Deformations of Group and Groupoid Representations

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Dedicated to my uncle and teacher Mr. David Karayan, who has inspired me to delve deeper into Mathematics

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Introduction

Deformation theory appears in various areas of Mathematics. In particular, deformations of complex manifolds and complex structures have been extensively investigated, notably in the work [13] of Kodaira and Spencer. This has inspired many mathematicians to study deformations of geometric objects in different contexts. One approach to understanding how geometric objects or structures behave under deformations is to look at the variation of the object at hand at an infinitesimal level. This usually gives rise to a cocycle in an appropriate complex. The vanishing of the associated cohomology leads to rigidity results and this highlights the importance of some cohomology theories controlling deformations (see [7, 8, 18, 20, 24, 28]).

In this thesis, we primarily examine deformations of Lie group representations. Recall that elements of a Lie group G can be represented via automorphisms of a finite-dimensional real vector space V given by a Lie group homomorphism $G \to \mathrm{GL}(V)$, which in turn is equivalent to a smooth linear action of G on V. Hence, the first natural approach that we take is understanding deformations of Lie group actions on smooth manifolds, as studied by Palais and Stewart in [23, 24]. The following rigidity result is obtained from [23] which we prove with further illustration.

Theorem. *Every smooth deformation of an action of a compact Lie group on a compact smooth manifold is trivial.*

The next natural step is to study deformations of Lie group homomorphisms, inspired by the work [20] of Nijenhuis and Richardson. In the attempt of understanding how a Lie group homomorphism behaves under deformations, we will see how the variation of the homomorphism gives rise to a differentiable 1-cocycle. The discussion will lead to an important rigidity result, which is only stated in [20]. In this thesis, we prove the statement by using our construction of the cocycle.

Theorem. *Let $\phi : G \to H$ be a Lie group homomorphism between two connected Lie groups G and H. Consider the representation of G on the Lie algebra $\mathfrak{h}$ of H given by $g \mapsto \mathrm{Ad}_{\phi(g)}$, where Ad is the adjoint representation of H. If $H^1(G, \mathfrak{h}) = 0$, then every smooth deformation of ϕ is locally trivial.*

This theorem has the following immediate consequence on rigidity of representations.

Theorem. *Let G be a compact and connected Lie group and V be a finite-dimensional real vector space. Let $\psi : G \longrightarrow \mathrm{GL}(V)$ be a representation of G on V. Then, every smooth deformation of ψ is locally trivial.*

Furthermore, a significant part of this thesis is devoted to understanding deformations of so-called Lie groupoids and their representations. Intrinsically, a groupoid is a group-like geometric structure, which has many identities, arrows between them and a multiplication defined on composable arrows. As such, a Lie groupoid is an extension of the notion of a Lie group. In essence, it can be also thought of as a category with objects the identities, and where each arrow is invertible. The theory of Lie groupoids comes together with its infinitesimal counterpart, namely Lie algebroids, which are generalizations of the notion of Lie algebras. For a historical note on groupoids, one may refer to [3] by Brown or [29] by Weinstein.

Our analysis of deformations of Lie groupoids is mainly based on the paper [7] by Crainic, Mestre and Struchiner, which is further elaborated in the PhD thesis [18] by Mestre. The authors have defined the so-called deformation cohomology $H^*_{\mathrm{def}}(G)$ of a Lie groupoid G and thoroughly investigated how it controls deformations of G. More precisely, they have shown that deformations give rise to 2-cocycles and that the vanishing of $H^2_{\mathrm{def}}(G)$ leads to rigidity results of the underlying geometric structure. This paper comes parallel to a previous one [8] by Crainic and Moerdijk, who have introduced the deformation cohomology $H^*_{\mathrm{def}}(A)$ of a Lie algebroid A and shown how it controls deformations of A and its bracket. Two of the rigidity results from [18] are as follows, which we prove with more details.

Theorem.

(i) *Every (s,t)-constant deformation of a proper Lie groupoid is trivial.*

(ii) *Every s-constant deformation of a compact Lie groupoid is trivial.*

In addition, we explore the generalization of the notion of representations to the case of Lie groupoids. This has proved to be more subtle and challenging due to the very definition of Lie groupoids, and hence there are only few representations defined in a natural way. In this thesis, we examine two natural representations of regular Lie groupoids, namely the isotropy $\mathfrak{i}$ and normal $\mathfrak{v}$ representations, and their relation with the deformation cohomology in low degrees as first studied in [7, 18]. We arrive to this exact sequence from [18], which we prove with further details.

Proposition. *For any Lie groupoid G, there is an exact sequence:*

$$0 \to H^1(G,\mathfrak{i}) \to H^1_{\mathrm{def}}(G) \to H^0(G,\mathfrak{v}) \to H^2(G,\mathfrak{i}) \to H^2_{\mathrm{def}}(G)$$

In contrast to Lie groups, some representations, such as the adjoint representation, do not have natural generalizations to Lie groupoids. Nevertheless, Arias Abad and Crainic have introduced and studied so-called representations up to homotopy of a Lie algebroid in [1], and by a parallel construction, representations up to homotopy of Lie groupoids in [2]. The adjoint representation of a Lie groupoid will be shown to be a well-defined representation up to homotopy up to isomorphism. Although the choice of a connection on the groupoid will be crucial in defining the adjoint representation, different connections will yield isomorphic representations as shown in [2]. The significance of the adjoint representation in deformations is depicted in the following isomorphism, stated and proved in [18].

Theorem. *Given an Ehresmann connection σ on a Lie groupoid G, $H^*_{\mathrm{def}}(G) \cong H(G, \mathrm{Ad}_\sigma)^*$.*

This result in fact generalizes a similar result in the case of groups, where the usual adjoint representation is considered. A proof is provided in this thesis.

Theorem. *For a Lie group G, $H^*_{\mathrm{def}}(G) \cong H^*(G, \mathrm{Ad})$.*

Lastly, an approach to deformations of Lie groupoid representations will be presented. Similar to representations of Lie groups, we will show that a representation of a Lie groupoid G can be given by a Lie groupoid morphism from G to the so-called general linear groupoid $\mathrm{GL}(E)$ of a vector bundle E. For this reason, we first try to understand deformations of groupoid morphisms, which will require the notion of cohomology of groupoid morphisms as introduced in [7]. In this thesis, we generalize the theorem on rigidity of Lie group homomorphisms from [20] to the case of Lie groupoid morphisms.

Theorem. *Let G and H be compact Lie groupoids and let F be a Lie groupoid morphism from G to H. If $H^1_{\mathrm{def}}(F) = 0$, then every smooth deformation of F is locally trivial.*

In terms of groupoid representations, the theorem can be stated as follows.

Corollary. *Let G be a compact Lie groupoid and let (E, φ) be a representation of G. Denote by ψ the corresponding Lie groupoid morphism from G to $\mathrm{GL}(E)$. If $H^1_{\mathrm{def}}(\psi) = 0$, then every smooth deformation of ψ is locally trivial.*

As a conclusion, we make the following observation. Actions of Lie groups on smooth manifolds give rise to a special kind of Lie groupoids, namely action groupoids. Hence, it is natural to ask how deformations of action groupoids and deformations of the underlying representations (viewed as smooth linear actions) could be related. In particular, one can ask if the rigidity of one would lead to the rigidity of the other. In this thesis, we try to establish some connections between them. Nonetheless, this topic is yet to be studied and is open for future research and investigation.

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1 Preliminaries

1.1 Lie groups and Lie algebras

Lie groups, which are simultaneously smooth manifolds and groups together with smooth group operations, are fundamental tools for studying symmetries. Moreover, the theory of Lie groups comes in parallel with its infinitesimal counterpart, the theory of Lie algebras. This section provides a brief recall on the basic notions of a Lie group and a Lie algebra along with some examples. For a more detailed exposition, one can for instance refer to [12, 14].

Definition 1.1.1 (Lie group)**.** A **Lie group** G is a smooth manifold which comes equipped with a group structure such that the group operations

(i) $G \times G \longrightarrow G, \quad (g,h) \longmapsto gh$ (multiplication)

(ii) $G \longrightarrow G, \quad g \longmapsto g^{-1}$ (inversion)

are smooth maps.

Example 1.1.2.

- The real and complex euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Lie groups under addition.
- $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ are Lie groups under multiplication.
- The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a Lie group under matrix multiplication.
- Let $G_1, ..., G_n$ be Lie groups. Then, their direct product $G_1 \times \cdots \times G_n$ is a Lie group as well.

Many other examples of Lie groups can be constructed as Lie subgroups of existing Lie groups. Recall that a **Lie subgroup** of a Lie group G is a subgroup H of G such that it comes equipped with a topology and smooth structure making it into an immersed submanifold of G and a Lie group on its own.

Given a Lie group G and an element $g \in G$, one can define the following maps

$$L_g : G \longrightarrow G, \quad h \longmapsto gh$$

$$R_g : G \longrightarrow G, \quad h \longmapsto hg$$

which are called **left translation** and **right translation** respectively. These maps are diffeomorphisms with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$ respectively.

Definition 1.1.3 (Lie group homomorphism)**.** A **Lie group homomorphism** is a smooth map $F : G \longrightarrow H$ between two Lie groups G and H, which is also a group homomorphism.

The main application of Lie groups is through their actions on smooth manifolds as well as through their representations, which will be recalled in section 1.5.

Definition 1.1.4 (Lie algebra)**.** A **Lie algebra** V is a real vector space together with a map

$$[\cdot,\cdot]\colon V \times V \longrightarrow V, \quad (u,v) \mapsto [u,v]$$

called the Lie bracket, satisfying the following conditions:

(i) $[c_1u_1 + c_2u_2, v] = c_1[u_1, v] + c_2[u_2, v]$
$[u, c_1v_1 + c_2v_2] = c_1[u, v_1] + c_2[u, v_2]$ (bilinearity)

(ii) $[u, v] = -[v, u]$ (antisymmetry)

(iii) $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ (Jacobi identity)

$\forall\ u, u_1, u_2, v, v_1, v_2, w \in V$ and $\forall\ c_1, c_2 \in \mathbb{R}$.

