The global q-de Rham complex

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Let p be a prime. In [BS19, §16], Bhatt and Scholze construct a functorial $(p, q-1)$-complete q-de Rham complex relative to any q-PD pair (D, I). This verifies Scholze's conjecture [Sch17, Conjecture 3.1] after p-completion, but leaves open the global case. There are (at least) two strategies to tackle the global case:

(*a*) One can glue the global q-de Rham complex from its p-completions and its rationalisation using an arithmetic fracture square.

(*b*) Following Kedlaya [Ked21, §29], one can construct the global q-de Rham complex as the cohomology of a global q-crystalline site.

Strategy (*a*) is what Bhatt and Scholze originally had in mind, but they never published the argument. The argument is essentially straightforward, but not entirely trivial, and it may be useful for other constructions with q-de Rham cohomology for which a site-theoretic interpretation as in strategy (*b*) is not (yet) available. So we'll use this short note to provide the missing details. The goal is to prove the following theorem.

0.1. Theorem. — *For a $\mathbb{Z}$-torsion free Λ-ring A there exists a functor*

$$q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_A \longrightarrow \mathrm{CAlg}\Big(\widehat{\mathcal{D}}_{(q-1)}\big(A\llbracket q-1\rrbracket\big)\Big)$$

from the category of smooth A-algebras into the ∞-category of $(q-1)$-complete $\mathbb{E}_\infty$-algebras over $A\llbracket q-1\rrbracket$, such that $q\text{-}\Omega_{-/A}\otimes^{\mathrm{L}}_{\mathbb{Z}\llbracket q-1\rrbracket}\mathbb{Z}\simeq\Omega_{-/A}$ agrees with the de Rham complex functor and for every framed smooth $\mathbb{Z}$-algebra $(R,\square)$, the underlying object of $q\text{-}\Omega_{R/A}$ in the derived ∞-category of $A\llbracket q-1\rrbracket$ can be represented as

$$q\text{-}\Omega_{R/A}\simeq q\text{-}\Omega^*_{R/A,\square}\,,$$

where the coordinate-dependent q-de Rham complex $q\text{-}\Omega^*_{R/A,\square}$ is constructed as in [Sch17, §3].

§1. Rationalised q-crystalline cohomology

Fix a prime p. Then $(\widehat{A}_p\llbracket q-1\rrbracket,(q-1))$ is a q-PD pair as in [BS19, Definition 16.1]. We'll show that, after rationalisation, q-crystalline cohomology becomes a base change of crystalline cohomology.

1.1. Lemma. — *For all p-completely smooth $\widehat{A}_p$-algebras R, there is a functorial equivalence of $\mathbb{E}_\infty$-$(\widehat{A}_p\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\llbracket q-1\rrbracket$-algebras*

$$\Big(\mathrm{R}\Gamma_{q\text{-crys}}\big(R/\widehat{A}_p\llbracket q-1\rrbracket\big)\otimes^{\mathrm{L}}_{\mathbb{Z}_p}\mathbb{Q}_p\Big)^\wedge_{(q-1)}\simeq\Big(\mathrm{R}\Gamma_{\mathrm{crys}}\big(R/\widehat{A}_p\big)\otimes^{\mathrm{L}}_{\mathbb{Z}_p}\mathbb{Q}_p\Big)\llbracket q-1\rrbracket\,.$$

Proof. Let $P \twoheadrightarrow R$ be a surjection from a p-completely ind-smooth δ-$\widehat{A}_p$-algebra. Extend the δ-structure on P to $P[\![q-1]\!]$ via $\delta(q) \coloneqq 0$. Let J be the kernel of $P \twoheadrightarrow R$ and let $D \coloneqq D_P(J)_p^\wedge$ be its p-completed PD-envelope. Finally, let q-D denote the q-PD-envelope of $J[\![q-1]\!] \subseteq P[\![q-1]\!]$ as defined in [BS19, Lemma 16.10].

