma 16.20], we may further extend γ_i to a δ-ring map $\gamma_i\colon q\text{-}D\rightarrow q\text{-}D$. Then

$$\nabla_{q,i}(x)=\frac{\gamma_i(x)-x}{(q-1)U_i}\,\mathrm{d}U_i\,.$$

for all $x \in q\text{-}D$. Using that γ_i is compatible with the Frobenius on $q\text{-}D$ (it is a δ-ring map after all), we compute

$$\begin{aligned}\phi_\square\bigl(\nabla_{q,i}(x)\bigr) = \frac{\phi_\square\bigl(\gamma_i(x)\bigr)-\phi_\square(x)}{\phi_\square(q-1)\phi_\square(U_i)}\phi_\square(\mathrm{d}U_i) &= \frac{\gamma_i\bigl(\phi_\square(x)\bigr)-\phi_\square(x)}{(q^p-1)U_i^p}[p]_qU_i^{p-1}\,\mathrm{d}U_i\\ &= \frac{\gamma_i\bigl(\phi_\square(x)\bigr)-\phi_\square(x)}{(q-1)U_i}\,\mathrm{d}U_i\,,\end{aligned}$$

and the right-hand side coincides with $\nabla_{q,i}(\phi_\square(x))$, as required. This shows that $\phi_\square$ is a map of cochain complexes.

To prove commutativity of the diagram above, consider the cosimplicial complex $q\text{-}M^{\bullet,*}$ from the proof of Corollary 2.13. It's clear from the constructions that $\phi_\square$ induces a map $\phi_\square\colon q\text{-}M^{\bullet,*}\to q\text{-}M^{\bullet,*}$ of cosimplicial complexes. The induced map on 0^{th} columns is precisely the map $\phi_\square\colon q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square}\to q\text{-}\widehat{\Omega}^*_{q\text{-}D/\mathbb{Z}_p[\![q-1]\!],\square}$ under consideration, whereas the induced map on 0^{th} rows computes $\phi_{q\text{-crys}}\colon R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])\to R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])$ since it is given by the Frobenii of the δ-rings involved. □

3.3. Construction. — Let $\phi\colon \mathbb{Z}_p[\![q-1]\!]\to\mathbb{Z}_p[\![q-1]\!]$, $q\mapsto q^p$ be the Frobenius associated to the usual δ-structure. The explicit description from Lemma 3.2 implies that the n-fold iterated Frobenius $\phi^n_{q\text{-crys}}=\phi_{q\text{-crys}}\circ\dots\circ\phi_{q\text{-crys}}$ factors through $L\eta_{[p^n]_q}R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])$. Moreover, $\phi^n_{q\text{-crys}}$ is ϕ^n-linear. That is, it induces a $\mathbb{Z}_p[\![q-1]\!]$-linear map

$$\phi_{q\text{-crys}}\colon R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p[\![q-1]\!]\bigr)\longrightarrow(\phi^n)_*L\eta_{[p^n]_q}R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p[\![q-1]\!]\bigr)\,,$$

where the right-hand side has the same underlying complex as $L\eta_{[p^n]_q}R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])$, but we equip it with the $\mathbb{Z}_p[\![q-1]\!]$-module structure obtained via $\phi^n\colon\mathbb{Z}_p[\![q-1]\!]\to\mathbb{Z}_p[\![q-1]\!]$ rather than the standard module structure.

3.4. Proposition. — *Let R be a p-completely smooth $\mathbb{Z}_p$-algebra. For all $n\geqslant 0$, the map from Construction 3.3 induces a quasi-isomorphism*

$$R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p[\![q-1]\!]\bigr)\otimes^L_{\mathbb{Z}_p[\![q-1]\!],\phi^n}\mathbb{Z}_p[\![q-1]\!]\xrightarrow{\ \sim\ }L\eta_{[p^n]_q}R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p[\![q-1]\!]\bigr)\,.$$

Proof. We use induction on n. The case $n=0$ is trivial. Now assume the assertion is true for $n\geqslant 0$; let's show it is true for $n+1$ as well. Using the induction hypothesis, the right-hand side of the map in question can be rewritten as

$$\begin{aligned}L\eta_{[p^{n+1}]_q}R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p[\![q-1]\!]\bigr)&\simeq L\eta_{[p]_{q^{p^n}}}\bigl(L\eta_{[p^n]_q}R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])\bigr)\\&\simeq L\eta_{[p]_{q^{p^n}}}\bigl(R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])\otimes^L_{\mathbb{Z}_p[\![q-1]\!],\phi^n}\mathbb{Z}_p[\![q-1]\!]\bigr)\\&\simeq\bigl(L\eta_{[p]_q}R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])\bigr)\otimes^L_{\mathbb{Z}_p[\![q-1]\!],\phi^n}\mathbb{Z}_p[\![q-1]\!]\,.\end{aligned}$$

In the last quasi-isomorphism we used the fact that $[p]_{q^{p^{n-1}}}=\phi^{n-1}([p]_q)$ and that the functor $\eta_{[p]_q}$ commutes with base change along the flat morphism $\phi^{n-1}\colon\mathbb{Z}_p[\![q-1]\!]\to\mathbb{Z}_p[\![q-1]\!]$. Hence it suffices to check that

$$R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p[\![q-1]\!]\bigr)\otimes^L_{\mathbb{Z}_p[\![q-1]\!],\phi}\mathbb{Z}_p[\![q-1]\!]\longrightarrow L\eta_{[p]_q}R\Gamma_{q\text{-crys}}\bigl(R/\mathbb{Z}_p[\![q-1]\!]\bigr)$$

is a quasi-isomorphism; that is, it suffices to do the case $n=1$ of the proposition. Observe that both sides are derived $(p,q-1)$-complete. Indeed, for the left-hand side this is clear as ϕ^n is finite, and for the right-hand side [Stacks, Tag 0F7P] implies that the cohomology groups of $L\eta_{[p]_q} R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])$ are derived $(p, q-1)$-complete again, hence so is the object itself. Therefore, to show that the morphism in question is a quasi-isomorphism, it suffices to do so after applying $(-)/^L(q-1)$ on both sides. But Lemma 3.5 below makes sure that applying $(-)/^L(q-1)$ leaves us with the quasi-isomorphism

$$R\Gamma_{\text{crys}}\big((R/p)/\mathbb{Z}_p\big) \xrightarrow{\sim} L\eta_p R\Gamma_{\text{crys}}\big((R/p)/\mathbb{Z}_p\big)$$

from [BO78, Section 8]; see also [BLM21, Section 4] for a treatment from a different perspective. $\square$

3.5. Lemma. — *There is a functorial map* $(L\eta_{[p]_q} M)/^L(q-1) \to L\eta_p(M/^L(q-1))$ *for all* $M \in D(\mathbb{Z}[\![q-1]\!])$. *Applied to the q-crystalline cohomology of R, it yields a quasi-isomorphism*

$$\big(L\eta_{[p]_q} R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])\big)/^L(q-1) \xrightarrow{\sim} L\eta_p R\Gamma_{\text{crys}}\big((R/p)/\mathbb{Z}_p\big)\,.$$

Proof. On K-flat representatives it's clear how to construct the functorial map in question, so we only need to show the second assertion. As both sides are derived $[p]_q$-complete, it suffices to show that

$$\big(L\eta_{[p]_q} R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])\big)/^L\big([p]_q, q-1\big) \longrightarrow \big(L\eta_p R\Gamma_{\text{crys}}((R/p)/\mathbb{Z}_p)\big)/^L[p]_q$$

is a quasi-isomorphism. By [Stacks, Tag 0F7T] and the q-analogue of the Cartier isomorphism in [Sch17, Proposition 3.4], we have

$$\begin{aligned}\big(L\eta_{[p]_q} R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])\big)/^L[p]_q &\simeq H^*\big(R\Gamma_{q\text{-crys}}(R/\mathbb{Z}_p[\![q-1]\!])/^L[p]_q\big)\\ &\simeq \widehat{\Omega}^*_R \otimes_R R[\![q-1]\!]/[p]_q\,.\end{aligned}$$

Furthermore, $[p]_q$ acts like p on $L\eta_p R\Gamma_{\text{crys}}((R/p)/\mathbb{Z}_p)$, and so applying $(-)/^L[p]_q$ to it is the same as applying $(-)/^L p$. Finally, using the same trick as above as well as the actual Cartier isomorphism, we obtain

$$\big(L\eta_p R\Gamma_{\text{crys}}((R/p)/\mathbb{Z}_p)\big)/^L p \simeq H^*\big(R\Gamma_{\text{crys}}((R/p)/\mathbb{Z}_p)/^L p\big) \simeq \Omega^*_{(R/p)/\mathbb{F}_p}\,.$$

Thus, we're done if we can show that $(\widehat{\Omega}^*_R \otimes_R R[\![q-1]\!]/[p]_q)/^L(q-1) \to \Omega^*_{(R/p)/\mathbb{F}_p}$ is a quasi-isomorphism. This is now obvious. $\square$

§4. Cohomology of the q-Hodge Complex I: The p-Complete Case

The goal of §4 and §5 is to give a complete and functorial description of the cohomology groups $H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ for every framed smooth $\mathbb{Z}$-algebra $(R,\square)$ and all $m \geqslant 1$. As a consequence, we will see that these cohomology groups are independent of the choice of the framing $\square$.

Although our final result will be completely global, the bulk of the work goes into computing the cohomology groups after completion at an arbitrary prime. So throughout §4 let's fix a prime p as well as a p-completely smooth $\mathbb{Z}_p$-algebra R together with a p-completely étale framing $\square\colon \mathbb{Z}_p[T_1,\dots,T_d] \to R$. Our goal for now is to compute the cohomology groups $H^*(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1))$ for all $n \geqslant 0$, and to explain how these can be arranged into a system of commutative differential-graded algebras, which resembles the de Rham–Witt pro-complex of an $\mathbb{F}_p$-algebra.

This section is organised as follows: In §4.1, we'll investigate the $\mathbb{Z}_p[\![q-1]\!]/(q^{p^n}-1)$-module structure on $H^*(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1))$. In particular, we'll show that it is degree-wise p-torsion free, which will be crucial later on. In §4.2 and §4.3 we'll construct q-versions of Witt vectors and the de Rham–Witt pro-complex and show that they come with a comparison map into the cohomology we're interested in. Finally, in §4.4 we'll show that this comparison map is an isomorphism.

§4.1. The Additive Structure

Our strategy will be to construct a certain decomposition of $q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)$ according to a Frobenius lift on R. This is an old trick: A similar decomposition is used in [Sch17, Proposition 3.4] to compute $H^*(q\text{-}\widehat{\Omega}^*_{R,\square}/[p]_q)$, and much earlier in Katz's proof of the Cartier isomorphism in [Kat70, Theorem (7.2)].

4.1. The Frobenius Lift. — We always let $\phi\colon R \to R$ denote the Frobenius lift on R, which is defined as the unique extension of the Frobenius lift on $\mathbb{Z}_p[T_1,\dots,T_d]$ given by $T_i \mapsto T_i^p$ along the p-completely étale map $\square\colon \mathbb{Z}_p[T_1,\dots,T_d] \to R$.

We observe that ϕ is injective. Indeed, ϕ is injective modulo p, since R/p is a smooth $\mathbb{F}_p$-algebra and thus reduced. Hence every $x \in R$ with $\phi(x) = 0$ must be divisible by p; say $x = px'$. But then $0 = \phi(px') = p\phi(x')$ implies $\phi(x') = 0$. Now the argument can be iterated to see that $x \in \bigcap_{m\geqslant 1} p^m R$. But R is p-complete and thus p-adically separated, so $x = 0$, as required.

Futhermore, we observe that for all $n \geqslant 0$, the ring R is a free module over its subring $\phi^n(R)$, with a basis given by $T_1^{\alpha_1}\cdots T_d^{\alpha_d}$ for all multi-indices $\alpha = (\alpha_1,\dots,\alpha_d)$ satisfying $0 \leqslant \alpha_i \leqslant p^n - 1$. To see why this is true, it suffices to show that

$$\begin{tikzcd}
\mathbb{Z}_p[T_1,\dots,T_d] \arrow[r] \arrow[d, "T_i \mapsto T_i^{p^n}"'] & R \arrow[d, "\phi^n"] \\
\mathbb{Z}_p[T_1,\dots,T_d] \arrow[r] & R
\end{tikzcd}$$

is a derived pushout square of rings. But both the derived pushout and R are derived p-complete, so by the derived Nakayama lemma it's enough to check that we get a derived pushout square after applying $-\otimes^L_{\mathbb{Z}_p}\mathbb{F}_p$ everywhere. This is proved in [Stacks, Tag 0EBS].

4.2. Frobenius Decompositions. — For every multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ as in 4.1, let $\widehat{\Omega}_R^{*,\alpha} \subseteq \widehat{\Omega}_R^*$ be the p-complete graded sub-$\phi^n(R)$-module generated by the elements

$$\prod_{i\in I} T_i^{\alpha_i} \bigwedge_{j\in J} T_j^{\alpha_j - 1}\,\mathrm{d}T_j$$

for all disjoint decompositions $I \sqcup J = \{1, \dots, d\}$. If $\alpha_j = 0$ for some j, we use the convention that $T_j^{\alpha_j-1}\,\mathrm{d}T_j \coloneqq T_j^{p^n-1}\,\mathrm{d}T_j$. Then we obtain a decomposition

$$\widehat{\Omega}_R^* \cong \bigoplus_\alpha \widehat{\Omega}_R^{*,\alpha}\,. \tag{4.2.1}$$

In the same way, we obtain decompositions

$$q\text{-}\widehat{\Omega}^*_{R,\square} \cong \bigoplus_\alpha q\text{-}\widehat{\Omega}^{*,\alpha}_{R,\square} \quad\text{and}\quad q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square} \cong \bigoplus_\alpha q\text{-}\widehat{\mathrm{Hdg}}^{*,\alpha}_{R,\square}\,. \tag{4.2.2}$$

as graded $\phi^n(R)[\![q-1]\!]$-modules. For brevity, the summand corresponding to $\alpha = (0, \dots, 0)$ will be denoted $\widehat{\Omega}_R^{*,0}$, $q\text{-}\widehat{\Omega}^{*,0}_{R,\square}$, and $q\text{-}\widehat{\mathrm{Hdg}}^{*,0}_{R,\square}$, respectively.

4.3. Lemma. — *Each of the decompositions from 4.2 is a decomposition of complexes (rather than just a decomposition of their underlying graded modules).*

Proof. It suffices to prove the assertion for $q\text{-}\widehat{\Omega}^*_{R,\square}$, as it implies the other two. We must show that ∇_q restricts to a map

$$\nabla_q\colon q\text{-}\widehat{\Omega}^{k,\alpha}_{R,\square} \longrightarrow q\text{-}\widehat{\Omega}^{k+1,\alpha}_{R,\square}$$

for all k and all α. Using the q-Leibniz rule, this is easily reduced to the case $k = 0$ and $\alpha = (0, \dots, 0)$. That is, we must check that for $x \in \phi^n(R)[\![q-1]\!]$ one has $\nabla_q(x) \in \widehat{\Omega}_R^{1,0}[\![q-1]\!]$. By definition of ∇_q, it suffices to check that

$$(q-1)T_i\nabla_{q,i}(x) = \gamma_i(x) - x \in T_i^{p^n}\phi^n(R)[\![q-1]\!]\,. \tag{4.3.1}$$

for all $i = 1, \dots, d$. This will be done in two steps.

First we prove that $\gamma_i(x) \in \phi^n(R)[\![q-1]\!]$. For that we may assume that x is already contained in $\phi^n(R)$, i.e. x is a constant power series in $(q-1)$ whose only nonzero coefficient is in the image of ϕ^n. We can extend ϕ to a Frobenius lift $\Phi\colon R[\![q-1]\!] \to R[\![q-1]\!]$ by putting $\Phi(q-1) = q^p - 1$. Now γ_i commutes with Φ. Indeed, this can be checked first after restriction along the $(p, q-1)$-completely étale map $\mathbb{Z}_p[T_1, \dots, T_d][\![q-1]\!] \to R[\![q-1]\!]$, and second after composition with the $(p, q-1)$-complete pro-infinitesimal thickening $R[\![q-1]\!] \to R/p$; both cases follow from a simple inspection. By assumption on x, it is contained in the image of Φ^n. Hence $\gamma_i(x)$ is contained in the image of Φ^n as well, which is in turn contained in $\phi^n(R)[\![q-1]\!]$, as claimed.

This shows that $\gamma_i(x) - x \in \phi^n(R)[\![q-1]\!]$. To finish the proof of (4.3.1), we must show that the canonical projection

$$\pi\colon \phi^n(R)[\![q-1]\!] \longrightarrow \phi^n(R)[\![q-1]\!]/T_i^{p^n}$$

coincides with $\gamma_i \circ \pi$. Again, this may be checked after restriction along the $(p, q-1)$-completely étale map $\mathbb{Z}_p[T_1^{p^n}, \dots, T_d^{p^n}][\![q-1]\!] \to \phi^n(R)[\![q-1]\!]$, and after composition with the $(p, q-1)$-complete pro-infinitesimal thickening $\phi^n(R)[\![q-1]\!]/T_i^{p^n} \to \phi^n(R)/(p, T_i^{p^n})$, where it becomes clear. $\square$

4.4. Lemma. — *For all $n \geqslant 0$ and all multi-indices α, the complex $q\text{-}\widehat{\mathrm{Hdg}}_{R,\square}^{*,\alpha}/(q^{p^n}-1)$ is linear over the ring $\phi^n(R)\llbracket q-1\rrbracket/(q^{p^n}-1)$.*

Proof. We must check that $(q-1)\nabla_q(x)$ is divisible by $q^{p^n}-1$ for all $x \in \phi^n(R)\llbracket q-1\rrbracket$, or equivalently, that the canonical projection

$$\pi\colon \phi^n(R)\llbracket q-1\rrbracket \longrightarrow \phi^n(R)\llbracket q-1\rrbracket/(q^{p^n}-1)$$

agrees with $\gamma_i \circ \pi$ for all $i = 1, \dots, d$. As in Lemma 4.3, this may be checked after restriction along the $(p, q-1)$-completely étale map $\mathbb{Z}_p[T_1^{p^n}, \dots, T_d^{p^n}]\llbracket q-1\rrbracket \to \phi^n(R)\llbracket q-1\rrbracket$ and after composition with the pro-infinitesimal thickening $\phi^n(R)\llbracket q-1\rrbracket/(q^{p^n}-1) \to \phi^n(R)/p$. $\square$

4.5. More Decompositions. — Using Lemma 4.4, we can write

$$q\text{-}\widehat{\mathrm{Hdg}}_{R,\square}^{*,\alpha}/(q^{p^n}-1) \cong \phi^n(R)\llbracket q-1\rrbracket \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} K^{*,\alpha}(d)\,, \tag{4.5.1}$$

where $K^{*,\alpha}(d)$ is the complex of free $\mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^n}-1)$-modules with basis the elements

$$\prod_{i\in I} T_i^{\alpha_i} \bigwedge_{j\in J} T_j^{\alpha_j-1}\,\mathrm{d}T_j$$

for all disjoint decompositions $I \sqcup J = \{1, \dots, d\}$, and differentials given by $(q-1)\nabla_q$. As in 4.2, we use the convention that $T_j^{\alpha_j-1}\,\mathrm{d}T_j \coloneqq T_j^{p^n-1}\,\mathrm{d}T_j$ if $\alpha_j = 0$. The complex $K^{*,\alpha}(d)$ can be decomposed into a tensor product

$$K^{*,\alpha}(d) \cong K_n^{*,\alpha_1}(1) \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} \dots \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} K_n^{*,\alpha_d}(1)\,, \tag{4.5.2}$$

where $K^{*,\alpha_i}(1)$ is the complex

$$K^{*,\alpha_i}(1) = \left(T_i^{\alpha_i} \cdot \mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^n}-1) \xrightarrow{(q-1)\nabla_q} T_i^{\alpha_i-1}\,\mathrm{d}T_i \cdot \mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^n}-1)\right)$$

concentrated in degrees 0 and 1. If $\alpha_i \geqslant 1$, then we can write $\alpha_i = p^e\alpha_i'$, where $e = v_p(\alpha_i)$ is the exponent of p in the prime factorisation of α_i. The differential $(q-1)\nabla_q$ of $K^{*,\alpha_i}(1)$ sends the generator $T_i^{\alpha_i}$ in degree zero to

$$(q-1)\nabla_q(T_i^{\alpha_i}) = (q^{\alpha_i}-1)T_i^{\alpha_i-1}\,\mathrm{d}T_i = [\alpha_i']_{q^{p^e}}(q^{p^e}-1)T_i^{\alpha_i-1}\,\mathrm{d}T_i\,.$$

Now observe that $[\alpha_i']_{q^{p^e}}$ is a unit in $\mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^n}-1)$. Indeed, it can be written as a sum of α_i', which is a unit, and a multiple of the topologically nilpotent element $q-1$. Hence $K^{*,\alpha_i}(1)$ is isomorphic to the complex $K_{n,e}^*$ given by

$$K_{n,e}^* = \left(\mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^n}-1) \xrightarrow{(q^{p^e}-1)} \mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^n}-1)\right)\,, \tag{4.5.3}$$

again concentrated in degrees 0 and 1. If $\alpha_i = 0$, then similarly $K^{*,0}(1) \cong K_{n,n}^*$, where the differential of $K_{n,n}^*$ is multiplication with $q^{p^n}-1$, hence zero.

Combining these considerations with (4.5.2), we see that $K^{*,\alpha}(d)$ can be written as a tensor product of complexes of the form K_{n,e_i}^* for some $0 \leqslant e_1, \dots, e_d \leqslant n$. Fortunately, such a tensor product is easy to compute:

4.6. Lemma. — *Suppose $e_1 \geqslant e_2 \geqslant 0$. Then there is an isomorphism of complexes*

$$K_{n,e_1}^* \otimes_{\mathbb{Z}_p[\![q-1]\!]} K_{n,e_2}^* \cong K_{n,e_2}^*[-1] \oplus K_{n,e_2}^* \,.$$

Proof. An explicit isomorphism $K_{n,e_1}^* \otimes_{\mathbb{Z}_p[\![q-1]\!]} K_{n,e_2}^* \xrightarrow{\sim} K_{n,e_2}^*[-1] \oplus K_{n,e_2}^*$ is given by the diagram (in which we write $A \coloneqq \mathbb{Z}_p[\![q-1]\!]$ for short, or it wouldn't fit the page)

$$\begin{tikzcd}
A/(q^{p^n}-1) \arrow[r, "\begin{pmatrix} q^{p^{e_1}}-1 \\ q^{p^{e_2}}-1 \end{pmatrix}"] \arrow[d, equal] & A/(q^{p^n}-1) \oplus A/(q^{p^n}-1) \arrow[r, "\left(-(q^{p^{e_2}}-1),(q^{p^{e_1}}-1)\right)"] \arrow[d] & A/(q^{p^n}-1) \arrow[d, equal] \\
A/(q^{p^n}-1) \arrow[r, "\begin{pmatrix} 0 \\ q^{p^{e_2}}-1 \end{pmatrix}"] & A/(q^{p^n}-1) \oplus A/(q^{p^n}-1) \arrow[r, "\left(-(q^{p^{e_2}}-1),0\right)"] & A/(q^{p^n}-1)
\end{tikzcd}$$

where the vertical arrow in the middle sends $(a,b) \mapsto \left(a - \frac{q^{p^{e_1}}-1}{q^{p^{e_2}}-1}b, b\right)$. $\square$

This now implies:

4.7. Proposition. — *Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a multi-index as before, and let's write $\alpha_i = p^{e_i}\alpha_i'$ as in 4.5 (with the convention that $e_i \coloneqq n$ in the case $\alpha_i = 0$). If we denote $e \coloneqq \min\{e_1, \dots, e_d\}$, then there is an isomorphism of complexes*

$$q\text{-}\widehat{\operatorname{Hdg}}_{R,\square}^{*,\alpha}/(q^{p^n}-1) \cong \phi^n(R)[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} \left(\bigoplus_{k=0}^{d-1} \left(K_{n,e}^*[-k]\right)^{\oplus \binom{d-1}{k}} \right) .$$

Proof. Use 4.5, Lemma 4.6, and induction on d. $\square$

From Proposition 4.7 one can easily deduce a description of $q\text{-}\widehat{\operatorname{Hdg}}_{R,\square}^{*,\alpha}/(q^{p^n}-1)$. However, for us only the following consequence will be relevant.

4.8. Corollary. — *For all $n \geqslant 0$, the cohomology groups $H^*(q\text{-}\widehat{\operatorname{Hdg}}_{R,\square}^{*}/(q^{p^n}-1))$ are p-torsion free.*

Proof. By Proposition 4.7, each cohomology group of $q\text{-}\widehat{\operatorname{Hdg}}_{R,\square}^{*}/(q^{p^n}-1)$ is a direct sum of terms isomorphic to

$$\phi^n(R)[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} H^0(K_{n,e}^*) \quad \text{or} \quad \phi^n(R)[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} H^1(K_{n,e}^*)$$

for some $e \geqslant 1$. But $H^0(K_{n,e}^*) = [p^{n-e}]_{q^{p^e}}(\mathbb{Z}_p[\![q-1]\!]/(q^{p^n}-1)) \cong \mathbb{Z}_p[\![q-1]\!]/(q^{p^e}-1)$ and also $H^1(K_{n,e}^*) \cong \mathbb{Z}_p[\![q-1]\!]/(q^{p^e}-1)$, so everything is indeed p-torsion free. $\square$

§4.2. q-Witt Vectors

In this subsection, we will show that the 0^{th} cohomology $H^0(q\text{-}\widehat{\operatorname{Hdg}}_{R,\square}^{*}/(q^{p^n}-1))$ can be identified with a certain ring $q\text{-}W_{n+1}(R)$, resembling the ring of truncated Witt vectors of length $n+1$ over R. These rings can be defined in quite some generality. So for the next few pages, we forget about R and consider an arbitrary commutative but not necessarily unital ring S instead. We let $W_{n+1}(S)$ denote the usual truncated p-typical Witt vectors of length $n+1$ over S for all $n \geqslant 0$. Furthermore,

$$F_{n+1}\colon W_{n+1}(S) \longrightarrow W_n(S) \quad \text{and} \quad V_n\colon W_n(S) \longrightarrow W_{n+1}(S)$$

denote the Frobenius and Verschiebung maps. Usually we drop the indices and just write F and V for these. We'll even abuse notation and write V^{n-i} instead of the $(n-i)$-fold composition $V_n \circ V_{n-1} \circ \dots \circ V_{i+1}\colon W_{i+1}(S) \to W_{n+1}(S)$, and similarly for F. Finally,

$$[-]\colon S \longrightarrow W_{n+1}(S)$$

denotes the Teichmüller lift (this collides somewhat with our notation $[m/d]_{q^d} = \frac{q^m-1}{q^d-1}$ from 1.10, but it shouldn't cause any ambiguities).

4.9. Definition. — Let S be a commutative, but not necessarily unital ring. The ring of *q-Witt vectors of length $n+1$ over S* is the ring

$$q\text{-}W_{n+1}(S) \coloneqq W_{n+1}(S)[\![q-1]\!]/I_{n+1}\,,$$

where I_{n+1} is the ideal generated by:

- $(q^{p^i}-1)\operatorname{im} V^{n-i}$ for all $0 \leqslant i \leqslant n$, and
- $\operatorname{im}([p^{n-j}]_{q^{p^i}} V^{j-i} - V^{n-i}F^{n-j})$ for all $0 \leqslant i \leqslant j < n$.

4.10. Remark. — Despite the suggestive name, q-$W_{n+1}(S)$ is no q-deformation of $W_{n+1}(S)$ in general. For example, if S is p-torsion free and the Frobenius on S/p is injective, then one can show using Proposition 4.15 below that

$$q\text{-}W_{n+1}(S)/(q-1) \cong S^{p^n} + pS^{p^{n-1}} + \ldots + p^n S \subseteq S$$

is isomorphic to the image of the n^{th} ghost map $w_n\colon W_{n+1}(S) \to S$. Usually (e.g. for $S = \mathbb{Q}$ and $n \geqslant 1$) this image is not even abstractly isomorphic to $W_{n+1}(S)$.

However, if S is an $\mathbb{F}_p$-algebra, then it is true that q-$W_{n+1}(S)$ is a q-deformation of $W_{n+1}(S)$. Indeed, in that case the Frobenius and Verschiebung satisfy $V \circ F = p = F \circ V$ and so

$$[p^{n-j}]_{q^{p^i}} V^{j-i} - V^{n-i}F^{n-j} \equiv 0 \mod q-1\,.$$

This implies that $I_{n+1} \subseteq W_{n+1}(S)[\![q-1]\!]$ is already contained in the ideal generated by $q-1$, and thus the canonical morphism $W_{n+1}(S) \to q\text{-}W_{n+1}(S)/(q-1)$ is indeed an isomorphism.