Example 1.1.5.

- Any vector space V is a Lie algebra with the zero bracket.
- Let M be a smooth manifold. The space $\mathfrak{X}(M)$ of all smooth vector fields on M is a Lie algebra under the commutator as the Lie bracket.

Recall that a **Lie subalgebra** of a Lie algebra V is a linear subspace U of V such that it is closed under the Lie bracket. Then, U would be a Lie algebra on its own with the restriction of the bracket of V.

Each Lie group has an associated Lie algebra, defined via its left-invariant vector fields. Given a Lie group G, recall that a vector field $X \in \mathfrak{X}(G)$ on G is called left-invariant if it is invariant under all left translations, i.e. $(dL_g)_h(X_h) = X_{gh}$ for all $g, h \in G$. It can be shown that the set of all left-invariant vector fields on G is a Lie subalgebra of the space $\mathfrak{X}(G)$ of vector fields on G, called **the Lie algebra of the Lie group** G, and denoted by $\mathrm{Lie}(G) = \mathfrak{g}$.

Remark 1.1.6. *Given a Lie group G, the map*

$$\mathfrak{g} \longrightarrow T_eG, \quad X \longmapsto X_e$$

is an isomorphism of vector spaces.

Definition 1.1.7 (Lie algebra homomorphism)**.** A linear map $f : V \longrightarrow W$ between two Lie algebras V and W is called a **Lie algebra homomorphism** if it preserves the Lie bracket, i.e. if $f([v_1, v_2]) = [f(v_1), f(v_2)]$ for all $v_1, v_2 \in V$.

1.2 Lie groupoids

This section will introduce one of the main geometric objects under study of this thesis, mainly Lie groupoids, as well as discuss their main properties and provide fundamental examples. Lie groupoids are primarily understood as generalizations of Lie groups [3] and therefore describe symmetry of geometrical structures in more general cases. One may refer to [4, 15, 29] for further understanding of Lie groupoids, their structures and their contribution to symmetry.

Definition 1.2.1 (Groupoid)**.** A **groupoid** over a set M is a set G together with the following structure maps:

(i) $s : G \longrightarrow M$, called the **source**
$t : G \longrightarrow M$, called the **target**

(ii) $m : G^{(2)} \longrightarrow G$, called the **multiplication**, which is defined on the set of composable pairs

$$G^{(2)} := \{(g, h) \in G \times G \mid s(g) = t(h)\}$$

and commonly denoted by $m(g, h) = gh$, such that $\forall\ (g, h), (h, k) \in G^{(2)}$:

- $(gh)k = g(hk)$ (associativity)
- $s(gh) = s(h)$ and $t(gh) = t(g)$

(iii) $u : M \longrightarrow G$, called the **unit**, such that $\forall\ g \in G,\ x \in M$:

- $gu(s(g)) = u(t(g))g = g$
- $s(u(x)) = t(u(x)) = x$

(iv) $i : G \longrightarrow G$, called the **inversion**, commonly denoted by $i(g) = g^{-1}$, such that $\forall\ g \in G$:

- $gg^{-1} = u(t(g))$ and $g^{-1}g = u(s(g))$
- $s(g^{-1}) = t(g)$ and $t(g^{-1}) = s(g)$.

Definition 1.2.2 (Lie groupoid)**.** A **Lie groupoid** is a groupoid G over M such that G and M are both smooth manifolds (with M Hausdorff), all the structure maps are smooth and the source and target maps are surjective submersions.

A groupoid G over a set M is usually denoted by $G \rightrightarrows M$ where the two maps represent the source and target maps. G is commonly called the **space of arrows**, whereas M is called the **space of objects**, as we can visualize an element g of G as an arrow from its source $s(g)$ to its target $t(g)$.

$$\begin{tikzcd} t(g) & s(g) \arrow[l, bend right, "g"'] \end{tikzcd}$$

Moreover, multiplication between two composable arrows g and h can be understood as the composition between them as depicted below.

$$\begin{tikzcd} t(gh) = t(g) & s(g) = t(h) \arrow[l, bend right, "g"'] & s(h) = s(gh) \arrow[l, bend right, "h"'] \arrow[ll, bend right=50, "gh"'] \end{tikzcd}$$

The multiplication naturally induces a division map on G defined by:

$$\bar{m} : G{\times_s}G \longrightarrow G, \quad (g, h) \mapsto m(g, i(h)) = gh^{-1}$$

where $G{\times_s}G := \{(g, h) \in G \times G \mid s(g) = s(h)\}$.

One can also view a groupoid as a category whose objects are the elements of M and the morphisms are the arrows between these objects, hence the elements of G, such that each morphism is invertible.

In this thesis, we will be dealing mostly with Lie groupoids unless stated otherwise. For all $x \in M$, the sets $s^{-1}(x)$ and $t^{-1}(x)$ are called the **source fiber** and the **target fiber** of x respectively, and are closed embedded submanifolds of G since s and t are submersions. Moreover, the inversion map $i : G \to G$ turns out to be a diffeomorphism of G which exchanges the source and target fibers (see [15, Proposition 1.1.5]). Additionally, the unit

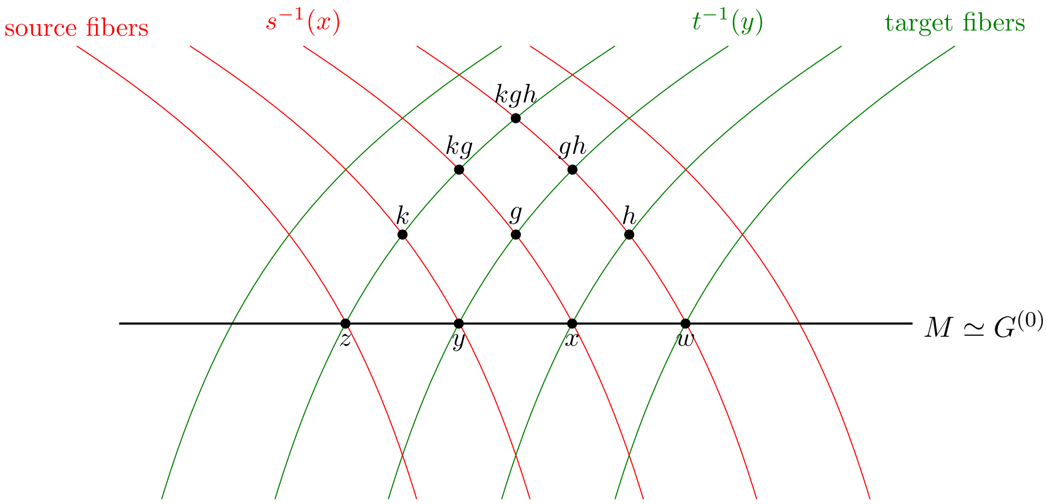

Figure 1: Lie groupoid

map $u : M \hookrightarrow G$ is an embedding and hence M can be viewed as a closed embedded submanifold of G. One may refer to [15, 19] for further properties of a groupoid.

Figure 1, which is inspired from [4, p. 86], is extremely useful in understanding Lie groupoids and the operations associated to it.

Let us now consider few examples of groupoids.

Example 1.2.3.

- A **Lie group** can be viewed as a Lie groupoid over a point.
- Let M be a smooth manifold. The product manifold $M \times M$ can be seen as a groupoid over M, called the **pair groupoid** with the following structure maps. For all $x, y \in M$, $s(x, y) = y$, $t(x, y) = x$, $m((x, y), (y, z)) = (x, z)$, $u(x) = (x, x)$, $i(x, y) = (y, x)$.
- Every groupoid $G \rightrightarrows M$ induces a groupoid structure on the tangent bundle of G over the tangent bundle of M, $TG \rightrightarrows TM$, with structure maps given by the differentials of the structure maps of G. This induced groupoid is called the **tangent groupoid**.

Let $G \rightrightarrows M$ be a given Lie groupoid. One can define an equivalence relation $\sim$ on M by:

$$x \sim y \;\Leftrightarrow\; \exists\, g \in G \;\text{ s.t. }\; s(g) = x \text{ and } t(g) = y.$$

Remark 1.2.4. *The relation $\sim$ indeed defines an equivalence relation on M.*

Proof.

- **Reflexive**: $x \sim x$ since the unit $u(x) \in G$ is s.t. $s(u(x)) = t(u(x)) = x$.
- **Symmetric**: $x \sim y \Rightarrow \exists\, g \in G$ s.t. $s(g) = x$ and $t(g) = y$, by the inversion map $g^{-1} \in G$ is s.t. $s(g^{-1}) = t(g) = y$ and $t(g^{-1}) = s(g) = x \Rightarrow y \sim x$.
- **Transitive**: $x \sim y$ and $y \sim z \Rightarrow \exists\, g : y \curvearrowleft x, h : z \curvearrowleft y \in G$, which implies that (h, g) is a composable pair where $s(hg) = s(g) = x$ and $t(hg) = t(h) = z \Rightarrow x \sim z$.

$\square$

The **orbits** $\{\mathrm{Orb}(x)\}_{x\in M}$ of the groupoid are defined to be the equivalence classes of this relation. It can be proved that $\mathrm{Orb}(x)$ is a submanifold of M for all x in M (see [15, Theorem 1.5.11] or [19, Theorem 5.4]).

The groupoid $G \rightrightarrows M$ is called a **regular groupoid** if its orbits have the same dimension. Moreover, it is said to be a **proper groupoid** if G is Hausdorff and

$$(s,t) : G \to M \times M, \quad g \mapsto (s(g), t(g))$$

is a proper map. The notion of properness can be seen as a generalization of the notion of compactness. For instance, a Lie group H being compact is equivalent to the Lie groupoid $H \rightrightarrows \{*\}$ being proper. Also, $G \rightrightarrows M$ is called a **compact groupoid** if G is Hausdorff and compact as a manifold.

