The negative Pell equation, governing fields and beyond

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History of Pell's equation

For a fixed squarefree integer $d > 0$, the equation

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Unbeknownst, Fermat challenged English mathematicians Brouncker and Wallis to solve the notorious case $d = 61$. The smallest non-trivial solution is

$$1766319049^2 - 61 \cdot 226153980^2 = 1.$$

Lagrange was the first to give an algorithm with proof of correctness.

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Question: as we vary d, how often is the negative Pell equation soluble?

A criterion for solubility

Recall that the narrow class group $\mathrm{Cl}^+(K)$ is defined as the quotient of the ideal group I_K by the principal ideals P_K^+ admitting a totally positive generator, while the class group is the quotient by the principal ideals P_K.

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There is a fundamental exact sequence

$$1 \to \frac{P_K}{P_K^+} \to \mathrm{Cl}^+(K) \to \mathrm{Cl}(K) \to 1$$

with $\left|\frac{P_K}{P_K^+}\right| \in \{1, 2\}$ and $\frac{P_K}{P_K^+}$ generated by $(\sqrt{d})$.

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Goal: study joint distribution of $(\mathrm{Cl}^+(K)[2^\infty], \mathrm{Cl}(K)[2^\infty])$.

Previous work on negative Pell

Define $\mathcal{D}$ to be the set of squarefree integers d such that $p \mid d$ implies $p \equiv 1, 2 \bmod 4$.

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By the Hasse-Minkowski Theorem we have

$$d \in \mathcal{D} \Leftrightarrow x^2 - dy^2 = -1 \text{ is soluble with } x, y \in \mathbb{Q}$$
$$\Leftrightarrow \mathrm{rk}_2 \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = \mathrm{rk}_2 \mathrm{Cl}(\mathbb{Q}(\sqrt{d})).$$

Example:

$$\mathrm{rk}_4\ \mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} = 2$$
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Fouvry and Klüners (2010) computed the asymptotic density of $d \in \mathcal{D}$ satisfying

$$\mathrm{rk}_4 \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = 0$$

and also those satisfying

$$\mathrm{rk}_4 \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = 1 + \mathrm{rk}_4 \mathrm{Cl}(\mathbb{Q}(\sqrt{d})).$$

Further improvements on negative Pell

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From their works they were able to deduce that

$$\frac{5\alpha}{4} \leq \liminf_{X \to \infty} \frac{|\mathcal{D}^-_{\leq X}|}{|\mathcal{D}_{\leq X}|} \leq \limsup_{X \to \infty} \frac{|\mathcal{D}^-_{\leq X}|}{|\mathcal{D}_{\leq X}|} \leq \frac{2}{3},$$

where $\alpha = \prod_{j=1}^{\infty}(1 + 2^{-j})^{-1} \approx 0.41942$.

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Together with S. Chan, D. Milovic and C. Pagano I computed the density of $d \in \mathcal{D}$ with

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Corollary 1 (Chan, K., Milovic, Pagano)

We have

$$\beta\alpha \leq \liminf_{X \to \infty} \frac{|\mathcal{D}^-_{\leq X}|}{|\mathcal{D}_{\leq X}|}, \quad \beta := \sum_{n=0}^{\infty} 2^{-n(n+3)/2} \approx 1.28325.$$

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Further improvements to upper and lower bounds in recent work of K. and Pagano. Both these results use recent ideas of A. Smith.

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If p divides the discriminant of $\mathbb{Q}(\sqrt{d})$, then p ramifies, so

$$\begin{array}{ccc} \mathbb{Q}(\sqrt{d}) & \mathfrak{p} & \mathfrak{p}^2 = (p). \\ | & | & \\ \mathbb{Q} & p & \end{array}$$

There is precisely one relation between the ramified primes.

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Goal: to compute 4-rank, it is enough to understand Art_1. We start by describing $\mathrm{Cl}^{+,\vee}(K)[2]$.