The functorial way to think about q-Witt vectors, and in particular the strange ideal I_{n+1}, is as follows:

4.11. Lemma. — *The sequence $(q\text{-}W_{n+1}(S))_{n\geqslant 0}$ from Definition 4.9 is the universal sequence of rings satisfying the following two conditions:*

(a) *q-$W_{n+1}(S)$ is a $W_{n+1}(S)[\![q-1]\!]/(q^{p^n}-1)$-algebra for all $n \geqslant 0$.*

(b) *The Frobenius and Verschiebung maps on the ordinary Witt vectors of S extend to $\mathbb{Z}[\![q-1]\!]$-linear maps $F_{n+1}\colon q\text{-}W_{n+1}(S) \to q\text{-}W_n(S)$ and $V_n\colon q\text{-}W_n(S) \to q\text{-}W_{n+1}(S)$ satisfying the relations*

$$F_{n+1} \circ V_n = p \quad \text{and} \quad V_n \circ F_{n+1} = [p]_{q^{p^{n-1}}}\,.$$

To show Lemma 4.11, we prove an auxiliary lemma first.

4.12. Lemma. — *The $\mathbb{Z}[\![q-1]\!]$-linear maps $F\colon W_{n+1}(S)[\![q-1]\!] \to W_n(S)[\![q-1]\!]$ and $V\colon W_n(S)[\![q-1]\!] \to W_{n+1}(S)[\![q-1]\!]$ satisfy $F(I_{n+1}) \subseteq I_n$ and $V(I_n) \subseteq I_{n+1}$. Hence they descend to maps on q-Witt vectors.*

Proof. The condition on V holds by construction. So we only need to check the condition on F. That is, we need to verify that F sends all the generators of I_{n+1} from Definition 4.9 into I_n. For $i \leqslant n-1$, we can use $F \circ V = p$ to see that

$$F\big((q^{p^i}-1)V^{n-i}x\big) = p(q^{p^i}-1)V^{n-1-i}x$$

is contained in I_n for all x. For $i = n$, we see that $F((q^{p^n}-1)x) = (q^{p^n}-1)Fx$ is divisible by $q^{p^{n-1}}-1$ and thus also contained in I_n. This deals with the first kind of generators of I_{n+1}. On to the second kind of generators. For $j - i \geqslant 1$ we can use $FV = p$ again to see that

$$F\big([p^{n-j}]_{q^{p^i}}V^{j-i}y - V^{n-i}F^{n-j}y\big) = p\big([p^{n-1-(j-1)}]_{q^{p^i}}V^{j-1-i}y - V^{n-1-i}F^{n-1-(j-1)}y\big)$$

is contained in I_n for all y. For $j = i$, we use $FV = p$ once again to compute

$$F\big([p^{n-i}]_{q^{p^i}}y - V^{n-i}F^{n-i}y\big) = [p^{n-i}]_{q^{p^i}}Fy - pV^{n-1-i}F^{n-1-i}(Fy)\,.$$

Now $[p^{n-i}]_{q^{p^i}}Fy \equiv p[p^{n-1-i}]_{q^{p^i}}Fy \mod q^{p^{n-1}}-1$ and thus the right-hand side of the equation above is contained in I_n, as desired. This finishes the proof that $F(I_{n+1}) \subseteq I_n$. $\square$

Proof of Lemma 4.11. First note that if the construction $q\text{-}W_{n+1}(S) = W_{n+1}(S)[\![q-1]\!]/I_{n+1}$ satisfies (a) and (b), then it will automatically be the universal choice. Indeed, if the ordinary Verschiebung V^{n-i} extends to a map $q\text{-}W_{i+1}(S) \to q\text{-}W_{n+1}(S)$, then the image of V^{n-i} in $q\text{-}W_{n+1}(S)$ must be $(q^{p^i}-1)$-torsion, so we have to mod out at least $(q^{p^i}-1)\operatorname{im} V^{n-i}$ for all $0 \leqslant i \leqslant n$. Furthermore, to ensure $V_n \circ F_{n+1} = [p]_{q^{p^{n-1}}}$ for all $n \geqslant 1$, we have to mod out at least $\operatorname{im}([p^{n-j}]_{q^{p^i}}V^{j-i} - V^{n-i}F^{n-j})$ for all $0 \leqslant i \leqslant j < n$.

It remains to check that our q-Witt vectors do indeed satisfy (a) and (b). For (a) this is trivial. For (b), Lemma 4.12 ensures that the ordinary Witt vector Frobenius and Verschiebung extend to well-defined $\mathbb{Z}[\![q-1]\!]$-linear maps $F\colon q\text{-}W_{n+1}(S) \to q\text{-}W_n(S)$ and $V\colon q\text{-}W_n(S) \to q\text{-}W_{n+1}(S)$. These satisfy $F \circ V = p$, because this is already true for ordinary Witt vectors, and $V \circ F = [p]_{q^{p^{n-1}}}$ because $\operatorname{im}([p]_{q^{p^{n-1}}} - VF)$ is contained in I_{n+1}. Hence our q-Witt vectors satisfy the desired conditions $\square$

4.13. Remark. — $F_{n+1}\colon q\text{-}W_{n+1}(S) \to q\text{-}W_n(S)$ and $V_n\colon q\text{-}W_n(S) \to q\text{-}W_{n+1}(S)$ will be called *Frobenius* and *Verschiebung* again. Furthermore, the composition

$$[-]\colon S \longrightarrow W_{n+1}(S) \longrightarrow q\text{-}W_{n+1}(S)$$

will also be called the *Teichmüller lift*.

4.14. Remark. — What about the restrictions though? As it turns out, the naive attempt of extending the restriction maps $\operatorname{Res}\colon W_{n+1}(S) \to W_n(S)$ to a map on q-Witt vectors by putting $\operatorname{Res}(q-1) = q-1$ doesn't work. Indeed, such a map would necessarily commute with V and thus induce an $S[\![q-1]\!]$-linear map $q\text{-}W_{n+1}(S)/\operatorname{im} V \to q\text{-}W_n(S)/\operatorname{im} V$. By Proposition 4.15 below, this would provide us with an $S[\![q-1]\!]$-linear map

$$S[\![q-1]\!]/[p]_{q^{p^{n-1}}} \longrightarrow S[\![q-1]\!]/[p]_{q^{p^{n-2}}}$$

which doesn't exist in general. Nevertheless, one can still construct an analogue of the restriction maps, but it is rather unsatisfying. If we change the variable q to $q^{1/p}$, then $q\text{-}W_{n+1}(S)/\operatorname{im} V^n$ can be identified with $q\text{-}W_n(S)[\![q^{1/p}-1]\!]/[p^n]_{q^{1/p}}$. The quotient map

$$q\text{-}W_{n+1}(S) \longrightarrow q\text{-}W_{n+1}(S)/\operatorname{im} V^n \cong q\text{-}W_n(S)[\![q^{1/p}-1]\!]/[p^n]_{q^{1/p}}$$

(which isn't $\mathbb{Z}[\![q-1]\!]$-linear, as it sends $q \mapsto q^{1/p}$ instead) can then be regarded as the restriction we're looking for. That being said, these restrictions won't play any role in the theory we're going to develop.

Our main tool for dealing with q-Witt vectors will be the following short exact sequence.

4.15. Proposition. — *Suppose the derived p-completion of S is discrete (for example if S has bounded p^∞-torsion). Then for all $n \geqslant 1$, there is a functorial short exact sequence*

$$0 \longrightarrow q\text{-}W_n(S) \xrightarrow{V} q\text{-}W_{n+1}(S) \longrightarrow S[\![q-1]\!]/[p]_{q^{p^{n-1}}} \longrightarrow 0$$

compatible with the analogous sequence for the ordinary Witt vectors.

The somewhat weird condition that the derived p-completion of S be discrete is explained by the following lemma.

4.16. Lemma. — *The following conditions are equivalent:*

(a) *The derived p-completion of S is discrete.*

(b) *For all $n \geqslant 1$, $[p]_{q^{p^{n-1}}}$ is a nonzerodivisor in $S[\![q-1]\!]$.*

(c) *For some $n \geqslant 1$, $[p]_{q^{p^{n-1}}}$ is a nonzerodivisor in $S[\![q-1]\!]$.*

Proof. Suppose some nonzero power series $x = \sum_{i \geqslant 0} x_i (q-1)^i$ is annihilated by $[p]_{q^{p^{n-1}}}$. Without restriction we may assume $x_0 \neq 0$. It's well-known that $[p]_{q^{p^{n-1}}}$ is an Eisenstein polynomial in $q-1$ with degree $N = (p-1)p^{n-1}$ and constant coefficient p. We may therefore write

$$[p]_{q^{p^{n-1}}} = (q-1)^N - p\big(a_1(q-1)^{N-1} + \dots + a_N\big)\,,$$

where $a_1, \dots, a_N$ are integers and $a_N = -1$. By comparing constant coefficients, $[p]_{q^{p^{n-1}}} x = 0$ implies $px_0 = 0$, and by comparing coefficients of $(q-1)^{i+N}$, we obtain

$$x_i = p(a_1 x_{i+1} + \dots + a_N x_{i+N})$$

for all $i \geqslant 0$. Now define a double sequence $(y_{i,j})_{i,j \geqslant 0}$ recursively as follows: We put $y_{i,0} \coloneqq x_i$ and $y_{i,j+1} \coloneqq a_1 y_{i+1,j} + \dots + a_N y_{i+N,j}$ for all $j \geqslant 0$. By a simple induction, it follows that $py_{i,j+1} = y_{i,j}$. Combining this with the fact that x_0 is a nonzero p-torsion element, we see that the sequence $(y_{0,j})_{j \geqslant 0}$ defines a nonzero element in the limit $\lim_{j \geqslant 0} S[p^{j+1}]$, where the transition maps are multiplication with p. But then $\widehat{S}_p$ is not discrete by Lemma A.5(a). This shows (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) being trivial, it remains to show (c) $\Rightarrow$ (a). Suppose the derived p-completion of S isn't discrete and choose a nonzero element $(y_j)_{j \geqslant 0} \in \lim_{j \geqslant 0} S[p^{j+1}]$. Let u denote the power series $p \cdot ([p]_{q^{p^{n-1}}})^{-1} \in \mathbb{Z}\big[\tfrac{1}{p}\big][\![q-1]\!]$. Let $x \in S[\![q-1]\!]$ denote the power series obtained from the formal product $y_0 u$ by replacing every occurrence of y_0/p^j by y_j. Then x is nonzero because its constant coefficient is y_0 and it satisfies $[p]_{q^{p^{n-1}}} x = py_0 = 0$. $\square$

Before we can prove Proposition 4.15, we need yet another auxiliary lemma.

4.17. Lemma. — *Suppose S has discrete derived p-completion. Let $n \geqslant 1$ and consider a power series $x = \sum_{i\geqslant 0} x_i(q-1)^i \in W_n(S)[\![q-1]\!]$. If the element*

$$Vx = \sum_{i\geqslant 0} Vx_i(q-1)^i \in W_{n+1}(S)[\![q-1]\!]$$

is divisible by $[p]_{q^{p^{n-1}}}$, then x must already be divisible by $[p]_{q^{p^{n-1}}}$.

Proof. We must show that $V\colon W_n(S)[\![q-1]\!]/[p]_{q^{p^{n-1}}} \to W_{n+1}(S)[\![q-1]\!]/[p]_{q^{p^{n-1}}}$ is injective. Using the Verschiebung sequence for ordinary Witt vectors, we see that

$$0 \longrightarrow W_n(S)[\![q-1]\!] \xrightarrow{V} W_{n+1}(S)[\![q-1]\!] \longrightarrow S[\![q-1]\!] \longrightarrow 0$$

is exact. To show that V remains injective after modding out $[p]_{q^{p^{n-1}}}$, it suffices to check that $\operatorname{Tor}_1^{\mathbb{Z}[\![q-1]\!]}(S[\![q-1]\!], \mathbb{Z}[\![q-1]\!]/[p]_{q^{p^{n-1}}})$ vanishes, which it does since $S[\![q-1]\!]$ is $[p]_{q^{p^{n-1}}}$-torsion free by Lemma 4.16. $\square$

Proof of Proposition 4.15. Let's first show that $V\colon q\text{-}W_n(S) \to q\text{-}W_{n+1}(S)$ is injective. Let $x \in W_n(S)[\![q-1]\!]$ and assume that the element $Vx \in W_{n+1}(S)[\![q-1]\!]$ (defined as in Lemma 4.17) vanishes in $q\text{-}W_{n+1}(S)$, i.e., Vx is contained in I_{n+1}. This implies that we can write

$$Vx = \sum_{0\leqslant i\leqslant n}(q^{p^i}-1)V^{n-i}(y_i) + \sum_{0\leqslant i\leqslant j<n}\Big([p^{n-j}]_{q^{p^i}}V^{j-i}z_{i,j} - V^{n-i}F^{n-j}(z_{i,j})\Big) \tag{4.17.1}$$

for some $y_i \in W_{i+1}[\![q-1]\!]$ and some $z_{i,j} \in W_{n-(j-i)+1}[\![q-1]\!]$. We're free to change x by elements from I_n, so let's do that to simplify the equation above. If we replace x by $x-(q^{p^i}-1)V^{n-1-i}(y_i)$ for some $0 \leqslant i < n$, then the corresponding summand in (4.17.1) cancels. So we may assume $y_i = 0$ for all $0 \leqslant i < n$. Furthermore, for $j - i \geqslant 1$ we may replace x by $x - [p^{n-1-(j-1)}]_{q^{p^i}}V^{j-1-i}z_{i,j} - V^{n-1-i}F^{n-1-(j-1)}(z_{i,j})$ to assume $z_{i,j} = 0$. Finally, if we replace x by $x - \sum_{1\leqslant i<n}([p^{n-1-i}]_{q^{p^i}}Fz_{i,i} - V^{n-1-i}F^{n-1-i}(Fz_{i,i}))$, then the summands corresponding to $z_{i,i}$ won't quite cancel, but at least (4.17.1) can be simplified to

$$Vx = (q^{p^n}-1)y + [p]_{q^{p^{n-1}}}z - VFz\,, \tag{4.17.2}$$

where $y = y_n$ and $z = \sum_{0\leqslant i<n}[p^{n-1-i}]_{q^{p^i}}z_{i,i}$.

This is now much easier to work with. We see that $V(x-Fz) \in W_{n+1}(S)[\![q-1]\!]$ is divisible by $[p]_{q^{p^{n-1}}}$. By Lemma 4.17, we can write $x - Fz = [p]_{q^{p^{n-1}}}w$ for some $w \in W_n(S)[\![q-1]\!]$. Then

$$[p]_{q^{p^{n-1}}}Vw = V(x-Fz) = [p]_{q^{p^{n-1}}}\big((q^{p^{n-1}}-1)y + z\big)\,.$$

Since $[p]_{q^{p^{n-1}}}$ is a nonzerodivisor in $W_{n+1}(S)[\![q-1]\!]$ (this follows inductively from Lemma 4.16 and the short exact sequence from the proof of Lemma 4.17), we get $Vw = (q^{p^{n-1}}-1)y + z$. Since $q\text{-}W_n(S)$ is $(q^{p^{n-1}}-1)$-torsion, we have

$$[p]_{q^{p^{n-1}}}w = pw = FVw = (q^{p^{n-1}}-1)Fy + Fz = Fz$$

in $q\text{-}W_n(S)$, which implies $x = 0$ in $q\text{-}W_n(S)$, as desired. This completes the proof that $V\colon q\text{-}W_n(S) \to q\text{-}W_{n+1}(S)$ is injective.

It remains to analyse its cokernel. Clearly $q\text{-}W_{n+1}/\operatorname{im} V \cong W_{n+1}[\![q-1]\!]/(\operatorname{im} V, I_{n+1})$. The ideal $(\operatorname{im} V, I_{n+1})$ coincides with $(\operatorname{im} V, [p]_{q^{p^{n-1}}})$, whence

$$q\text{-}W_{n+1}/\operatorname{im} V \cong (W_{n+1}/\operatorname{im} V)[\![q-1]\!]/[p]_{q^{p^{n-1}}} \cong S[\![q-1]\!]/[p]_{q^{p^{n-1}}}\,,$$

as desired. $\square$

4.18. Corollary. — *If S is p-torsion free or has bounded p^∞-torsion, then the same is true for all $q\text{-}W_{n+1}(S)$.*

Proof. To show that $q\text{-}W_{n+1}(S)$ is p-torsion free or has bounded p^∞-torsion, we can argue by induction on n. The case $n = 0$ is the respective assumption on S. The inductive step follows immediately from Proposition 4.15 together with the fact that if S is p-torsion free or has bounded p^∞-torsion, then the same holds for $S[\![q-1]\!]/[p]_{q^{p^{n-1}}}$ for all $n \geqslant 1$. To see why the latter is true, observe that Lemma 4.16 provides a short exact sequence

$$0 \longrightarrow S[\![q-1]\!] \xrightarrow{[p]_{q^{p^{n-1}}}} S[\![q-1]\!] \longrightarrow S[\![q-1]\!]/[p]_{q^{p^{n-1}}} \longrightarrow 0\,.$$

This can be used to compute the p^m-torsion part $T_m = \operatorname{Tor}_1^{\mathbb{Z}}(S[\![q-1]\!]/[p]_{q^{p^{n-1}}}, \mathbb{Z}/p^m)$. Since the map $[p]_{q^{p^{n-1}}}\colon (S/p^m)[\![q-1]\!] \to (S/p^m)[\![q-1]\!]$ is injective by Lemma 4.16, we see that

$$\operatorname{Tor}_1^{\mathbb{Z}}\left(S[\![q-1]\!], \mathbb{Z}/p^m\right) \xrightarrow{[p]_{q^{p^{n-1}}}} \operatorname{Tor}_1^{\mathbb{Z}}\left(S[\![q-1]\!], \mathbb{Z}/p^m\right) \longrightarrow T_m \longrightarrow 0$$

is exact. If S is p-torsion free, then the same is true for $S[\![q-1]\!]$ and the above sequence shows immediately that $T_1 = 0$, as desired. If S has bounded p^∞-torsion, then the same is true for $S[\![q-1]\!]$ and the above sequence shows that $(T_m)_{m\geqslant 0}$ eventually stabilises, as desired. $\square$

4.19. Corollary. — *On rings with discrete derived p-completion, the functors $q\text{-}W_{n+1}(-)$ commute with derived p-completion.*

Proof. Using induction, Proposition 4.15, and Lemma A.2(b), it's clear that $q\text{-}W_{n+1}(\widehat{S}_p)$ is derived p-complete again, hence the canonical map $q\text{-}W_{n+1}(S) \to q\text{-}W_{n+1}(\widehat{S}_p)$ factors over a map $q\text{-}W_{n+1}(S)_p^\wedge \to q\text{-}W_{n+1}(\widehat{S}_p)$. We'll show by induction on n that this is an isomorphism. The case $n = 0$ is clear. For the inductive step, we obtain a diagram

$$\begin{tikzcd}
q\text{-}W_n(S)_p^\wedge \arrow[r] \arrow[d, "\simeq"'] & q\text{-}W_{n+1}(S)_p^\wedge \arrow[r] \arrow[d] & \left(S[\![q-1]\!]/[p]_{q^{p^{n-1}}}\right)_p^\wedge \arrow[d, "\simeq"] \\
q\text{-}W_n(\widehat{S}_p) \arrow[r] & q\text{-}W_{n+1}(\widehat{S}_p) \arrow[r] & \widehat{S}_p[\![q-1]\!]/[p]_{q^{p^{n-1}}}
\end{tikzcd}$$

in $D(\mathbb{Z}_p[\![q-1]\!])$. The top row is a cofibre sequence by Proposition 4.15 and the fact that derived completion is exact. The bottom row is a cofibre sequence by Proposition 4.15. The left vertical arrow is a quasi-isomorphism by the inductive hypothesis. The right vertical arrow is a quasi-isomorphism since $S[\![q-1]\!]/[p]_{q^{p^{n-1}}} \simeq S[\![q-1]\!]/^L[p]_{q^{p^{n-1}}}$ is also a derived quotient by Lemma 4.16 and derived completion commutes with derived quotients. hence the middle vertical arrow must be a quasi-isomorphism as well. $\square$

4.20. Corollary. — *If S has discrete derived p-completion, then each of the rings q-$W_{n+1}(S)$ has bounded $(q-1)^\infty$-torsion. In particular, they are all (underived) $(q-1)$-complete.*

Proof. We argue via induction on n. The case $n=0$ is clear. The inductive step follows immediately from Proposition 4.15, if we can show that the $(q-1)^\infty$-torsion of $S\llbracket q-1\rrbracket/[p]_{q^{p^{n-1}}}$ is bounded. Using the exact sequence

$$0 \longrightarrow S\llbracket q-1\rrbracket \xrightarrow{[p]_{q^{p^{n-1}}}} S\llbracket q-1\rrbracket \longrightarrow S\llbracket q-1\rrbracket/[p]_{q^{p^{n-1}}} \longrightarrow 0\,.$$

provided by Lemma 4.16 to see that the $(q-1)^m$-torsion in $S\llbracket q-1\rrbracket/[p]_{q^{p^{n-1}}}$ is isomorphic to the kernel of the multiplication map $[p]_{q^{p^{n-1}}}\colon S\llbracket q-1\rrbracket/(q-1)^m \to S\llbracket q-1\rrbracket/(q-1)^m$. Let $x=\sum_{i=0}^{m-1} x_i(q-1)^i$ represent an element of $S\llbracket q-1\rrbracket/(q-1)^m$ annihilated by $[p]_{q^{p^{n-1}}}$. By writing $[p]_{q^{p^{n-1}}}$ as an Eisenstein polynomial in $q-1$ as in the proof of Lemma 4.16, we find inductively that each x_i must be p-torsion. But then x is annihilated by $[p]_{q^{p^{n-1}}}$ iff it is annihilated by its leading term $(q-1)^N$, where $N=(p-1)p^{n-1}$. This shows that for $m\geqslant N$ the $[p]_{q^{p^{n-1}}}$-torsion part of $S\llbracket q-1\rrbracket/(q-1)^m$ is represented precisely of those polynomials which are divisible by $(q-1)^{m-N}$ and for which each coefficient is p-torsion. Upon inspection, this shows that the $(q-1)^m$-torsion part of $S\llbracket q-1\rrbracket/[p]_{q^{p^{n-1}}}$ stabilises for $m\geqslant N$, as desired.

The additional assertion follows from Lemma A.5(a) and the fact that all q-$W_{n+1}(S)$ are derived $(q-1)$-complete, because they can be written as a cokernel of a map of $(q-1)$-complete objects as we'll see in the proof of Lemma 4.22 below. $\square$

We'll show yet another application of Proposition 4.15. In contrast to the previous three corollaries, this one won't be used in the rest of the text, but it is perhaps nice to know.

4.21. Corollary. — *Let $S\to S'$ be a surjection of rings with discrete derived p-completion and let J be its kernel. Then for all $n\geqslant 0$ we have a canonical exact sequence*

$$0\longrightarrow q\text{-}W_{n+1}(J)\longrightarrow q\text{-}W_{n+1}(S)\longrightarrow q\text{-}W_{n+1}(S')\longrightarrow 0\,.$$

Proof. We use induction on n. The case $n=0$ is clear. For the inductive step, consider

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & q\text{-}W_n(J) \arrow[r, "V"] \arrow[d] & q\text{-}W_{n+1}(J) \arrow[r] \arrow[d] & J\llbracket q-1\rrbracket/[p]_{q^{p^{n-1}}} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & q\text{-}W_n(S) \arrow[r, "V"] \arrow[d] & q\text{-}W_{n+1}(S) \arrow[r] \arrow[d] & S\llbracket q-1\rrbracket/[p]_{q^{p^{n-1}}} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & q\text{-}W_n(S') \arrow[r, "V"] \arrow[d] & q\text{-}W_{n+1}(S') \arrow[r] \arrow[d] & S'\llbracket q-1\rrbracket/[p]_{q^{p^{n-1}}} \arrow[r] \arrow[d] & 0\\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By Proposition 4.15, each row is exact; for the top row we should mention that J is still a (non-unital) ring with discrete derived p-completion, since $\widehat{J}_p\to\widehat{S}_p\to\widehat{S}'_p$ is a cofibre sequence. Hence, if we regard each column as a complex, then the diagram can be interpreted as a short exact sequence of complexes. The left column is acyclic by the induction hypothesis and the right column is acyclic because $[p]_{q^{p^{n-1}}}$ is a nonzerodivisor on each of $J[\![q-1]\!]$, $S[\![q-1]\!]$, and $S'[\![q-1]\!]$ by Lemma 4.16, so each quotient is also a derived quotient. Hence the middle column must be acyclic as well. $\square$

Last but not least, we'll prove a q-Witt vector analogue of a theorem proved by van der Kallen [Kal86, Theorem (2.4)] and, independently, by Borger [Bor11, Theorem B].

4.22. Lemma. — *Suppose $S \to S'$ is an étale map of (commutative unital) rings whose derived p-completions are discrete. Then the canonical morphism*

$$q\text{-}W_{n+1}(S) \mathbin{\widehat{\otimes}}_{W_{n+1}(S)} W_{n+1}(S') \xrightarrow{\sim} q\text{-}W_{n+1}(S')$$

(where the tensor product is derived $(q-1)$-completed) is an isomorphism. In particular, $q\text{-}W_{n+1}(S) \to q\text{-}W_{n+1}(S')$ is $(q-1)$-completely étale.

Proof. Let's first check that the base change assertion implies the "in particular". By van der Kallen's theorem, $W_{n+1}(S) \to W_{n+1}(S')$ is étale again. Therefore, assuming the base change assertion, we see that $q\text{-}W_{n+1}(S) \to q\text{-}W_{n+1}(S')$ is the derived $(q-1)$-completion of a base change of an étale morphism and thus $(q-1)$-completely étale.

Now for the first part. Let

$$M \coloneqq \bigoplus_{0 \leqslant i \leqslant n} W_{i+1}(S)[\![q-1]\!] \quad \text{and} \quad N \coloneqq \bigoplus_{0 \leqslant i \leqslant j < n} W_{n-(j-i)+1}(S)[\![q-1]\!]\,.$$

By definition, we can write $q\text{-}W_{n+1}(S) \cong \operatorname{coker}(M \oplus N \to W_{n+1}(S)[\![q-1]\!])$, where the map in question is given as follows: For $0 \leqslant i \leqslant n$, the i^{th} component of $M \to W_{n+1}(S)[\![q-1]\!]$ is given by

$$(q^{p^i}-1)V^{n-i}\colon W_{i+1}(S)[\![q-1]\!] \longrightarrow W_{n+1}(S)[\![q-1]\!]\,,$$

and for $0 \leqslant i \leqslant j < n$, the $(i,j)^{\text{th}}$ component of $N \to W_{n+1}(S)[\![q-1]\!]$ is given by

$$[p^{n-i}]_{q^{p^i}} V^{j-i} - V^{n-i}F^{n-j}\colon W_{n-(j-i)+1}(S)[\![q-1]\!] \longrightarrow W_{n+1}(S)[\![q-1]\!]\,.$$

Observe that the well-known relation $V(F(x)y) = xV(y)$ for Witt vectors shows that these maps are morphism of $W_{n+1}(S)[\![q-1]\!]$-modules, if we equip $W_{i+1}(S)$ with the module structure obtained through the Frobenius morphisms $F^{n-i}\colon W_{n+1}(S) \to W_{i+1}(S)$ rather than the one obtained via the restrictions.

With this module structure (but also with the one via the restrictions), the canonical map

$$W_{i+1}(S) \otimes_{W_{n+1}(S)} W_{n+1}(S') \xrightarrow{\sim} W_{i+1}(S')$$

is an isomorphism. Indeed, in the case where S and S' are F-finite $\mathbb{Z}_{(p)}$-algebras (which is all we ever need), this was proved by Langer and Zink in [LZ04, Corollary A.18]. But together with their base change result, their argument actually covers all $\mathbb{Z}_{(p)}$-algebras, even though they don't state it. By Zariski descent of Witt vectors, it remains to check the case where S and S' are $\mathbb{Z}[\frac{1}{p}]$-algebras, which is trivial because then $W_{n+1}(S) \cong S^{n+1}$ via the ghost maps.

Now write $q\text{-}W_{n+1}(S') \cong \operatorname{coker}(M' \oplus N' \to W_{n+1}(S')[\![q-1]\!])$, with M' and N' defined in an analogous way to M and N. The above discussion yields

$$M' \simeq M \mathbin{\widehat{\otimes}}_{W_{n+1}(S)} W_{n+1}(S') \quad \text{and} \quad N' \simeq N \mathbin{\widehat{\otimes}}_{W_{n+1}(S)} W_{n+1}(S')$$

where the tensor products are (derived or underived) $(q-1)$-completed. So if we can show that the derived $(q-1)$-completion of $q\text{-}W_{n+1}(S)\otimes_{W_{n+1}(S)}W_{n+1}(S')$ is discrete again, then it will automatically coincide with $\operatorname{coker}(M'\oplus N'\to W_{n+1}(S')[\![q-1]\!])\cong q\text{-}W_{n+1}(S')$ and we'll be done. To show discreteness, recall from Corollary 4.20 that $q\text{-}W_{n+1}(S)$ has bounded $(q-1)^\infty$-torsion. Since $W_{n+1}(S')$ is étale and thus flat over $W_{n+1}(S)$, the base change $q\text{-}W_{n+1}(S)\otimes_{W_{n+1}(S)}W_{n+1}(S')$ still has bounded $(q-1)^\infty$-torsion and then Lemma A.5(a) does the rest. $\square$

This finishes our discussion of q-Witt vectors in general and we return to the situation at hand. So from now on, R is again as specified at the beginning of §4.