For each $x \in M$, we let G_x denote the set of arrows starting and ending at x

$$G_x := \{g \in G \mid s(g) = t(g) = x\} = s^{-1}(x) \cap t^{-1}(x)$$

and call it the **isotropy group** of x.

Remark 1.2.5. *For all $x \in M$, G_x is a Lie group.*

Proof. We first show that it satisfies the properties of a group.

- **Binary operation**: Let $G_x \times G_x \longrightarrow G_x,\ (g,h) \mapsto m(g,h) = gh$ where m is the multiplication of the groupoid. This is well-defined since all arrows starting and ending at the same object x are composable, and their composition starts and ends at x as well.

- **Associativity**: Follows from that inside the groupoid.

- **Identity**: We show that $u(x)$ is the identity of G_x. Firstly, $s(u(x)) = t(u(x)) = x$ and hence $u(x) \in G_x$. Now, $\forall\ g \in G_x,\ m(g, u(x)) = m(g, u(s(g))) = g$ and $m(u(x), g) = m(u(t(g)), g) = g$ hold.

- **Inverse**: For all $g \in G_x$, $g^{-1} \in G_x$ too since $s(g^{-1}) = t(g) = x$ and $t(g^{-1}) = s(g) = x$. Moreover, $m(g, g^{-1}) = u(t(g)) = u(x)$ and $m(g^{-1}, g) = u(s(g)) = u(x)$.

To see that G_x is a smooth submanifold of G, one can for instance refer to [19, Theorem 5.4], where the authors provide a proof using some foliation theory. Smoothness of the multiplication and inversion maps follow directly from that of G. ☐

Definition 1.2.6 (Groupoid morphism)**.** A **groupoid morphism** between two groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$, with structure maps $s, t, m, \bar{m}, u, i$ and $s', t', m', \bar{m}', u', i'$ respectively, is a pair of maps $F : G \to G'$ and $f : M \to M'$ such that the following diagrams commute:

$$\begin{tikzcd}
G \arrow[r, "F"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & G' \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \\
M \arrow[r, "f"] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
G^{(2)} \arrow[r, "m"] \arrow[d, "F \times F|_{G^{(2)}}"'] & G \arrow[d, "F"] \\
G'^{(2)} \arrow[r, "m'"] & G'
\end{tikzcd}$$

That is, $s' \circ F = f \circ s$, $t' \circ F = f \circ t$ and $F(m(g,h)) = m'(F(g), F(h))$ for all $(g,h) \in G^{(2)}$.

A **Lie groupoid morphism** between two Lie groupoids is a groupoid morphism (F, f) with smooth maps. Moreover, a Lie groupoid morphism (F, f) is an **isomorphism** of Lie groupoids if the maps F and f are diffeomorphisms.

Remark 1.2.7. *Let (F, f) be a groupoid morphism between two groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$. Then,*

(i) $F(u(x)) = u'(f(x)) \quad \forall\, x \in M$,

(ii) $F(g^{-1}) = (F(g))^{-1} \quad \forall\, g \in G$.

Proof. For (i): Let $x \in M$. Then, $F(u(x))$ is an element of G' with

$$F(u(x)) = F(m(u(x), u(x))) = m'(F(u(x)), F(u(x))).$$

This makes sense since $s'(F(u(x))) = f(s(u(x))) = f(x)$ and $t'(F(u(x))) = f(t(u(x))) = f(x)$. Therefore, $F(u(x)) \in G'_{f(x)}$, the isotropy group of $f(x)$, with the condition that $F(u(x)) = m'(F(u(x)), F(u(x)))$. But, $G'_{f(x)}$ is a group, which implies that $F(u(x))$ can only be the identity of the group. Therefore, $F(u(x)) = u'(f(x))$ true for all $x \in M$ as x was chosen arbitrarily.

For (ii): We show that $F(g^{-1})$ is the inverse of $F(g)$ by showing that it satisfies the conditions of the inversion map. Using the commutativity relations,

$$s'(F(g^{-1})) = f(s(g^{-1})) = f(t(g)) = t'(F(g)) \quad \text{and}$$

$$t'(F(g^{-1})) = f(t(g^{-1})) = f(s(g)) = s'(F(g)).$$

Moreover, we get that

$$\begin{aligned} m'(F(g), F(g^{-1})) &= F(m(g, g^{-1})) \\ &= F(u(t(g))) \\ &= u'(f(t(g))) \\ &= u'(t'(F(g))). \end{aligned}$$

Similarly, $m'(F(g^{-1}), F(g)) = u'(s'(F(g)))$ and therefore $F(g^{-1}) = (F(g))^{-1} \ \ \forall g \in G$. □

Proposition 1.2.8. *Let $G \rightrightarrows M$ and $G' \rightrightarrows M'$ be two groupoids and let $F : G \to G'$ and $f : M \to M'$ be two maps. Then, the following are equivalent:*

(i) *(F, f) is a groupoid morphism*

(ii) *$s' \circ F = f \circ s$ and $\bar{m}'(F(g), F(h)) = F(\bar{m}(g, h))$*

for all $(g, h) \in G \times_s G$.

Proof. (i) $\Rightarrow$ (ii): By the definition of groupoid morphism, $s'(F(g)) = f(s(g))$ holds $\forall\, g \in G$. For the second equality,

$$\begin{aligned} \bar{m}'(F(g), F(h)) &= m'(F(g), (F(h))^{-1}) \\ &= m'(F(g), F(h^{-1})) \\ &= F(m(g, h^{-1})) \\ &= F(\bar{m}(g, h)) \end{aligned}$$

$\forall\, (g, h) \in G \times_s G$ by using the properties of a groupoid morphism and Remark 1.2.7.

(ii) $\Rightarrow$ (i): First note that for all g in G, $t(g) = s(\bar{m}(g,g))$ since $s(\bar{m}(g,g)) = s(m(g,g^{-1})) = s(u(t(g))) = t(g)$. Hence, for all $g \in G$, we get

$$\begin{aligned} t'(F(g)) &= s'(\bar{m}'(F(g),F(g))) \\ &= s'(F(\bar{m}(g,g))) \\ &= f(s(\bar{m}(g,g))) \\ &= f(t(g)). \end{aligned}$$

To show commutativity of multiplication, observe that for all $g \in G$ and $x \in M$:

$$F(u(x)) = u'(f(x)) \quad \text{and} \quad F(g^{-1}) = (F(g))^{-1}$$

by using commutativity of the source, target and division maps and similar arguments as in Remark 1.2.7.

Therefore, $\forall\ (g,h) \in G^{(2)}$, these observations lead to:

$$\begin{aligned} F(m(g,h)) &= F(\bar{m}(g,h^{-1})) \\ &= \bar{m}'(F(g),F(h^{-1})) \\ &= \bar{m}'(F(g),(F(h))^{-1}) \\ &= m'(F(g),F(h)). \end{aligned}$$

□

Definition 1.2.9 (Lie subgroupoid)**.** A **Lie subgroupoid** of a Lie groupoid $G \rightrightarrows M$ is a Lie groupoid $H \rightrightarrows N$ such that there exist injective immersions $H \to G$ and $N \to M$ which form a Lie groupoid morphism.

1.3 Lie algebroids

Similar to the notion of the infinitesimal counterpart of a Lie group G, namely its Lie algebra $\mathfrak{g}$, we can construct the infinitesimal object associated to a Lie groupoid $G \rightrightarrows M$, its Lie algebroid Lie(G). First of all, let us consider the definition of an abstract Lie algebroid.

Definition 1.3.1 (Lie algebroid)**.** A **Lie algebroid** over a manifold M is defined to be a vector bundle $\pi : A \longrightarrow M$ together with:

(i) a Lie bracket $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ on its space of smooth sections

(ii) a vector bundle map $\rho : A \longrightarrow TM$, called the **anchor**, where TM is the tangent bundle of M, such that the Leibniz rule is satisfied:

$$[\alpha, f\beta] = f[\alpha,\beta] + (\rho(\alpha)f)\beta \tag{1}$$

$\forall\ \alpha,\beta \in \Gamma(A),\ f \in C^\infty(M)$.

The following remark shows that (1) makes sense.

Remark 1.3.2. *The anchor induces a Lie algebra map between the space of sections, also denoted by ρ:*

$$\rho : \Gamma(A) \longrightarrow \mathfrak{X}(M).$$

Proof. Note that the induced map is naturally defined by $\rho(\alpha)(x) := \rho(\alpha(x))$ for $\alpha \in \Gamma(A)$ and $x \in M$.

- Linear:
For all $\alpha, \beta \in \Gamma(A)$, $x \in M$, we have $\rho(\alpha+\beta)(x) = \rho((\alpha+\beta)(x)) = \rho(\alpha(x)+\beta(x))$ where $\alpha(x)$, $\beta(x) \in \pi^{-1}(x)$, the fiber of A over x. As $\rho : A \longrightarrow TM$ is a vector bundle map, and hence linear at the level of fibers, we get that $\rho(\alpha(x)+\beta(x)) = \rho(\alpha(x)) + \rho(\beta(x))$ and thus linearity.