To construct the desired identification between q-crystalline and crystalline cohomology after rationalisation, it will be enough to construct a functorial equivalence

$$\left(q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right)_{(q-1)}^\wedge \simeq \left(D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right)[\![q-1]\!] .$$

If $D^\circ \coloneqq D_P(J)$ denotes the uncompleted PD-envelope, then $P \to q\text{-}D \to (q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{(q-1)}^\wedge$ uniquely factors through $D^\circ \to (q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{(q-1)}^\wedge$. The tricky part is to show that this map extends over the p-completion. Since D° is p-torsion free, its p-completion agrees with $D^\circ[\![t]\!]/(t-p)$. By Lemma 1.3 below, for every fixed $n \geqslant 0$, every p-power series in D° converges in the natural topology on $(q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(q-1)^n$, so we get indeed our desired extension $D \to (q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{(q-1)}^\wedge$. Extending further, we get a map $(D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)[\![q-1]\!] \to (q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{(q-1)}^\wedge$ of the desired form. Whether this is an equivalence can be checked modulo $(q-1)$ by the derived Nakayama lemma. Then the base change property from [BS19, Lemma 16.10(3)] finishes the proof, up to verifying convergence for p-power series in D°. □

To complete the proof of Lemma 1.1, we need to prove two technical lemmas about (q-)divided powers. Let's fix the following notation: According to [BS19, Lemmas 2.15 and 2.17], we may uniquely extend the δ-structure from q-D to $(q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{q-1}^\wedge$. We still let ϕ and δ denote the extended Frobenius and δ-map. Furthermore, we denote by

$$\gamma(x) = \frac{x^p}{p} \quad \text{and} \quad \gamma_q(x) = \frac{\phi(x)}{\Phi_p(q)} - \delta(x)$$

the maps defining a PD-structure and a q-PD structure, respectively. Note that $\gamma(x)$ and $\gamma_q(x)$ make sense for all $x \in (q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{q-1}^\wedge$ since p and $\Phi_p(q)$ are invertible.

1.2. Lemma. — *With notation as above, the following is true for the self-maps δ and γ_q of $(q\text{-}D \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{(q-1)}^\wedge$:*

(a) *For all $n \geqslant 1$ and all $\alpha \geqslant 1$, the map δ sends $(q-1)^n q\text{-}D$ into itself, and $p^{-\alpha}(q-1)^n q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^n q\text{-}D$.*

(b) *For all $n \geqslant 1$ and all $\alpha \geqslant 1$, the map γ_q sends $(q-1)^n q\text{-}D$ into $(q-1)^{n+1} q\text{-}D$, and $p^{-\alpha}(q-1)^n q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^{n+1} q\text{-}D$.*

Proof. Let's prove (a) first. Let $x = p^{-\alpha}(q-1)^n y$ for some $y \in q\text{-}D$. Since q-D is flat over $\mathbb{Z}_p[\![q-1]\!]$ and thus p-torsion free, we can compute

$$\delta(x) = \frac{\phi(x) - x^p}{p} = \frac{(q^p-1)^n \phi(y)}{p^{\alpha+1}} - \frac{(q-1)^{pn} y^p}{p^{p\alpha+1}} .$$

As $q^p - 1$ is divisible by $q-1$, the right-hand side lies in $p^{-(p\alpha+1)}(q-1)^n q\text{-}D$. If $\alpha = 0$, then the right-hand side must also be contained in q-D. But $q\text{-}D \cap p^{-1}(q-1)^n q\text{-}D = (q-1)^n q\text{-}D$ by flatness again. This proves both parts of (a). Now for (b), we first compute

$$\gamma_q(q-1) = \frac{\phi(q-1)}{\Phi_p(q)} - \delta(q-1) = -(q-1)^2 \sum_{i=2}^{p-1} \frac{1}{p}\binom{p}{i}(q-1)^{i-2} .$$

Hence $\gamma_q(q-1)$ is divisible by $(q-1)^2$. In the following, we'll repeatedly use the relation $\gamma_q(xy)=\phi(y)\gamma_q(x)-x^p\delta(y)$ from [BS19, Remark 16.6] repeatedly. First off, it shows that

$$\gamma_q\big((q-1)^nx\big)=\phi\big((q-1)^{n-1}x\big)\gamma_q(q-1)-(q-1)^p\delta\big((q-1)^{n-1}x\big)$$