4.23. Proposition. — *For all $n\geqslant 0$, there are isomorphisms*

$$q\text{-}W_{n+1}(R)\xrightarrow{\sim}H^0\bigl(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)\bigr)\,.$$

Under these isomorphisms, the Frobenius $F\colon q\text{-}W_{n+1}(R)\to q\text{-}W_{n+1}(R)$ gets identified with the map induced by the canonical projection

$$q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)\longrightarrow q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^{n-1}}-1)\,,$$

and the Verschiebung $V\colon q\text{-}W_n(R)\to q\text{-}W_{n+1}(R)$ gets identified with the map induced by the scalar multiplication map

$$[p]_{q^{p^{n-1}}}\colon q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^{n-1}}-1)\longrightarrow q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)\,.$$

To prove Proposition 4.23, we first construct a rather unexpected ring morphism.

4.24. Construction. — Let A be any δ-ring over $\mathbb{Z}_{(p)}$ and let $\phi_A\colon A\to A$ denote its Frobenius. Let furthermore ε_i be the inverse Joyal operations from Lemma A.12. We define a map (of sets) $c_n\colon W_{n+1}(A)\to A[\![q-1]\!]/(q^{p^n}-1)$ by the formula

$$c_n\bigl([x_0,\dots,x_n]\bigr)\coloneqq\sum_{i=0}^n[p^i]_{q^{p^{n-i}}}\phi_A^{n-i}\bigl(\varepsilon_i(x_0,\dots,x_i)\bigr)\,.$$

4.25. Lemma. — *The map $c_n\colon W_{n+1}(A)\to A[\![q-1]\!]/(q^{p^n}-1)$ from Construction 4.24 is a morphism of rings.*

Proof. By taking a suitable surjective map $A'\twoheadrightarrow A$ of δ-rings, we may replace A by a δ-ring which is flat over $\mathbb{Z}_{(p)}$. Then

$$A[\![q-1]\!]/(q^{p^n}-1)\longrightarrow\prod_{i=1}^n A[\![q-1]\!]/[p]_{q^{p^{n-i}}}\times A$$

is an injective ring map. Indeed, in the case $A=\mathbb{Z}_{(p)}$ this follows from the fact that $q^{p^n}-1=(q-1)\prod_{i=1}^n[p]_{q^{p^{n-i}}}$ is the prime factorisation of $q^{p^n}-1$ in the factorial ring $\mathbb{Z}_{(p)}[\![q-1]\!]$, and the general case follows since $A[\![q-1]\!]$ is $(q-1)$-completely flat and thus flat on the nose over the noetherian ring $\mathbb{Z}_{(p)}[\![q-1]\!]$; see Lemma A.7.

Therefore, it's enough to check that the induced maps $c_{n,j}\colon W_{n+1}(A)\to A[\![q-1]\!]/[p]_{q^{p^{n-j}}}$ for $j=1,\dots,n$ and $c_{n,n+1}\colon W_{n+1}(A)\to A$ are ring morphisms. Observe that

$$[p^i]_{q^{p^{n-i}}}\equiv\begin{cases}0 & \text{if } 1\leqslant j\leqslant i\\ p^i & \text{if } i<j\end{cases}\quad\operatorname{mod}\,[p]_{q^{p^{n-j}}}\,.$$

Indeed, the first case follows from the factorisation $[p^i]_{q^{p^{n-i}}} = [p]_{q^{p^{n-1}}} \cdots [p]_{q^{p^{n-i}}}$, whereas in the second case we have that $q^{p^{n-i}} \equiv 1 \mod [p]_{q^{p^{n-j}}}$, so that $[p^i]_{q^{p^{n-i}}} \equiv p^i \mod [p]_{q^{p^{n-j}}}$. Hence

$$\begin{aligned} c_{n,j}\big([x_0,\dots,x_n]\big) = \sum_{i=0}^{j-1} p^i \phi_A^{n-i}\big(\varepsilon_i(x_0,\dots,x_i)\big) &= \phi_A^{n-j+1}\left(\sum_{i=0}^{j-1} p^i \phi^{j-1-i}\big(\varepsilon_i(x_0,\dots,x_i)\big)\right) \\ &= \phi_A^{n-j+1}\left(x_0^{p^{j-1}} + \dots + p^{j-1}x_{j-1}\right) \\ &= \phi_A^{n-j+1} w_{j-1}\big([x_0,\dots,x_{j-1}]\big)\,, \end{aligned}$$

for all $j = 1,\dots,n$, where $w_{j-1}\colon W_{n+1}(A) \to A$ denotes the $(j-1)^{\text{th}}$ ghost map. Now w_{j-1} is a ring morphism by definition of the ordinary Witt vectors and thus the same is true for $c_{n,j} = \phi_A^{n-j+1} \circ w_{j-1}$. A similar calculation shows that $c_{n,n+1} = w_n$ is also a ring morphism. This finishes the proof. $\square$

4.26. Construction. — We can extend the map $c_n\colon W_{n+1}(A) \to A[\![q-1]\!]/(q^{p^n}-1)$ canonically over q-$W_{n+1}(A)$. We only need to check that the diagrams

$$\begin{tikzcd} W_{n+1}(A) \arrow[r, "c_n"] \arrow[d, "F"'] & A[\![q-1]\!]/(q^{p^n}-1) \arrow[d] \\ W_n(A) \arrow[r, "c_{n-1}"] & A[\![q-1]\!]/(q^{p^{n-1}}-1) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W_n(A) \arrow[r, "c_n"] \arrow[d, "V"'] & A[\![q-1]\!]/(q^{p^{n-1}}-1) \arrow[d, "{[p]_{q^{p^{n-1}}}}"] \\ W_{n+1}(A) \arrow[r, "c_{n-1}"] & A[\![q-1]\!]/(q^{p^n}-1) \end{tikzcd}$$

commute, for then the universal property from Lemma 4.11 will provide us with the desired extension. By definition of the Witt vector Frobenius, we have $w_{j-1}(Fx) = w_j(x)$ and thus the equation

$$c_{n-1,j}(Fx) = \phi_A^{n-j} w_{j-1}(Fx) = \phi_A^{n-j} w_j(x) = c_{n,j+1}(x)$$

holds for all $j = 1,\dots,n$. If A is flat over $\mathbb{Z}_{(p)}$, this is enough to show commutativity of the diagram on the left by the same trick as in the proof of Lemma 4.25. The general case can again be reduced to this special case by taking a surjection $A' \twoheadrightarrow A$ of δ-rings with A flat over $\mathbb{Z}_{(p)}$. A similar calculation shows that the diagram on the right commutes as well, as desired.

Proof of Proposition 4.23. Recall from 4.1 that the framing $\square\colon \mathbb{Z}_p[T_1,\dots,T_d] \to R$ determines a Frobenius lift $\phi\colon R \to R$ and thus, by p-torsion freeness, a δ-structure. Let's denote $H^0_{R,\square}(n) \coloneqq H^0(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1))$ for short. It follows from Proposition 4.7 that $H^0_{R,\square}(n) \subseteq R[\![q-1]\!]/(q^{p^n}-1)$ is the subring

$$\sum_{i=0}^{n} [p^i]_{q^{p^{n-i}}} \big(\phi^{n-i}(R)[\![q-1]\!]/(q^{p^n}-1)\big) \subseteq R[\![q-1]\!]/(q^{p^n}-1)\,;$$

recall from 4.2 that $\phi^i(R)$ is a direct summand of R as $\mathbb{Z}_p$-modules, so $\phi^i(R)[\![q-1]\!]/(q^{p^n}-1)$ can indeed be regarded as being embedded into $R[\![q-1]\!]/(q^{p^n}-1)$, as the formula above suggests. By inspection, the map $c_n\colon q\text{-}W_{n+1}(R) \to R[\![q-1]\!]/(q^{p^n}-1)$ from Construction 4.26 lands inside $H^0_{R,\square}(n)$.

We'll show that $c_{n+1}\colon q\text{-}W_{n+1}(R) \rightarrow H^0_{R,\square}(n)$ is an isomorphism using induction on n. The case $n = 0$ is trivial, as $q\text{-}W_1(R) \cong R \cong H^0_{R,\square}(0)$. For the inductive step, we use that we have a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & q\text{-}W_n(R) \arrow[r, "V"] \arrow[d, "c_{n-1}"', "\cong"] & q\text{-}W_{n+1}(R) \arrow[r] \arrow[d, "c_n"] & R[\![q-1]\!]/[p]_{q^{p^{n-1}}} \arrow[r] \arrow[d, "\phi^n"', "\cong"] & 0 \\
0 \arrow[r] & H^0_{R,\square}(n-1) \arrow[r, "{[p]_{q^{p^{n-1}}}}"] & H^0_{R,\square}(n) \arrow[r] & \phi^n(R)[\![q-1]\!]/[p]_{q^{p^{n-1}}} \arrow[r] & 0
\end{tikzcd}$$

in which the top row is the sequence from Proposition 4.15 and the bottom follows immediately from our explicit description of $H^0_{R,\square}(n) \subseteq R[\![q-1]\!]/(q^{p^n}-1)$. The left square commutes by what we've checked in Construction 4.26, and the right square commutes by a simple inspection. Hence the five lemma finishes the inductive step and we're done. $\square$

§4.3. The q-de Rham Witt Complex

Having studied the q-Witt vectors in the previous subsection, we'll now proceed to construct a sequence of commutative differential-graded algebras $(q\text{-}W_n\widehat{\Omega}^*_R)_{n\geqslant 1}$, which satisfy a universal property similar to that of the de Rham–Witt pro-complex of an $\mathbb{F}_p$-algebra. This will provide us with a comparison map

$$q\text{-}W_{n+1}\widehat{\Omega}^*_R \longrightarrow H^*\left(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)\right),$$

which is in fact an isomorphism, as we'll show in §4.4. To motivate the definition of $(q\text{-}W_n\widehat{\Omega}^*_R)_{n\geqslant 1}$, let's first recall how the usual de Rham–Witt complex works.

4.27. De Rham–Witt Complexes. — Classically, de Rham–Witt complexes were defined as a certain left adjoint functor in [Ill79, Théorème I.1.3]. For us it will be convenient to work with an equivalent variant of that definition, which appears in [BLM21, Definition 4.4.1] and which we'll recall here. Let B be an $\mathbb{F}_p$-algebra. A *B-framed V-pro-complex* consists of the following data:

- A pro-system of commutative differential-graded algebras
$$(M_n^*)_{n\geqslant 1} = \left(\dots \xrightarrow{\mathrm{Res}} M_3^* \xrightarrow{\mathrm{Res}} M_2^* \xrightarrow{\mathrm{Res}} M_1^*\right).$$
- Maps $V\colon M_n^* \rightarrow M_{n+1}^*$ of graded abelian groups for all $n \geqslant 1$.
- A $W_n(B)$-algebra structure on M_n^0 for all $n \geqslant 1$.

This data is required to satisfy the following three conditions:

(a) For all $n \geqslant 1$, the structure maps $W_n(B) \rightarrow M_n^0$ are compatible with the restriction and Verschiebung on both sides. Furthermore, $V \circ \mathrm{Res} = \mathrm{Res} \circ V$.

(b) For all $n \geqslant 1$ and all $x, y \in M_n^*$, one has $V(x\,\mathrm{d}y) = Vx\,\mathrm{d}Vy$.

(c) For all $n \geqslant 1$ and all $b \in B$, $y \in M_n^0$, one has $Vy\,\mathrm{d}[b]_{n+1} = V(y[b]_n^{p-1})\,\mathrm{d}V[b]_n$, where $[b]_n$ and $[b]_{n+1}$ denote the images of the respective Teichmüller lifts of b under $W_n(B) \rightarrow M_n^0$ and $W_{n+1}(B) \rightarrow M_{n+1}^0$.

There is an obvious category $V\mathrm{PC}_B$ of B-framed V-pro-complexes, and the classical de Rham–Witt pro-complex $(W_n\Omega^*_B)_{n\geqslant 1}$ is its initial object. We remark that a priori there's no mention of any Frobenii, and in fact, the Frobenius operators on $(W_n\Omega_B^*)_{n\geqslant 1}$ are only constructed a posteriori.

In our situation, there seems to be no analogue of the restriction maps Res, as we've already seen in Lemma 4.11. So we simply leave out the restrictions, keep the other conditions and add a few technicalities to obtain the following definition:

4.28. Definition. — Let R be a (not necessarily framed) p-completely smooth $\mathbb{Z}_p$-algebra. An *R-framed p-complete q-V-sequence* consists of the following data:

- A sequence $(M_n^*)_{n\geqslant 1}$ of degree-wise p-torsion free and degree-wise $(p, q-1)$-complete commutative differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebras.
- Maps $V_n\colon M_n^* \to M_{n+1}^*$ of graded $\mathbb{Z}_p[\![q-1]\!]$-modules for all $n \geqslant 1$. We'll usually drop the index and just write V if n is clear from the context.
- An q-$W_n(R)$-algebra structure on M_n^0 for all $n \geqslant 1$.

These data are required to satisfy the following conditions:

(a) For all $n \geqslant 1$, the structure maps $q\text{-}W_n(R) \to M_n^0$ are compatible with the Verschiebung on both sides.

(b) For all $n \geqslant 1$ and all $x, y \in M_n^*$, one has $V(x\,\mathrm{d}y) = Vx\,\mathrm{d}Vy$.

(c) For all $n \geqslant 1$ and all $r \in R$, $y \in M_n^0$, one has $Vy\,\mathrm{d}[r]_{n+1} = V(y[r]_n^{p-1})\,\mathrm{d}V[r]_n$, where $[r]_n$ and $[r]_{n+1}$ denote the images of the respective Teichmüller lifts of r under $q\text{-}W_n(R) \to M_n^0$ and $q\text{-}W_{n+1}(R) \to M_{n+1}^0$. From now on we'll omit the indices and just write $[r]$ for both Teichmüller lifts.

There is an obvious category $q\text{-}V\widehat{\mathrm{Seq}}_R$ of R-framed p-complete q-V-sequences.

As for the classical situation, the Frobenii should be additional structure rather than part of the definition.

4.29. Definition. — Let $(M_n^*)_{n\geqslant 1}$ be an R-framed p-complete q-V-sequence. A set of *Frobenius operators* on $(M_n^*)_{n\geqslant 1}$ consists of maps $F_{n+1}\colon M_{n+1}^* \to M_n^*$ of graded $\mathbb{Z}_p[\![q-1]\!]$-algebras, which satisfy the following conditions:

(a) For all $n \geqslant 1$, the structure maps $q\text{-}W_n(R) \to M_n^0$ are compatible with the Frobenius on both sides.

(b) For all $n \geqslant 1$, one has $F_{n+1} \circ V_n = p$ and $V_n \circ F_{n+1} = [p]_{q^{p^{n-1}}}$.

As before, we'll drop the index and just write F whenever n is clear from the context. An R-framed p-complete q-V-sequence equipped with a choice of Frobenius operators will be called a *p-complete q-Witt sequence over R*, and the corresponding category will be denoted $q\text{-}\widehat{\mathrm{Witt}}_R$.

4.30. Remark. — The axioms from Definition 4.28 and Definition 4.29 formally imply number of further relations, which we'll summarise here. First of all, the condition from Definition 4.28(b) yields

$$V(a\,\mathrm{d}x_1\cdots\mathrm{d}x_k) = Va\,\mathrm{d}Vx_1\cdots\mathrm{d}Vx_k\,. \tag{4.30.1}$$

for all $n, k \geqslant 0$ and all $a, x_1, \dots, x_k \in M_n^*$. If M_n^* is generated by elements of this form, then Definition 4.28(b) is even equivalent to (4.30.1). Also, the condition from Definition 4.28(a) implies that $V_n(1) = [p]_{q^{p^{n-1}}}$, so using Definition 4.28(b) with $x = 1 \in M_n^0$ shows

$$V_n \circ \mathrm{d} = [p]_{q^{p^{n-1}}}(\mathrm{d} \circ V_n) = \mathrm{d} \circ V_n \circ [p]_{q^{p^{n-1}}}\,.$$

However, the source of V_n is M_n^*, which is $(q^{p^{n-1}}-1)$-torsion, so $\mathrm{d}\circ V_n\circ [p]_{q^{p^{n-1}}} = d\circ V_n\circ p$ and we obtain the relation

$$V\circ \mathrm{d} = p(\mathrm{d}\circ V)\,. \tag{4.30.2}$$

In particular, the graded $\mathbb{Z}_p[\![q-1]\!]$-module map $V\colon M_m^*\rightarrow M_{n+1}*$ is neither compatible with the multiplicative nor with the differential-graded structure.

Furthermore, if $(M_n^*)_{n\geqslant 1}$ is equipped with Frobenius operators as in Definition 4.29, then $F\colon M_{n+1}^*\rightarrow M_n^*$ is compatible with the multiplicative structure (by assumption), but not with the differential-graded one. Instead, we get the relations

$$F\circ\mathrm{d}\circ V=\mathrm{d}\quad\text{and}\quad p(F\circ\mathrm{d}) = \mathrm{d}\circ F\,. \tag{4.30.3}$$

Indeed, for the first one, observe that $p(F\circ\mathrm{d}\circ V) = F\circ p(\mathrm{d}\circ V) = F\circ V\circ d = p\,\mathrm{d}$, which implies the desired relation by the p-torsion freeness assumption. For the second, first observe that $[p]_{q^{p^{n-1}}}F=pF$, because the target of F is M_n^*, which is $(q^{p^{n-1}}-1)$-torsion by condition (a). Hence we can compute $p(F\circ\mathrm{d}) = [p]_{q^{p^{n-1}}}(F\circ\mathrm{d}) = F\circ\mathrm{d}\circ[p]_{q^{p^{n-1}}} = F\circ\mathrm{d}\circ V\circ F=\mathrm{d}\circ F$, where we also use the first relation from (4.30.3).

Finally, if $(M_n^*)_{n\geqslant 1}$ is equipped with Frobenius operators, we get yet another two relations for free:

$$xVy = V\bigl(F(x)y\bigr)\quad\text{and}\quad pV(yz) = V(y)V(z) \tag{4.30.4}$$

for all $x\in M_{n+1}^*$, $y,z\in M_n^*$ (these relations are automatically satisfied by Definition 4.28(a) if x, y, and z are in the images of q-$W_{n+1}(R)\rightarrow M_{n+1}^0$ and q-$W_n(R)\rightarrow M_n^0$ respectively, but for arbitrary x, y, and z we really need to show something). Indeed, we compute

$$pV\bigl(F(x)y\bigr) = V\bigl(F(x)py\bigr) = V\bigl(F(x)FV(y)\bigr) = VF\bigl(xV(y)\bigr) = [p]_{q^{p^{n-1}}}xVy\,.$$

Now Vy is a $(q^{p^{n-1}}-1)$-torsion element, whence $[p]_{q^{p^{n-1}}}xVy=pxVy$. By p-torsion freeness, this proves the first relation from (4.30.4). The second relation formally follows from the first one by plugging in Vz instead of x.

4.31. Remark. — Observe that the Teichmüller relation $V(y)\,\mathrm{d}[r] = V(y[r]^{p-1})\,\mathrm{d}V[r]$ from Definition 4.28(c) is redundant if y is contained in the image of q-$W_n(R)\rightarrow M_n^0$ or if $(M_n^*)_{n\geqslant 1}$ can be equipped with Frobenius operators. Indeed, in either case we have

$$p^pV\bigl(y[r]^{p-1}\bigr) = pV(y)V\bigl([r]\bigr)^{p-1}$$

by a repeated application of (4.30.4). As d is a derivation, we can conclude

$$p^pV\bigl(y[r]^{p-1}\bigr)\,\mathrm{d}V[r] = pV(y)V\bigl([r]\bigr)^{p-1} = V(y)\,\mathrm{d}\bigl((V[r])^p\bigr) = p^{p-1}V(y)\,\mathrm{d}\bigl(V([r]^p)\bigr)\,.$$

Note that $[r]^p=[r^p]=F[r]$ holds in $W_n(R)$ by a standard relation for ordinary Witt vectors. Thus $V([r]^p) = VF[r] = [p]_{q^{p^{n-1}}}[r]$ holds in q-$W_{n+1}(R)$. Plugging this into the equation above, we obtain

$$p^pV\bigl(y[r]^{p-1}\bigr)\,\mathrm{d}V[r] = p^{p-1}[p]_{q^{p^{n-1}}}V(y)\,\mathrm{d}[r]\,.$$

But $V(y)$ is $(q^{p^{n-1}}-1)$-torsion, hence the right-hand side agrees with $p^pV(y)\,\mathrm{d}[r]$. So the Teichmüller relation holds up to multiplication with p^p. But then it must hold on the nose, since everything is p-torsion free.

We can now explain the connection to the cohomology groups we're interested in. In the following we'll denote $H^*_{R,\square}(n) \coloneqq H^*(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1))$ for short.

4.32. Lemma. — *The graded $\mathbb{Z}_p[\![q-1]\!]$-modules $H^*_{R,\square}(n)$ for $n \geqslant 1$ can be equipped with the structure of an R-framed p-complete q-V-sequence in a natural way. Furthermore, there is a natural choice of Frobenius operators.*

Before we can prove Lemma 4.32, we have to explain where all the additional structure comes from.

4.33. The Graded Algebra Structure. — The complex $q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}$ itself can be given the structure of a non-commutative differential-graded algebra as follows: For homogeneous generators $\omega = x\,\mathrm{d}T_{i_1} \wedge \dots \wedge \mathrm{d}T_{i_k} \in q\text{-}\widehat{\mathrm{Hdg}}^k_{R,\square}$ and $\eta = y\,\mathrm{d}T_{j_1} \wedge \dots \wedge \mathrm{d}T_{j_\ell} \in q\text{-}\widehat{\mathrm{Hdg}}^\ell_{R,\square}$ we put

$$\omega \wedge \eta \coloneqq x\gamma_{i_1}\bigl(\gamma_{i_2}(\cdots\gamma_{i_k}(y)\cdots)\bigr)\,\mathrm{d}T_{i_1} \wedge \dots \wedge \mathrm{d}T_{i_k} \wedge \mathrm{d}T_{j_1} \wedge \dots \wedge \mathrm{d}T_{j_\ell}\,.$$

More succinctly, we use the good old wedge product and impose the additional non-commutative rule $\mathrm{d}T_i \wedge x \coloneqq \gamma_i(x) \wedge \mathrm{d}T_i$ for all $x \in R[\![q-1]\!]$ and all $i = 1, \dots, d$.

From the q-Leibniz rule, we easily get $\nabla_q(\omega \wedge \eta) = \nabla_q(\omega) \wedge \eta + (-1)^k \omega \wedge \nabla_q(\eta)$, so this multiplication does indeed define a differential-graded algebra structure. Hence $q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)$ inherits a differential-graded algebra structure, and so its cohomology $H^*_{R,\square}(n)$ comes equipped with the structure of a graded algebra.

However, a priori it might not be commutative. But it is! Indeed, if $x \in q\text{-}\widehat{\mathrm{Hdg}}^0_{R,\square}$ represents an element in $H^0_{R,\square}(n)$, then $\gamma_i(x) - x \equiv 0 \mod q^{p^n}-1$ holds by definition, and therefore $\mathrm{d}T_i \wedge x = x \wedge \mathrm{d}T_i$ in $H^*_{R,\square}(n)$.

4.34. Bockstein Differentials. — Since $q^{p^n}-1$ is a nonzerodivisor in $R[\![q-1]\!]$, we have a short exact sequence of complexes

$$0 \longrightarrow q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1) \xrightarrow{(q^{p^n}-1)} q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)^2 \longrightarrow q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1) \longrightarrow 0\,.$$

The associated connecting morphisms $\beta_n\colon H^*_{R,\square}(n) \to H^{*+1}_{R,\square}(n)$ are called *Bockstein differentials*. As the name suggests, β_n turns the graded $\mathbb{Z}_p[\![q-1]\!]$-module $H^*_{R,\square}(n)$ into a cochain complex (see [Stacks, Tag 0F7N] for example). This interacts well with the multiplicative structure:

4.35. Lemma. — *The Bockstein differential from 4.34 and the graded algebra structure from 4.33 make $(H^*_{R,\square}(n), \beta_n)$ a commutative differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebra.*

Proof. We only show that $\beta_n\colon H^0_{R,\square}(n) \to H^1_{R,\square}(n)$ is a derivation; the arguments in higher degrees are similar. Let $x, y \in R[\![q-1]\!]$ be elements whose images modulo $q^{p^n}-1$ are contained in $H^0_{R,\square}(n)$. Then $(q-1)\nabla_q(x) \in \widehat{\Omega}^1_R[\![q-1]\!]$ is divisible by $q^{p^n}-1$, so that $\nabla_q(f)$ is divisible by $[p^n]_q$. A quick unravelling then shows that $\beta_n(x)$ is the image of

$$\frac{(q-1)\nabla_q(x)}{q^{p^n}-1} = \frac{\nabla_q(x)}{[p^n]_q}$$

in $H^1_{R,\square}(n)$, and likewise for $\beta_n(y)$. Furthermore, $\nabla_{q,i}(x)$ being divisible by $[p^n]_q$ implies that $\gamma_i(x) - f = (qT_i - T_i)\nabla_{q,i}(x)$ is divisible by $q^{p^n}-1$. Thus, by the q-Leibniz rule,

$$\frac{\nabla_{q,i}(xy)}{[p^n]_q} = \gamma_i(x)\frac{\nabla_{q,i}(y)}{[p^n]_q} + y\frac{\nabla_{q,i}(x)}{[p^n]_q} \equiv x\frac{\nabla_{q,i}(y)}{[p^n]_q} + y\frac{\nabla_{q,i}(x)}{[p^n]_q} \mod q^{p^n}-1\,.$$

This shows $\beta_n(xy) = x\beta_n(y) + y\beta_n(x)$, as desired. $\square$

4.36. Remark. — In the special case $n = 0$, we see that the Bockstein differential sends x to the image of $\nabla_q(x)$ in $H^1_{R,\square}(0)$. But since $H^*_{R,\square}(0) = \widehat{\Omega}^*_R[\![q-1]\!]/(q-1) = \widehat{\Omega}^*_R$ holds as graded $\mathbb{Z}_p[\![q-1]\!]$-modules, we conclude that as a commutative differential-graded algebra, $(H^*_{R,\square}(0), \beta_0)$ coincides with the p-completed de Rham complex $(\widehat{\Omega}^*_R, \mathrm{d})$.

4.37. Frobenius and Verschiebung. — Proposition 4.23 suggests that the canonical projection and the multiplication map

$$F\colon H^*_{R,\square}(n) \longrightarrow H^*_{R,\square}(n-1) \quad\text{and}\quad V = [p]_{q^{p^{n-1}}}\colon H^*_{R,\square}(n-1) \longrightarrow H^*_{R,\square}(n)$$

should be thought of as Frobenius and Verschiebung. They clearly satisfy the relations $V \circ F = [p]_{q^{p^{n-1}}} = F \circ V$. But the target of F is $H^*_{R,\square}(n-1)$, which is $(q^{p^{n-1}}-1)$-torsion, hence also $F \circ V = p$. Thus, once Lemma 4.32 is proved (which we'll do in a moment), we see that all the relations from Remark 4.30 are satisfied.

Proof of Lemma 4.32. By Corollary 4.8 and Lemma 4.35, the $H^*_{R,\square}(n)$ are indeed degree-wise p-torsion free and degree-wise $(p, q-1)$-complete commutative differential-graded algebras. Define $V\colon H^*_{R,\square}(n-1) \to H^*_{R,\square}(n)$ and $F\colon H^*_{R,\square}(n) \to H^*_{R,\square}(n-1)$ as in 4.37 above. The condition from Definition 4.28(a) is satisfied by Proposition 4.23 (and its proof).

Let's check Definition 4.28(b) next. Suppose $\omega, \eta \in q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}$ are representatives of elements from $H^*_{R,\square}(n-1)$. Using the explicit descriptions of β_{n-1} and β_n from the proof of Lemma 4.35, we find that both $V(\omega\beta_{n-1}(\eta))$ and $V\omega\beta_n(V\eta)$ are represented by the element

$$[p]_{q^{p^{n-1}}}\left(\omega\frac{\nabla_q(\eta)}{[p^{n-1}]_q}\right) = \left([p]_{q^{p^{n-1}}}\omega\right)\cdot\frac{\nabla_q\left([p]_{q^{p^{n-1}}}\eta\right)}{[p^n]_q}\,,$$

as desired. It remains to check the Teichmüller condition from Definition 4.28, but this follows from the argument in Remark 4.31. $\square$

Let's now construct the q-de Rham–Witt pro-complex of any p-completely smooth $\mathbb{Z}_p$-algebra R, not necessarily equipped with a framing.