- Closed under the Lie bracket:
It is still required to show that $\rho[\alpha,\beta] = [\rho(\alpha), \rho(\beta)]\ \forall\ \alpha, \beta \in \Gamma(A)$. By using the Jacobi identity, bilinearity and antisymmetry on the Lie bracket as well as the Leibniz rule, one gets:

$$\begin{aligned}
0 &= [\alpha,[\beta,f\gamma]] + [\beta,[f\gamma,\alpha]] + [f\gamma,[\alpha,\beta]]\\
&= [\alpha, f[\beta,\gamma] + (\rho(\beta)f)\gamma] + [\beta, -f[\alpha,\gamma] - (\rho(\alpha)f)\gamma] - f[[\alpha,\beta],\gamma] - (\rho[\alpha,\beta]f)\gamma\\
&= [\alpha, f[\beta,\gamma]] + [\alpha, (\rho(\beta)f)\gamma] - [\beta, f[\alpha,\gamma]] - [\beta, (\rho(\alpha)f)\gamma] - f[[\alpha,\beta],\gamma] - (\rho[\alpha,\beta]f)\gamma\\
&= f[\alpha,[\beta,\gamma]] + \cancel{(\rho(\alpha)f)[\beta,\gamma]} + \cancel{(\rho(\beta)f)[\alpha,\gamma]} + (\rho(\alpha)\rho(\beta)f)\gamma\\
&\quad - f[\beta,[\alpha,\gamma]] - \cancel{(\rho(\beta)f)[\alpha,\gamma]} - \cancel{(\rho(\alpha)f)[\beta,\gamma]} - (\rho(\beta)\rho(\alpha)f)\gamma\\
&\quad - f[[\alpha,\beta],\gamma] - (\rho[\alpha,\beta]f)\gamma\\
&= \cancel{(f[\alpha,[\beta,\gamma]] + f[\beta,[\gamma,\alpha]] + f[\gamma,[\alpha,\beta]])} + (\rho(\alpha)\rho(\beta)f)\gamma - (\rho(\beta)\rho(\alpha)f)\gamma - (\rho[\alpha,\beta]f)\gamma\\
&= ([\rho(\alpha),\rho(\beta)]f)\gamma - (\rho[\alpha,\beta]f)\gamma
\end{aligned}$$

true for every section $\alpha, \beta, \gamma \in \Gamma(A)$ and for every function $f \in C^\infty(M)$, hence the result.

$\square$

Example 1.3.3.

- A Lie algebra can be viewed as a Lie algebroid over a point.
- Given a smooth manifold M, the tangent bundle TM over M is a Lie algebroid, together with the usual Lie bracket on the space $\mathfrak{X}(M)$ of vector fields on M, where the anchor is the identity on TM.

For the rest of the section, fix a Lie groupoid $G \rightrightarrows M$. Before looking at the details of the construction of the Lie algebroid of G, let us consider the following remark.

Remark 1.3.4. *Let $g : y \curvearrowleft x$ be an element of G.*

- *Right translations are defined only on source fibers. More precisely, one can apply right translation by g only to elements $h \in G$, such that $s(h) = t(g)$. This follows from the fact that multiplication in a groupoid is defined only on composable pairs.*

$$R_g : s^{-1}(y) \longrightarrow s^{-1}(x), \quad h \mapsto hg.$$

- *Left translations are defined only on target fibers by similar arguments.*

$$L_g : t^{-1}(x) \longrightarrow t^{-1}(y), \quad h \mapsto gh.$$

The Lie algebroid of $G \rightrightarrows M$ will be defined via right-invariant vector fields. However, we restrict the vector fields to be tangent to the source fibers, as right translation is defined only on source fibers. Let

$$T^sG := \ker(ds) = \bigcup_{x \in M} T(s^{-1}(x)) \subset TG$$

be the subbundle consisting of vectors tangent to the source fibers. Here, ds represents the differential of the source map, $T(s^{-1}(x))$ is the tangent space to source fibers and TG is the tangent bundle of G.

Definition 1.3.5. A vector field $X \in \mathfrak{X}(G)$ is called **right-invariant** if it is:

- tangent to the source fibers, i.e. $X \in \Gamma(T^sG)$
- invariant under right translations, i.e. $(dR_h)_g(X_g) = X_{gh}\ \forall\ (g,h) \in G^{(2)}$.

Left-invariant vector fields can be defined similarly, by being tangent to the target fibers and invariant under left translations. Denote the space of right-invariant vector fields of G by $\mathfrak{X}^s_{\text{inv}}(G)$, and that of left-invariant vector fields by $\mathfrak{X}^t_{\text{inv}}(G)$.

The Lie algebroid is now the vector bundle $A := u^*(T^sG)$, the pullback by the unit map of the subbundle $T^sG \subset TG$.

$$\begin{tikzcd}
A := u^*(T^sG) \arrow[d] & T^sG \subset TG \arrow[d] \\
M \arrow[r, "u"'] & G
\end{tikzcd}$$

Hence, the fibers of A are the tangent spaces to the source fibers at the units of the groupoid. That is, $A_x = T_{u(x)}s^{-1}(x)$ for all x in M, and $A = \ker(ds)|_{u(M)}$, as depicted in Figure 2.

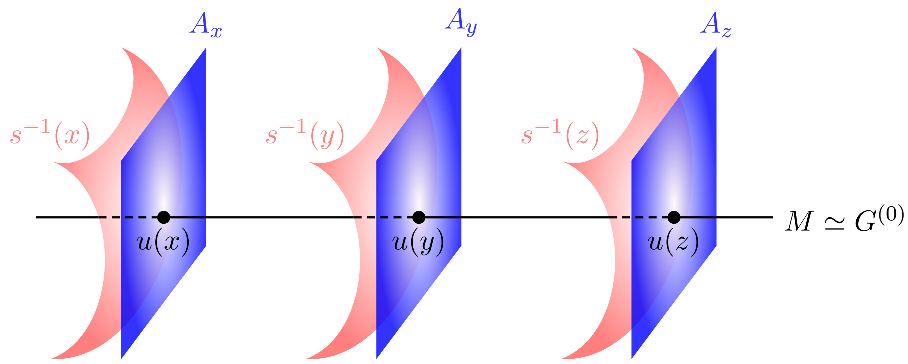

Figure 2: Fibers of the Lie algebroid of a Lie groupoid

The anchor is defined as the restriction of dt to A, where dt is the differential of the target map

$$\rho := dt|_A : A \longrightarrow TM.$$

It still remains to define the Lie bracket on the space $\Gamma(A)$ of sections of A. Note that $\Gamma(A)$ can be identified with the space $\mathfrak{X}^s_{\text{inv}}(G)$ of right-invariant vector fields on G via the following isomorphism

$$\Gamma(A) \longrightarrow \mathfrak{X}^s_{\text{inv}}(G), \quad \alpha \longmapsto \overrightarrow{\alpha}$$

where $\overrightarrow{\alpha}$ is a right-invariant vector field on G defined as

$$(g : y \curvearrowleft x) \longmapsto \overrightarrow{\alpha}_g := dR_g|_{u(y)}(\alpha_y) \ \in \ T_g s^{-1}(x)$$

as shown in Figure 3.

For any $\alpha, \beta \in \Gamma(A)$, let $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ be the induced right-invariant vector fields of α and β respectively. Since $\mathfrak{X}^s_{\text{inv}}(G)$ is closed under the usual Lie bracket on vector fields, we get that $[\overrightarrow{\alpha}, \overrightarrow{\beta}] \in \mathfrak{X}^s_{\text{inv}}(G)$. Hence, $[\alpha, \beta]$ can be defined such that $\overrightarrow{[\alpha, \beta]} = [\overrightarrow{\alpha}, \overrightarrow{\beta}]$.

For later use, note that every section $\alpha \in \Gamma(A)$ of A induces also a left-invariant vector field $\overleftarrow{\alpha} \in \mathfrak{X}^t_{\text{inv}}(G)$ on G, defined by

$$(g : y \curvearrowleft x) \longmapsto \overleftarrow{\alpha}_g := dL_g|_{u(x)} di(\alpha_x) \ \in \ T_g t^{-1}(y)$$

as illustrated in Figure 3.

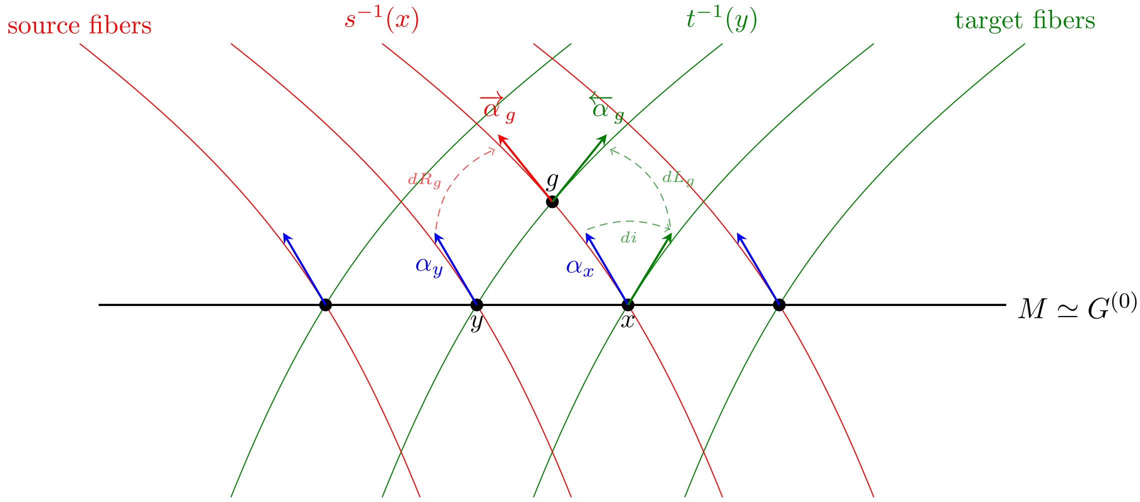

Figure 3: The right- and left-invariant vector fields induced from $\alpha \in \Gamma(A)$

Note that the anchor and the bracket defined above do satisfy the Leibniz identity (1) (see [6, Proposition 1.24]) and hence the Lie algebroid A of G is indeed a Lie algebroid. This finishes our discussion on the Lie algebroid associated to a Lie groupoid. We conclude by giving two basic examples.

Example 1.3.6.