The dual class group

Theorem 2 (Class field theory)

We have an isomorphism

$$\mathrm{Cl}^+(K) \cong \mathrm{Gal}(H^+(K)/K)$$

given by sending a prime ideal $\mathfrak{p}$ *to* $\mathrm{Art}(\mathfrak{p})$*. Furthermore, if* K *is Galois, this isomorphism respects the natural Galois action of* $\mathrm{Gal}(K/\mathbb{Q})$ *on both sides.*

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From this we get a bijection

$$\mathrm{Cl}^{+,\vee}(K)[2] \leftrightarrow \{\text{quadratic unramified extensions of } K\}.$$

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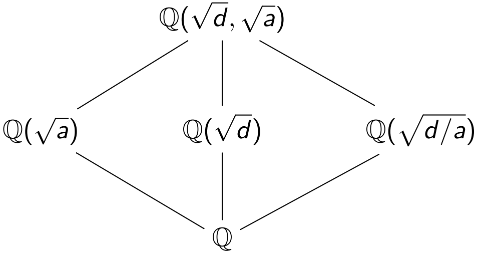

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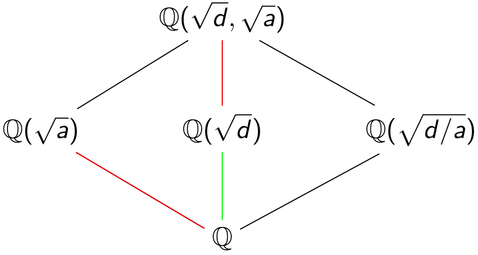

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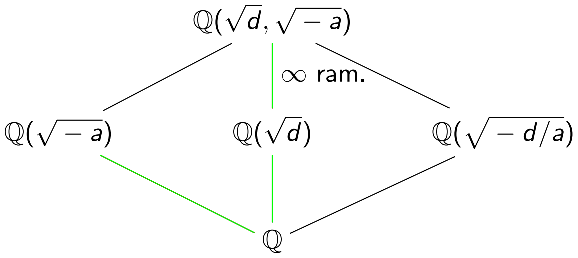

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The Artin pairing

Under the identifications, we have that

$$\mathrm{Art}_1 : \mathrm{Cl}^{+}(K)[2] \times \mathrm{Cl}^{+,\vee}(K)[2] \to \{\pm 1\}, \quad (\mathfrak{p}, \chi) \mapsto \chi(\mathrm{Art}\ \mathfrak{p}).$$

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Let $p_1, \ldots, p_t$ be the prime divisors of d. Define χ_m to be the quadratic character of $\mathbb{Q}(\sqrt{m})$. The Rédei matrix is

$$\begin{array}{ccccc} & \chi_{p_1} & \chi_{p_2} & \cdots & \chi_{p_t} \\ p_1 & * & \left(\frac{p_2}{p_1}\right) & \cdots & \left(\frac{p_t}{p_1}\right) \\ p_2 & \left(\frac{p_1}{p_2}\right) & * & \cdots & \left(\frac{p_t}{p_2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_t & \left(\frac{p_1}{p_t}\right) & \left(\frac{p_2}{p_t}\right) & \cdots & * \end{array}.$$

Left kernel gives generating set for $2\mathrm{Cl}^+(K)[4]$.

Interlude: Stevenhagen's conjecture

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Conjecture 1 (Stevenhagen's conjecture)

We have

$$\lim_{X\to\infty} \frac{|\mathcal{D}^-_{\leq X}|}{|\mathcal{D}_{\leq X}|} = \sum_{j=0}^{\infty} \frac{\mathbb{P}(4-\textit{rank of } d \in \mathcal{D} \textit{ equals } j)}{2^{j+1}-1} \approx 0.581.$$

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Furthermore,

$$\mathbb{P}(4\text{−rank of } d \in \mathcal{D} \text{ equals } j) = \lim_{t \to \infty} \mathbb{P}(t \times t\text{−symm. matrix ker. of dim. } j).$$

The second Artin pairing

There is a natural pairing

$$\mathrm{Art}_2 : 2A[4] \times 2A^\vee[4] \to \{\pm 1\}, \quad (a, \chi) \mapsto \psi(a),\ 2\psi = \chi.$$

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As before, class field theory gives that this pairing becomes

$$(\mathfrak{p}, \chi) \mapsto \psi(\mathrm{Art}\ \mathfrak{p}),\ 2\psi = \chi.$$

Goal: understand cyclic degree 4 unramified extensions of $\mathbb{Q}(\sqrt{d})$.