4.38. Proposition. — *Let R be a p-completely smooth $\mathbb{Z}_p$-algebra. The category $q\text{-}V\widehat{\mathrm{Seq}}_R$ has an initial object $(q\text{-}W_n\widehat{\Omega}^*_R)_{n\geqslant 1}$. It satisfies $q\text{-}W_n\widehat{\Omega}^0_R \cong q\text{-}W_n(R)$ for all $n \geqslant 1$. Moreover, $q\text{-}W_1\widehat{\Omega}^*_R \cong \widehat{\Omega}^*_R$ is the p-completed de Rham complex of R, and in general, $q\text{-}W_n\widehat{\Omega}^*_R$ is a quotient of the $(p, q-1)$-completed de Rham complex $\widehat{\Omega}^*_{q\text{-}W_n(R)/\mathbb{Z}_p[\![q-1]\!]}$ by a differential-graded ideal.*

4.39. Definition. — For all $n \geqslant 1$, the commutative differential-graded algebra $q\text{-}W_n\widehat{\Omega}^*_R$ from Proposition 4.38 is called the n^{th} *q-de Rham–Witt complex over R.*

4.40. Remark. — Throughout the proof of Proposition 4.38, we'll have to take various completions, so let's ensure that they behave well. We claim that for all $n \geqslant 0$ there exists a polynomial ring (in finitely many variables) P_{n+1} over $\mathbb{Z}_p$ together with a map $P_{n+1}[\![q-1]\!] \to q\text{-}W_{n+1}(R)$ that becomes surjective after $(p, q-1)$-completion. This will imply that $q\text{-}W_{n+1}(R)$ is noetherian and that the $(p, q-1)$-completed Kähler differential modules $\widehat{\Omega}^k_{q\text{-}W_{n+1}(R)/\mathbb{Z}_p[\![q-1]\!]}$ are finite modules over it. In particular, all their quotients will stay $(p, q-1)$-complete.

To prove the claim, let's first assume R admits a framing $\square$. By the explicit description of $H^0_{R,\square}(n)$ in the proof of Proposition 4.23, we can regard $q\text{-}W_{n+1}(R)$ both as a subring of $R[\![q-1]\!]$ and as an algebra over $\phi^n(R)[\![q-1]\!]$. Since $R[\![q-1]\!]$ is finite over $\phi^n(R)[\![q-1]\!]$ (see 4.1), so is $q\text{-}W_{n+1}(R)$, and the claim follows. For general R, choose a surjection $R' \twoheadrightarrow R$ from another p-completely smooth $\mathbb{Z}_p$-algebra R' such that R' admits a framing; for example, R' can be taken to be the p-completion of a suitable polynomial ring. Then $q\text{-}W_{n+1}(R)$ is a quotient of $q\text{-}W_{n+1}(R')$ and thus satisfies the desired condition as well.

Proof of Proposition 4.38. We first describe the construction of $(q\text{-}W_n\widehat{\Omega}^*_R)_{n\geqslant 1}$ and then we check that it's indeed initial. Put $q\text{-}W_1\widehat{\Omega}^*_R \coloneqq \widehat{\Omega}^*_R$ and observe that it is degree-wise p-torsion free as R is p-completely smooth over $\mathbb{Z}_p$. Now let $n \geqslant 1$ and suppose $q\text{-}W_i\widehat{\Omega}^*_R$ has already been constructed as a quotient of $\widehat{\Omega}^*_{q\text{-}W_i(R)/\mathbb{Z}_p[\![q-1]\!]}$ for $i = 1, \dots, n$. Furthermore, assume that for $i = 1, \dots, n-1$ we have already constructed suitable Verschiebung maps $V\colon q\text{-}W_i\widehat{\Omega}^*_R \to q\text{-}W_{i+1}\widehat{\Omega}^*_R$ satisfying the conditions from Definition 4.28.

We wish to construct $q\text{-}W_{n+1}\widehat{\Omega}^*_R$ and $V\colon q\text{-}W_n\widehat{\Omega}^*_R \to q\text{-}W_{n+1}\widehat{\Omega}^*_R$. To this end, let

$$N^*_{n+1} \subseteq \widehat{\Omega}^*_{q\text{-}W_{n+1}(R)/\mathbb{Z}_p[\![q-1]\!]}$$

denote the degree-wise $(p, q-1)$-complete differential-graded ideal generated by the following two kinds of generators:

- For all $r \in R$, $y \in q\text{-}W_n(R)$ we take the element $Vy\,\mathrm{d}[r] - V(y[r]^{p-1})\,\mathrm{d}V[r]$ in degree 1. Such generators will be called *generators of Teichmüller type*.
- For all $k \geqslant 1$, all finite indexing sets I, and all sequences $(a_i, x_{i,1}, \dots, x_{i,k})_{i\in I}$ of elements of $q\text{-}W_n(R)$ such that

  $$0 = \sum_{i\in I} a_i\,\mathrm{d}x_{i,1} \wedge \dots \wedge \mathrm{d}x_{i,k}$$

  holds in $q\text{-}W_n\widehat{\Omega}^k_R$ (which is a quotient of $\widehat{\Omega}^*_{q\text{-}W_n(R)/\mathbb{Z}_p[\![q-1]\!]}$, so the above sum makes indeed sense), we take the element

  $$\xi = \sum_{i\in I} Va_i\,\mathrm{d}Vx_{i,1} \wedge \dots \wedge \mathrm{d}Vx_{i,k}$$

  in degree k. Such generators will be called *generators of V-type*.

Now define $q\text{-}W'_{n+1}\widehat{\Omega}^*_R \coloneqq \widehat{\Omega}_{q\text{-}W_{n+1}(R)/\mathbb{Z}_p[\![q-1]\!]}/N^*_{n+1}$. As this might not yet be p-torsion free, we let $q\text{-}W_{n+1}\widehat{\Omega}^*_R$ be the quotient of $q\text{-}W'_{n+1}\widehat{\Omega}^*_R$ by its differential-graded ideal of p^∞-torsion. Furthermore, we define $V\colon q\text{-}W_n\widehat{\Omega}^*_R \to q\text{-}W_{n+1}\widehat{\Omega}^*_R$ by the formula

$$V\left(a\,\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_k\right) \coloneqq Va\,\mathrm{d}Vx_1 \wedge \dots \wedge \mathrm{d}Vx_k\,.$$

The generators of V-type make sure that this gives a well-defined $\mathbb{Z}_p[\![q-1]\!]$-linear map. Moreover, the conditions from Definition 4.28(a) and (c) are satisfied by construction, and so is the condition from Definition 4.28(b) by Remark 4.30. So we get a well-defined p-complete R-framed q-V-sequence $(q\text{-}W_n\widehat{\Omega}^*_R)_{n\geqslant 1}$.

It remains to show that $(q\text{-}W_n\widehat{\Omega}^*_R)_{n\geqslant 1}$ is initial. So let $(M^*_n)_{n\geqslant 1}$ be any p-complete R-framed q-V-sequence. Then M^0_1 is an R-algebra, so by the universal property of the de Rham complex, we get a unique map $f_1\colon q\text{-}W_1\widehat{\Omega}^*_R = \widehat{\Omega}^*_R \to M^*_1$ of degree-wise $(p, q-1)$-complete commutative differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebras. Now let $n \geqslant 1$ and assume

$f_i\colon q\text{-}W_i\widehat{\Omega}_R^* \rightarrow M_i^*$ have already been constructed for $i = 1, \dots, n$ in such a way that they are compatible with the Verschiebung maps on both sides. We show that there is a unique choice for $f_{n+1}\colon q\text{-}W_{n+1}\widehat{\Omega}_R^* \rightarrow M_{n+1}^*$. Since M_{n+1}^0 is a $q\text{-}W_{n+1}(R)$-algebra, we get a unique map

$$g_{n+1}\colon \widehat{\Omega}^*_{q\text{-}W_{n+1}(R)/\mathbb{Z}_p\llbracket q-1\rrbracket} \longrightarrow M_{n+1}^*$$

of of degree-wise $(p, q-1)$-complete commutative differential-graded $\mathbb{Z}_p\llbracket q-1\rrbracket$-algebras. So f_{n+1}, if it exists at all, is necessarily unique. To show existence, let's show that N_{n+1}^* is in the kernel of g_{n+1}. This can be done on generators. For generators of Teichmüller type, this is clear. So let's consider a generator of V-type. We put $\omega = \sum_{i\in I} a_i\, \mathrm{d}x_{i,1} \wedge \dots \wedge \mathrm{d}x_{i,k}$ for short. Then we need to show $g_{n+1}(V\omega) = 0$. Using Definition 4.28(a) and (b) for $(M_n)_{n\geqslant 1}$ we find that necessarily

$$g_{n+1}(V\omega) = V(f_n\omega)\,.$$

But ω vanishes in $q\text{-}W_n\widehat{\Omega}_R^k$ by assumption, and hence so does $V(f_n\omega)$ in M_{n+1}^k. This shows that g_{n+1} factors through $q\text{-}W'_{n+1}\widehat{\Omega}_R^*$. But since M_{n+1}^* is p-torsion free, the p^∞-torsion ideal of $q\text{-}W'_{n+1}\widehat{\Omega}_R^*$ must also be mapped to 0, so g_{n+1} descends indeed to a map

$$f_{n+1}\colon q\text{-}W_{n+1}\widehat{\Omega}_R^* \longrightarrow M_{n+1}^*\,.$$

As we've already noticed above, Definition 4.28(a) and (b) for $(M_n)_{n\geqslant 1}$ imply that $f_{n+1} \circ V = V \circ f_n$ and we're done. $\square$

4.41. Variants. — There are two more categories in which $(q\text{-}W_n\widehat{\Omega}_R^*)_{n\geqslant 1}$ happens to be the initial object.

- (a) The category of all $(M_n^*)_{n\geqslant 1}$ that satisfy all the conditions from Definition 4.28, except for possibly the Teichmüller condition.
- (b) The category $q\text{-}\widehat{\mathrm{Witt}}_R$ of all p-complete q-Witt sequences over R as defined in Definition 4.29.

To see why (a) is true, observe that the argument in Remark 4.31 implies that the generators of Teichmüller type are already p^∞-torsion elements in the quotient

$$\widehat{\Omega}^*_{q\text{-}W_n(R)/\mathbb{Z}_p\llbracket q-1\rrbracket}/(\text{generators of } V\text{-type})\,.$$

So $q\text{-}W_n\widehat{\Omega}_R^*$ could also be obtained by modding out all p^∞-torsion from that quotient. This shows that $(q\text{-}W_n\widehat{\Omega}_R^*)_{n\geqslant 1}$ is still initial if we start to denazify our category and drop the Teichmüller condition as a requirement.

For assertion (b), first note that by Theorem 4.43 below and Lemma 4.32 there is a choice of Frobenius operators on $(q\text{-}W_n\widehat{\Omega}_R^*)_{n\geqslant 1}$. Alternatively, these Frobenius operators can also be constructed by hand, which we'll have to do in the global case in §5. Furthermore, observe that any morphism $(q\text{-}W_n\widehat{\Omega}_R^*)_{n\geqslant 1} \rightarrow (M_n^*)_{n\geqslant 1}$ of p-complete q-V-sequences will automatically be compatible with any potential Frobenius operators on $(M_n^*)_{n\geqslant 1}$. Indeed, in degree 0 this is clear from Definition 4.29(a) and in degree 1 it follows from (4.30.3) and p-torsion freeness. In all higher degrees, compatibility is then automatic since the Frobenius operators are multiplicative and $q\text{-}W_n\widehat{\Omega}_R^*$ is $(p, q-1)$-completely generated by its elements in degree 0 and 1.

4.42. Relation to q-Crystalline Cohomology. — Since the usual de Rham–Witt complex computes crystalline cohomology, one might ask whether the q-de Rham–Witt complex computes q-crystalline cohomology. The answer is "yes, but actually no". Using Theorem 4.43 below and [Stacks, Tag 0F7T], we get a sequence of quasi-isomorphisms

$$q\text{-}W_{n+1}\widehat{\Omega}^*_R \simeq H^*\big(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)\big) \simeq L\eta_{q^{p^n}-1}\big(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}\big)/^L(q^{p^n}-1)\,.$$

Now the right-hand side can be identified with

$$L\eta_{[p^n]_q}\big(q\text{-}\widehat{\Omega}^*_{R,\square}\big)/^L(q^{p^n}-1) \simeq R\Gamma_{q\text{-crys}}\big(R/\mathbb{Z}_p[\![q-1]\!]\big) \mathbin{\widehat{\otimes}}^L_{\mathbb{Z}_p[\![q-1]\!],\phi^n} \mathbb{Z}_p[\![q-1]\!]/(q^{p^n}-1)$$

using Proposition 3.4. So yes, $q\text{-}W_{n+1}\widehat{\Omega}^*_R$ computes a base change of q-crystalline cohomology. But no, since $\phi^n\colon \mathbb{Z}_p[\![q-1]\!] \rightarrow \mathbb{Z}_p[\![q-1]\!]/(q^{p^n}-1)$ factors over $\mathbb{Z}_p$, this is actually just a base change of the crystalline cohomology of R.

§4.4. The Main Theorem in the p-Complete Case

Finally, we can formulate and prove the main result of §4.

4.43. Theorem. — *Let $(R,\square)$ be a framed p-completely smooth $\mathbb{Z}_p$-algebra. For all $n \geqslant 0$, the unique map induced by Lemma 4.32 and Proposition 4.38 is an isomorphism*

$$q\text{-}W_{n+1}\widehat{\Omega}^*_R \xrightarrow{\sim} H^*\big(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)\big)\,.$$

4.44. Outline of the Strategy, Part I. — The proof of Theorem 4.43 will occupy the rest of this subsection. We'll keep the shorthand notation $H^*_{R,\square}(n) \coloneqq H^*(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1))$. We first consider the case where $R = \mathbb{Z}_p\langle T\rangle$ is the p-completion of a polynomial ring in one variable. In this case, we understand $H^*_{\mathbb{Z}_p\langle T\rangle,\square}(n)$ well enough to verify the desired universal property directly. Next, we'll handle the case $R = \mathbb{Z}_p\langle T_1,\dots,T_d\rangle$: The main idea is to show that $H^*_{\mathbb{Z}_p\langle T_1,\dots,T_d\rangle,\square}(n)$ is "not too far" from being the $(p,q-1)$-completed tensor product

$$H^*_{\mathbb{Z}_p\langle T_1\rangle,\square}(n) \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p[\![q-1]\!]} \cdots \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p[\![q-1]\!]} H^*_{\mathbb{Z}_p\langle T_d\rangle,\square}(n)\,.$$

This will enable us to reduce the case $R = \mathbb{Z}_p\langle T_1,\dots,T_d\rangle$ to the one-variable case. Finally, we'll reduce the general case to the case of a polynomial ring via an étale base change argument that uses Lemma 4.22

4.45. Lemma. — *Theorem 4.43 is true in the case where $R = \mathbb{Z}_p\langle T\rangle$ is the p-completion of a polynomial ring in one variable and $\square\colon \mathbb{Z}_p[T] \rightarrow \mathbb{Z}_p\langle T\rangle$ is the obvious framing.*

Proof sketch. Let $(M^*_n)_{n\geqslant 0}$ be any R-framed p-complete q-V-sequence. As $H^*_{R,\square}(0) = \widehat{\Omega}^*_R$ by Remark 4.36, we get a unique morphism $H^*_{R,\square}(0) \rightarrow M^*_1$. Now let $n \geqslant 1$ and assume we've already constructed morphisms $f_i\colon H^*_{R,\square}(i) \rightarrow M^*_{i+1}$ for $i = 0,\dots,n-1$ in such a way that they are compatible with the Verschiebung maps on both sides. We must show that there is a unique choice for $f_n\colon H^*_{R,\square}(n) \rightarrow M^*_{n+1}$.

By Proposition 4.23, we already have a map $f^0_n\colon H^0_{R,\square}(n) \rightarrow M^0_{n+1}$ as part of the data of $(M^*_n)_{n\geqslant 1}$. So we only need to construct the degree-1 part f^1_n. Denote $R_n \coloneqq \mathbb{Z}_p\langle T^{p^n}\rangle$ for short. By Proposition 4.7, we can write

$$H^1_{R,\square}(n) \cong R_n[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} \bigoplus_{\alpha=1}^{p^n} T^{\alpha-1}\,\mathrm{d}T \cdot \mathbb{Z}_p[\![q-1]\!]/(q^{p^{e_\alpha}}-1)\,,$$

where we define $e_\alpha \coloneqq v_p(\alpha)$ to be the exponent of p in the prime factorisation of α. It follows that f_n^1 is already determined by the values of $f_n^1(T^{\alpha-1}\,\mathrm{d}T)$. Write $\alpha = p^{e_\alpha}\alpha'$. As observed in 4.5, $[\alpha']_{q^{p^{e_\alpha}}}$ is a unit. From the explicit description of the Bockstein differential β_n in the proof of Lemma 4.35, we find that

$$T^{\alpha-1}\,\mathrm{d}T = \frac{1}{[\alpha']_{q^{p^{e_\alpha}}}}\beta_n\left([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}T^\alpha\right),$$

so $f_n^1(T^{\alpha-1}\,\mathrm{d}T)$ is already uniquely determined by $f_n^0([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}T^\alpha)$. Also observe that $[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}T^\alpha \in H^0_{R,\square}(n)$ is $(q^{p^{e_\alpha}}-1)$-torsion, hence so is the prescribed value of $f_n(T^{\alpha-1}\,\mathrm{d}T)$. Therefore, we can extend these values to a unique $R_n[\![q-1]\!]$-linear morphism

$$f_n^1\colon H^1_{R,\square}(n) \longrightarrow M^1_{n+1}\,.$$

To finish the proof, we must verify that f_n defined as above is indeed a morphism of commutative differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebras. This leads to some straightforward but tedious calculations, which have been moved to the appendix, §A.3. $\square$

4.46. Remark. — A quick reality check: In the proof of Lemma 4.45, including the calculations that have been outsourced to the appendix, we never use the Teichmüller condition from Definition 4.28(c). However, this doesn't mean the proof is fishy; on the contrary, in the light of 4.41 it would be suspicious if we had to use the Teichmüller condition.

This finishes the case $R = \mathbb{Z}_p\langle T\rangle$. Onward to arbitrary polynomial rings!

4.47. Outline of the Strategy, Part II. — For the next few pages, $R = \mathbb{Z}_p\langle T_1,\dots,T_d\rangle$ is the p-completion of a polynomial ring in d variables, equipped with its obvious framing $\square\colon \mathbb{Z}_p[T_1,\dots,T_d] \to \mathbb{Z}_p\langle T_1,\dots,T_d\rangle$, and assume that Theorem 4.43 is true for $P \coloneqq \mathbb{Z}_p\langle T_1\rangle$ and $Q \coloneqq \mathbb{Z}_p\langle T_2,\dots,T_d\rangle$. It follows from Lemma 4.32 and functoriality of the q-Witt vectors that $(H^*_{R,\square}(n-1))_{n\geqslant 1}$ is both a P-framed and a Q-framed p-complete q-V-sequence. Therefore we get morphisms of commutative differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebras

$$H^*_{P,\square}(n)\,\widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]}\,H^*_{Q,\square}(n) \longrightarrow H^*_{R,\square}(n)$$

for all $n \geqslant 0$. To prove that $(H^*_{R,\square}(n-1))_{n\geqslant 1}$ has the desired universal property, we'll carefully analyse how far the above map is from being an isomorphism. To do this, we'll use a decomposition of $q\text{-}\widehat{\mathrm{Hdg}}_{R,\square}/(q^{p^n}-1)$ similar to Proposition 4.7. This will allow us to work with small manageable complexes and we can get away with only a tiny bit of calculation. So let's set up these decompositions first.

4.48. The Return of the Frobenius Decompositions. — In the following we'll denote $R_n \coloneqq \phi^n(R) = \mathbb{Z}_p\langle T_1^{p^n},\dots,T_d^{p^n}\rangle$ for short; also define P_n and Q_n similarly. Recall that Proposition 4.7 allows us to write

$$q\text{-}\widehat{\mathrm{Hdg}}^*_{Q,\square}/(q^{p^{n+1}}-1) \cong Q_{n+1}[\![q-1]\!]\otimes_{\mathbb{Z}_p[\![q-1]\!]}\bigoplus_j K^*_{n+1,e_j}[-k_j]\,.$$

where $0 \leqslant e_j \leqslant n+1$ and $0 \leqslant k_j \leqslant d-2$; the indexing set doesn't matter for our considerations. Since $q\text{-}\widehat{\mathrm{Hdg}}^*_{Q,\square}/(q^{p^n}-1)$ is a quotient of $q\text{-}\widehat{\mathrm{Hdg}}^*_{Q,\square}/(q^{p^{n+1}}-1)$ and Q_{n+1} is flat over $\mathbb{Z}_p$, it follows that

$$H^*_{Q,\square}(n) = Q_{n+1}[\![q-1]\!]\otimes_{\mathbb{Z}_p[\![q-1]\!]}\bigoplus_j H^{*-k_j}(\overline{K}^*_{n+1,e_j})\,, \tag{4.48.1}$$

where we put $\overline{K}^*_{n+1,e} \coloneqq K^*_{n+1,e}/(q^{p^n}-1)$. Of course, we could have applied Proposition 4.7 directly to $q\text{-}\widehat{\mathrm{Hdg}}^*_{Q,\square}/(q^{p^n}-1)$ to obtain a similar decomposition with Q_n in place of Q_{n+1}. The reason why we didn't do that is somewhat technical and will become apparent in 4.51. Also, we note that $\overline{K}^*_{n+1,e_i} = K^*_{n,\min\{e_i,n\}}$, but we'll stick to the former notation to emphasise which decomposition it is we're using.

Similarly, we can write

$$H^*_{P,\square}(n) = P_{n+1}\llbracket q-1\rrbracket \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} \bigoplus_i H^*\bigl(\overline{K}^*_{n+1,e_i}\bigr) \tag{4.48.2}$$

(no shifts occur since P is the p-completion of a polynomial ring in only one variable). Furthermore, since $q\text{-}\widehat{\mathrm{Hdg}}_{P,\square}/(q^{p^n}-1) \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p\llbracket q-1\rrbracket} q\text{-}\widehat{\mathrm{Hdg}}_{Q,\square}/(q^{p^n}-1) \cong q\text{-}\widehat{\mathrm{Hdg}}_{R,\square}/(q^{p^n}-1)$, we see that also $H^*_{R,\square}(n)$ has a decomposition

$$H^*_{R,\square}(n) = R_{n+1}\llbracket q-1\rrbracket \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} \bigoplus_{i,j} H^{*-k_j}\bigl(\overline{K}^*_{n+1,e_i} \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} \overline{K}_{n+1,e_j}\bigr)\,. \tag{4.48.3}$$

By construction, the decompositions from (4.48.1), (4.48.2), and (4.48.3) are compatible with the morphism $H^*_{P,\square}(n) \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p\llbracket q-1\rrbracket} H^*_{Q,\square}(n) \to H^*_{R,\square}(n)$ from 4.47. This has some nice consequences.

4.49. Lemma. — *The morphism from 4.47 is injective; moreover, it induces an isomorphism after inverting p. That is,*

$$\bigl(H^*_{P,\square}(n) \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p\llbracket q-1\rrbracket} H^*_{Q,\square}(n)\bigr)\bigl[\tfrac{1}{p}\bigr] \xrightarrow{\sim} H^*_{R,\square}(n)\bigl[\tfrac{1}{p}\bigr]\,.$$

Proof. By 4.48, it suffices to check that

$$H^*\bigl(\overline{K}^*_{n+1,e_i}\bigr) \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} H^*\bigl(\overline{K}^*_{n+1,e_j}\bigr) \longrightarrow H^*\bigl(\overline{K}^*_{n+1,e_i} \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} \overline{K}^*_{n+1,e_j}\bigr) \tag{4.49.1}$$

is injective, and an isomorphism after inverting p, for all i and j. This is a straightforward calculation: $H^*(\overline{K}^*_{n+1,e_i})$ is a degree-wise free $\mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^{\overline{e}_i}}-1)$-module generated by

$$\omega_0 = [p^{n-\overline{e}_i}]_{q^{p^{\overline{e}_i}}} \in \overline{K}^0_{n+1,e_i} \quad\text{and}\quad \omega_1 = 1 \in \overline{K}^1_{n+1,e_i}$$

in degrees 0 and 1, respectively. Here we put $\overline{e}_i \coloneqq \min\{e_i, n\}$ to get a consistent notation in the case $e_i = n+1$ (we brought this case upon ourselves by working with the decomposition from 4.48 rather than the one from Proposition 4.7, but as mentioned before, it will come in handy later on). Likewise, $H^*(\overline{K}^*_{n+1,e_j})$ is degree-wise free over $\mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^{\overline{e}_j}}-1)$ generated by

$$\eta_0 = [p^{n-\overline{e}_j}]_{q^{p^{\overline{e}_j}}} \in \overline{K}^0_{n+1,e_i} \quad\text{and}\quad \eta_1 = 1 \in \overline{K}^1_{n+1,e_i}\,.$$

Assume $e_i \geqslant e_j$; the other case is entirely analogous. Since both sides of (4.49.1) are degree-wise free $\mathbb{Z}_p\llbracket q-1\rrbracket/(q^{p^{\overline{e}_j}}-1)$-modules, it suffices to show that generators are mapped to generators up to powers of p.

Degree 0. By Lemma 4.6, $H^0(\overline{K}^*_{n+1,e_i} \otimes_{\mathbb{Z}_p\llbracket q-1\rrbracket} \overline{K}^*_{n+1,e_j})$ is generated by the element $\xi_0 \coloneqq 1 \otimes [p^{n-\overline{e}_j}]_{q^{p^{\overline{e}_j}}}$. Using the fact that $[p^{n-\overline{e}_i}]_{q^{p^{\overline{e}_i}}}$ acts like $p^{n-\overline{e}_i}$ on any $(q^{p^{\overline{e}_j}}-1)$-torsion element, we see that $\omega_0 \otimes \eta_0$ is mapped to

$$[p^{n-\overline{e}_i}]_{q^{p^{\overline{e}_i}}} \otimes [p^{n-\overline{e}_j}]_{q^{p^{\overline{e}_j}}} = p^{n-\overline{e}_i}\bigl(1 \otimes [p^{n-\overline{e}_j}]_{q^{p^{\overline{e}_j}}}\bigr) = p^{n-\overline{e}_i}\xi_0\,.$$

The left-hand side differs from ξ_0 only by a power of p. Thus the degree-0 part of (4.49.1) is indeed injective and an isomorphism after inverting p.

Degree 1. By Lemma 4.6, we have

$$H^1\big(\overline{K}^*_{n+1,e_i}\otimes_{\mathbb{Z}_p[\![q-1]\!]}\overline{K}^*_{n+1,e_j}\big)\cong H^1\big(\overline{K}^*_{n+1,e_j}[-1]\oplus\overline{K}^*_{n+1,e_j}\big)\,,$$

and a basis of the right-hand side is given by $\xi_1 \coloneqq ([p^{n-\overline{e}_j}]_{q^{p^{\overline{e}_j}}},0)$ and $\xi_2\coloneqq(0,1)$. Now condider $\omega_1\otimes\eta_0$ and $\omega_1\otimes\eta_0$. Under the isomorphism from Lemma 4.6, they are mapped to ξ_0 and $(-[p^{n-\overline{e}_j}]_{q^{p^{\overline{e}_j}}},[p^{n-\overline{e}_i}]_{q^{p^{\overline{e}_i}}})$, respectively. Hence $\omega_1\otimes\eta_0+\omega_1\otimes\eta_0$ is mapped to $p^{n-\overline{e}_i}\xi_2$, where we use the same torsion trick as in degree 0 above. So again a basis is mapped to a basis up to powers of p and thus (4.49.1) in degree 1 is as desired.

Degree 2. $\omega_1\otimes\eta_1$ is a generator of $H^1(\overline{K}^*_{n+1,e_i})\otimes_{\mathbb{Z}_p[\![q-1]\!]}H^1(\overline{K}^*_{n+1,e_j})$ and is mapped to a generator of $H^2(\overline{K}^*_{n+1,e_i}\otimes_{\mathbb{Z}_p[\![q-1]\!]}\overline{K}^*_{n+1,e_j})$. Since all other degrees vanish, we're done. $\square$

4.50. Lemma. — *Let $D^*\subseteq H^*_{R,\square}(n)\big[\frac{1}{p}\big]$ be the $(p,q-1)$-complete sub-differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebra generated by:*

- *the image of $H^*_{P,\square}(n)\mathbin{\widehat{\otimes}}_{\mathbb{Z}_p[\![q-1]\!]}H^*_{Q,\square}(n)$, and*
- *the sub-$\mathbb{Z}_p[\![q-1]\!]$-modules $p^{-i}\operatorname{im}(V^i\otimes V^i)\subseteq H^*_{R,\square}(n)\big[\frac{1}{p}\big]$ for $i=1,\dots,n$.*

Then $D^*=H^*_{R,\square}(n)$.

Lemma 4.50 will be the heart of the proof of Theorem 4.43 in the polynomial ring case. Before we can prove it, we need to go on a brief digression about the Bockstein differential β_n on $H^*_{R,\square}(n)$.

4.51. The Bockstein Differential modulo p. — Unfortunately, β_n doesn't seem to interact well with the decomposition of $q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)$ constructed in 4.48. However, this difficulty goes away if we reduce things modulo p.