- The Lie algebroid of a Lie group $G \rightrightarrows \{*\}$ is precisely $\mathfrak{g} \rightrightarrows \{*\}$, where $\mathfrak{g}$ is the Lie algebra of G.
- Given a smooth manifold M, the Lie algebroid of the pair groupoid $M \times M$ is isomorphic to the tangent bundle TM of M.

1.4 Integral curves, flows and multiplicative vector fields

The aim of this section is to first present the definitions of integral curves of vector fields and flows on smooth manifolds as well as highlight the fundamental theorem on flows. Intuitively, flows, which are the collection of all integral curves of a vector field on a manifold, represent movements of the manifold along the integral curves at different times, and hence will be important in the discussion of deformations. In addition, we will introduce multiplicative vector fields and study their flows. For a thorough understanding of these geometric objects and the background behind their definitions, one may for instance refer to [14, 15].

Let M be a smooth manifold and I be an open interval containing zero throughout the subsection.

Definition 1.4.1 (Integral curve)**.** An **integral curve** of a vector field $V \in \mathfrak{X}(M)$ is a smooth curve $\gamma : I \to M$ such that

$$\frac{d}{d\varepsilon}\gamma(\varepsilon) = V_{\gamma(\varepsilon)}, \quad \forall\, \varepsilon \in I.$$

That is, the velocity vector at each point of the integral curve is equal to the value of the vector field at that point.

Definition 1.4.2 (Global flow)**.** A smooth **global flow** on the smooth manifold M is a smooth map $\phi : \mathbb{R} \times M \to M$ such that $\forall\, x \in M$, $\varepsilon, \delta \in \mathbb{R}$

- $\phi(0, x) = x$
- $\phi(\varepsilon, \phi(\delta, x)) = \phi(\varepsilon + \delta, x)$.

For a given flow ϕ on M, consider the curve defined by $\phi^{(x)} : \mathbb{R} \to M$, $\varepsilon \mapsto \phi(\varepsilon, x)$, for each $x \in M$. It is a known result that every smooth global flow ϕ on M gives rise to a smooth vector field V defined by $V_x := \dot{\phi}^{(x)}(0) \in T_xM$ for every $x \in M$, where the curves $\phi^{(x)}$ are precisely the integral curves of V starting at each x in M. However, it is not in general true that every vector field generates a global flow since not all integral curves of smooth vector fields are defined on the whole $\mathbb{R}$. This leads to the definition of local flows, which are flows defined only on open subsets of $\mathbb{R} \times M$.

Definition 1.4.3 (Local flow)**.** A smooth **local flow** of M is a smooth map $\phi : D \to M$ where $D \subseteq \mathbb{R} \times M$ is an open subset such that $\forall\, x \in M$, $D^{(x)} := \{\varepsilon \in \mathbb{R} \mid (\varepsilon, x) \in D\} \subseteq \mathbb{R}$ is an open interval containing zero, and such that

- $\phi(0, x) = x$
- $\phi(\varepsilon, \phi(\delta, x)) = \phi(\varepsilon + \delta, x)$

hold $\forall\, x \in M$, $\varepsilon \in D^{(\phi(\delta,x))}$, $\delta \in D^{(x)}$, $(\varepsilon + \delta) \in D^{(x)}$.

Let us introduce some terminology and notation. D as in the previous definition is usually called a **flow domain** for M. A vector field V on M is said to generate a flow $\phi_V : D \to M$ on M if $V_x = \dot{\phi}_V^{(x)}(0)\ \forall\, x \in M$ for some flow domain D and where $\phi_V^{(x)} : D^{(x)} \longrightarrow M$ is defined by $\phi_V^{(x)}(\varepsilon) = \phi_V(\varepsilon, x)$. By a **maximal integral curve**, we mean an integral curve which cannot be extended to a larger interval, and by a **maximal flow**, we mean a flow which cannot be extended to a flow on a larger flow domain. The following theorem from [14, Theorem 9.12] is of great importance.

Theorem 1.4.4 (Fundamental Theorem on Flows)**.** *Every smooth vector field $V \in \mathfrak{X}(M)$ on M generates a unique smooth maximal flow $\phi_V : D \to M$ on M. Moreover, the curves $\phi_V^{(x)}$ are the unique maximal integral curves of V starting at each x in M.*

Given a vector field $V \in \mathfrak{X}(M)$, let $D_\varepsilon = D_\varepsilon(V) := \{x \in M \mid (\varepsilon, x) \in D\} \subseteq M$ be the set of all elements $x \in M$ such that the integral curve of V at ε starting at x is defined. Also, denote the flow of V at time ε by $\phi_V^\varepsilon : D_\varepsilon(V) \to M,\ x \mapsto \phi_V(\varepsilon, x)$.

The fundamental theorem on flows can be generalized to the time-dependent case in a slightly different manner. Recall that a smooth **time-dependent vector field** on M is a family $\{V(\varepsilon)\}_{\varepsilon \in I}$ of vector fields on M which is smoothly parametrized by ε, that is

$$M \times I \to TM, \quad (x, \varepsilon) \mapsto V(\varepsilon)_x \in T_x M$$

is a smooth map. One can identify such a vector field with the vector field V^* on $M \times I$ defined by $V^*_{(x,\varepsilon)} = \big(V(\varepsilon)_x, \frac{\partial}{\partial \varepsilon}\big)$. Similar to the time-independent vector fields, an integral curve of a time-dependent vector field $\{V(\varepsilon)\}_{\varepsilon \in I}$ on M is a smooth curve $\gamma : I \to M$ such that

$$\frac{d}{d\varepsilon}\gamma(\varepsilon) = V(\varepsilon)_{\gamma(\varepsilon)}, \quad \forall\, \varepsilon \in I.$$

This is equivalent to the condition that $I \to M \times I,\ \varepsilon \mapsto (\gamma(\varepsilon), \varepsilon)$ is an integral curve of the induced vector field V^* on $M \times I$. The generalization of the previous theorem is as follows (see [14, Theorem 9.48]).

Theorem 1.4.5 (Fundamental Theorem on Time-Dependent Flows)**.** *Let $V = \{V(\varepsilon)\}_{\varepsilon \in I}$ be a smooth time-dependent vector field on M. Then, there exists a smooth map*

$$\psi_V : \Delta \to M$$

for some $\Delta \subseteq I \times I \times M$ open, such that

(i) *$\forall\ x \in M,\ \varepsilon_0 \in I,\ \Delta^{(\varepsilon_0, x)} := \{\varepsilon \in I \mid (\varepsilon, \varepsilon_0, x) \in \Delta\} \subseteq I$ is an open interval containing ε_0.*

(ii) *the smooth curves $\psi_V^{(\varepsilon_0, x)} : \Delta^{(\varepsilon_0, x)} \to M,\ \varepsilon \mapsto \psi_V(\varepsilon, \varepsilon_0, x)$ are the unique maximal integral curves of V starting at time $\varepsilon = \varepsilon_0$ at each $x \in M$, i.e. $\psi_V(\varepsilon_0, \varepsilon_0, x) = x$.*

(iii) *$\forall\ (\varepsilon_1, \varepsilon_2) \in I \times I$, the map $\psi_V^{(\varepsilon_1, \varepsilon_2)} : M \to M,\ x \mapsto \psi_V(\varepsilon_1, \varepsilon_2, x)$ is locally a diffeomorphism.*

(iv) *For $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in I$, $\psi_V^{(\varepsilon_1, \varepsilon_2)} \circ \psi_V^{(\varepsilon_2, \varepsilon_3)} = \psi_V^{(\varepsilon_1, \varepsilon_3)}$ whenever defined.*

Given a time-dependent vector field $V = \{V(\varepsilon)\}$ on M, the induced map ψ_V as in Theorem 1.4.5 is usually called the **time-dependent flow** of V. Similar to the time-independent case, where vector fields on compact manifolds generate global flows, there is a desirable result in the time-dependent case which will be crucial in proving some rigidity results for deformations of group actions.

Proposition 1.4.6. *If M is a compact smooth manifold, then every time-dependent vector field $V = \{V(\varepsilon)\}_{\varepsilon \in I}$ on M generates a time-dependent flow ψ_V with domain the whole of $I \times I \times M$.*

Corollary 1.4.7. *Let M be a compact smooth manifold and let $V = \{V(\varepsilon)\}_{\varepsilon \in I}$ be a time-dependent vector field on M. Then, the unique maximal integral curves $\psi_V^{(\varepsilon_0, x)}$ of V at ε_0 starting at each $x \in M$ are defined on the whole interval I.*

Corollary 1.4.8. *Let M be a compact smooth manifold and let $V = \{V(\varepsilon)\}_{\varepsilon \in I}$ be a time-dependent vector field on M. Then, the map $\psi_V^{(\varepsilon_1, \varepsilon_2)} : M \to M,\ x \mapsto \psi_V(\varepsilon_1, \varepsilon_2, x)$ is a diffeomorphism defined on the entire M for all $(\varepsilon_1, \varepsilon_2) \in I \times I$.*

Given a time-dependent vector field $V = \{V(\varepsilon)\}$ on M, note that the associated time-dependent flow $\psi_V^{(\varepsilon_1,\varepsilon_2)}$ depends on two time parameters. In relation to the flow ψ_{V^*} generated by V^*, which is the induced vector field on $M \times I$, one has

$$\psi_{V^*}^{\varepsilon_1}(x, \varepsilon_2) = \left(\psi_V^{(\varepsilon_1+\varepsilon_2,\varepsilon_2)}(x), \varepsilon_1 + \varepsilon_2\right) \tag{2}$$

whenever defined. When we deal with $\psi_V^{(\varepsilon_1,\varepsilon_2)}$ in the case where the parameters are close to zero, then we will consider the flow given by a single time parameter, by setting the other one to be equal to zero:

$$\psi_V^{\varepsilon} := \psi_V^{(\varepsilon,0)}.$$

The subsection will be concluded by introducing multiplicative vector fields on Lie groupoids and by showing that the flows generated by such vector fields preserve the groupoid structure.