Cyclic degree 4 extensions

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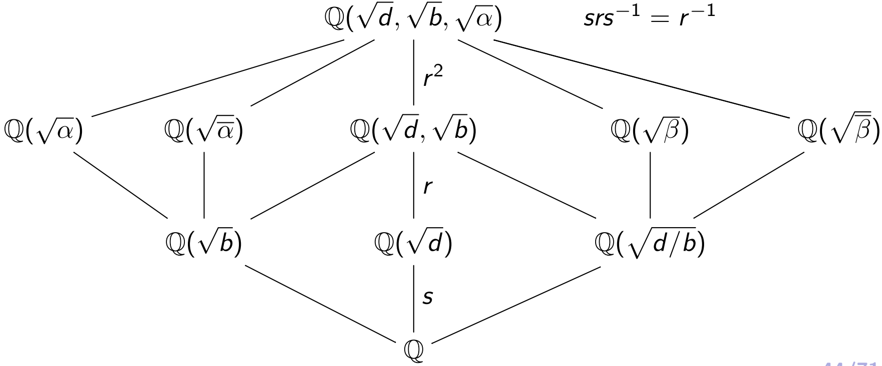

Unramified degree 4 extensions

To make the extension unramified, we need to find a *primitive* solution

$$x^2 = by^2 + \frac{d}{b}z^2 \text{ with } x, y, z \in \mathbb{Z}, \gcd(x, y, z) = 1.$$

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To understand the splitting in dihedral extensions, let us work in greater generality. Suppose that

$$(a, b)_v = (b, c)_v = (a, c)_v = 1, \quad \gcd(a, b, c) = 1.$$

We define the Rédei symbol

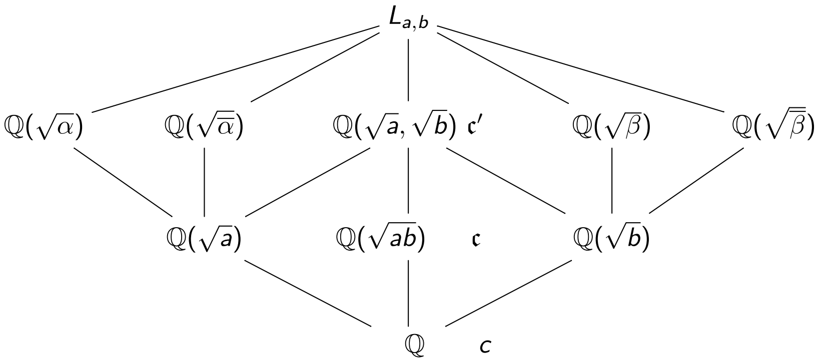
$$[a, b, c] \in \mathbb{F}_2 \cong \mathrm{Gal}(L_{a,b}/\mathbb{Q}(\sqrt{a}, \sqrt{b}))$$

to be the splitting of $\mathfrak{c}$ in a *minimally ramified* degree 4 cyclic extension $L_{a,b}$ of $\mathbb{Q}(\sqrt{ab})$, where $\mathfrak{c}$ is an ideal in $\mathbb{Q}(\sqrt{ab})$ of norm c.