First note that since both the complex $q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)$ and its cohomology are degree-wise p-torsion free, we get

$$H^*_{R,\square}(n)/p=H^*\big(q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(p,q^{p^n}-1)\big)\,.$$

Next, observe that the ideal $(p,q^{p^{n+1}}-1)=(p,(q^{p^n}-1)^p)$ is contained in $(p,(q^{p^n}-1)^2)$. Hence for every multi-index α as in 4.2 we get quotient maps of complexes

$$q\text{-}\widehat{\mathrm{Hdg}}^{*,\alpha}_{R,\square}/(p,q^{p^{n+1}}-1)\longrightarrow q\text{-}\widehat{\mathrm{Hdg}}^{*,\alpha}_{R,\square}/\big(p,(q^{p^n}-1)^2\big)\twoheadrightarrow q\text{-}\widehat{\mathrm{Hdg}}^{*,\alpha}_{R,\square}/\big(p,(q^{p^n}-1)\big)\,.$$

Now Proposition 4.7 shows that $q\text{-}\widehat{\mathrm{Hdg}}^{*,\alpha}_{R,\square}/(p,q^{p^{n+1}}-1)$ can be written as a direct sum of shifts of complexes of the form $R_{n+1}[\![q-1]\!]\otimes_{\mathbb{Z}_p[\![q-1]\!]}K^*_{n+1,e}/p$. This induces similar decompositions for the other two complexes above. Therefore, β_n modulo p can be computed using the Bockstein differentials $\overline{\beta}_{n+1,e}$ associated to the short exact sequences

$$0\longrightarrow K^*_{n+1,e}/(p,q^{p^n}-1)\xrightarrow{(q^{p^n}-1)}K^*_{n+1,e}/\big(p,(q^{p^n}-1)^2\big)\longrightarrow K^*_{n+1,e}/(p,q^{p^n}-1)\longrightarrow 0$$

for $e=0,\dots,n-1$.

Explicitly, this means the following: Suppose ω is an element of $q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)$, which is contained in a subcomplex of the form $R_{n+1}[\![q-1]\!]\otimes_{\mathbb{Z}_p[\![q-1]\!]}K^*_{n+1,e}/(q^{p^n}-1)[-k]$ for some

k and some e. Suppose furthermore that ω represents an element of the cohomology $H^*_{R,\square}$. By flatness of R_{n+1}, we may then write ω as a sum $\omega = \sum_i r_i \otimes x_i$, where $r_i \in R_{n+1}[\![q-1]\!]$ and $x_i \in K^{k_i}_{n+1,e}/(q^{p^n}-1)$ represent elements in the cohomology. Then

$$\beta_n(r_i \otimes x_i) \equiv r_i \otimes \overline{\beta}_{n+1,e}(x_i) \mod p\,.$$

If $k_i \neq 0$, then automatically $\overline{\beta}_{n+1,e}(x_i) = 0$, so that $\beta_n(r_i \otimes x_i) \equiv 0 \mod p$. Also, if $e = n+1$, then one checks that $\overline{\beta}_{n+1,n+1}$ vanishes, so again $\beta_n(r_i \otimes x_i) \equiv 0 \mod p$. So let's suppose $k_i = 0$ and $e \leqslant n$. In that case, for x_i to represent an element of the cohomology, we must have $(q^{p^e}-1)x_i = 0$ in $\mathbb{Z}_p[\![q-1]\!]/(p, q^{p^n}-1)$, so we can write $x_i = [p^{n-e}]_{q^{p^e}} y_i$ for some y_i, which is unique up to multiples of $(q^{p^e}-1)$. Then it's easy to check that $\overline{\beta}_{n+1,e}(x_i) = y_i$, so that $\beta_n(r_i \otimes x_i) \equiv r_i \otimes y_i \mod p$.

Proof of Lemma 4.50. If $\omega \in H^*_{P,\square}(n-i)$ and $\eta \in H^*_{Q,\square}(n-i)$ are any elements, then $H^*_{R,\square}(n)$ already contains the element

$$V^i(\omega \otimes \eta) = \frac{1}{p^i}\big(V^i(\omega) \otimes V^i(\eta)\big)\,.$$

This already proves $D^* \subseteq H^*_{R,\square}(n)$. To show equality, it then suffices to show that the induced morphism modulo p is degree-wise surjective, because both sides are degree-wise p-complete.

For this, we'll use the decompositions from 4.48 and argue very similar to the proof of Lemma 4.49. Consider an arbitrary summand

$$R_{n+1}[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} H^{*-k_j}\big(\overline{K}^*_{n+1,e_i} \otimes_{\mathbb{Z}_p[\![q-1]\!]} \overline{K}^*_{n+1,e_j}\big)$$

of $H^*_{R,\square}(n)$ as well as the corresponding summands $P_{n+1}[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} H^*(\overline{K}_{n+1,e_i})$ of $H^*_{P,\square}(n)$ and $Q_{n+1}[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} H^{*-k_j}(\overline{K}_{n+1,e_j})$. We assume $e_i \geqslant e_j$, the other case being analogous, and we put $\overline{e}_i \coloneqq \min\{e_i, n\}$, $\overline{e}_j = \min\{e_j, n\}$. Let furthermore ω_0, ω_1, η_0, η_1, ξ_0, ξ_1, ξ_2 be defined as in the proof of Lemma 4.49. We'll do a degree-wise case distinction again, except this time everything is shifted by k_j.

Degree k_j. $R_{n+1}[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} H^0(\overline{K}^*_{n+1,e_i} \otimes_{\mathbb{Z}_p[\![q-1]\!]} \overline{K}^*_{n+1,e_j})$ is generated by $1 \otimes \xi_0$ as an $R_{n+1}[\![q-1]\!]$-module. As in the proof of Lemma 4.49, we see that $(1 \otimes \omega_0) \otimes (1 \otimes \eta_0)$ is mapped to $p^{n-\overline{e}_i}(1 \otimes \xi_0)$. Now observe that

$$1 \otimes \omega_0 \in \operatorname{im} V^{n-\overline{e}_i} \quad \text{and} \quad 1 \otimes \eta_0 \in \operatorname{im} V^{n-\overline{e}_j}$$

Indeed, in $q\text{-}\widehat{\mathrm{Hdg}}^*_{P,\square}/(q^{p^n}-1)$ the element $1 \otimes \omega_0 = 1 \otimes [p^{n-\overline{e}_i}]_{q^{p^{\overline{e}_i}}}$ is divisible by $[p^{n-\overline{e}_i}]_{q^{p^{\overline{e}_i}}} = [p]_{q^{p^{n-1}}} \cdots [p]_{q^{p^{\overline{e}_i}}}$, and thus it is contained in the image of $V^{n-\overline{e}_i}$ on the level of complexes. Furthermore, any preimage will automatically represent an element in $H^*_{P,\square}(n-\overline{e}_i)$, hence $1 \otimes \omega_0$ also lies in the image of $V^{n-\overline{e}_i}$ on the level of cohomology. The same argument works for $1 \otimes \eta_0$. Since $e_i \geqslant e_j$, we can conclude that $1 \otimes \xi_0$ is contained in $p^{-(n-\overline{e}_i)} \operatorname{im}(V^{n-\overline{e}_i} \otimes V^{n-\overline{e}_i})$ and thus in D^*.

Degree $k_j + 1$. As we've seen in the proof of Lemma 4.49 $(1 \otimes \omega_1) \otimes (1 \otimes \eta_0)$ is mapped to $1 \otimes \xi_1$ and $(1 \otimes \omega_1) \otimes (1 \otimes \eta_0) + (1 \otimes \omega_0) \otimes (1 \otimes \eta_1)$ is mapped to $p^{n-\overline{e}_i}(1 \otimes \xi_2)$. If $e_i = n+1$, then $p^{n-\overline{e}_i} = 1$ and we obtain a generating system of the $R_{n+1}[\![q-1]\!]$-module $R_{n+1}[\![q-1]\!] \otimes_{\mathbb{Z}_p[\![q-1]\!]} H^0(\overline{K}^*_{n+1,e_i} \otimes_{\mathbb{Z}_p[\![q-1]\!]} \overline{K}^*_{n+1,e_j})$. So suppose $e_i \leqslant n$. Then also $e_j \leqslant n$. Thus 4.51 shows

$$\beta_n(\xi_0) \equiv 1 \otimes (0,1) \equiv 1 \otimes \xi_2 \mod p\,,$$

so in this case $\beta_n(\xi_0)$ and $\omega_1 \otimes \eta_0$ form a generating system modulo p. In either case we find a generating system modulo p which is contained in D^*.

Degree k_j+2. As we've seen in the proof of Lemma 4.49, $(1\otimes\omega_1)\otimes(1\otimes\eta_1)$ is mapped to a generator of the $R_{n+1}[\![q-1]\!]$-module $R_{n+1}[\![q-1]\!]\otimes_{\mathbb{Z}_p[\![q-1]\!]} H^2(\overline{K}^*_{n+1,e_i}\otimes_{\mathbb{Z}_p[\![q-1]\!]}\overline{K}^*_{n+1,e_j})$. Since all other degrees vanish, we are done. $\square$

4.52. Corollary. — *Theorem 4.43 is true in the case where $R=\mathbb{Z}_p\langle T_1,\dots,T_d\rangle$ is the p-completion of an arbitrary polynomial ring and $\square\colon \mathbb{Z}_p[T_1,\dots,T_d]\to\mathbb{Z}_p\langle T_1,\dots,T_d\rangle$ is the obvious framing.*

Proof. We do induction on d. The case $d=1$ is Lemma 4.45. For $d\geqslant 2$, define P and Q as in 4.47. Let furthermore $(M_n^*)_{n\geqslant 1}$ be any R-framed p-complete q-V-sequence. Since $H^*_{R,\square}(0)=\widehat{\Omega}^*_R$ by Remark 4.36, we get a unique map $f_0\colon H^*_{R,\square}(0)\to M_1^*$. Now let $n\geqslant 1$ and assume $f_i\colon H^*_{R,\square}(i)\to M^*_{i+1}$ have already been constructed for $i=0,\dots,n-1$ in such a way that they are compatible with the Verschiebung maps on both sides. We need to show that there exists choice for $f_n\colon H^*_{R,\square}(n)\to M^*_{n+1}$. Using the induction hypothesis for P and Q, we get a unique extension $g_n\colon H^*_{P,\square}(n)\mathbin{\widehat{\otimes}}_{\mathbb{Z}_p[\![q-1]\!]}H^*_{Q,\square}(n)\to M^*_{n+1}$, which, by Lemma 4.49, induces a map

$$g_n\left[\tfrac{1}{p}\right]\colon H^*_{R,\square}(n)\left[\tfrac{1}{p}\right]\longrightarrow M^*_{n+1}\left[\tfrac{1}{p}\right].$$

Since M^*_{n+1} is degree-wise p-torsion free, it can be regarded as a sub-differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebra of $M_{n+1}\left[\tfrac{1}{p}\right]$. Therefore, there's at most one choice for f_n. But Lemma 4.50 makes sure that the restriction of $g_n\left[\tfrac{1}{p}\right]$ to $H^*_{R,\square}(n)$ lands in M^*_{n+1}, as desired. $\square$

Now that we've dealt with the case of polynomial rings, R is again allowed to be an arbitrary p-completely smooth $\mathbb{Z}_p$-algebra. We only need one more preparatory lemma.

4.53. Lemma. — *Let R' be another p-completely smooth $\mathbb{Z}_p$-algebra and let $R\to R'$ be p-completely étale. Then $\square\colon\mathbb{Z}_p[T_1,\dots,T_d]\to R'$ is also a p-completely étale framing of R', and the induced morphism*

$$q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q^{p^n}-1)\mathbin{\widehat{\otimes}}_{q\text{-}W_{n+1}(R)}q\text{-}W_{n+1}(R')\xrightarrow{\ \sim\ }q\text{-}\widehat{\mathrm{Hdg}}^*_{R',\square}/(q^{p^n}-1)$$

(*where the tensor product is degree-wise $(p,q-1)$-completed*) *is an isomorphism.*

Proof. Since the two q-Hodge complexes are degree-wise free over $R[\![q-1]\!]/(q^{p^n}-1)$ and $R'[\![q-1]\!]/(q^{p^n}-1)$ respectively and have compatible bases by construction, it's enough to check that

$$R[\![q-1]\!]/(q^{p^n}-1)\mathbin{\widehat{\otimes}}_{q\text{-}W_{n+1}(R)}q\text{-}W_{n+1}(R')\xrightarrow{\ \sim\ }R'[\![q-1]\!]/(q^{p^n}-1)$$

is an isomorphism. By Elkik's algebraisation results [Elk73], R' is the derived p-completion of an étale R-algebra. Combining this with Lemma 4.22 and Corollary 4.19 shows that $q\text{-}W_{n+1}(R)\to q\text{-}W_{n+1}(R')$ is $(p,q-1)$-completely étale. So we may as well check that

$$R[\![q-1]\!]/(q^{p^n}-1)\mathbin{\widehat{\otimes}}^L_{q\text{-}W_{n+1}(R)}q\text{-}W_{n+1}(R')\xrightarrow{\ \sim\ }R'[\![q-1]\!]/(q^{p^n}-1)$$

is a quasi-isomorphism. By derived Nakayama ([Stacks, Tag 0G1U]), this can be checked after applying the functor $R\mathbin{\widehat{\otimes}}^L_{R[\![q-1]\!]/(q^{p^n}-1)}-$ on both sides. The left-hand side then becomes

$R \mathbin{\widehat{\otimes}}^L_{q\text{-}W_{n+1}(R)} q\text{-}W_{n+1}(R')$, where R is considered as an $q\text{-}W_{n+1}(R)$-algebra via the map $c_{n,n+1}\colon q\text{-}W_{n+1}(R) \to R$ from the proof of Lemma 4.25. But this map coincides with the n-fold Frobenius $F^n\colon q\text{-}W_{n+1}(R) \to q\text{-}W_1(R)$. By Lemma 4.22 and the analogous assertion for ordinary Witt vectors, we have

$$q\text{-}W_1(R) \mathbin{\widehat{\otimes}}^L_{q\text{-}W_{n+1}(R)} q\text{-}W_{n+1}(R') \simeq q\text{-}W_1(R') \simeq R'\,.$$

But also $R \mathbin{\widehat{\otimes}}^L_{R\llbracket q-1\rrbracket/(q^{p^n}-1)} R'\llbracket q-1\rrbracket/(q^{p^n}-1) \simeq R'$ by $(p,q-1)$-complete flatness. This finishes the proof. $\square$

Proof of Theorem 4.43. Let $P \coloneqq \mathbb{Z}_p\langle T_1, \dots, T_d\rangle$, so that Theorem 4.43 is true for P by Corollary 4.52. The framing $\square$ induces a p-completely étale map $P \to R$, and so $q\text{-}W_{n+1}(P) \to q\text{-}W_{n+1}(R)$ is $(p,q-1)$-completely étale by the argument in the proof of Lemma 4.53. Furthermore, that lemma shows that

$$H^*_{R,\square}(n) \cong H^*_{P,\square}(n) \mathbin{\widehat{\otimes}}_{q\text{-}W_{n+1}(P)} q\text{-}W_{n+1}(R)$$

as graded-commutative $\mathbb{Z}_p\llbracket q-1\rrbracket$-algebras, and then also as commutative differential-graded $\mathbb{Z}_p\llbracket q-1\rrbracket$-algebras by a general fact about étale base change of differential-graded algebras; see Lemma A.16. Thus, we may reduce the desired universal property for the R-framed p-complete q-V-sequence $(H^*_{R,\square}(n-1))_{n\geqslant 1}$ to the corresponding universal property for the P-framed p-complete q-V-sequence $(H^*_{P,\square}(n-1))_{n\geqslant 1}$. $\square$

§5. Cohomology of the q-Hodge Complex II: The Global Case

After dealing with the p-complete case in Theorem 4.43, we can now attack the global problem of computing the cohomology groups $H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ for any $m \geqslant 1$.

§5.1. Big q-Witt Vectors

We start with a brief recollection about truncated big Witt vectors, following Hesselholt's exposition in [Hes15, Section 1].

5.1. Big Witt Vectors. — Let S be an arbitrary commutative, but not necessarily unital ring, and $\Sigma \subseteq \mathbb{N}$ a subset which is closed under divisors (a *truncation set* in Hesselholt's terminology). As a set, the big Witt ring $\mathbb{W}_\Sigma(S)$ is given by S^Σ. Its ring structure is uniquely determined by the condition that for all $n \in \Sigma$ the *ghost map* $w_n\colon \mathbb{W}_\Sigma(S) \rightarrow S$ given by

$$w_n\big((a_i)_{i\in\Sigma}\big) \coloneqq \sum_{d\mid n} d a_d^{n/d}$$

is a morphism of rings and functorial in S.

For us, Σ will always be the set Σ_m of positive divisors of some integer m, and we'll write $\mathbb{W}_m(S) = \mathbb{W}_{\Sigma_m}(S)$ for short. By [Hes15, Lemmas 1.3–1.5], for every divisor $d \mid m$ there are *Frobenius* and *Verschiebung* maps

$$F_{m/d}\colon \mathbb{W}_m(S) \longrightarrow \mathbb{W}_d(S) \quad \text{and} \quad V_{m/d}\colon \mathbb{W}_d(S) \longrightarrow \mathbb{W}_d(S)$$

such that $F_{m/d}$ is a ring map and $V_{m/d}$ is a map of abelian groups. If $n = m/d$ and the numbers m and d are clear from the context (or irrelevant), we abuse notation and write just $F_n \coloneqq F_{m/d}$ and $V_n \coloneqq V_{m/d}$. These maps fulfil the following relations: For all chains of divisors $e \mid d \mid m$ we have

$$F_{d/e} \circ F_{m/d} = F_{m/e} \quad \text{and} \quad V_{m/d} \circ V_{d/e} = V_{m/e}\,.$$

Furthermore, if $n \geqslant 1$ is arbitrary and k is coprime to n, then

$$F_n \circ V_n = n \quad \text{and} \quad F_n \circ V_k = V_k \circ F_n\,,$$

where we use the abuse of notation we just warned about. Finally, there's a multiplicative section of $w_m\colon \mathbb{W}_m(S) \rightarrow S$, called the *Teichmüller lift*

$$[-]_m\colon S \longrightarrow \mathbb{W}_m(S)\,.$$

We usually drop the index if no confusion can occur (and to avoid further confusion with the notation $[m/d]_{q^d} = \frac{q^m-1}{q^d-1}$ from 1.10). It interacts with the Frobenius and the Verschiebung via the formulas $F_n[s] = [s^n]$ for all $s \in S$ and $x = \sum_{d\mid m} V_{m/d}[x_d]$ for all $x = (x_d)_{d\mid m} \in \mathbb{W}_m(S)$.

5.2. Remark. — If $m = p^n$ is a prime power, then $\mathbb{W}_{p^n}(S) \cong W_{n+1}(S)$ equals the ring of truncated p-typical Witt vectors of length $n+1$. Furthermore, the Frobenius and Verschiebung maps F_p and V_p coincide with their p-typical namesakes F and V, as does the Teichmüller lift.

Now there's a natural way to generalise our p-typical Definition 4.9 to the big world: In addition to $F_n \circ V_n = n$, we would like to enforce the condition that $V_n \circ F_n$ is a q-analogue of n; more precisely, we want that $V_{m/d} \circ F_{m/d} = [m/d]_{q^d}$. This leads to the following.

5.3. Definition. — Let S be a commutative, but not necessarily unital ring. The ring of *m-truncated big q-Witt vectors over S* is the ring

$$q\text{-}\mathbb{W}_m(S) \coloneqq \mathbb{W}_m(S)[\![q-1]\!]/\mathbb{I}_m\,,$$

where $\mathbb{I}_m$ is the ideal generated by:

- $(q^d-1)\operatorname{im} V_{m/d}$ for all divisors $d \mid m$, and
- $\operatorname{im}([d/e]_{q^e}V_{m/d} - V_{m/e}F_{d/e})$ for all chains of divisors $e \mid d \mid m$.

5.4. Remark. — Observe that if $m = p^n$ is a prime power, then $q\text{-}\mathbb{W}_{p^n}(S) \cong q\text{-}W_{n+1}(S)$ by inspection and Remark 5.2.

5.5. Lemma. — *The system $(q\text{-}\mathbb{W}_m(S))_{m\in\mathbb{N}}$ from Definition 5.3 is the universal system of rings satisfying the following two conditions:*

(*a*) *$q\text{-}\mathbb{W}_m(S)$ is a $\mathbb{W}_m(S)[\![q-1]\!]/(q^m-1)$-algebra for all $m \in \mathbb{N}$.*

(*b*) *For all divisors $d \mid m$, the Frobenius and Verschiebung maps on the ordinary big Witt vectors of S extend to $\mathbb{Z}[\![q-1]\!]$-linear maps $F_{m/d}\colon q\text{-}\mathbb{W}_m(S) \to q\text{-}\mathbb{W}_d(S)$ and $V_{m/d}\colon q\text{-}\mathbb{W}_d(S) \to q\text{-}\mathbb{W}_m(S)$ satisfying the relations*

$$F_{m/d} \circ V_{m/d} = m/d \quad \textit{and} \quad V_{m/d} \circ F_{m/d} = [m/d]_{q^d}\,.$$

Proof. The only non-obvious point is that the Frobenius maps extend, the rest can be done as in Lemma 4.11. It suffices to show that $F_p\colon \mathbb{W}_m(S)[\![q-1]\!] \to \mathbb{W}_{m/p}(S)[\![q-1]\!]$ satisfies $F_p(\mathbb{I}_m) \subseteq \mathbb{I}_{m/p}$ for all prime factors $p \mid m$.

Let's first consider generators of the form $(q^d-1)V_{m/d}x$ for $x \in \mathbb{W}_d(S)$. Depending on whether $n = m/d$ is coprime to p or not, the relations from 5.1 yield, respectively,

$$F_p\big((q^d-1)V_n x\big) = (q^d-1)V_n(F_p x) \quad \text{or} \quad F_p\big((q^d-1)V_n x\big) = p(q^d-1)V_{(m/p)/d}x\,.$$

In either case, we get an element of $\mathbb{I}_{m/p}$. Now let's consider generators of the form $[d/e]_{q^e}V_{m/d}x - V_{m/e}F_{d/e}x$ for some $x \in \mathbb{W}_d(S)$. If p divides both m/d and m/e, we can use an easy computation as above to show that F_p sends the element into $\mathbb{I}_{m/p}$. Next, let's assume p is coprime to both m/d and m/e. Let's write $m_0 \coloneqq m/p$, $d_0 \coloneqq d/p$, and $e_0 \coloneqq e/p$ for short. Using the relations from 5.1, we can compute

$$\begin{aligned}F_p\big([d/e]_{q^e}V_{m/d}x - V_{m/e}F_{d/e}x\big) &= [d/e]_{q^e}V_{m_0/d_0}(F_p x) - V_{m_0/e_0}F_{d_0/e_0}(F_p x)\\ &= [d_0/e_0]_{q^{e_0}}V_{m_0/d_0}(F_p x) - V_{m_0/e_0}F_{d_0/e_0}(F_p x) + \big([d/e]_{q^e} - [d_0/e_0]_{q^{e_0}}\big)V_{m_0/d_0}(F_p x)\end{aligned}$$

The first summand is contained in $\mathbb{I}_{m/p}$ by definition. Regarding the second summand, observe that our assumptions imply that p is coprime to $d/e = d_0/e_0$ and therefore the sequences $1, q^{e/p}, (q^{e/p})^2 \dots, (q^{e/p})^{d/e-1}$ and $1, q^e, (q^e)^2 \dots, (q^e)^{d/e-1}$ coincide modulo $q^{d/p}-1$ up to permutation. Thus $[d/e]_{q^e} - [d_0/e_0]_{q^{e_0}}$ is divisible by $q^{d/p}-1 = q^{d_0}-1$ and so the second summand is also contained in $\mathbb{I}_{m/p}$.

It remains to consider the case where m/d is coprime to p, but m/e is not. Put $m_0 \coloneqq m/p$ and $d_0 \coloneqq d/p$ again. Using the relations from 5.1, we can compute

$$\begin{aligned}F_p\bigl([d/e]_{q^e} V_{m/d}x - V_{m/e}F_{d/e}x\bigr) &= [d/e]_{q^e} V_{m_0/d_0}(F_p x) - pV_{m_0/e}F_{d/e}x\\ &= p\bigl([d_0/e]_{q^e} V_{m_0/d_0}(F_p x) - V_{m_0/e}F_{d_0/e}(F_p x)\bigr) + \bigl([p]_{q^{d_0}} - p\bigr)[d_0/e]_{q^e} V_{m_0/d_0}(F_p x)\,.\end{aligned}$$

The first summand is again contained in $\mathbb{I}_{m/p}$ by definition. Regarding the second summand, we observe $[p]_{q^{d_0}} \equiv p \mod q^{d_0}-1$ and so $[p]_{q^{d_0}}[d_0/e]_{q^e} \equiv p[d_0/e]_{q^e} \mod q^{d_0}-1$. Hence the second summand is contained in $(q^{d_0}-1)\operatorname{im} V_{m_0/d_0}$, which is in turn contained in $\mathbb{I}_{m/p}$ by definition. This finishes the proof. $\square$

5.6. Lemma. — *Let S be a commutative, but not necessarily unital ring, and let $m \in \mathbb{N}$ be arbitrary.*

(*a*) *Let p be a prime, let $(-)_p^\wedge$ denote derived p-completion and let $v_p(m)$ be the p-adic valuation of m. Assume that $\widehat{S}_p$ is discrete. Then*

$$q\text{-}\mathbb{W}_m(S)_p^\wedge \simeq q\text{-}W_{v_p(m)+1}(\widehat{S}_p)\,.$$

(*b*) *Let N be an integer divisible by m and let $(-)_{q-1}^\wedge$ denote derived $(q-1)$-completion. Then*

$$q\text{-}\mathbb{W}_m(S)\bigl[\tfrac{1}{N}\bigr]_{q-1}^\wedge \simeq S\bigl[\tfrac{1}{N}\bigr]\,.$$

Proof. For (*a*), observe that $n \coloneqq m/p^{v_p(m)}$ and $[n]_{q^{m/n}}$ are units in $\mathbb{Z}_p[\![q-1]\!]$. Hence, using Lemma 5.5, the Frobenius and Verschiebung induce quasi-isomorphisms

$$F_n\colon q\text{-}\mathbb{W}_m(S)_p^\wedge \xrightarrow{\sim} q\text{-}\mathbb{W}_{p^{v_p(m)}}(S)_p^\wedge \quad\text{and}\quad V_n\colon q\text{-}\mathbb{W}_{p^{v_p(m)}}(S)_p^\wedge \xrightarrow{\sim} q\text{-}\mathbb{W}_m(S)_p^\wedge$$

which are mutually inverse (up to unit). Combining this with Remark 5.4 and Corollary 4.19 proves (*a*). For (*b*), a similar argument shows that

$$F_m\colon q\text{-}\mathbb{W}_m(S)\bigl[\tfrac{1}{N}\bigr]_{q-1}^\wedge \xrightarrow{\sim} q\text{-}\mathbb{W}_1(S)\bigl[\tfrac{1}{N}\bigr]_{q-1}^\wedge \simeq S\bigl[\tfrac{1}{N}\bigr]$$

is a quasi-isomorphism, with inverse (up to unit) given by V_m. $\square$

Combining Lemma 5.6 with the derived Beauville–Laszlo theorem (Lemma A.8, Remark A.9) provides a convenient method to reduce assertions about big q-Witt vectors to their p-typical counterparts. For example, that method can be used to show analogues of Proposition 4.15, Corollary 4.18, and Corollary 4.21 for big q-Witt vectors, but also to deduce the global version of Theorem 4.43 (as we will see soon). With that being said, we can start computing the cohomology of the q-Hodge complex.

5.7. Proposition. — *Let $(R,\square)$ be a framed smooth $\mathbb{Z}$-algebra. Then for all $m \in \mathbb{N}$, there are isomorphisms*

$$q\text{-}\mathbb{W}_m(R) \xrightarrow{\sim} H^0\bigl(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)\bigr)\,.$$

Under these isomorphisms, the Frobenius $F_{m/d}\colon q\text{-}\mathbb{W}_m(R) \to q\text{-}\mathbb{W}_d(R)$ gets identified with the map induced by the canonical projection

$$q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1) \longrightarrow q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^d-1)\,,$$

and the Verschiebung $V_{m/d}\colon q\text{-}\mathbb{W}_d(R) \to q\text{-}\mathbb{W}_m(R)$ gets identified with the map induced by the scalar multiplication map

$$[m/d]_{q^d}\colon q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^d-1) \longrightarrow q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)\,.$$

5.8. Outline of the Strategy. — We put $H^*_{R,\square}(m) \coloneqq H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ for short. To prove Proposition 5.7 we'll see the combination of Lemma 5.6 and derived Beauville–Laszlo in action for the first time. Unfortunately though, the argument will be somewhat convoluted, due to the following complication: For our Beauville–Laszlo arguments to work, we need to ensure that the q-Hodge complex and its cohomology interact well with various completions. However, in order to ensure that, we already need to prove a part of Proposition 5.7.

So we'll proceed as follows: First, we construct a map $q\text{-}\mathbb{W}_m(R) \to R[\![q-1]\!]/(q^m-1)$ and check that it lands in $H^0_{R,\square}(m)$. Then we use this to show that $q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)$ is a complex of finite modules over the noetherian ring $q\text{-}\mathbb{W}_m(R)$. Once this is clear, all completions till the end of this section will behave nicely and we can finish the proof of Proposition 5.7.