Definition 1.4.9 (Multiplicative vector fields)**.** Let $G \rightrightarrows M$ be a Lie groupoid. A **multiplicative vector field** on G is a pair (X, V) of vector fields where $X \in \mathfrak{X}(G)$ and $V \in \mathfrak{X}(M)$, such that the pair of maps $X : G \to TG$ and $V : M \to TM$ make a groupoid morphism between the groupoid $G \rightrightarrows M$ and the induced tangent groupoid $TG \rightrightarrows TM$.

$$\begin{tikzcd} G \arrow[r, "X"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & TG \arrow[d, shift right, "dt"'] \arrow[d, shift left, "ds"] \\ M \arrow[r, "V"] & TM \end{tikzcd}$$

Denote the set of all multiplicative vector fields on G by $\mathfrak{X}_{\mathrm{mult}}(G)$. According to the situation, sometimes $X \in \mathfrak{X}(G)$ alone will be called a multiplicative vector field on G and the corresponding vector field $V \in \mathfrak{X}(M)$ will be called the base field associated to X.

The following terminology will be commonly used in the next sections. We say that X in $\mathfrak{X}(G)$ is s-projectable to V in $\mathfrak{X}(M)$ if $ds(X_g) = V_{s(g)}$ for all arrows $g \in G$. Similarly, X is called t-projectable to V if $dt(X_g) = V_{t(g)}\ \forall\, g \in G$. Naturally, X is called (s,t)-projectable if it is both s- and t-projectable. In this case, X is also called an (s,t)-lift of V. In light of these notations, a vector field X on G is called multiplicative if it is (s,t)-projectable to some vector field V on M and if it commutes with multiplication.

Remark 1.4.10. *The multiplication dm on the tangent groupoid has the following explicit formula [15, p. 6]*

$$dm|_{(g,h)}(X_g, X_h) = dR_h|_g(X_g) + dL_g|_h(X_h) \tag{3}$$

for all $(g,h) \in G^{(2)}$ and whenever $ds(X_g) = dt(X_h) = 0$.

Example 1.4.11.

- A multiplicative vector field (X, V) on a Lie group G, which is a Lie groupoid over a point, is necessarily of type $(X, 0)$ such that

  $$X_{m(g,h)} = dm(X_g, X_h) = dR_h(X_g) + dL_g(X_h) \quad \text{using (3)}$$

  for all composable pairs $(g,h) \in G^{(2)}$.

- Let M be a smooth manifold. A multiplicative vector field on the pair groupoid $M \times M \to M$ is of the form $(V \times V, V)$ for $V \in \mathfrak{X}(M)$.

Remark 1.4.12. *If the given pairs (X, V) and (X', V') are multiplicative vector fields on a Lie groupoid $G \rightrightarrows M$, then so is $(X + X', V + V')$.*

Proof. By straight forward calculation, one gets for all g in G:

$$\begin{aligned} ds(X + X')(g) &= ds(X_g + X'_g) \\ &= ds(X_g) + ds(X'_g) \\ &= V_{s(g)} + V'_{s(g)} \\ &= (V + V')_{s(g)}. \end{aligned}$$

Similarly, $dt(X + X')(g) = (V + V')_{t(g)} \; \forall \; g \in G$. Moreover, for every $(g, h) \in G^{(2)}$, we get

$$\begin{aligned} dm(X_g + X'_g, X_h + X'_h) &= dm(X_g, X_h) + dm(X'_g, X'_h) \\ &= X_{m(g,h)} + X'_{m(g,h)} \\ &= (X + X')_{m(g,h)}. \end{aligned}$$

□

The following important result from [15, Proposition 9.8.3] shows that the flows generated by multiplicative vector fields preserve the groupoid structure.

Proposition 1.4.13. *Let (X, V) be a multiplicative vector field on a Lie groupoid $G \rightrightarrows M$. Then, the flow ϕ_X of X preserves the groupoid structure. That is, the pair of flows $\phi^\varepsilon_X : D_\varepsilon(X) \to G$ and $\phi^\varepsilon_V : D_\varepsilon(V) \to M$ of X and V respectively form a groupoid morphism $\forall \; \varepsilon \geq 0$.*

Proof. It is required to prove that the maps ϕ^ε_X and ϕ^ε_V commute with the source, target and multiplication maps of the groupoid. Without loss of generality, assume that the flows ϕ^ε_X and ϕ^ε_V of X and V respectively are defined globally. Let $g \in G$. Due to the multiplicativity of (X, V), we know that X projects to V by s and hence $ds(X_g) = V_{s(g)}$. Also, $X_g = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \phi^{(g)}_X(\varepsilon)$ where $\phi^{(g)}_X$ is the unique maximal integral curve of X starting at g. The projection of this curve under s is a maximal integral curve of V in M starting at $s(g)$, since

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} s(\phi^{(g)}_X(\varepsilon)) = ds\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi^{(g)}_X(\varepsilon)\right) = ds(X_g) = V_{s(g)}.$$

This is true for all g in G and so by uniqueness of maximal integral curves, $s \circ \phi^\varepsilon_X = \phi^\varepsilon_V \circ s$. Similarly, $dt(X_g) = V_{t(g)} \; \forall \; g \in G$ implies that $t \circ \phi^\varepsilon_X = \phi^\varepsilon_V \circ t$ holds.

To prove commutativity of multiplication, observe that

$$X * X : G^{(2)} \to TG^{(2)}, \quad (g, h) \mapsto (X_g, X_h)$$

is a vector field on $G^{(2)}$, where X_g and X_h are indeed composable for $(g, h) \in G^{(2)}$, since $ds(X_g) = V_{s(g)} = V_{t(h)} = dt(X_h)$. Moreover, the flow generated by $X * X$ in $G^{(2)}$ is exactly $\phi^\varepsilon_{X*X}(g, h) = (\phi^\varepsilon_X(g), \phi^\varepsilon_X(h))$. Now, due to the multiplicativity of X we have that $dm(X_g, X_h) = X_{m(g,h)}$ for all $(g, h) \in G^{(2)}$, and hence $X * X$ projects to X under m. By similar arguments, $m \circ \phi^\varepsilon_{X*X} = \phi^\varepsilon_X \circ m$ and hence $m(\phi^\varepsilon_X(g), \phi^\varepsilon_X(h)) = \phi^\varepsilon_X(m(g, h))$ for all $(g, h) \in G^{(2)}$. □

Observe that $D_\varepsilon(X) \rightrightarrows D_\varepsilon(V)$ is an open subgroupoid of $G \rightrightarrows M$. It is clear from [15, Proposition 9.8.3] that the converse of this result is true as well. Hence, the multiplicativity of a vector field $X \in \mathfrak{X}(G)$ is equivalent to the condition that its flow ϕ_X preserves the groupoid structure. In the proper case, the authors in [7] have proved the following important result [7, Lemma 4.4].

Remark 1.4.14. *If $G \rightrightarrows M$ is additionally proper, the flow $\phi^\varepsilon_X(g)$ of X is defined precisely when the flows $\phi^\varepsilon_V(s(g))$ and $\phi^\varepsilon_V(t(g))$ are, that is $D_\varepsilon(X) = G|_{D_\varepsilon(V)}$.*

1.5 Actions and representations

Actions of Lie groups on smooth manifolds as well as representations of Lie groups are of primary importance and are considered to be the heart of Lie group theory. In this section, the notions of Lie group actions and representations will be recalled. Moreover, these notions will be further generalized to the case of Lie groupoids in their own context. A Lie groupoid will act on a space which is fibered over its base, the space of objects. Furthermore, the idea of representing elements of a group via automorphisms of a vector space will be generalized to representing elements of a groupoid via linear isomorphisms between fibers of a vector bundle over the base of the groupoid.

Definition 1.5.1 (Lie group action)**.** A **smooth (left) action** of a Lie group G on a smooth manifold M is a smooth map $\varphi : G \times M \longrightarrow M,\ (g, x) \longmapsto \varphi(g, x) := g \cdot x$ such that

- $(gh) \cdot x = g \cdot (h \cdot x)$
- $e \cdot x = x$

$\forall\ g, h \in G,\ x \in M$ and where e is the identity of G.

Right actions can be defined in a similar manner.

Example 1.5.2.

- A group G acts on itself naturally by conjugation, i.e. $g \cdot h := ghg^{-1}\ \forall\ g, h \in G$.
- Left translations as defined in section 1.1 are actions of groups on themselves.
- A smooth global flow ϕ on a manifold M is a smooth left action of $\mathbb{R}$ on M.

The concept of equivariant maps is useful for us, especially because it will be needed in defining equivalent deformations of group actions.

Definition 1.5.3 (Equivariant map)**.** Let G be a Lie group acting on two smooth manifolds M and N. A map $f : M \to N$ is said to be G**-equivariant** if $f(g \cdot x) = g \cdot f(x)\ \forall\ g \in G$, $x \in M$.

If φ_1 and φ_2 denote the actions of G on M and N respectively, then f is G-equivariant if the following diagram commutes.

$$\begin{tikzcd} G \times M \arrow[r, "\varphi_1"] \arrow[d, "\mathrm{Id}_G \times f"'] & M \arrow[d, "f"] \\ G \times N \arrow[r, "\varphi_2"] & N \end{tikzcd}$$

Definition 1.5.4 (Linear action)**.** An action φ of a Lie group G on a finite-dimensional real vector space V is said to be **linear** if for all g in G, the map $\varphi_g : V \to V,\ v \mapsto \varphi(g, v) = g \cdot v$ is a linear map.

Definition 1.5.5 (Representation of a Lie group)**.** Let G be a Lie group and V be a finite-dimensional real vector space. A **representation** of G on V is a Lie group homomorphism

$$\psi : G \longrightarrow \mathrm{GL}(V) = \mathrm{Aut}(V), \quad g \mapsto \psi(g)$$

where $\mathrm{GL}(V)$ is the general linear group of V, composed of all automorphisms of V.