It follows from (*a*) that $\delta((q-1)^{n-1}x)$ and $\phi((q-1)^{n-1}x)$ are divisible by $(q-1)^{n-1}$. Hence $\gamma_q((q-1)^nx)$ is indeed divisible by $(q-1)^{n+1}$. Moreover, we obtain

$$\gamma_q\big(p^{-\alpha}(q-1)^nx\big)=\phi(p^{-\alpha})\gamma_q\big((q-1)^nx\big)-(q-1)^{np}x^p\delta(p^{-\alpha})\,.$$

Now $\phi(p^{-\alpha})=p^{-\alpha}$ and $\delta(p^{-\alpha})$ is contained in $p^{-(p\alpha+1)}q\text{-}D$, hence $\gamma_q(p^{-\alpha}(q-1)^nx)$ is contained in $p^{-(p\alpha+1)}(q-1)^nq\text{-}D$. This finishes the proof of (*b*). □

1.3. Lemma. — *Let $x\in J$. For every $n\geqslant 1$, there are elements $y_0,\dots,y_n\in q\text{-}D$ such that y_0 admits q-divided powers in $q\text{-}D$ and*

$$\gamma^n(x)=y_0+\sum_{i=1}^n p^{-2(p^{i-1}+\dots+p+1)}(q-1)^{(p-2)+i}y_i$$

holds in $q\text{-}D\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, where $\gamma^n=\gamma\circ\dots\circ\gamma$ denotes the n-fold iteration of γ.

Proof. We use induction on n. For $n=1$, we compute

$$\gamma(x)=\frac{x^p}{p}=\gamma_q(x)+\frac{\Phi_p(q)-p}{p}\big(\gamma_q(x)+\delta(x)\big)\,.$$

Note that x admits q-divided powers in $q\text{-}D$ since we assume $x\in J$. Then $\gamma_q(x)$ admits q-divided powers again by [BS19, Lemma 16.7]. Moreover, writing $\Phi_p(q)=p(1+(q-1)u)+(q-1)^{p-1}$, we find that $(\Phi_p(q)-p)/p=(q-1)u+p^{-1}(q-1)^{p-1}$. Then $(q-1)u(\gamma_q(x)+\delta(x))$ admits q-divided powers since it is a multiple of $(q-1)$. This settles the case $n=1$. We also remark that the above equation for $\gamma(x)$ remains true without the assumption $x\in J$ as long as the expression $\gamma_q(x)$ makes sense.

Now assume γ^n can be written as above. We put $z_i=p^{-2(p^{i-1}+\dots+p+1)}(q-1)^{(p-2)+i}y_i$ for short, so that $\gamma^n(x)=y_0+z_1+\dots+z_n$. Recall the relations

$$\gamma_q(a+b)=\gamma_q(a)+\gamma_q(b)+\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}a^ib^{p-i}\,,\ \ \delta(a+b)=\delta(a)+\delta(b)-\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}a^ib^{p-i}\,.$$

The first relation implies that $\gamma_q(y_0+z_1+\dots+z_n)$ is equal to $\gamma_q(y_0)+\gamma_q(z_1)+\dots+\gamma_q(z_n)$ plus a linear combination of terms of the form $y_0^{v_0}z_1^{v_1}\cdots z_n^{v_n}$ with $0\leqslant v_i<p$ and $v_0+\dots+v_n=p$. Now $\gamma_q(y_0)$ admits q-divided powers again. Moreover, Lemma 1.2(*b*) makes sure that each $\gamma_q(z_i)$ is contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{(p-2)+i+1}q\text{-}D$. It remains to consider monomials $y_0^{v_0}z_1^{v_1}\cdots z_n^{v_n}$. Put $m\coloneqq\max\{i\mid v_i\neq 0\}$. If $v_0=p-1$, then all other v_i must vanish except $v_m=1$. In this case, the monomial is contained in $p^{-2(p^{m-1}+\dots+p+1)}(q-1)^{(p-2)+m}q\text{-}D$. If $v_0<p-1$, then we get at least one more factor $(q-1)$ and the monomial $y_0^{v_0}z_1^{v_1}\cdots z_n^{v_n}$ is contained in $p^{-2(p^m+\dots+p+1)}(q-1)^{(p-2)+m+1}q\text{-}D$.