Proof of Proposition 5.7 (first half). Let's first construct a map

$$q\text{-}\mathbb{W}_m(R) \longrightarrow R[\![q-1]\!]/(q^m-1)\,.$$

Using Lemma 5.6 as well as derived Beauville–Laszlo in the form of Remarks A.9 and A.10, we get a pullback square of $\mathbb{Z}[\![q-1]\!]$-modules (and then also of $\mathbb{Z}[\![q-1]\!]$-algebras)

$$\begin{tikzcd} q\text{-}\mathbb{W}_m(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_{p\mid N} q\text{-}W_{v_p(m)+1}(\widehat{R}_p) \arrow[d] \\ R\left[\tfrac{1}{N}\right] \arrow[r] & \prod_{p\mid N} \widehat{R}_p\left[\tfrac{1}{p}\right] \end{tikzcd}$$

where N is any integer divisible my m. A similar pullback diagram exists for $R[\![q-1]\!]/(q^m-1)$. We note that $R[\![q-1]\!]$ is noetherian, so all derived completions that appear in the following will be discrete and coincide with their underived counterparts. As in Lemma 5.6, we see that $R[\![q-1]\!]/(q^m-1) \to R[\![q-1]\!]/(q^{p^{v_p(m)}}-1)$ becomes an isomorphism after $(-)^\wedge_p$ for all prime factors $p \mid N$. Similarly, the quotient map $R[\![q-1]\!]/(q^m-1) \to R$ becomes an isomorphism after $(-)\left[\tfrac{1}{N}\right]^\wedge_{q-1}$. Thus, we get a pullback

$$\begin{tikzcd} R[\![q-1]\!]/(q^m-1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_{p\mid N} \widehat{R}_p[\![q-1]\!]/(q^{p^{v_p(m)}}-1) \arrow[d] \\ R\left[\tfrac{1}{N}\right] \arrow[r] & \prod_{p\mid N} \widehat{R}_p\left[\tfrac{1}{p}\right] \end{tikzcd}$$

of $\mathbb{Z}[\![q-1]\!]$-algebras. The maps $q\text{-}W_{v_p(m)+1}(\widehat{R}_p) \to \widehat{R}_p[\![q-1]\!]/(q^{p^{v_p(m)}}-1)$ from Proposition 4.23 induce a map between these pullback squares, which gives us the desired map $q\text{-}\mathbb{W}_m(R) \to R[\![q-1]\!]/(q^m-1)$.

This map doesn't depend on the choice of N by a standard argument. It's also immediately clear that it lands in $H^0_{R,\square}(m)$. Indeed, $q\text{-}\mathrm{Hdg}^1_{R,\square}/(q^m-1)$ can be written as a similar pullback above, and then, by construction, $q\text{-}\mathbb{W}_m(R)$ is sent to 0 in each of the factors. This finishes the first half of the proof. $\square$

5.9. Lemma. — *For all $m \in \mathbb{N}$, the ring $q\text{-}\mathbb{W}_m(R)$ is noetherian, and the map from Proposition 5.7 makes $q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1)$ into a complex of finite $q\text{-}\mathbb{W}_m(R)$-modules.*

Proof. Let $P \coloneqq \mathbb{Z}[T_1, \dots, T_d]$ and $P_m \coloneqq \mathbb{Z}[T_1^m, \dots, T_d^m]$. Replacing R by P in the above, we get a map $q\text{-}\mathbb{W}_m(P) \to P[\![q-1]\!]/(q^m-1)$, which is injective because so are the maps on all the pullback factors. Moreover, its image contains $P_m[\![q-1]\!]/(q^m-1)$, because each pullback factor contains T_i^m for all $i = 1, \dots, d$ by inspection and Proposition 4.23. Now $q\text{-}\mathbb{W}_m(P)$ is finite over the noetherian ring $P_m[\![q-1]\!]/(q^m-1)$ since it is a submodule of the finite module $P[\![q-1]\!]/(q^m-1)$. This proves that $q\text{-}\mathbb{W}_m(P)$ itself is noetherian. To show the same for $q\text{-}\mathbb{W}_m(R)$, choose a surjection $P' \twoheadrightarrow R$ from another polynomial ring P' to see that $q\text{-}\mathbb{W}_m(R)$ is a quotient of $q\text{-}\mathbb{W}_m(P')$, which is noetherian by the same argument as for P.

The complex $q\text{-}\mathrm{Hdg}_{R,\square}^*/(q^m-1)$ is degree-wise finite free over $R[\![q-1]\!]/(q^m-1)$, hence it suffices to show that $R[\![q-1]\!]/(q^m-1)$ is finite over $q\text{-}\mathbb{W}_m(R)$. In the case $R = P$, this follows from the arguments above. In general, the étale framing $\square\colon P \to R$ induces a quasi-isomorphism

$$P[\![q-1]\!]/(q^m-1) \mathbin{\widehat{\otimes}}^L_{q\text{-}\mathbb{W}_m(P)} q\text{-}\mathbb{W}_m(R) \xrightarrow{\sim} R[\![q-1]\!]/(q^m-1)\,.$$

Indeed, using Lemma 5.6 and derived Beauville–Laszlo (Remark A.9), this can be reduced to an analogous assertion for p-typical q-Witt vectors, which has already been shown in the proof of Lemma 4.53. $\square$

5.10. Remark. — The proof of Lemma 5.9 shows slightly more: For all $m \in \mathbb{N}$, there exists a Polynomial ring (in finitely many variables) P_m over $\mathbb{Z}$ and surjection $P_m[\![q-1]\!] \twoheadrightarrow q\text{-}\mathbb{W}_m(R)$. Furthermore, the argument didn't use that $(R, \square)$ is a framed smooth $\mathbb{Z}$-algebra; all we needed was that R has finite type over $\mathbb{Z}$.

Proof of Proposition 5.7 (second half). On finite modules over a noetherian ring, p-completion is exact. Therefore, Lemma 5.9 ensures

$$H^0_{R,\square}(m)_p^\wedge \cong H^0\left(\left(q\text{-}\mathrm{Hdg}_{R,\square}^*/(q^m-1)\right)_p^\wedge\right) \cong H^0\left(q\text{-}\widehat{\mathrm{Hdg}}_{\hat{R}_p,\square}^*/(q^{p^{v_p(m)}}-1)\right)$$

and so $q\text{-}\mathbb{W}_m(R) \to H^0_{R,\square}(m)$ is an isomorphism after $(-)_p^\wedge$ for all prime factors $p \mid N$ by Lemma 5.6 and Proposition 4.23. A similar argument shows that it is an isomorphism after $(-)\left[\frac{1}{N}\right]_{q-1}^\wedge$, whence we are done by derived Beauville–Laszlo. $\square$

§5.2. The Big q-de Rham–Witt Complex

Hesselholt–Madsen [HM03] and Hesselholt [Hes15] have introduced a big de Rham–Witt complex over any ring A. However, the (yet to be defined) object that we will confidently call *big q-de Rham Witt complex over R* isn't really a q-version of their big de Rham–Witt complex. Instead, it's probably better to think of it as a big version of the q-de Rham–Witt complex from Definition 4.39.

On that note, let's first give big versions of Definitions 4.28 and 4.29.

5.11. Definition. — Let R be a (not necessarily framed) smooth $\mathbb{Z}$-algebra. An *R-framed q-V-system* consists of the following data:

- A system $(M_m^*)_{m\in\mathbb{N}}$ of degree-wise $\mathbb{Z}$-torsion free and degree-wise $(q-1)$-complete commutative differential-graded $\mathbb{Z}[\![q-1]\!]$-algebras.
- Maps $V_{m/d}\colon M_d^* \to M_m^*$ of graded $\mathbb{Z}_p[\![q-1]\!]$-modules for all divisors $d \mid m$. Occasionally we'll use the same abuse of notation that we warned about in 5.1.

- An q-$\mathbb{W}_m(R)$-algebra structure on M_m^0 for all $m \in \mathbb{N}$.

This data is required to satisfy the following four conditions:

(a) For all $m \in \mathbb{N}$, the structure maps $q\text{-}\mathbb{W}_m(R) \rightarrow M_m^0$ are compatible with the Verschiebung on both sides.

(b) For all chains $e \mid d \mid m$ of divisors, one has $V_{m/d} \circ V_{d/e} = V_{m/e}$.

(c) For all $d \mid m$ and all $x, y \in M_d^*$, one has $V_{m/d}(x\, \mathrm{d}y) = V_{m/d}x\, \mathrm{d}V_{m/d}y$.

(d) For all $d \mid m$ and all $r \in R$, $y \in M_d^0$, one has

$$V_{m/d}y\, \mathrm{d}[r]_m = V_{m/d}\bigl(y[r]_d^{m/d-1}\bigr)\, \mathrm{d}V_{m/d}[r]_d\,,$$

where $[r]_d$ and $[r]_m$ denote the images of the respective Teichmüller lifts of r under $q\text{-}\mathbb{W}_d(R) \rightarrow M_d^0$ and $q\text{-}\mathbb{W}_m(R) \rightarrow M_m^0$.

There is an obvious category $q\text{-}V\mathrm{Sys}_R$ of R-framed q-V-systems.

5.12. Definition. — Let $(M_m^*)_{m\in\mathbb{N}}$ be an R-framed q-V-system. A set of *Frobenius operators* on $(M_m^*)_{m\in\mathbb{N}}$ consists of maps $F_{m/d}\colon M_m^* \rightarrow M_d^*$ of graded $\mathbb{Z}[\![q-1]\!]$-algebras for all $d \mid m$, which satisfy the following conditions:

(a) For all $m \in \mathbb{N}$, the structure maps $q\text{-}W_m(R) \rightarrow M_m^0$ are compatible with the Frobenius on both sides.

(b) For all chains $e \mid d \mid m$ of divisors, one has $F_{d/e} \circ F_{m/d} = F_{m/e}$.

(c) For all $d \mid m$, one has $F_{m/d} \circ V_{m/d} = m/d$ and $V_{m/d} \circ F_{m/d} = [m/d]_{q^d}$.

An R-framed q-V-system equipped with a choice of Frobenius operators will be called a *q-Witt system over R*, and the corresponding category will be denoted $q\text{-}\mathrm{Witt}_R$.

5.13. Remark. — Just as in Remark 4.30 (only that we have to replace p-torsion freeness by general $\mathbb{Z}$-torsion freeness), we get a whole zoo of relations that hold for any R-framed q-V-system $(M_m)_{m\in\mathbb{N}}$. First of all,

$$V_n \circ \mathrm{d} = n(\mathrm{d} \circ V_n) \tag{5.13.1}$$

holds for all $n \geqslant 1$. Furthermore, if $(M_m)_{m\in\mathbb{N}}$ is equipped with Frobenius operators, then

$$F_n \circ \mathrm{d} \circ V_n = \mathrm{d} \quad \text{and} \quad n(F_n \circ \mathrm{d}) = \mathrm{d} \circ F_n \tag{5.13.2}$$

holds for all $n \geqslant 1$, along with the projection formulas

$$xV_ny = V_n\bigl(F_n(x)y\bigr) \quad \text{and} \quad nV_n(y)V_n(z) = V_n(yz) \tag{5.13.3}$$

for all $y, z \in M_m^*$, $x \in M_{mn}^*$ (but note that these formulas also hold without the existence of Frobenius operators as long as x, y, and z are contained in the image of the structure maps $q\text{-}\mathbb{W}_{mn}(R) \rightarrow M_{mn}^0$ and $q\text{-}\mathbb{W}_m(R) \rightarrow M_m^0$, respectively). Finally, if n and k are coprime, then the relation

$$F_n \circ V_k = V_k \circ F_n \tag{5.13.4}$$

holds automatically. Indeed, by $\mathbb{Z}$-torsion freeness, this can be checked after applying $n = F_n \circ V_n$ on both sides, hence also after applying V_n. For $x \in M_m^*$, we obtain $V_nF_nV_kx =$

$[n]_{q^{km/n}}V_kx$ and $V_nV_kF_nx = V_kV_nF_nx = [n]_{q^{m/n}}V_kx$. Since k and n are coprime, the sequences $1, q^{m/n}, (q^{m/n})^2, \dots, (q^{m/n})^{n-1}$ and $1, q^{km/n}, (q^{km/n})^2, \dots, (q^{km/n})^{n-1}$ coincide modulo $q^m - 1$ up to permutation. Now V_kx is a (q^m-1)-torsion element and thus $[n]_{q^m/n}V_kx = [n]_{q^{km/n}}V_kx$, as desired.

5.14. Remark. — As in Remark 4.31, one can show that the Teichmüller condition from Definition 5.11(d) is redundant for y contained in the image of $q\text{-}\mathbb{W}_d(R) \to M_d^0$, and in the presence of Frobenius operators even for arbitrary y.

5.15. Lemma. — *Let $(R, \square)$ be a framed smooth $\mathbb{Z}$-algebra. The graded $\mathbb{Z}[\![q-1]\!]$-modules $H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ for $m \in \mathbb{N}$ can be equipped with the structure of a q-Witt system over R (Definition 5.12) in a natural way.*

Proof. The differential-graded algebra structures along with the Frobenius and Verschiebung maps can be constructed as in 4.33, 4.34, and 4.37, and to show that they satisfy the desired conditions one can argue as in the proof of Lemma 4.32. It only remains to show that $H^*(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))$ is $\mathbb{Z}$-torsion free in every degree. But for every prime p, being p-torsion free can be tested after derived p-completion (as long as everything stays discrete, see Lemma A.5(b)), and

$$\left(H^i(q\text{-}\mathrm{Hdg}^*_{R,\square}/(q^m-1))\right)^\wedge_p \cong H^i\left(q\text{-}\widehat{\mathrm{Hdg}}^*_{\widehat{R}_p,\square}/(q^{p^{v_p(m)}}-1)\right)$$

holds for all i by Lemma 5.9, so Corollary 4.8 finishes the job. ☐

5.16. Proposition. — *Let R be a smooth $\mathbb{Z}$-algebra. The category $q\text{-}V\mathrm{Sys}_R$ has an initial object $(q\text{-}\mathbb{W}_m\Omega^*_R)_{m\in\mathbb{N}}$. It satisfies $q\text{-}\mathbb{W}_m\Omega^0_R \cong q\text{-}\mathbb{W}_m(R)$ for all $m \in \mathbb{N}$. Moreover, $q\text{-}\mathbb{W}_1\Omega^*_R \cong \Omega^*_R$ is de Rham complex of R, and in general, $q\text{-}\mathbb{W}_m\Omega^*_R$ is a quotient of the degree-wise $(q-1)$-completed de Rham complex $\widehat{\Omega}^*_{q\text{-}\mathbb{W}_m(R)/\mathbb{Z}[\![q-1]\!]}$ by a differential-graded ideal.*

5.17. Definition. — For all $m \in \mathbb{N}$, the commutative differential-graded algebra $q\text{-}\mathbb{W}_m\Omega^*_R$ from Proposition 5.16 is called the m^{th} *big q-de Rham–Witt complex over R.*

Proof of Proposition 5.16. Remark 5.10 implies that $\widehat{\Omega}^*_{q\text{-}\mathbb{W}_m(R)/\mathbb{Z}[\![q-1]\!]}$ is degree-wise finite over the noetherian ring $q\text{-}\mathbb{W}_m(R)$. Hence all completions in the following will be nicely behaved.

We first describe the construction. Let $m > 1$ and suppose the $q\text{-}\mathbb{W}_i\Omega^*_R$ for $i \leqslant m-1$ along with the Verschiebung maps between them have already been constructed as desired. We wish to construct $q\text{-}\mathbb{W}_m\Omega^*_R$ and $V_{m/d}\colon q\text{-}\mathbb{W}_d\Omega^*_R \to q\text{-}\mathbb{W}_m\Omega^*_R$ for all divisors $d \mid m$. To this end, let

$$\mathbb{N}^*_m \subseteq \widehat{\Omega}^*_{q\text{-}\mathbb{W}_m(R)/\mathbb{Z}[\![q-1]\!]}$$

denote the smallest degree-wise $(q-1)$-complete differential-graded ideal with the following property: For all divisors $d \mid m$, $d \neq m$, all $k \geqslant 1$, all finite indexing sets I, and all sequences $(a_i, x_{i,1}, \dots, x_{i,k})_{i\in I}$ of elements of $q\text{-}\mathbb{W}_d(R)$ such that

$$0 = \sum_{i\in I} a_i\,\mathrm{d}x_{i,1} \wedge \dots \wedge \mathrm{d}x_{i,k}$$

holds in $q\text{-}\mathbb{W}_d\Omega^k_R$ (which is a quotient of $\widehat{\Omega}^*_{q\text{-}\mathbb{W}_d(R)/\mathbb{Z}[\![q-1]\!]}$, so the above sum makes indeed sense), the degree-k element

$$\xi = \sum_{i\in I} V_{m/d}a_i\,\mathrm{d}V_{m/d}x_{i,1} \wedge \dots \wedge \mathrm{d}V_{m/d}x_{i,k}$$

is contained in $\mathbb{N}_m^*$ (here $V_{m/d}$ denotes the Verschiebung on big q-Witt vectors, so the definition isn't circular). Now define $q\text{-}\mathbb{W}'_m\Omega_R^* \coloneqq \widehat{\Omega}_{q\text{-}\mathbb{W}_m(R)/\mathbb{Z}[\![q-1]\!]}/\mathbb{N}_m^*$. As this might not yet be $\mathbb{Z}$-torsion free, we let $q\text{-}\mathbb{W}_m\Omega_R^*$ be the quotient of $q\text{-}\mathbb{W}'_m\Omega_R^*$ by its differential-graded ideal of $\mathbb{Z}$-torsion. Furthermore, we define $V_{m/d}\colon q\text{-}\mathbb{W}_d\Omega_R^* \to q\text{-}\mathbb{W}_m\Omega_R^*$ by the formula

$$V_{m/d}\left(a\,\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_k\right) \coloneqq V_{m/d}a\,\mathrm{d}V_{m/d}x_1 \wedge \dots \wedge \mathrm{d}V_{m/d}x_k$$

for all divisors $d \mid m$. By definition of $\mathbb{N}_m^*$, the map $V_{m/d}$ is well-defined. Hence the above construction yields indeed a system $(q\text{-}\mathbb{W}_m\Omega_R^*)_{m\in\mathbb{N}}$ that satisfies all conditions from Definition 5.11 except for possibly the Teichmüller condition. But since $q\text{-}\mathbb{W}_m\Omega_R^0 \cong q\text{-}\mathbb{W}_m(R)$, Remark 5.14 ensures that the Teichmüller condition holds true automatically. Finally, proving that $(q\text{-}\mathbb{W}_m\Omega_R^*)_{m\in\mathbb{N}}$ is indeed initial in $q\text{-}V\mathrm{Sys}_R$ can then be done by a straightforward argument as in the proof of Proposition 4.38. ☐

§5.3. The Main Theorem in the Global Case

5.18. Theorem. — *Let $(R,\square)$ be a framed smooth $\mathbb{Z}$-algebra. For all $m \in \mathbb{N}$, the unique map induced by Lemma 5.15 and Proposition 5.16 is an isomorphism*

$$q\text{-}\mathbb{W}_m\Omega_R^* \xrightarrow{\sim} H^*\left(q\text{-}\mathrm{Hdg}_{R,\square}^*/(q^m-1)\right).$$

In contrast to the p-complete case, this time we won't get around proving the existence of Frobenius operators on $(q\text{-}\mathbb{W}_m\Omega_R^*)_{m\in\mathbb{N}}$. Instead, their existence will play a crucial role in the proof. So that will be our first task.

5.19. Proposition. — *For any smooth $\mathbb{Z}$-algebra R, the big q-de Rham–Witt complex $(q\text{-}\mathbb{W}_m\Omega_R^*)_{m\in\mathbb{N}}$ admits a unique choice of Frobenius operators. Therefore, it is also an initial object in the category $q\text{-}\mathrm{Witt}_R$ from Definition 5.12.*

We send an auxiliary lemma in advance.

5.20. Lemma. — *Let $m \in \mathbb{N}$ and $n \geqslant 1$. Then for all $x \in q\text{-}\mathbb{W}_{mn}(R)$, the element $\mathrm{d}F_n(x) \in q\text{-}\mathbb{W}_m\Omega_R^1$ is divisible by n.*

Proof. Since all maps in sight are linear over $\mathbb{Z}[\![q-1]\!]$ and $(q-1)$-adically continuous, we may as well assume $x \in \mathbb{W}_{mn}(R)$. Let $x = (x_d)_{d\mid mn}$, so that $x = \sum_{d\mid mn} V_{mn/d}[x_d]$ by 5.1. Then we may further reduce to the case $x = V_{mn/d}[x_d]$. Putting $k = \gcd(n, mn/d)$, we compute

$$F_n(x) = F_n\left(V_{mn/d}[x_d]\right) = kV_{(mn)/(dk)}\left(F_{n/k}[x_d]\right) = kV_{(mn)/(dk)}\left([x_d]^{n/k}\right).$$

Since d is a derivation, $\mathrm{d}([x_d]^{n/k})$ is divisible by n/k. Hence also $V_{(mn)/(dk)}\,\mathrm{d}([x_d]^{n/k}) = (mn)/(dk)\,\mathrm{d}V_{(mn)/(dk)}([x_d]^{n/k})$ is divisible by n/k. But $(mn)/(dk)$ and n/k are coprime by definition of k, so already $\mathrm{d}V_{(mn)/(dk)}([x_d]^{n/k})$ is divisible by n/k. Then the calculation above shows that $\mathrm{d}F_n(x)$ is indeed divisible by n. ☐

Proof of Proposition 5.19. It suffices to construct $F_p\colon q\text{-}\mathbb{W}_m\Omega_R^* \to q\text{-}\mathbb{W}_{m/p}\Omega_R^*$ for all prime factors $p \mid m$. The Frobenius on big q-Witt vectors induces a map of differential-graded algebras on $(q-1)$-completed de Rham complexes over $\mathbb{Z}[\![q-1]\!]$. Combining this with Lemma 5.20 shows that we can define a map of graded rings $F_p\colon \widehat{\Omega}^*_{q\text{-}\mathbb{W}_m(R)/\mathbb{Z}[\![q-1]\!]} \to q\text{-}\mathbb{W}_{m/p}\Omega_R^*$ via

$$F_p(a\,\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_k) \coloneqq \frac{F_pa\,\mathrm{d}F_px_1 \wedge \dots \wedge \mathrm{d}F_px_k}{p^k}$$

for all $a, x_1, \dots, x_k \in q\text{-}\mathbb{W}_m(R)$; here we also use $\mathbb{Z}$-torsion freeness to ensure that the quotient is well-defined. We're certainly done if we can show that this map descends to $q\text{-}\mathbb{W}_m\Omega_R^*$.

To this end, first note that F_p satisfies $\mathrm{d} \circ F_p = p(F_p \circ \mathrm{d})$. By $\mathbb{Z}$-torsion freeness of $q\text{-}\mathbb{W}_{m/p}\Omega_R^*$, this implies that the kernel of F_p is a differential-graded ideal, even though F_p itself is not a map of differential-graded algebras. Hence it suffices to show that F_p annihilates the generators of the differential-graded ideal $\mathbb{N}_m^*$ from the proof of Proposition 5.16 (because it also automatically annihilates any remaining $\mathbb{Z}$-torsion).

So let ξ be as in the proof of Proposition 5.16. Let's first assume p divides m/d. Then $F_p \circ V_{m/d} = pV_{(m/p)/d}$ and therefore

$$F_p(\xi) = pV_{(m/p)/d}\left(\sum_{i \in I} a_i \, \mathrm{d}x_{i,1} \wedge \dots \wedge \mathrm{d}x_{i,k}\right).$$

But $\sum_{i \in I} a_i \, \mathrm{d}x_{i,1} \wedge \dots \wedge \mathrm{d}x_{i,k}$ vanishes in $q\text{-}\mathbb{W}_d\Omega_R^*$ by assumption, and so $F_p(\xi) = 0$, as desired. Now assume p is coprime to m/d; let $m_0 \coloneqq m/p$ and $d_0 \coloneqq d/p$. Then $F_p \circ V_{m/d} = V_{m_0/d_0} \circ F_p$. Furthermore, arguing inductively, we may assume that $F_p\colon q\text{-}\mathbb{W}_d\Omega_R^* \to q\text{-}\mathbb{W}_{d/p}\Omega_R^*$ has already been constructed. This allows us to compute

$$\begin{aligned} F_p(\xi) &= \sum_{i \in I} \frac{(V_{m_0/d_0}F_p a_i)\, \mathrm{d}(V_{m_0/d_0}F_p x_{i,1}) \wedge \dots \wedge \mathrm{d}(V_{m_0/d_0}F_p x_{i,k})}{p^k} \\ &= V_{m_0/d_0}F_p\left(\sum_{i \in I} a_i \, \mathrm{d}x_{i,1} \wedge \dots \wedge dx_{i,k}\right). \end{aligned}$$

But $\sum_{i \in I} a_i \, \mathrm{d}x_{i,1} \wedge \dots \wedge \mathrm{d}x_{i,k}$ vanishes in $q\text{-}\mathbb{W}_d\Omega_R^*$ again and therefore we get $F_p(\xi) = 0$ in the second case too. This finishes the proof that $(q\text{-}\mathbb{W}_m\Omega_R^*)_{m \in \mathbb{N}}$ can be equipped with Frobenius operators. Uniqueness and the fact that we get an initial object in $q\text{-}\mathrm{Witt}_R$ can be deduced as in 4.41. $\square$

We need only one final lemma before we can prove Theorem 5.18.

5.21. Lemma. — *The full subcategory of $q\text{-}\mathrm{Witt}_R$ spanned by those q-Witt systems $(M_m^*)_{m \in \mathbb{N}}$ over R such that each M_m^* is degree-wise p-complete is equivalent to the category $\widehat{q\text{-}\mathrm{Witt}}_{\widehat{R}_p}$ of p-complete q-Witt sequences over $\widehat{R}_p$ (Definition 4.29).*

Proof. Let $(M_m^*)_{m \in \mathbb{N}}$ be an element of $q\text{-}\mathrm{Witt}_R$ such that each M_m^* is degree-wise p-complete. If we denote $n \coloneqq m/p^{v_p(m)}$, then Frobenius and Verschiebung

$$F_n\colon M_m^* \longrightarrow M_{p^{v_p(m)}}^* \quad \text{and} \quad V_n\colon M_{p^{v_p(m)}}^* \longrightarrow M_m^*$$

are mutually inverse (up to unit) isomorphisms by the same argument as in Lemma 5.6. Hence $(M_m^*)_{m \in \mathbb{N}} \mapsto (M_{p^n}^*)_{n \geqslant 1}$ defines the desired equivalence. $\square$

Proof of Theorem 5.18. Let N be any integer divisible by m. By the derived Beauville–Laszlo theorem (Lemma A.8, Remark A.9), whether $q\text{-}\mathbb{W}_m\Omega_R^* \to H^*(q\text{-}\mathrm{Hdg}_{R,\square}^*/(q^m-1))$ is an isomorphism can be checked after applying the functors $(-)_p^\wedge$ for $p \mid N$ and $(-)\left[\frac{1}{N}\right]_{q-1}^\wedge$ in every degree.

Let's consider the p-completions first. Using Proposition 5.19, we see that the degree-wise p-completion of $(q\text{-}\mathbb{W}_m\Omega_R^*)_{m \in \mathbb{N}}$ is the initial object in the category of all q-Witt systems

$(M_m^*)_{m\in\mathbb{N}}$ such that each M_m^* is degree-wise p-complete. By Lemma 5.21, this provides us with isomorphisms

$$(q\text{-}\mathbb{W}_m\Omega_R^i)_p^\wedge \xrightarrow{\sim} q\text{-}W_{v_p(m)+1}\widehat{\Omega}_{\widehat{R}_p}^i$$

for all i. But thanks to Lemma 5.9, we also have an isomorphism

$$\left(H^i(q\text{-}\mathrm{Hdg}_{R,\square}^*/(q^m-1))\right)_p^\wedge \xrightarrow{\sim} H^i\left(q\text{-}\widehat{\mathrm{Hdg}}_{\widehat{R}_p,\square}^*/(q^{p^{v_p(m)}}-1)\right).$$

Hence Theorem 4.43 finishes the job after p-completion.

Now let's consider the situation after $(-)\left[\frac{1}{N}\right]_{q-1}^\wedge$ has been applied. By the same argument as in Lemma 5.6, the Frobenius

$$F_m\colon q\text{-}\mathbb{W}_m\Omega_R^i\left[\tfrac{1}{N}\right]_{q-1}^\wedge \longrightarrow q\text{-}\mathbb{W}_1\Omega_R^i\left[\tfrac{1}{N}\right]_{q-1}^\wedge \cong \Omega_R^i\left[\tfrac{1}{N}\right]$$

is an isomorphism for all i, with inverse (up to unit) given by the Verschiebung V_m. But the same argument again, together with Lemma 5.9 to ensure that $(q-1)$-completion commutes with cohomology, shows that

$$\left(H^i(q\text{-}\mathrm{Hdg}_{R,\square}^*/(q^m-1))\right)\left[\tfrac{1}{N}\right]_{q-1}^\wedge \longrightarrow H^i\left(q\text{-}\mathrm{Hdg}_{R,\square}^*/(q-1)\right)\left[\tfrac{1}{N}\right]_{q-1}^\wedge \cong \Omega_R^i\left[\tfrac{1}{N}\right]$$

is an isomorphism as well. We are done! $\square$

§6. Why Functoriality Fails

Fix a prime p. In this section, we'll attempt to explain why there is no p-completed q-Hodge complex functor

$$q\text{-}\widehat{\mathrm{Hdg}}_{(-)}\colon \widehat{\mathrm{Sm}}_{\mathbb{Z}_p} \longrightarrow \mathcal{D}_{\mathrm{comp}}\big(\mathbb{Z}_p[\![q-1]\!]\big)$$

from the category of p-completely smooth $\mathbb{Z}_p$-algebras into the full sub-∞-category of derived $(p, q-1)$-complete objects in the derived ∞-category $\mathcal{D}(\mathbb{Z}_p[\![q-1]\!])$; or, at least, we'll explain why the properties we would reasonably expect from this functor are inconsistent. For the sake of contradiction, assume such a functor exists.