Equivalently, a representation of G on V is a smooth linear action of G on V

$$\varphi : G \times V \longrightarrow V, \quad (g, v) \mapsto \varphi(g, v) = g \cdot v.$$

This equivalence of definitions is not difficult to prove and it mainly follows from the axioms of a group action and the properties of a group homomorphism (see [14, Proposition 7.37]).

One of the most important representations of a Lie group in this thesis will be the so-called adjoint representation, which gives a way of representing the elements of the group as automorphisms of the Lie algebra of the group through conjugation.

Example 1.5.6 (Adjoint representation)**.** Let G be a Lie group with $\mathfrak{g}$ its Lie algebra. For all $g \in G$, define the group automorphism through conjugation

$$\Psi_g : G \longrightarrow G, \quad h \mapsto ghg^{-1}$$

and let $\mathrm{Ad}_g := d(\Psi_g)_e : T_eG \cong \mathfrak{g} \longrightarrow T_eG \cong \mathfrak{g}$. Now,

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g$$

is called the **adjoint representation** of G.

This is indeed a representation of G since $\mathrm{Ad}_{g_1g_2} = d(\Psi_{g_1g_2})_e = d(\Psi_{g_1} \circ \Psi_{g_2})_e = d(\Psi_{g_1})_e \circ d(\Psi_{g_2})_e = \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}$ and hence a group homomorphism.

Next, we present the generalization of actions and representations to Lie groupoids.

Definition 1.5.7 (Action of a Lie groupoid)**.** Let $G \rightrightarrows M$ be a Lie groupoid and let P be a smooth manifold such that $\mu : P \longrightarrow M$ is a smooth surjective map. A **smooth (left) action** of G on P is a smooth map:

$$G \times_M P \longrightarrow P, \quad (g, p) \longmapsto g \cdot p$$

where $G \times_M P := \{(g, p) \in G \times P \mid \mu(p) = s(g)\}$ and such that

- $\mu(g \cdot p) = t(g), \quad \forall\ (g, p) \in G \times_M P$
- $(gh) \cdot p = g \cdot (h \cdot p), \quad \forall\ (g, h) \in G^{(2)},\ (h, p) \in G \times_M P$
- $u(\mu(p)) \cdot p = p, \quad \forall\ p \in P.$

Note that the definition of an action of a groupoid implies that $g : y \curvearrowleft x \in G$ maps elements in the fiber of P over x to elements in the fiber over y. Also, for the second condition, (g, h) being a composable pair and choosing $p \in P$ such that $\mu(p) = s(h)$ make these actions and equality well-defined (using the fact that $s(gh) = s(h)$, $t(gh) = t(g)$, $s(g) = t(h)$). The map μ is usually called the moment map.

Example 1.5.8. Let $G \rightrightarrows M$ be a Lie groupoid.

- G acts on itself via left translation when viewed as a space fibered over M through the target map.
- Let $G^{(k)}$ denote the set of k-composable arrows for $k > 0$. That is,

$$G^{(k)} := \{(g_1, ..., g_k) \in G \times ... \times G \quad | \quad s(g_i) = t(g_{i+1}) \quad \forall\, i \in \{1, k-1\}\}\,.$$

  Then, $G^{(k)}$ can be viewed as a space fibered over M by

$$\mu : G^{(k)} \to M, \quad (g_1, ..., g_k) \mapsto t(g_1).$$

  There is a left action of the groupoid G on $G^{(k)}$ defined by

$$g \cdot (g_1, g_2, ..., g_k) := (gg_1, g_2, ..., g_k) \quad \forall\, (g, g_1) \in G^{(2)}.$$

Definition 1.5.9 (Representation of a Lie groupoid)**.** Let $G \rightrightarrows M$ be a Lie groupoid. A **representation** of G is a vector bundle $\pi : E \to M$, together with a linear action of G on E. That is, for every arrow $g : y \curvearrowleft x \in G$, $g : E_x \to E_y$ is a linear isomorphism.

$$\begin{tikzcd} G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \curvearrowright & E \arrow[dll, "\pi"] \\ M & & \end{tikzcd}$$

Example 1.5.10.

- If $M = \{*\}$ is a point, the groupoid $G \rightrightarrows \{*\}$ can be viewed as a Lie group G, the vector bundle $E \to \{*\}$ can be viewed as a vector space E, and each $g \in G$ induces automorphisms of E. We then recover the usual definition of a representation of a Lie group.
- Let M be a smooth manifold and $G = M \times M \rightrightarrows M$ be the pair groupoid. Then, a representation of G is a vector bundle E over M with an identification of fibers E_y and E_x for all $(x, y) \in M \times M$. That is, it is precisely a trivialization of E.

In contrast to Lie groups, the nature of groupoids is more subtle and hence there are in general only few canonical representations of Lie groupoids. In sections 4.1.1-4.1.2, some important representations of regular Lie groupoids will be studied. In particular, the isotropy $\mathfrak{i}$ and normal $\mathfrak{v}$ representations will be defined in terms of the kernel and cokernel of the anchor ρ of the Lie algebroid associated to the Lie groupoid respectively, which are useful in the analysis of deformations of groupoids.

On the other hand, this generalization of representations to groupoids fails to make sense for some well-defined representations of groups, such as the adjoint representation. This leads to the notion of representations up to homotopy of groupoids introduced in [2], which will be studied in section 4.2. The main example of representations up to homotopy that we will consider is the adjoint representation Ad which also has direct implications on the deformation theory of groupoids similar to the case of groups.

In the context of representations up to homotopy, the notion of quasi-actions will be needed. Quasi-actions are operations which behave like actions but do not necessarily satisfy the identity and associativity axioms, as defined below:

Definition 1.5.11 (Quasi-action)**.** Let $G \rightrightarrows M$ be a Lie groupoid and let $E \to M$ be a vector bundle over M. A **quasi-action** of G on E is a smooth map:

$$G \times_M E \longrightarrow E, \quad (g, v) \longmapsto g \cdot v =: \lambda_g(v) \in E_{t(g)}$$

such that for all $g \in G$, $\lambda_g : E_{s(g)} \longrightarrow E_{t(g)}$ is linear and where λ_g varies smoothly with g.

Clearly, a quasi-action of G on E is an action whenever the identity and associativity axioms hold.

2 Cohomology theory

Cohomology theory plays a significant role in determining the behavior of geometric objects that undergo deformations. In [7], Crainic, Mestre and Struchiner have defined the deformation cohomology of a Lie groupoid, and meticulously investigated how this cohomology controls deformations of Lie groupoids mainly by showing that deformations of groupoids give rise to deformation 2-cocycles. In section 3.2, we will see how deformations of Lie group homomorphisms give rise to differentiable 1-cocycles. In later sections, it will be clear how the vanishing of cohomologies yield rigidity results.

2.1 Cohomology of Lie groups

In this subsection, we examine two types of cohomologies of a Lie group G. Firstly, the usual differentiable cohomology with values in a representation of G will be recalled. Next, we define the deformation cohomology of G, which will turn out to be isomorphic to the differentiable cohomology with values in the adjoint representation $\mathfrak{g}$ of G.

Let G be a Lie group and $G^k = \underbrace{G \times ... \times G}_{k}$ be the direct product of k copies of G.

Definition 2.1.1 (Differentiable cohomology of a Lie group)**.** Let V be a representation of G. The **differentiable cohomology** $H^*(G, V)$ of G with coefficients in V is the cohomology of the complex $(C^*(G, V), \delta)$, where the cochains are defined as:

- $k \geq 1$: the k-cochains $w \in C^k(G, V)$ are the smooth maps
$$w : G^k \longrightarrow V, \quad (g_1, ..., g_k) \mapsto w(g_1, ..., g_k) \in V$$
where the differential is defined as $\delta : C^k(G, V) \longrightarrow C^{k+1}(G, V)$
$$\begin{aligned}(\delta w)(g_1, ..., g_{k+1}) := g_1 \cdot w(g_2, ..., g_{k+1}) \\ + \sum_{i=1}^{k} (-1)^i w(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\ + (-1)^{k+1} w(g_1, ..., g_k).\end{aligned}$$

- $k = 0$: the 0-cochains $w \in C^0(G, V)$ are the elements of V, i.e. $C^0(G, V) = V$. The differential is defined as $\delta : C^0(G, V) \longrightarrow C^1(G, V)$ with
$$(\delta w)(g) := g \cdot w - w.$$

Definition 2.1.2 (Deformation cohomology of a Lie group)**.** The **deformation** cohomology $H^*_{\text{def}}(G)$ of G is the cohomology of the so-called **deformation complex** $(C^*_{\text{def}}(G), \delta)$ of G, where the cochains are defined as:

- $k \geq 1$: the k-cochains $c \in C^k_{\text{def}}(G)$ are the smooth maps
$$c : G^k \longrightarrow TG, \quad (g_1, ..., g_k) \mapsto c(g_1, ..., g_k) \in T_{g_1} G$$
and the differential is defined as $\delta : C^k_{\text{def}}(G) \longrightarrow C^{k+1}_{\text{def}}(G)$
$$\begin{aligned}(\delta c)(g_1, ..., g_{k+1}) := & - d\bar{m}(c(g_1 g_2, ..., g_{k+1}), c(g_2, ..., g_{k+1})) \\ & + \sum_{i=2}^{k} (-1)^i c(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\ & + (-1)^{k+1} c(g_1, ..., g_k)\end{aligned}$$
where $\bar{m}(g, h) = gh^{-1}$ for all $g, h \in G$.

- $k=0$: the 0-cochains $c \in C^0_{\text{def}}(G)$ are the elements of $\mathfrak{g} \cong T_eG$, i.e. $C^0_{\text{def}}(G) = \mathfrak{g}$. The differential is defined as $\delta : C^0_{\text{def}}(G) \longrightarrow C^1_{\text{def}}(G)$ with

$$(\delta c)(g) := dR_g(c) - dL_g(c).$$

Note that $(C^*_{\text{def}}(G), \delta)$ as in Definition 2.1.2 is indeed a cochain complex (see [7, Lemma 2.2]).