A similar analysis, using the second of the above relations as well as Lemma 1.2(*a*), shows that $(q-1)u\delta(y_0+z_1+\dots+z_n)$ and $p^{-1}(q-1)^{p-1}\delta(y_0+z_1+\dots+z_n)$ can be decomposed into a bunch of terms, each of which is either a multiple of $(q-1)$ in q-D, so that it admits q-divided powers, or contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{i+1}q$-$D$ for some $1\leqslant i\leqslant n+1$. We conclude that

$$\gamma^{n+1}(x)=\gamma_q\bigl(\gamma^n(x)\bigr)+\frac{\Phi_p(q)-p}{p}\bigl(\gamma_q(\gamma^n(x))+\delta(\gamma^n(x))\bigr)$$

can be written in the desired form. □

1.4. The equivalence on q-de Rham complexes. — Suppose the p-completely smooth $\widehat{A}_p$-algebra R is equipped with a p-completely étale framing $\square\colon \widehat{A}_p\langle T_1,\dots,T_d\rangle\to R$. In this case, the q-crystalline cohomology can be computed as a q-de Rham complex

$$\mathrm{R}\Gamma_{q\text{-crys}}\bigl(R/\widehat{A}_p[\![q-1]\!]\bigr)\simeq q\text{-}\Omega^*_{R/\widehat{A}_p,\square}$$

by [BS19, Theorem 16.22]. Similarly, it's well-known that the crystalline cohomology is given by the ordinary de Rham complex $\Omega^*_{R/\widehat{A}_p}$ (here and throughout the rest of §1, all (q-)de Rham complexes are implicitly p-completed). In this case, an explicit isomorphism of complexes

$$\Bigl(q\text{-}\Omega^*_{R/\widehat{A}_p,\square}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\Bigr)^\wedge_{(q-1)}\cong\Bigl(\Omega^*_{R/\widehat{A}_p}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\Bigr)[\![q-1]\!]$$

can be constructed as explained in [Sch17, Lemma 4.1]: One first observes that, after rationalisation, the partial q-derivatives q-∂_i can be computed in terms of the usual partial derivative ∂_i via the formula

$$q\text{-}\partial_i=\left(\frac{\log(q)}{q-1}+\sum_{n\geqslant 1}\frac{\log(q)^n}{n!(q-1)}(\partial_iT_i)^{(n-1)}\right)\partial_i\,;$$

see [BMS18, Lemma 12.4]. Here $\log(q)$ refers to the usual Taylor series for the logarithm around $q=1$. Noticing that the first factor is an invertible automorphism, one can then appeal to the general fact that for any abelian group M together with commuting endomorphisms $g_1,\dots,g_d$ and commuting automorphisms $h_1,\dots,h_d$ such that h_i commutes with g_j for $i\neq j$ one always has a canonical isomorphism of Koszul complexes $\mathrm{Kos}^*(M,(g_1,\dots,g_d))\cong\mathrm{Kos}^*(M,(h_1g_1,\dots,h_dg_d))$. Observe that we don't require h_i to commute with g_i (and it's not true in the case at hand).

We would like to show that this explicit isomorphism is compatible with the one constructed in Lemma 1.1. To this end, let's put ourselves in a slightly more general situation: Instead of a p-completely étale framing $\square$ as above, let's assume we're given a surjection $P\twoheadrightarrow R$ from a p-completely ind-smooth $\widehat{A}_p$-algebra P, which is in turn equipped with a p-completely ind-étale framing $\square\colon \widehat{A}_p\langle\{T_s\}_{s\in S}\rangle\to P$ for some (possible infinite) set S. Then $\widehat{A}_p\langle\{T_i\}_{i\in I}\rangle$ carries a δ-$\widehat{A}_p$-algebra structure characterised by $\delta(T_i)=0$ for all $i\in I$. By [BS19, Lemma 2.18], this extends uniquely to a δ-$\widehat{A}_p$-algebra structure on P. If J denotes the kernel of $P\twoheadrightarrow R$, we can form the usual PD-envelope $D\coloneqq D_P(J)^\wedge_p$ and the q-PD-envelope q-D as before. Furthermore, we let $\breve{\Omega}^*_{D/\widehat{A}_p}$ and $q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}$ denote the usual PD-de Rham complex and the q-PD-de Rham complex from [BS19, Construction 16.20], respectively (both are implicitly p-completed).