6.1. The Derived q-Hodge Complex. — Let $\widehat{\mathrm{Ani}}(\mathrm{Ring})$ denote the full sub-∞-category of derived p-complete objects in the ∞-category of animated rings. Here, the *∞-category of animated rings* refers to the animation of the category of rings as defined in [ČS21, Section 5.1]; for example, it can be obtained as the ∞-categorical localisation of the category of simplicial rings at those morphisms that are weak homotopy equivalences of underlying simplicial sets. Also an animated ring A is called *derived p-complete* iff $A \simeq R\lim_{n\geqslant 1} A/^L p^n$.

If a functor as above exists, we can study its left Kan extension along $\widehat{\mathrm{Sm}}_{\mathbb{Z}_p} \to \widehat{\mathrm{Ani}}(\mathrm{Ring})$ (where the left Kan extension is necessarily taken in the ∞-categorical sense; see [HTT, §4.3]). This left Kan extension exists because $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p[\![q-1]\!])$, being a Bousfield localisation of $\mathcal{D}(\mathbb{Z}_p[\![q-1]\!])$, has all colimits. We still let

$$q\text{-}\widehat{\mathrm{Hdg}}_{(-)}\colon \widehat{\mathrm{Ani}}(\mathrm{Ring}) \longrightarrow \mathcal{D}_{\mathrm{comp}}\big(\mathbb{Z}_p[\![q-1]\!]\big)$$

denote the extended functor. This notation makes sense, because the value $q\text{-}\widehat{\mathrm{Hdg}}_R$ on p-completely smooth $\mathbb{Z}_p$-algebras R remains unchanged; indeed, by derived $(q-1)$-completeness and the filtration of $q\text{-}\widehat{\mathrm{Hdg}}_R/^L(q-1)$ from Lemma 6.3 below, this reduces to the observation that $\bigwedge^i \widehat{L}_R \simeq \widehat{\Omega}^i_R$, where $\widehat{L}_R$ denotes the p-completed cotangent complex of R.

6.2. q-De Rham–Witt Filtrations. — For all $n \geqslant 0$ and all i, let

$$\mathrm{Fil}_i^{q\text{-}W\Omega}\big(q\text{-}\widehat{\mathrm{Hdg}}_{(-)}/^L(q^{p^n}-1)\big)$$

denote the left Kan extension of the functor $R \mapsto \tau^{\leqslant i}(q\text{-}\widehat{\mathrm{Hdg}}_R/^L(q^{p^n}-1))$ on p-completely smooth $\mathbb{Z}_p$-algebras. For any derived p-complete animated $\mathbb{Z}_p$-algebra A, this defines an increasing filtration

$$\Big\{\mathrm{Fil}_i^{q\text{-}W\Omega}\big(q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1)\big) \to q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1)\Big\}_{i\geqslant 0}$$

on $q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1)$ by functoriality of left Kan extensions; we call it the *q-de Rham–Witt filtration*. This terminology is due to the following observation: Left Kan extension (being a left adjoint) preserves colimits of functors (which are computed pointwise), and hence in light of Theorem 4.43 we would expect that the filtration quotients, i.e., the cofibres $\mathrm{gr}_i^{q\text{-}W\Omega} \simeq \mathrm{cofib}(\mathrm{Fil}_{i-1}^{q\text{-}W\Omega} \to \mathrm{Fil}_i^{q\text{-}W\Omega})$ in $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p[\![q-1]\!])$, are given by

$$\mathrm{gr}_i^{q\text{-}W\Omega}\big(q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1)\big) \simeq Lq\text{-}W_{n+1}\widehat{\Omega}^i_A[-i]\,,$$

where the right-hand side denotes the left Kan extension (shifted by $-i$) of the functor $R \mapsto q\text{-}W_{n+1}\widehat{\Omega}^i_R$ on p-completely smooth $\mathbb{Z}_p$-algebras.

Furthermore, preservation of colimits shows that the filtration is exhaustive, that is, we have an equivalence

$$q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1) \simeq \left(\operatorname*{colim}_{i\geqslant 0} \mathrm{Fil}_i^{q\text{-}W\Omega}\left(q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1)\right)\right)^\wedge_{(p,q-1)},$$

where the right-hand side is the derived $(p,q-1)$-completion of the colimit in $\mathcal{D}(\mathbb{Z}_p[\![q-1]\!])$ and thus agrees with the colimit taken in $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p[\![q-1]\!])$. Finally, by functoriality of the smart truncation $\tau^{\leqslant i}$, the projection map

$$q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1) \longrightarrow q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^{n-1}}-1)$$

should be compatible with the q-de Rham–Witt filtrations on both sides. In view of Theorem 4.43, we should moreover assume that the map $Lq\text{-}W_{n+1}(A) \to Lq\text{-}W_n(A)$ induced on the 0^{th} filtration quotients is induced by the Frobenius $F\colon q\text{-}W_{n+1}(A) \to q\text{-}W_n(A)$, and similarly in higher degrees.

6.3. Lemma. — *In the case $n=0$, the filtration quotients from 6.2 can be explicitly determined: For every derived p-complete animated $\mathbb{Z}_p$-algebra, one has*

$$Lq\text{-}W_1\widehat{\Omega}^i_A \simeq \left(\textstyle\bigwedge^i_A \widehat{L}_A\right)^\wedge_p,$$

where $\widehat{L}_A$ denotes the derived p-completed cotangent complex of A and $\bigwedge^i_A$ denotes the derived exterior power functor; see [SAG, Construction 25.2.2.2] for example.

Proof. We need to show that $\left(\bigwedge^i_A \widehat{L}_A\right)^\wedge_p$ is the left Kan extension of the functor $R \mapsto \widehat{\Omega}^i_R$ on p-completely smooth $\mathbb{Z}_p$-algebras. Unfortunately, this isn't immediately clear from the construction of the cotangent complex, which we now recall: Let $\mathrm{Poly}_{\mathbb{Z}}$ denote the category of polynomial rings in finitely many variables. Then the functor $\mathrm{Ani}(\mathrm{Ring}) \to \mathcal{D}(\mathbb{Z})$ given by $A \mapsto \bigwedge^i_A L_A$ is the non-abelian left derived functor, or *animation*, of the functor $\mathrm{Poly}_{\mathbb{Z}} \to \mathcal{D}(\mathbb{Z})$ given by $P \mapsto \Omega^i_P$. That is, it's the contractibly unique sifted colimits-preserving extension guaranteed by [HTT, Proposition 5.5.8.15].

Then $A \mapsto \left(\bigwedge^i_A \widehat{L}_A\right)^\wedge_p$ is the animation of $\mathrm{Poly}_{\mathbb{Z}} \to \mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p)$ given by $P \mapsto \widehat{\Omega}^i_P$, because the derived p-completion functor $(-)^\wedge_p\colon \mathcal{D}(\mathbb{Z}) \to \mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p)$ preserves all colimits. Hence the technical Lemma 6.4 below (together with the derived Nakayama lemma to verify the required condition) finishes the proof. $\square$

6.4. Lemma. — *Let $F\colon \mathrm{Poly}_{\mathbb{Z}} \to \mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p)$ be a functor whose animation*

$$LF\colon \mathrm{Ani}(\mathrm{Ring}) \longrightarrow \mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p)$$

satisfies $LF(A) \simeq LF(\widehat{A}_p)$ for all animated rings A. Then the restriction of LF to the full sub-∞-category of derived p-complete animated rings agrees with the left Kan extension of the restriction of LF to $\widehat{\mathrm{Sm}}_{\mathbb{Z}_p}$.

Proof. Note that the forgetful functor $\widehat{\mathrm{Ani}}(\mathrm{Ring}) \to \mathrm{Ani}(\mathrm{Ring})$ has a left adjoint given by derived p-completion; moreover, the counit is evidently an equivalence as all derived p-complete animated rings A satisfy $\widehat{A}_p \simeq A$. Hence $\widehat{\mathrm{Ani}}(\mathrm{Ring})$ can be regarded as a Bousfield localisation (and thus indeed as a full sub-∞-category) of $\mathrm{Ani}(\mathrm{Ring})$.

The unit of the adjunction above induces a natural transformation $LF \Rightarrow LF \circ (-)_p^\wedge$ which is an equivalence by our assumption on LF. Hence LF factors over a functor

$$L'F\colon \widehat{\mathrm{Ani}}(\mathrm{Ring}) \longrightarrow \mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p)\,.$$

By general nonsense about localisations, $L'F$ is necessarily the left Kan extension of LF. Hence by general nonsense about Kan extensions, $L'F$ is also the left Kan extension of F along $(-)_p^\wedge\colon \mathrm{Poly}_{\mathbb{Z}} \to \widehat{\mathrm{Ani}}(\mathrm{Ring})$. But then $L'F$ can also be computed by first left Kan extending F to a functor

$$F'\colon \widehat{\mathrm{Sm}}_{\mathbb{Z}_p} \longrightarrow \mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p)$$

and then left Kan extending F' along $\widehat{\mathrm{Sm}}_{\mathbb{Z}_p} \to \widehat{\mathrm{Ani}}(\mathrm{Ring})$. So it remains to show that F' is the restriction of LF, or equivalently of $L'F$. But that's a general property of Kan extensions along fully faithful functors, and $\widehat{\mathrm{Sm}}_{\mathbb{Z}_p} \to \widehat{\mathrm{Ani}}(\mathrm{Ring})$ is indeed fully faithful because the discrete derived p-complete rings form a full sub-∞-category of all derived p-complete animated rings (and even a Bousfield localisation, with left adjoint given by π_0). $\square$

6.5. Remark. — Given that $q\text{-}\widehat{\mathrm{Hdg}}_R/^L(q-1) \simeq q\text{-}\widehat{\mathrm{Hdg}}^*_{R,\square}/(q-1)$ for every framed p-completely smooth $\mathbb{Z}_p$-algebra $(R,\square)$, we might even expect that the filtration for $n=0$ is split, that is,

$$q\text{-}\widehat{\mathrm{Hdg}}_R/^L(q-1) \simeq \left(\bigoplus_{i\geqslant 0} \textstyle\bigwedge_A^i \widehat{L}_A\right)_p^\wedge .$$

However, we won't need that to derive our contradiction.

6.6. Proposition. — *There is no functor*

$$q\text{-}\widehat{\mathrm{Hdg}}_{(-)}\colon \widehat{\mathrm{Ani}}(\mathrm{Ring}) \longrightarrow \mathcal{D}_{\mathrm{comp}}\big(\mathbb{Z}_p[\![q-1]\!]\big)$$

such that $q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q^{p^n}-1)$ *has a filtration as in 6.2 for all derived p-complete animated rings A and all $n \geqslant 0$.*

Our strategy to prove Proposition 6.6 will be to evaluate $q\text{-}\widehat{\mathrm{Hdg}}_{(-)}$ at a perfectoid ring R and show that it has contradictory properties. We send two lemmas in advance.

6.7. Lemma. — *If A is a discrete derived p-complete ring, then the q-Witt vectors of A coincide with their derived variant. That is*

$$Lq\text{-}W_n(A) \simeq q\text{-}W_n(A)\,.$$

Proof. By construction of left Kan extensions, [HTT, Definition 4.3.2.2], $Lq\text{-}W_n(A)$ is a colimit over $q\text{-}W_n(R)$, indexed by maps $R \to A$ with R a p-completely smooth $\mathbb{Z}_p$-algebra. By functoriality of $q\text{-}W_n(-)$, this provides a contractibly unique map $Lq\text{-}W_n(A) \to q\text{-}W_n(A)$. We use induction on n to show that it is an equivalence. The case $n=1$ is clear, as then both sides are just A. For the inductive step, Proposition 4.15 and its derived variant provide a map of cofibre sequences

$$\begin{tikzcd} Lq\text{-}W_n(A) \arrow[r] \arrow[d, "\simeq"'] & Lq\text{-}W_{n+1}(A) \arrow[r] \arrow[d] & A[\![q-1]\!]/^L[p]_{q^{p^{n-1}}} \arrow[d, "\simeq"] \\ q\text{-}W_n(A) \arrow[r] & q\text{-}W_{n+1}(A) \arrow[r] & A[\![q-1]\!]/[p]_{q^{p^{n-1}}} \end{tikzcd}$$

in $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p[\![q-1]\!])$. The left vertical arrow is an equivalence by the induction hypothesis, and the right vertical arrow is an equivalence because $[p]_{q^{p^{n-1}}}$ is a nonzerodivisor on $A[\![q-1]\!]$ by Lemma 4.16. Hence the middle arrow must be an equivalence as well. $\square$

6.8. Lemma. — *Let R be a perfectoid ring and let $A_{\mathrm{inf}} \coloneqq A_{\mathrm{inf}}(R)$. Then*

$$q\text{-}\widehat{\mathrm{Hdg}}_{A_{\mathrm{inf}}} \simeq A_{\mathrm{inf}}[\![q-1]\!]\,.$$

Proof. Since A_{inf} is p-torsion free, its Frobenius defines a unique δ-structure. Then Construction 4.26 provides maps $q\text{-}W_{n+1}(A_{\mathrm{inf}}) \to A_{\mathrm{inf}}[\![q-1]\!]/(q^{p^n}-1)$ for all $n \geqslant 0$. Using the same inductive argument as in the proof of Proposition 4.23, plus the fact that the Frobenius $\phi\colon A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ is an isomorphism, shows that $q\text{-}W_{n+1}(A_{\mathrm{inf}}) \cong A_{\mathrm{inf}}[\![q-1]\!]/(q^{p^n}-1)$ for all $n \geqslant 0$. Using 6.2, Lemma 6.7, and Lemma A.11, we get maps

$$A_{\mathrm{inf}}[\![q-1]\!] \longrightarrow R\lim_{n\geqslant 0} q\text{-}W_{n+1}(A_{\mathrm{inf}}) \longrightarrow R\lim_{n\geqslant 0} q\text{-}\widehat{\mathrm{Hdg}}_{A_{\mathrm{inf}}}/^L(q^{p^n}-1) \simeq q\text{-}\widehat{\mathrm{Hdg}}_{A_{\mathrm{inf}}}\,.$$

Whether the composite map $A_{\mathrm{inf}}[\![q-1]\!] \to q\text{-}\widehat{\mathrm{Hdg}}_{A_{\mathrm{inf}}}$ is an equivalence can be checked after applying $(-)/^L(q-1)$ on both sides. But $\widehat{L}_{A_{\mathrm{inf}}} \simeq 0$ by a standard argument and so Lemma 6.3 shows that we get indeed an equivalence. $\square$

Proof of Proposition 6.6. Let R be a perfectoid ring in the sense of [BMS18, §3]. We assume that R is p-torsion free for convenience. First we wish to show that $q\text{-}\widehat{\mathrm{Hdg}}_R$ is discrete and nonzero. Applying Lemma A.5(b) twice, it suffices to show that $q\text{-}\widehat{\mathrm{Hdg}}_R/^L(p,q-1)$ is discrete and nonzero. By 6.2, we can write

$$q\text{-}\widehat{\mathrm{Hdg}}_R/^L(p,q-1) \simeq \operatorname*{colim}_{i\geqslant 0} \mathrm{Fil}_i^{q\text{-}W\Omega}\left(q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q-1)\right)/^L p\,.$$

Because this is a filtered colimit, it suffices to show that each $\mathrm{Fil}_i^{q\text{-}W\Omega}(q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q-1))/^L p$ is discrete and that the transition maps are injective. Moreover, both assertions will follow at once if we can show that each filtration quotient $\mathrm{gr}_i^{q\text{-}W\Omega}(q\text{-}\widehat{\mathrm{Hdg}}_A/^L(q-1))/^L p$ is discrete. These filtration quotients can be evaluated using Lemma 6.3: We have $\widehat{L}_R \simeq R[1]$; see [BMS19, Proposition 4.19] for example. Hence a result of Illusie (see [Ill71, Proposition 4.3.2.1] or [SAG, Proposition 25.2.4.2]) shows

$$\left(\textstyle\bigwedge_R^i \widehat{L}_R\right)[-i] \simeq \left(\textstyle\bigwedge_R^i (R[1])\right)[-i] \simeq \Gamma_R^i(R) \simeq R\,,$$

where $\Gamma_R^i(-)$ denotes the i^{th} divided powers functor. Since we assume R to be p-torsion free, we see that $R/^L p$ is indeed discrete, as desired.

By now we know that $q\text{-}\widehat{\mathrm{Hdg}}_R$ is discrete. Moreover, it is $(p,q-1)$-completely flat over $\mathbb{Z}_p[\![q-1]\!]$ since $q\text{-}\widehat{\mathrm{Hdg}}_R/^L(p,q-1)$ is discrete and thus automatically flat over $\mathbb{F}_p$. Together with Lemma A.7, this implies that $q\text{-}\widehat{\mathrm{Hdg}}_R/^L(q^p-1)$ and $q\text{-}\widehat{\mathrm{Hdg}}_R/^L p$ are discrete as well. So all these derived quotients can also be written as ordinary quotients, which we'll do in the following.

We're ready to derive the final contradiction. Write $R \cong A_{\mathrm{inf}}/\xi$, where ξ is a distinguished nonzerodivisor, and consider the map

$$A_{\mathrm{inf}}[\![q-1]\!] \simeq q\text{-}\widehat{\mathrm{Hdg}}_{A_{\mathrm{inf}}} \longrightarrow q\text{-}\widehat{\mathrm{Hdg}}_R\,.$$

The element $\xi \in A_{\inf}[\![q-1]\!]$ vanishes under $q\text{-}W_1(A_{\inf}) \to q\text{-}W_1(R)$, hence it vanishes in $q\text{-}\widehat{\mathrm{Hdg}}_R/(q-1)$. Thus the image of ξ in $q\text{-}\widehat{\mathrm{Hdg}}_R$ must be divisible by $q-1$. Similarly, the element $\phi(\xi) - [p]_q\delta(\xi) \in A_{\inf}[\![q-1]\!]$ vanishes under $q\text{-}W_2(A_{\inf}) \to q\text{-}W_2(R)$, because $\phi(\xi) - [p]_q\delta(\xi)$ is the image of the Teichmüller lift $[\xi] \in q\text{-}W_2(A_{\inf})$ under Construction 4.24. Hence the element $\phi(\xi) - [p]_q\delta(\xi)$ vanishes in $q\text{-}\widehat{\mathrm{Hdg}}_R/(q^p-1)$, which means that its image in $q\text{-}\widehat{\mathrm{Hdg}}_R$ must be divisible by q^p-1.

We conclude that in $q\text{-}\widehat{\mathrm{Hdg}}_R/p$ both ξ^p and $\phi(\xi) - [p]_q\delta(\xi) \equiv \xi^p - (q-1)^{p-1}\delta(\xi) \mod p$ are divisible by $(q-1)^p$. Hence $(q-1)^{p-1}\delta(\xi)$ is divisible by $(q-1)^p$. But $q\text{-}\widehat{\mathrm{Hdg}}_R/p$ is $(q-1)$-torsion free as $q\text{-}\widehat{\mathrm{Hdg}}_R/^L(p, q-1)$ discrete. Hence $\delta(\xi)$ is divisible by $(q-1)$ and thus vanishes in $q\text{-}\widehat{\mathrm{Hdg}}_R/(p, q-1)$. Then it must already vanish in the 0^{th} filtration step $q\text{-}W_1(R)/p \cong R/p$. However, this forces $R/p = 0$, since ξ is distinguished and therefore $\delta(\xi)$ is a unit. This is clearly a contradiction. $\square$

§A. Appendix

§A.1. Derived Complete Algebra

Completions, both derived and underived, are ubiquitous throughout the text. For the reader's convenience, we collect all the necessary facts in this subsection.

A.1. Derived Completeness. — Let A be a ring and $f \in A$. Recall that an element $M \in D(A)$ of the derived category of A is called *derived f-complete* if

$$R\lim\left(\dots \longrightarrow M \xrightarrow{f} M \xrightarrow{f} M\right) \simeq 0\,.$$

The $R\lim$ on the left-hand side is denoted $T(M, f)$ in [Stacks, Tag 091N], where T probably stands for *telescope*.

More generally, if $I \subseteq A$ is any ideal, we say that M is *derived I-complete* iff it is derived f-complete for all $f \in I$. Equivalently, by [Stacks, Tag 091Q] it suffices to require that M is derived f-complete for all f in a generating system of I. We let $D_{I\text{-comp}}(A)$ denote the full subcategory of $D(A)$ spanned by the derived I-complete objects. Furthermore, if $\mathcal{D}(A)$ denotes the derived ∞-category of A, then $\mathcal{D}_{I\text{-comp}}(A)$ denotes the full sub-∞-category spanned by the derived I-complete objects. If I is clear from the context, we often just write $D_{\mathrm{comp}}(A)$ and $\mathcal{D}_{\mathrm{comp}}(A)$.

A.2. Lemma. — *Let A be a ring and $I \subseteq A$ an ideal.*

(a) *The sub-∞-category $\mathcal{D}_{\mathrm{comp}}(A) \subseteq \mathcal{D}(A)$ is closed under finite colimits and arbitrary limits.*

(b) *An object $M \in D(A)$ is derived I-complete iff its cohomology $H^i(M)$ is derived I-complete in every degree i. Thus, the category of all A-modules which are derived I-complete as objects of $D(A)$ is closed under kernels, cokernels, and extensions.*

Proof. Part (a) is clear as $R\lim$, being the internal limit in the stable ∞-category $\mathcal{D}(A)$, commutes with finite colimits and arbitrary limits. For the first half of (b) see [Stacks, Tag 091P]. Together with (a), this formally implies the second half of (b). □

A.3. Koszul Complexes and Derived Completion. — If the ideal I is finitely generated, there's yet another criterion for derived I-completeness. Let $f_1, \dots, f_r$ be arbitrary generators of I and let

$$K_n^* \coloneqq \operatorname{Kos}^*\left(A, (f_1^n, \dots, f_r^n)\right) = \left(A \xrightarrow{f_1^n} A\right) \otimes_A \dots \otimes_A \left(A \xrightarrow{f_r^n} A\right)$$

denote the cohomological Koszul complex of A with respect to $f_1^n, \dots, f_r^n$, where each of the complexes in the tensor product on the right-hand side is concentrated in degrees -1 and 0 (so that K_n^* is concentrated in degrees $-r, \dots, 0$). Then [Stacks, Tag 091Z] shows that $M \in D(A)$ is derived I-complete iff the canonical map

$$M \xrightarrow{\sim} \underset{n \geqslant 1}{R\lim}\, M \otimes_A^L K_n^*$$

is a quasi-isomorphism. Furthermore, $M \mapsto R\lim_{n\geqslant 1} M \otimes_A^L K_n^*$ defines a left adjoint to the inclusion $\mathcal{D}_{\mathrm{comp}}(A) \subseteq \mathcal{D}(A)$, thus making $\mathcal{D}_{\mathrm{comp}}(A)$ a Bousfield localisation of $\mathcal{D}(A)$. We'll usually denote this left adjoint by $(-)_I^\wedge$ and call it *derived I-completion*. It follows formally that $(-)_I^\wedge$ commutes with colimits, or in other words, that colimits in $\mathcal{D}_{\mathrm{comp}}(A)$ are computed by taking the colimit in $\mathcal{D}(A)$ and then I-completing it.

A.4. Lemma (Derived Nakayama lemma)**.** — *Let A be a ring and let $M \in D(A)$.*

(*a*) *If M is derived f-complete for some $f \in A$, then $M/^L f \simeq 0$ implies $M \simeq 0$.*

(*b*) *If $I \subseteq A$ is a finitely generated ideal and $M \otimes_A^L A/I \simeq 0$, then $M \simeq 0$. Moreover, if M is discrete, then already $M/IM = 0$ implies $M = 0$.*

Proof. If $M/^L f \simeq 0$, then multiplication by f induces a quasi-isomorphism $f\colon M \to M$. Thus $T(M, f) \simeq M$. But if M is derived f-complete, then $T(M, f) \simeq 0$ by definition. Hence part (*a*) follows. For (*b*) see [Stacks, Tags 0G1U and 09B9] □

Next we'll discuss various questions of discreteness.

A.5. Lemma. — *Let $f \in A$ and $M \in D(A)$.*

(*a*) *If M is discrete, then the derived f-completion of M is discrete iff $\lim_{n \geqslant 1} M[f^n] = 0$, where $M[f^n]$ denotes the f^n-torsion part and the transition maps $M[f^{n+1}] \to M[f^n]$are multiplication by f. In particular, if M is f^∞-torsion free, then its derived f-completion is discrete and coincides with the underived f-completion.*

(*b*) *If M is discrete, then M is f-torsion free iff $\widehat{M}_f$ is discrete and f-torsion free.*

(*c*) *If M is derived f-complete and $M/^L f$ is discrete, then M is discrete.*

Proof. Part (*a*) follows from [Stacks, Tag 0BKG], which implies that $\widehat{M}_f$ is concentrated in degrees 0 and -1, with $H^{-1}(\widehat{M}_f) \cong \lim_{n \geqslant 1} M[f^n]$.

For part (*b*), first note that $M/^L f$ is derived f-complete (because its cohomology is f-torsion, so Lemma A.2(*b*) applies) and that derived f-completion commutes with $(-)/^L f$. Hence $M/^L f \simeq \widehat{M}_f/^L f$. Since the f-torsion part of M is isomorphic to $H^{-1}(M/^L f)$, the claim follows.

Finally, we prove (*c*). Observe that we have exact sequences

$$0 \longrightarrow H^i(M)/f \longrightarrow H^i(M/^L f) \longrightarrow H^{i+1}(M)[f] \longrightarrow 0$$

for all i. If $M/^L f$ is discrete, then $H^i(M)/f = 0$ for all $i \neq 0$. But $H^i(M)$ is derived f-complete by Lemma A.2(*b*), hence $H^i(M)/f = 0$ implies $H^i(M) = 0$ by the derived Nakayama lemma. □

A.6. Lemma. — *Let A be noetherian and $I \subseteq A$ any ideal. Then derived and underived I-completion coincide for all finite A-modules.*

Proof. If I is principal, this follows directly from Lemma A.5(*a*). For the general case see [Stacks, Tag 0EEU]. □

A.7. Lemma. — *Let A be a noetherian ring and $I \subseteq A$ an ideal. Let $M \in D(A)$ be derived I-complete and I-completely flat. Then M is discrete and a flat A module.*

Proof. We may assume that A is I-complete, for if not, we can replace A by its I-completion $\widehat{A}$ (which is flat as A is noetherian). It suffices to show that $M \otimes_A^L N$ is discrete whenever N is finite over A. Note that $M \otimes_A^L N \simeq M \mathbin{\widehat{\otimes}}_A^L N$ by [Stacks, Tag 0EEV]. Also, if we choose generators $f_1, \dots, f_r$ of I and let $K_n^* = \operatorname{Kos}^*(A, (f_1^n, \dots, f_r^n))$ denote the corresponding homological Koszul complexes, then

$$M \mathbin{\widehat{\otimes}}_A^L N \simeq \underset{n \geqslant 1}{R\lim} \left((M \otimes_A^L N) \otimes_A^L K_n^*\right) \simeq \underset{n \geqslant 1}{R\lim} \left(M \otimes_A^L (N \otimes_A K_n^*)\right)$$

by definition of derived I-completion and the fact that K_n^* is degree-wise free. Now the pro-objects $\{N \otimes_A K_n^*\}_{n\geqslant 1}$ and $\{N/(f_1^n, \dots, f_r^n)\}_{n\geqslant 1}$ in $D(A)$ are isomorphic; this can be shown by virtually the same argument as in [Stacks, Tag 0921]. Hence

$$M \mathbin{\widehat{\otimes}}_A^L N \simeq R\lim_{n\geqslant 1} \left(M \otimes_A^L N/(f_1^n, \dots, f_r^n)\right).$$

Now each $M \otimes_A^L N/(f_1^n, \dots, f_r^n)$ is discrete as M is I-completely flat, hence it coincides with $(M \otimes_A N)/(f_1^n, \dots, f_r^n)$. This shows that the transition maps are surjective and thus the $R\lim$ above must be discrete. □

The formalism of derived completion allows us to prove a derived analogue of the Beauville–Laszlo theorem ([BL95]; see also [Stacks, Tag 0BNI]). This result is used on several occasions throughout the text, both in the construction of the q-de Rham complex over $\mathbb{Z}$ and in the computation of the cohomology groups of the q-Hodge complex.