Rédei symbols in a diagram

Facts:

- $L_{a,b}$ minimally ramified means $L_{a,b}/\mathbb{Q}(\sqrt{a},\sqrt{b})$ unramified outside the primes dividing a and b;
- can change such $L_{a,b}$ only by twisting α to $p\alpha$ with p dividing ab;
- every $p \mid c$ splits or ramifies in $\mathbb{Q}(\sqrt{ab})$, hence $\mathfrak{c}$ exists;
- every $\mathfrak{p}$ dividing $\mathfrak{c}$ splits in $\mathbb{Q}(\sqrt{a},\sqrt{b})$;
- $[a,b,c] := \mathrm{Art}(L_{a,b}/\mathbb{Q}(\sqrt{ab}),\mathfrak{c}) \in \mathrm{Gal}(L_{a,b}/\mathbb{Q}(\sqrt{a},\sqrt{b}))$.



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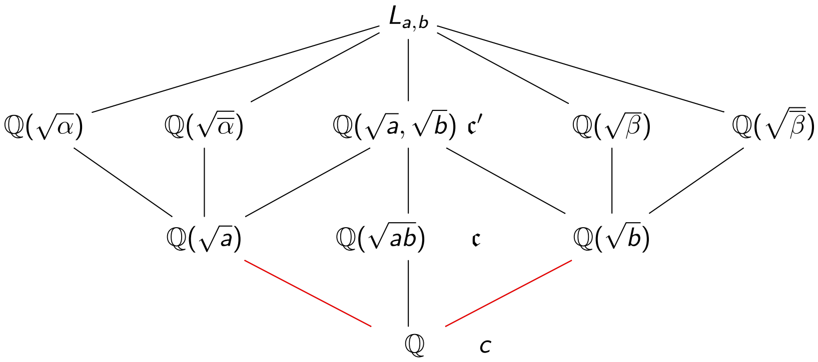

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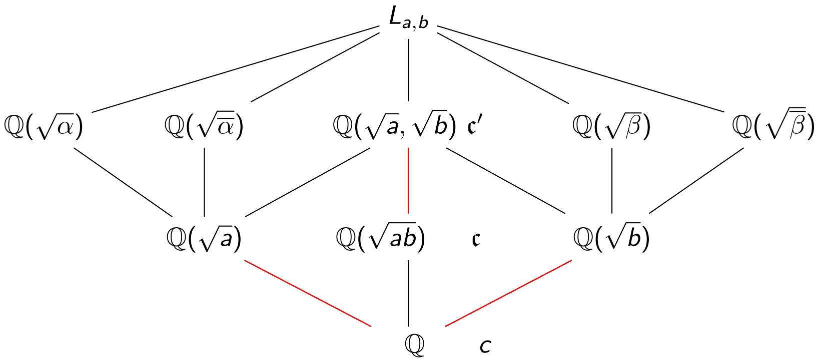

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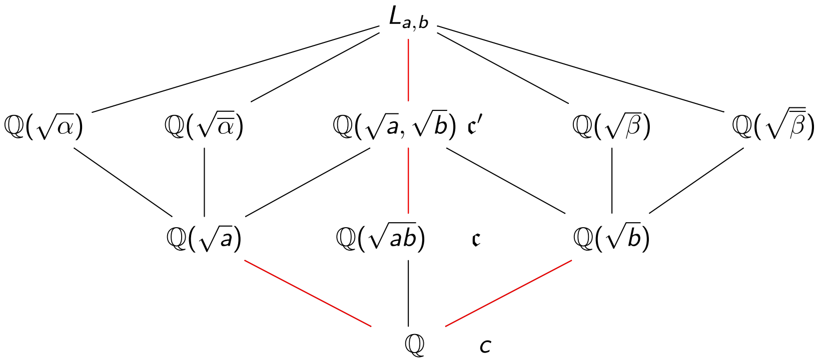

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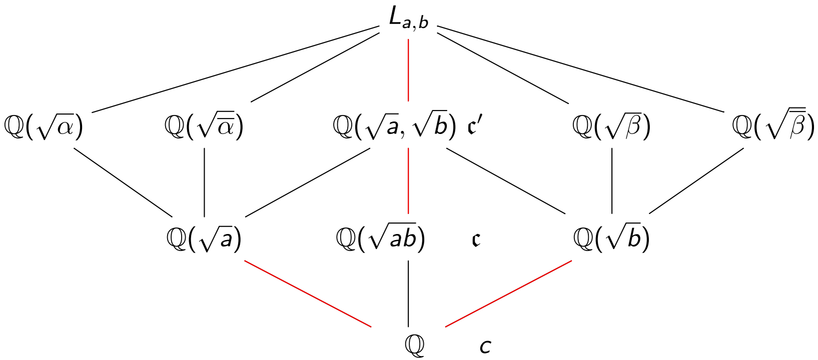
Facts:

- $L_{a,b}$ minimally ramified means $L_{a,b}/\mathbb{Q}(\sqrt{a},\sqrt{b})$ unramified outside the primes dividing a and b;
- can change such $L_{a,b}$ only by twisting α to $p\alpha$ with p dividing ab;
- every $p \mid c$ splits or ramifies in $\mathbb{Q}(\sqrt{ab})$, hence $\mathfrak{c}$ exists;
- every $\mathfrak{p}$ dividing $\mathfrak{c}$ splits in $\mathbb{Q}(\sqrt{a},\sqrt{b})$;
- $[a,b,c] := \mathrm{Art}(L_{a,b}/\mathbb{Q}(\sqrt{ab}),\mathfrak{c}) \in \mathrm{Gal}(L_{a,b}/\mathbb{Q}(\sqrt{a},\sqrt{b}))$.



An example

Take $a = 5$, $b = 41$ and $c = 59$. We have

$$11^2 = 5 \cdot 4^2 + 41 \cdot 1^2, \quad \alpha := -11 + 4\sqrt{5}.$$

To compute the splitting of 59 in $L_{a,b}$ (or equivalently in $\mathbb{Q}(\sqrt{\alpha})$ or in $\mathbb{Q}(\sqrt{\overline{\alpha}})$), need to compute if

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This is independent of the choice of $\sqrt{5}$ in $\mathbb{Z}/59\mathbb{Z}$, since

$$(-11 + 4\sqrt{5}) \cdot (-11 - 4\sqrt{5}) = 41 \equiv \square \bmod 59$$

by the assumptions. The choices of $\sqrt{5}$ are $\{8, 51\}$, so need to check

$$29 \equiv \square \bmod 59 \text{ or equivalently } 21 \equiv \square \bmod 59.$$

Answer is yes.

Rédei reciprocity

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Theorem 3 (Rédei reciprocity)

The Rédei symbol is trilinear and symmetric in all its entries

$$[a, b, c] = [b, a, c] = [a, c, b].$$

Governing fields

We will use Rédei reciprocity to study the 8-rank. Fix a squarefree integer d, and look at the family $\mathbb{Q}(\sqrt{dp})$ as p varies over primes.

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Then if we have two primes p and p' with $p \equiv p' \bmod 8d$, we have

$$\begin{aligned}\mathrm{Art}_{2,\mathbb{Q}(\sqrt{dp})}(b,\chi_a) + \mathrm{Art}_{2,\mathbb{Q}(\sqrt{dp'})}(b,\chi_a) &= [a, dp/a, b] + [a, dp'/a, b] \\ &= [a, b, pp'].\end{aligned}$$

Idea: the splitting of p in the compositum of the $L_{a,b}$ determines the 8-rank. Now apply the Chebotarev density theorem.

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Instead vary two primes, say p and q. Then we get that the sum of the four Artin pairings

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If p and q are small, we can apply Chebotarev. However, we no longer have direct control over Art_2. Use combinatorial ideas to overcome this.

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Theorem 4 (K., Pagano)

We have

$$\limsup_{X \to \infty} \frac{|\mathcal{D}^-_{\leq X}|}{|\mathcal{D}_{\leq X}|} \leq 0.61.$$

Further beyond the 8-rank

Milovic discovered that the 16-rank in the family $\mathbb{Q}(\sqrt{-p})$ is determined by a spin symbol, i.e.

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where K is a fixed Galois extension, $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ fixed and π an odd prime above p.

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However, effectivity in terms of d is currently not good enough to transition to the squarefree integers.

That's it!

Thank you for your attention!