1.5. Lemma. — *With notation as above, there is again an explicit isomorphism of complexes*

$$\Bigl(q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\Bigr)^\wedge_{(q-1)}\cong\Bigl(q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\Bigr)[\![q-1]\!]\,.$$

Proof. This follows from the same recipe as in 1.4, provided we can show that the formula for q-∂_i in terms of ∂_i remains true under the identification $(q\text{-}D\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^\wedge_{(q-1)}\cong(D\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)[\![q-1]\!]$ from the proof of Lemma 1.1. But for every fixed n, the images of

$$\begin{array}{ccc} & (P\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)[\![q-1]\!] & \\ \swarrow & & \searrow \\ (q\text{-}D\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)/(q-1)^n & \xrightarrow{\ \cong\ } & (D\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)[\![q-1]\!]/(q-1)^n \end{array}$$

are dense for the p-adic topology and for elements of $(P\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)[\![q-1]\!]$ the formula is clear. ☐

1.6. Lemma. — *With notation as above, the following diagram commutes:*

$$\begin{array}{ccc} \left(\mathrm{R}\Gamma_{q\text{-crys}}\big(R/\widehat{A}_p[\![q-1]\!]\big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right)^\wedge_{(q-1)} & \xrightarrow[\text{Lemma 1.1}]{\cong} & \left(\mathrm{R}\Gamma_{\mathrm{crys}}\big(R/\widehat{A}_p\big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right)[\![q-1]\!] \\ \simeq\downarrow & & \downarrow\simeq \\ \left(q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right)^\wedge_{(q-1)} & \xrightarrow[\text{Lemma 1.5}]{\cong} & \left(\breve{\Omega}^*_{D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\right)[\![q-1]\!] \end{array}$$

Here the left vertical arrow is the quasi-isomorphism from [BS19, Theorem 16.22] and the right vertical arrow is the usual quasi-isomorphism between crystalline cohomology and PD-de Rham complexes.

Proof. Let $P^\bullet$ be the degree-wise p-completed Čech nerve of $\widehat{A}_p\to P$ and let $J^\bullet\subseteq P^\bullet$ be the kernel of the augmentation $P^\bullet\twoheadrightarrow R$. Let $D^\bullet\coloneqq D_{P^\bullet}(J^\bullet)^\wedge_p$ be the PD-envelope and let q-$D^\bullet$ be the corresponding q-PD-envelope. Finally, form the cosimplicial complexes

$$M^{\bullet,*}\coloneqq\breve{\Omega}^*_{D^\bullet/\widehat{A}_p}\quad\text{and}\quad q\text{-}M^{\bullet,*}\coloneqq q\text{-}\breve{\Omega}^*_{q\text{-}D^\bullet/\widehat{A}_p,\square}\,.$$

In the proof of [BS19, Theorem 16.22] it's shown that the totalisation $\operatorname{Tot}(q\text{-}M^{\bullet,*})$ of q-$M^{\bullet,*}$ is quasi-isomorphic to the 0^{th} column $q\text{-}M^{0,*}\cong q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}$, but also to the totalisation of the 0^{th} row $\operatorname{Tot}(q\text{-}M^{\bullet,0})\cong\operatorname{Tot}(q\text{-}D^\bullet)$. This provides the desired quasi-isomorphism

$$q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\simeq\operatorname{Tot}(q\text{-}M^{\bullet,*})\simeq\operatorname{Tot}(q\text{-}D^\bullet)\simeq\mathrm{R}\Gamma_{q\text{-crys}}\big(R/\widehat{A}[\![q-1]\!]\big)\,.$$