A.8. Lemma (Derived Beauville–Laszlo). — *Let A be a ring and $f \in A$ any element. We write $(-)\left[\frac{1}{f}\right]\colon D(A) \to D(A)$ for the f-localisation functor and $(-)_f^\wedge\colon D(A) \to D(A)$ for derived f-completion.*

(*a*) *The functors $(-)\left[\frac{1}{f}\right]$ and $(-)_f^\wedge$ are jointly conservative.*

(*b*) *For any $M \in D(A)$ we have the following pullback square in the derived ∞-category $\mathcal{D}(A)$:*

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \widehat{M}_f \arrow[d] \\ M\left[\frac{1}{f}\right] \arrow[r] & \widehat{M}_f\left[\frac{1}{f}\right] \end{tikzcd}$$

Proof. In contrast to the Beauville–Laszlo theorem for modules, the derived version is almost trivial. For (*a*), recall from [Stacks, Tag 091V] that $\widehat{M}_f$ can also be computed as $R\mathrm{Hom}_A(C, M)$, where C denotes the complex $A \to A\left[\frac{1}{f}\right]$ in degrees 0 and 1. Thus, $\widehat{M}_f \simeq 0$ implies

$$R\mathrm{Hom}_A\left(A\left[\tfrac{1}{f}\right], M\right) \simeq R\mathrm{Hom}_A(A, M) \simeq M\,.$$

But then M is already f-local, so $M\left[\frac{1}{f}\right] \simeq M$. Therefore $M\left[\frac{1}{f}\right] \simeq 0$ implies $M \simeq 0$.

For (*b*), we use (*a*) to see that the homotopy pullback property may be checked after applying $(-)\left[\frac{1}{f}\right]$ and $(-)_f^\wedge$. If we apply $(-)\left[\frac{1}{f}\right]$ to the diagram above, both columns become quasi-isomorphisms and the pullback property is obvious. Similarly, derived f-completion vanishes on f-local objects, hence after $(-)_f^\wedge$ the bottom row becomes 0, whereas the top row becomes a quasi-isomorphism, so again the pullback property is trivially satisfied. □

A.9. Remark. — Whenever Lemma A.8(*b*) is used in the text, it will appear in the following formulation: Let $M \in D_{\mathrm{comp}}(\mathbb{Z}[\![q-1]\!])$ be derived $(q-1)$-complete and let $N > 0$ be an integer. Then

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \prod_{p\mid N} \widehat{M}_p \arrow[d] \\ M\left[\frac{1}{N}\right]_{q-1}^\wedge \arrow[r] & \prod_{p\mid N} \widehat{M}_p\left[\frac{1}{p}\right]_{q-1}^\wedge \end{tikzcd}$$

is a pullback diagram in $\mathcal{D}_{\mathrm{comp}}(\mathbb{Z}[\![q-1]\!])$. Indeed, this can be seen as follows: First apply Lemma A.8(b) in the case $A = \mathbb{Z}[\![q-1]\!]$ and $f = N$ and observe that the derived N-completion of M coincides with the product over the derived p-completions of M for all prime factors $p \mid N$; for example, this follows from the Chinese remainder theorem. Then apply derived $(q-1)$-completion to the ensuing pullback diagram. This preserves the pullback property and gives the pullback diagram above, as M and $\widehat{M}_p$ coincide with their derived $(q-1)$-completions because they are already derived $(q-1)$-complete by assumption and Lemma A.2(a).

A.10. Remark. — Let A be any ring. Observe that if we have a pullback diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

in the ∞-category $\mathcal{D}(A)$ for which all participants are discrete, then this diagram is also a pullback in the category of A-modules. Indeed, the pullback property is equivalent to $K \rightarrow M \oplus L \rightarrow N$ (where the maps are equipped with your favourite sign convention) being a fibre sequence in $\mathcal{D}(A)$. But then $0 \rightarrow K \rightarrow M \oplus L \rightarrow N \rightarrow 0$ must be a short exact sequence of A-modules, proving that the above square is indeed a pullback (and a pushout) of A-modules.

In particular, if M in Remark A.9 is discrete and all completions happen to stay discrete as well, then the pullback diagram given there is also a pullback diagram of $\mathbb{Z}[\![q-1]\!]$-modules. This observation is used several times in §5 to get some strictly functorial maps out of Remark A.9.

Finally, we prove a technical assertion needed in the proof of Lemma 6.8.

A.11. Lemma. — *Let $M \in D_{\mathrm{comp}}(\mathbb{Z}_p[\![q-1]\!])$ be a derived $(p, q-1)$-complete complex. Then the following canonical map is a quasi-isomorphism:*

$$M \xrightarrow{\sim} R\lim_{n \geqslant 0} M/^L(q^{p^n}-1)\,.$$

Proof. Both sides are derived $(q-1)$-complete, so it suffices to check that we get a quasi-isomorphism after applying $(-)/^L(q-1)$. Note that this commutes with the $R\lim_{n\geqslant 1}$. Also note that

$$\big(\mathbb{Z}_p[\![q-1]\!]/(q^{p^n}-1)\big)/^L(q-1) \simeq \mathbb{Z}_p/^L(q^{p^n}-1) \simeq \mathbb{Z}_p[1] \oplus \mathbb{Z}_p\,.$$

The transition maps $(\mathbb{Z}_p[\![q-1]\!]/(q^{p^{n+1}}-1))/^L(q-1) \rightarrow (\mathbb{Z}_p[\![q-1]\!]/(q^{p^n}-1))/^L(q-1)$ are the identity on the summand $\mathbb{Z}_p$ and multiplication by $[p]_{q^{p^n}} \equiv p \mod q-1$ on the summand $\mathbb{Z}_p[1]$. Putting everything together, we see that

$$R\lim_{n\geqslant 0} \big(M/^L(q^{p^n}-1)\big)/^L(q-1) \simeq T\big(M/^L(q-1), p\big)[1] \oplus R\lim_{n\geqslant 1} M/^L(q-1)\,.$$

But $M/^L(q-1)$ is derived p-complete and thus the telescope $T(M/^L(q-1), p)$ vanishes. Therefore, after applying $(-)/^L(q-1)$ to both sides of the original map we get indeed a quasi-isomorphism $M/^L(q-1) \simeq R\lim_{n\geqslant 0} M/^L(q-1)$, as claimed. $\square$

§A.2. Joyal's δ_n-Operations and their Inverses

Let A be a δ-ring. By a result of Joyal, [Joy85], there are functorial maps (of sets) $\delta_i\colon A \to A$, called *Joyal's δ_i-operations*, which satisfy

$$\phi^n(x) = \delta_0(x)^{p^n} + p\delta_1(x)^{p^{n-1}} + \dots + p^n\delta_n(x)$$

for all $x \in A$ and all $n \geqslant 0$. In particular, $\delta_0(x) = x$, $\delta_1(x) = \delta(x)$, and $x \mapsto (\delta_0(x), \delta_1(x), \dots)$ defines a ring map $A \to W(A)$ which is a section of the 0^{th} ghost map $w_0\colon W(A) \to A$.

In this subsection we'll construct maps $\varepsilon_i\colon A^{i+1} \to A$ which are inverse to Joyal's δ_i-operations in a certain sense.

A.12. Lemma. — *Let A be a δ-ring. There exist functorial maps (of sets) $\varepsilon_i\colon A^{i+1} \to A$ which are uniquely determined by the property that for all $n \geqslant 0$ and all $x_0, \dots, x_n \in A$,*

$$x_0^{p^n} + px_1^{p^{n-1}} + \dots + p^n x_n = \phi^n\bigl(\varepsilon_0(x_0)\bigr) + p\phi^{n-1}\bigl(\varepsilon_1(x_0, x_1)\bigr) + \ldots + p^n\varepsilon_n(x_0, \cdots, x_n)\,.$$

Proof. Uniqueness is clear by taking a surjection $A' \to A$ from a p-torsion free δ-ring A'. To show existence, first observe that it suffices to construct $\varepsilon_i(x) \coloneqq \varepsilon_i(x, 0, \dots, 0)$ for all $x \in R$, as then $\varepsilon_i(x_0, \dots, x_i) = \varepsilon_i(x_0) + \varepsilon_{i-1}(x_1) + \dots + \varepsilon_0(x_i)$ will do the job. Now consider the recursive definition $\varepsilon_0(x) \coloneqq x$, $\varepsilon_n(x) \coloneqq -\sum_{i=0}^{n-1} \varepsilon_i(\delta_{n-i}(x))$. We show, using induction on n, that it gives a sequence of maps $\varepsilon_i\colon A \to A$ satisfying $x^{p^n} = \phi^n(\varepsilon_0(x)) + \dots + p^n\varepsilon_n(x)$ for all $x \in A$. with the required properties. For $n = 0$, the assertion is trivial. Now let $n > 0$ and assume it holds for all smaller values. We compute

$$\begin{aligned}
\sum_{i=0}^{n} p^i\phi^{n-i}\bigl(\varepsilon_i(x)\bigr) &= \phi^n(x) - \sum_{i=1}^{n} p^i\phi^{n-i}\left(\sum_{j=0}^{i-1} \varepsilon_j\bigl(\delta_{i-j}(x)\bigr)\right)\\
&= \phi^n(x) - \sum_{k=1}^{n} p^k\left(\sum_{j=0}^{n-k} p^j\phi^{n-(k+j)}\bigl(\varepsilon_j(\delta_k(x))\bigr)\right)\\
&= \phi^n(x) - \Bigl(p\delta_1(x)^{p^{n-1}} + \dots + p^n\delta_n(x)\Bigr)\,,
\end{aligned}$$

which agrees with x^{p^n}. In the first equality plugged in the recursive definition of each ε_i. In the second equality, we reordered the sum by putting $k = i - j$. In the third equality, we applied the inductive hypothesis to each of the sums. This finishes the proof. $\square$

§A.3. Some Technical Calculations

In this subsection we complete the proof sketch of Lemma 4.45.

A.13. Lemma. — *For the morphism $f_n^1\colon H^1_{R,\square}(n) \to M^1_{n+1}$ constructed in the proof sketch of Lemma 4.45, the following relations hold:*

(a) $\mathrm{d} \circ f_n^1 = 0$.

(b) $f_n^1 \circ \beta_n = \mathrm{d} \circ f_n^0$.

(c) $f_n^1(x\omega) = xf_n^1(\omega)$ *for all* $x \in H^0_{R,\square}(n)$ *and all* $\omega \in H^1_{R,\square}(n)$.

Therefore, $f_n\colon H^*_{R,\square}(n) \to M^*_{n+1}$ is a morphism of commutative differential-graded $\mathbb{Z}_p[\![q-1]\!]$-algebras.

What makes the proof of this lemma so annoying is that while $[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha$ is an element of $H^0_{R,\square}(n)$, the monomial T^α alone is not. So we always have to carry these factors around to make sure get well-defined expressions ... Before we start the actual proof, we'll prove two auxiliary lemmas which will be used several times.

A.14. Lemma. — *For all $\alpha = 1, \dots, p^n$, we have*

$$[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} f_n^1(T^{\alpha-1}\,\mathrm{d}T) = f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\alpha-p^{e_\alpha}}\big)\,\mathrm{d}f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\big)\,.$$

Proof. Since M^*_{n+1} is degree-wise p-torsion free, it suffices to verify the desired equation after multiplying by some power of p. Moreover, as both sides are $(q^{p^{e_\alpha}}-1)$-torsion, we may as well multiply by powers of $[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}$. Now

$$\begin{aligned}
&[p^{n-e_\alpha}]^{\alpha'-1}_{q^{p^{e_\alpha}}} \cdot [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} f_n^1(T^{\alpha-1}\,\mathrm{d}T)\\
&\qquad = [p^{n-e_\alpha}]^{\alpha'}_{q^{p^{e_\alpha}}} \cdot \frac{1}{[\alpha']_{q^{p^{e_\alpha}}}}\,\mathrm{d}f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\alpha}\big)\\
&\qquad = [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} \cdot \frac{1}{[\alpha']_{q^{p^{e_\alpha}}}}\,\mathrm{d}\left(f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\big)^{\alpha'}\right)\\
&\qquad = [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} \cdot \frac{\alpha'}{[\alpha']_{q^{p^{e_\alpha}}}} f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\big)^{\alpha'-1}\,\mathrm{d}f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\big)\\
&\qquad = [p^{n-e_\alpha}]^{\alpha'-1}_{q^{p^{e_\alpha}}} \cdot f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\alpha-p^{e_\alpha}}\big)\,\mathrm{d}f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\big)\,.
\end{aligned}$$

In the third equality we used that d is a derivation. In the fourth equality we also used that α' and $[\alpha']_{q^{p^{e_\alpha}}}$ act the same, since everything is $(q^{p^{e_\alpha}}-1)$-torsion. $\square$

A.15. Lemma. — *For all $0 \leqslant j \leqslant i \leqslant n$ we have*

$$[p^{n-j}]_{q^{p^j}}\,\mathrm{d}f_n^0\big([p^{n-i}]_{q^{p^i}} T^{p^i}\big) = f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^i-p^j}\big)\,\mathrm{d}f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^j}\big)\,.$$

Proof. Note that the left-hand side can be rewritten as $[p^{n-i}]_{q^{p^i}}\,\mathrm{d}f_n^0([p^{n-j}]_{q^{p^j}} T^{p^i})$; this follows from the identity $[p^{n-j}]_{q^{p^j}} = [p^{i-j}]_{q^{p^j}} \cdot [p^{n-i}]_{q^{p^i}}$ by moving the factor $[p^{i-j}]_{q^{p^j}}$. Now a similar trick as in Lemma A.14 can be employed: By p-torsion freeness and the fact that both sides are $(q^{p^j}-1)$-torsion, it suffices to check the equation after multiplying both sides by a power of $[p^{n-j}]_{q^{p^j}}$. But then

$$\begin{aligned}
&[p^{n-j}]^{p^{i-j}-1}_{q^{p^j}} \cdot [p^{n-i}]_{q^{p^i}}\,\mathrm{d}f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^i}\big)\\
&\qquad = [p^{n-i}]_{q^{p^i}}\,\mathrm{d}f_n^0\left(\big([p^{n-j}]_{q^{p^j}} T^{p^j}\big)^{p^{i-j}}\right)\\
&\qquad = [p^{n-i}]_{q^{p^i}} \cdot p^{i-j} \cdot f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^j}\big)^{p^{i-j}-1}\,\mathrm{d}f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^j}\big)\\
&\qquad = [p^{n-j}]_{q^{p^j}} \cdot f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^j}\big)^{p^{i-j}-1}\,\mathrm{d}f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^j}\big)\\
&\qquad = [p^{n-j}]^{p^{i-j}-1}_{q^{p^j}} \cdot f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^i-p^j}\big)\,\mathrm{d}f_n^0\big([p^{n-j}]_{q^{p^j}} T^{p^j}\big)\,.
\end{aligned}$$

In the second equality we used that d is a derivation. In the third equality we used the fact that $[p^{n-i}]_{q^{p^i}} \cdot p^{i-j}$ and $[p^{i-j}]_{q^{p^j}} \cdot [p^{n-i}]_{q^{p^i}} = [p^{n-j}]_{q^{p^j}}$ act the same because everything is $(q^{p^j}-1)$-torsion. $\square$

Proof of Lemma A.13(b). It suffices to verify the relation $f_n^1\beta_n(x) = \mathrm{d}f_n^0(x)$ in the special case $x = T^{kp^n} \cdot [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha$ for some k and some α, because a general x can be written as a $(p, q-1)$-power series whose coefficients are finite $\mathbb{Z}_p[\![q-1]\!]$-linear combination of elements of the above form. We compute

$$\begin{aligned}
\beta_n\bigl(T^{kp^n} \cdot [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr) &= T^{kp^n} \beta_n\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr) + k[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\alpha+(k-1)p^n} \beta_n\bigl(T^{p^n}\bigr)\\
&= T^{kp^n} \beta_n\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr) + k[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\alpha+(k-1)p^n} \cdot T^{p^n-1}\,\mathrm{d}T\\
&= T^{kp^n} \beta_n\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr) + k[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{kp^n} \cdot T^{\alpha-1}\,\mathrm{d}T\,.
\end{aligned}$$

In the first equality we used that β_n is a derivation. In the second equality we used the explicit description of β_n that was given in the proof of Lemma 4.35. By construction of f_n^1 and by Lemma A.14, we conclude

$$\begin{aligned}
f_n^1\beta_n\bigl(T^{kp^n} \cdot [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr) &= f_n^0\bigl(T^{kp^n}\bigr)\,\mathrm{d}f_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr)\\
&\quad + kf_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{kp^n+\alpha-p^{e_\alpha}}\bigr)\,\mathrm{d}f_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\bigr)
\end{aligned}$$

So much about $f_n^1\beta_n(x)$; now we need to compute $\mathrm{d}f_n^0(x)$. We obtain

$$\begin{aligned}
\mathrm{d}f_n^0\bigl(T^{kp^n} \cdot [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr) &= f_n^0\bigl(T^{kp^n}\bigr)\,\mathrm{d}f_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr)\\
&\quad + kf_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\alpha+(k-1)p^n}\bigr)\,\mathrm{d}f_n^0\bigl(T^{p^n}\bigr)\\
&= f_n^0\bigl(T^{kp^n}\bigr)\,\mathrm{d}f_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^\alpha\bigr)\\
&\quad + kf_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\alpha+kp^n-p^{e_\alpha}}\bigr)\,\mathrm{d}f_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\bigr)\,.
\end{aligned}$$

The first equality follows as d is a derivation. Furthermore, Lemma A.15 for $i = n$ and $j = e_\alpha$ implies $[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}\,\mathrm{d}f_n^0(T^{p^n}) = f_n^0([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^n-p^{e_\alpha}})\,\mathrm{d}f_n^0([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}})$. Plugging this in, shows that the second equality holds after multiplication with $[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}$. But then our standard argument shows that it must hold on the nose as well. This justifies the calculation above. Upon comparing the expressions for $f_n^1\beta_n(x)$ and $\mathrm{d}f_n^0(x)$, we have thus finished the proof of part (b). ∎

Proof of Lemma A.13(a). By a similar argument as in (b), it suffices to show $\mathrm{d}f_n^1(\omega) = 0$ in the special case $\omega = T^{kp^n} \cdot T^{\alpha-1}\,\mathrm{d}T$ for some k and some α. If $k = 0$, this is clear because $f_n^1(T^{\alpha-1}\,\mathrm{d}T)$ is contained in the image of $\mathrm{d}\colon M_{n+1}^0 \to M_{n+1}^1$ by construction. So assume $k > 0$. Then the integer $kp^{n-e_\alpha} + \alpha'$ is positive and therefore a nonzerodivisor in M_n^*, which is after all a p-torsion free $\mathbb{Z}_p[\![q-1]\!]$-algebra. Furthermore, ω is $(q^{p^{e_\alpha}} - 1)$-torsion, and hence $[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}$ acts like p^{n-e_α} on it. Using Lemma A.14 and the definition of f_n^1, we can thus compute

$$\begin{aligned}
&[p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}^{kp^{n-e_\alpha}+\alpha'}\bigl(kp^{n-e_\alpha} + \alpha'\bigr) \cdot f_n^1\bigl(T^{kp^n} \cdot T^{\alpha-1}\,\mathrm{d}T\bigr)\\
&\quad = [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}^{kp^{n-e_\alpha}+\alpha'}\bigl(kp^{n-e_\alpha} + \alpha'\bigr) \cdot f_n^0\bigl(T^{kp^n}\bigr) f_n^1\bigl(T^{\alpha-1}\,\mathrm{d}T\bigr)\\
&\quad = [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}\bigl(kp^{n-e_\alpha} + \alpha'\bigr) \cdot f_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\bigr)^{kp^{n-e_\alpha}+\alpha'-1}\,\mathrm{d}f_n^0\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\bigr)\\
&\quad = [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}}\,\mathrm{d}f_n^0\left(\bigl([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\bigr)^{kp^{n-e_\alpha}+\alpha'}\right)\,.
\end{aligned}$$

This is clearly contained in the kernel of $\mathrm{d}\colon M_{n+1}^1 \to M_{n+1}^2$. Hence so is $f_n^1(\omega)$. This finishes the proof of part (a). ∎

Proof of Lemma A.13(c). We already know that f_n^1 is $R_n[\![q-1]\!]$-linear, so it suffices to verify the condition

$$f_n^0\big([p^{n-e_\beta}]_{q^{p^{e_\beta}}} T^\beta\big) f_n^1(T^{\alpha-1}\,\mathrm{d}T) = f_n^1\big([p^{n-e_\beta}]_{q^{p^{e_\beta}}} T^\beta \cdot T^{\alpha-1}\,\mathrm{d}T\big)$$

for all $\alpha = 1, \dots, p^n$ and all $\beta = 0, \dots, p^n - 1$. Let $\gamma \coloneqq \alpha + \beta$ and $e_\gamma = v_p(\gamma)$. The left-hand side of our desired equation is $(q^{p^{e_\alpha}} - 1)$-torsion, so by the same arguments as always together with Lemma A.14 we obtain

$$p^{n-e_\alpha} \cdot (\text{left-hand side}) = f_n^0\big([p^{n-e_\beta}]_{q^{p^{e_\beta}}} [p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{\beta+\alpha-p^{e_\alpha}}\big)\,\mathrm{d}f_n^0\big([p^{n-e_\alpha}]_{q^{p^{e_\alpha}}} T^{p^{e_\alpha}}\big)\,.$$

Similarly, Lemma A.14, together with a quick case distinction on whether $1 \leqslant \gamma \leqslant p^n$ or $p^n < \gamma \leqslant 2p^n - 1$, shows

$$p^{n-e_\gamma} \cdot (\text{right-hand side}) = f_n^0\big([p^{n-e_\beta}]_{q^{p^{e_\beta}}} [p^{n-e_\gamma}]_{q^{p^{e_\gamma}}} T^{\gamma-p^{e_\gamma}}\big)\,\mathrm{d}f_n^0\big([p^{n-e_\gamma}]_{q^{p^{e_\gamma}}} T^{p^{e_\gamma}}\big)\,.$$

If $e_\alpha = e_\gamma$, this is already sufficient. If not, then $\beta = \alpha - \gamma$ implies $e_\beta = \min\{e_\alpha, e_\gamma\}$. Hence we may Lemma A.15 for $i = e_\alpha$ and $j = e_\beta$ to the first equality, and for $i = e_\gamma$ and $j = e_\beta$ to the second equality, to obtain

$$\begin{aligned} p^{n-e_\beta} \cdot (\text{left-hand side}) &= f_n^0\big([p^{n-e_\beta}]^2_{q^{p^{e_\beta}}} T^{\gamma-p^{e_\beta}}\big)\,\mathrm{d}f_n^0\big([p^{n-e_\beta}]_{q^{p^{e_\beta}}} T^{p^{e_\beta}}\big)\\ &= p^{n-e_\beta} \cdot (\text{right-hand side})\,. \end{aligned}$$

We are done, at last. $\square$

§A.4. Étale Base Change of Differential-Graded Algebras

In this subsection we prove a technical lemma about extending differential-graded structures along étale maps, which was needed in the final step of the proof of Theorem 4.43. The author strongly suspects that this result is known already, since it seems like a rather natural question, but didn't succeed in locating a reference. In the following, *CDGA* means *strictly graded-commutative differential-graded algebra.*

A.16. Lemma. — *Let A^* be a CDGA concentrated in nonnegative cohomological degrees. Let $A^0 \to B^0$ be an étale ring map. Then the graded-commutative ring $B^0 \otimes_{A^0} A^*$ admits a natural CDGA-structure with the following universal property: For every map $A^* \to E^*$ of CDGAs together with a B^0-algebra structure on E^0, there is a unique map $B^0 \otimes_{A^0} A^* \to E^0$ such that the diagram*

$$\begin{tikzcd} A^* \arrow[r] \arrow[d] & E^* \\ B^0 \otimes_{A^0} A^* \arrow[ur] & \end{tikzcd}$$

commutes and consists of maps of CDGAs.

A.17. Remark. — Observe that we do not assume that the differentials on A^* are A^0-linear. So extending the differentials to $B^0 \otimes_{A^0} A^*$ will be a non-trivial part of the proof.

Proof of Lemma A.16. We will proceed in three steps: In Step 1, we'll construct linear maps $\mathrm{d}_B^n\colon B^0 \otimes_{A^0} A^n \to B^0 \otimes_{A^0} A^{n+1}$. In Step 2, we verify that these satisfy the Leibniz rule. In Step 3, we use this to show that $\mathrm{d}_B^{n+1} \circ \mathrm{d}_B^n = 0$.

Step 1. *Construction of the differentials.* The differentials on A^* define an A^0-linear map $\Omega^1_{A^0} \to A^1$. Using that $A^0 \to B^0$ is étale, this can be extended to a B^0-linear map

$$\Omega^1_{B^0} \cong B^0 \otimes_{A^0} \Omega^1_{A^0} \longrightarrow B^0 \otimes_{A^0} A^1\,,$$

which defines a derivation $\mathrm{d}^0_B \colon B^0 \to B^0 \otimes_{A^0} A^1$. To construct the higher differentials, consider the map $D\colon B^0 \times A^n \to B^0 \otimes_{A^0} A^{n+1}$ given by $D(b, a_n) \coloneqq \mathrm{d}^0_B(b)a_n + b\,\mathrm{d}^n_A(a_n)$, where the products are taken with respect to the graded ring structure on $B^0 \otimes_{A^0} A^*$. D is clearly $\mathbb{Z}$-bilinear. Moreover, it is A^0-balanced in the sense that $D(b, aa_n) = D(ba, a_n)$ for all $a \in A^0$; this is straightforward to check using the Leibniz rule for d^0_B and d^n_A. Hence D defines a $\mathbb{Z}$-linear map $\mathrm{d}^n_B\colon B^0 \otimes_{A^0} A^n \to B^0 \otimes_{A^0} A^{n+1}$ by a lesser-known universal property of the tensor product.

Step 2. *The Leibniz rule.* Next we check $\mathrm{d}^{m+n}_B(b_m b_n) = \mathrm{d}^m_B(b_m)b_n + (-1)^m b_n\,\mathrm{d}(b_n)$ for all $b_m \in B^0 \otimes_{A^0} A^m$ and $b_n \in B^0 \otimes_{A^0} A^n$. It suffices to do this on elementary tensors, so let's assume $b_m = b \otimes a_m$ and $b_n = c \otimes a_n$. Then

$$\begin{aligned} d^{m+n}_B\big((b\otimes a_m)(c \otimes a_n)\big) &= \mathrm{d}^{m+n}_B(bc \otimes a_m a_n)\\ &= \mathrm{d}^0_B(bc)a_m a_n + bc\,\mathrm{d}^{m+n}_A(a_m a_n)\\ &= \mathrm{d}^0_B(b)ca_m a_n + b\,\mathrm{d}^0_B(c)a_m a_n + bc\,\mathrm{d}^m_A(a_m)a_n + (-1)^m bca_m\,\mathrm{d}^n_A(a_n)\,. \end{aligned}$$

One down, one to go. We compute

$$\begin{aligned} &\mathrm{d}^m_B(b\otimes a_m)(c\otimes a_n) + (-1)^m(b\otimes a_m)\,\mathrm{d}^n_B(c\otimes a_n)\\ &\quad= \big(\mathrm{d}^0_B(b)a_m + b\,\mathrm{d}^m_A(a_m)\big)(c\otimes a_n) + (-1)^m(b\otimes a_m)\big(\mathrm{d}^0_B(c)a_n + c\,\mathrm{d}^n_A(a_n)\big)\\ &\quad= \mathrm{d}^0_B(b)a_m ca_n + b\,\mathrm{d}^m_A(a_m)ca_n + (-1)^m ba_m\,\mathrm{d}^0_B(c)a_n + (-1)^m ba_m c\,\mathrm{d}^n_A(a_n)\\ &\quad= \mathrm{d}^0_B(b)ca_m a_n + bc\,\mathrm{d}^m_A(a_m)a_n + b\,\mathrm{d}^0_B(c)a_m a_n + (-1)^m bca_m\,\mathrm{d}^n_A(a_n)\,; \end{aligned}$$

in the last line, we used graded commutativity, which ensures that c commutes with everything and $b\,\mathrm{d}^0_B(c)a_m a = (-1)^m ba_m\,\mathrm{d}^0_B(c)a_n$. Hence the graded Leibniz rule holds.

Step 3. d_B *is a differential.* Proving $\mathrm{d}^{n+1}_B \circ \mathrm{d}^n_B = 0$ can be reduced to the case $n = 0$ via the graded Leibniz rule. Using the graded Leibniz rule again, we compute

$$\begin{aligned} \mathrm{d}^1_B\,\mathrm{d}^0_B(bc) &= \mathrm{d}^1_B\big(\mathrm{d}^0_B(b)c + b\,\mathrm{d}^0_B(c)\big)\\ &= \mathrm{d}^1_B\,\mathrm{d}^0_B(b)c - \mathrm{d}^0_B(b)\,\mathrm{d}^0_B(c) + \mathrm{d}^0_B(b)\,\mathrm{d}^0_B(c) + b\,\mathrm{d}^1_B\,\mathrm{d}^0_B(c)\\ &= \mathrm{d}^1_B\,\mathrm{d}^0_B(b)c + b\,\mathrm{d}^1_B\,\mathrm{d}^0_B(c)\,. \end{aligned}$$

Hence $\mathrm{d}^1_B \circ \mathrm{d}^0_B\colon B^0 \to B^0 \otimes_{A^0} A^2$ is a derivation. It clearly vanishes on A^0, hence the induced map $\Omega^1_{A^0} \to B^0 \otimes_{A^0} A^2$ is zero. But then also $\Omega^1_{B^0} \cong B^0 \otimes_{A^0} \Omega^1_{A^0} \to B^0 \otimes_{A^0} A^2$ must vanish since $B^0 \to A^0$ is étale. This shows $\mathrm{d}^1_B \circ \mathrm{d}^0_B = 0$, as required.

We've thus completed the proof that $B^0 \otimes_{A^0} A^*$ can be equipped with a CDGA-structure. It remains to check the universal property, but this follows from the construction by an easy inspection. $\square$

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