In the exact same way, the quasi-isomorphism $\breve{\Omega}^*_{D/\widehat{A}_p}\simeq\mathrm{R}\Gamma_{\mathrm{crys}}(R/\widehat{A}_p)$ is constructed using the cosimplicial complex $M^{\bullet,*}$ in [Stacks, Tag 07LG]. Applying Lemma 1.5 column-wise gives an isomorphism of cosimplicial complexes $(q\text{-}M^{\bullet,*}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^\wedge_{(q-1)}\cong(M^{\bullet,*}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)[\![q-1]\!]$. On 0^{th} columns, this is the isomorphism from Lemma 1.5, whereas on 0^{th} rows it is the isomorphism from Lemma 1.1. This proves commutativity of the diagram. ☐

§2. Construction of the global q-de Rham complex

From now on, we no longer work in a p-complete setting.

2.1. Doing §1 for all primes at once. — Fix n and put $N_n\coloneqq\prod_{\ell\leqslant n}\ell^{2(\ell^{n-1}+\dots+\ell+1)}$, where the product is taken over all primes $\ell\leqslant n$. Now fix an arbitrary prime p. Lemma 1.3 shows (with notation as in §1) that the map $P \to q\text{-}D \to q\text{-}D/(q-1)^n$ can be extended to a map $D \to N_n^{-1} q\text{-}D/(q-1)^n$, no matter how our implicit prime p is chosen. This observation allows us to do all constructions from §1 "for all primes at once". For example, if R is a smooth A-algebra, then this observation allows us to construct a map

$$\left(\prod_p \mathrm{R}\Gamma_{q\text{-crys}}\big(\widehat{R}_p/\widehat{A}_p[\![q-1]\!]\big) \otimes_{\mathbb{Z}} \mathbb{Q}\right)^{\wedge}_{(q-1)} \xrightarrow{\simeq} \left(\prod_p \mathrm{R}\Gamma_{\mathrm{crys}}\big(\widehat{R}_p/\widehat{A}_p\big) \otimes_{\mathbb{Z}} \mathbb{Q}\right)[\![q-1]\!]\,.$$

compatible with the one from Lemma 1.1 constructed for each prime individually. This map is an equivalence as indicated, as one immediately checks modulo $q-1$.

2.2. Construction. — For all smooth A-algebras R, we construct the *q-de Rham complex of R over A* as the pullback

$$\begin{array}{ccc} q\text{-}\Omega_{R/A} & \longrightarrow & \prod_p \mathrm{R}\Gamma_{q\text{-crys}}\big(\widehat{R}_p/\widehat{A}_p[\![q-1]\!]\big) \\ \downarrow & \lrcorner & \downarrow \\ \big(\Omega_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q}\big)^{\wedge}_{(q-1)} & \longrightarrow & \left(\prod_p \mathrm{R}\Gamma_{\mathrm{crys}}\big(\widehat{R}_p/\widehat{A}_p\big) \otimes_{\mathbb{Z}} \mathbb{Q}\right)[\![q-1]\!] \end{array}$$

Here the bottom horizontal map comes from the comparison of de Rham and crystalline cohomology and the right vertical map comes from 2.1 above.

Proof of Theorem 0.1. We've constructed $q\text{-}\Omega_{R/A}$ in Construction 2.2. Functoriality is clear since all constituents of the pullback are functorial and so are the arrows between them. Modulo $q-1$, the pullback reduces to the usual arithmetic fracture square for $\Omega_{R/A}$, proving $q\text{-}\Omega_{-/A} \otimes^{\mathrm{L}}_{\mathbb{Z}[\![q-1]\!]} \mathbb{Z} \simeq \Omega_{-/A}$.

Finally, suppose R is equipped with an étale framing $\square\colon A[T_1,\dots,T_d] \to R$. The same argument as in 1.4 provides an isomorphism $(q\text{-}\Omega^*_{R/A,\square} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \cong (\Omega^*_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$. The compatibility check from Lemma 1.6 now allows us to identify the pullback square for $q\text{-}\Omega_{R/A}$ with the usual arithmetic fracture square for the complex $q\text{-}\Omega^*_{R/A,\square}$, completed at $(q-1)$. This shows $q\text{-}\Omega_{R/A} \simeq q\text{-}\Omega^*_{R/A,\square}$, finishing the proof. ☐

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