Lemma 2.1.3. *For all $g, h \in G$, $X_g \in T_gG$, $X_h \in T_hG$,*

$$d\bar{m}(X_g, X_h) = dR_{h^{-1}}(X_g) - dL_g dL_{h^{-1}} dR_{h^{-1}}(X_h).$$

Proof. We will mainly use the formula (3) for the differential of the multiplication map for groups. Firstly, note that for all $g \in G$, $X_g \in T_gG$

$$\begin{aligned} & dm(X_g, di(X_g)) = 0_e \\ \Rightarrow \quad & dR_{g^{-1}}(X_g) + dL_g(di(X_g)) = 0 \\ & \qquad \Rightarrow \quad di(X_g) = -dL_{g^{-1}} dR_{g^{-1}}(X_g) \end{aligned}$$

where i denotes the inversion in G. Now, we get that

$$\begin{aligned} d\bar{m}(X_g, X_h) &= dm(X_g, di(X_h)) \\ &= dR_{h^{-1}}(X_g) + dL_g di(X_h) \\ &= dR_{h^{-1}}(X_g) - dL_g dL_{h^{-1}} dR_{h^{-1}}(X_h). \end{aligned}$$

□

Theorem 2.1.4. *Consider the adjoint representation $\mathfrak{g}$ of the Lie group G. Then, the map*

$$f : C^k_{\text{def}}(G) \longrightarrow C^k(G, \mathfrak{g}), \quad (fc)(g_1, ..., g_k) := \left(dR_{g_1^{-1}}\right)_{g_1} (c(g_1, ..., g_k))$$

is an isomorphism of cochain complexes.

Proof. As right translations are diffeomorphisms, we will only show that f commutes with the differentials, i.e. the commutativity of the following diagram:

$$\begin{array}{ccc} C^k_{\text{def}}(G) & \xrightarrow{\quad f \quad} & C^k(G, \mathfrak{g}) \\ \delta \big\downarrow & & \big\downarrow \delta \\ C^{k+1}_{\text{def}}(G) & \xrightarrow{\quad f \quad} & C^{k+1}(G, \mathfrak{g}) \end{array}$$

Let $c \in C^k_{\text{def}}(G)$ and $(g_1, ..., g_{k+1}) \in G^{(k+1)}$.

- $$\begin{aligned} (\delta(fc))(g_1, ..., g_{k+1}) &= g_1 \cdot (fc)(g_2, ..., g_{k+1}) \\ &\quad + \sum_{i=1}^{k} (-1)^i (fc)(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\ &\quad + (-1)^{k+1} (fc)(g_1, ..., g_k) \\ &= dL_{g_1} dR_{g_1^{-1}} dR_{g_2^{-1}} c(g_2, ..., g_{k+1}) - dR_{(g_1 g_2)^{-1}} c(g_1 g_2, ..., g_{k+1}) \\ &\quad + \sum_{i=2}^{k} (-1)^i dR_{g_1^{-1}} c(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\ &\quad + (-1)^{k+1} dR_{g_1^{-1}} c(g_1, ..., g_k) \end{aligned}$$

- $$\begin{aligned}f(\delta(c))(g_1,...,g_{k+1}) &= dR_{g_1^{-1}}\delta(c)(g_1,...,g_{k+1})\\ &= -dR_{g_1^{-1}}d\bar{m}(c(g_1g_2,...,g_{k+1}),c(g_2,...,g_{k+1}))\\ &\quad + dR_{g_1^{-1}}\sum_{i=2}^{k}(-1)^i c(g_1,...,g_ig_{i+1},...,g_{k+1})\\ &\quad + (-1)^{k+1}dR_{g_1^{-1}}c(g_1,...,g_k)\\ &= -dR_{g_1^{-1}}dR_{g_2^{-1}}c(g_1g_2,...,g_{k+1})\\ &\quad + dR_{g_1^{-1}}dL_{g_1g_2}dL_{g_2^{-1}}dR_{g_2^{-1}}c(g_2,...,g_{k+1})\\ &\quad + \sum_{i=2}^{k}(-1)^i dR_{g_1^{-1}}c(g_1,...,g_ig_{i+1},...,g_{k+1})\\ &\quad + (-1)^{k+1}dR_{g_1^{-1}}c(g_1,...,g_k)\\ &= -dR_{(g_1g_2)^{-1}}c(g_1g_2,...,g_{k+1}) + dR_{g_1^{-1}}dL_{g_1}dR_{g_2^{-1}}c(g_2,...,g_{k+1})\\ &\quad + \sum_{i=2}^{k}(-1)^i dR_{g_1^{-1}}c(g_1,...,g_ig_{i+1},...,g_{k+1})\\ &\quad + (-1)^{k+1}dR_{g_1^{-1}}c(g_1,...,g_k)\end{aligned}$$

by straightforward calculation and using Lemma 2.1.3.

Hence, $\delta \circ f = f \circ \delta$ since $c \in C^k_{\text{def}}(G)$ and $(g_1,...g_{k+1}) \in G^{(k+1)}$ were chosen arbitrarily. □

Corollary 2.1.5. $H^*_{\text{def}}(G) \cong H^*(G,\mathfrak{g})$.

Hence, deformations of groups are mainly controlled by the differentiable cohomology with values in the adjoint representation.

2.2 Cohomology of Lie groupoids

We now generalize the notions of deformation and differentiable cohomologies of a Lie group to a Lie groupoid as first introduced by the authors in [7].

Consider a Lie groupoid $G \rightrightarrows M$ with its associated Lie algebroid $A = \text{Lie}(G)$, and let $G^{(k)} := \{(g_1,...,g_k) \in G \times ... \times G \mid s(g_i) = t(g_{i+1})\ \forall\ i \in \{1,k-1\}\}$ denote the set of k-composable arrows.

Definition 2.2.1 (Deformation cohomology of a Lie groupoid)**.** The **deformation cohomology** $H^*_{\text{def}}(G)$ of G is the cohomology of the **deformation complex** $(C^*_{\text{def}}(G),\delta)$ of G which is defined as:

- $k \geq 1$: the k-cochains $c \in C^k_{\text{def}}(G)$ are the smooth maps
$$c: G^{(k)} \longrightarrow TG, \quad (g_1,...,g_k) \mapsto c(g_1,...,g_k) \in T_{g_1}G$$
which are s-projectable, i.e. $ds \circ c(g_1,...,g_k)$ does not depend on g_1, and where the differential is defined as
$$\delta: C^k_{\text{def}}(G) \longrightarrow C^{k+1}_{\text{def}}(G)$$
$$\begin{aligned}(\delta c)(g_1,...,g_{k+1}) := &-d\bar{m}(c(g_1g_2,...,g_{k+1}),c(g_2,...,g_{k+1}))\\ &+\sum_{i=2}^{k}(-1)^i c(g_1,...,g_ig_{i+1},...,g_{k+1})\\ &+(-1)^{k+1}c(g_1,...,g_k).\end{aligned}$$

- $k = 0$: the 0-cochains $c \in C^0_{\text{def}}(G)$ are the smooth maps

$$c : G^{(0)} = M \longrightarrow A, \quad x \mapsto c(x) \in A_x$$

which means that $C^0_{\text{def}}(G) = \Gamma(A)$, the smooth sections of A. Let the differential be defined as $\delta : C^0_{\text{def}}(G) \longrightarrow C^1_{\text{def}}(G)$ with

$$(\delta c)(g) := \overrightarrow{c}_g + \overleftarrow{c}_g$$

where $\overrightarrow{c}$ and $\overleftarrow{c}$ are the induced right- and left-invariant vector fields of c respectively.

Note that $(C^*_{\text{def}}(G), \delta)$ as in Definition 2.2.1 is indeed a cochain complex as proven in [7, Lemma 2.2].

Definition 2.2.2 (Differentiable cohomology of a Lie groupoid)**.** The **differentiable cohomology** $H^*(G, E)$ of G with coefficients in a representation E of G is the cohomology of the complex $(C^*(G, E), \delta)$, where the cochains are defined by:

- $k \geq 1$: the k-cochains $w \in C^k(G, E)$ are the smooth maps

$$w : G^{(k)} \longrightarrow E, \quad (g_1, ..., g_k) \mapsto w(g_1, ..., g_k) \in E_{t(g_1)}$$

and where the differential is defined as $\delta : C^k(G, E) \longrightarrow C^{k+1}(G, E)$

$$\begin{aligned}(\delta w)(g_1, ..., g_{k+1}) := g_1 \cdot w(g_2, ..., g_{k+1})& \\ + \sum_{i=1}^{k} (-1)^i w(g_1, ..., g_i g_{i+1}, ..., g_{k+1})& \\ + (-1)^{k+1} w(g_1, ..., g_k).&\end{aligned}$$

- $k = 0$: the 0-cochains $w \in C^0(G, E)$ are the smooth maps

$$w : G^{(0)} = M \longrightarrow E, \quad x \mapsto w(x) \in E_x$$

That is, $C^0(G, E) = \Gamma(E)$, the smooth sections of E. Let the differential be defined as $\delta : C^0(G, E) \longrightarrow C^1(G, E)$ with

$$(\delta w)(g) := g \cdot w(s(g)) - w(t(g)).$$

Note that Definition 2.2.2 implies that in degree zero, $H^0(G, E) = \Gamma(E)^{\text{inv}}$ where $\Gamma(E)^{\text{inv}} := \{\alpha \in \Gamma(E) : g \cdot \alpha(s(g)) = \alpha(t(g)) \ \forall \ g \in G\}$ is the space of smooth section of E invariant under the action of G.

In the special case where E is the trivial line bundle, that is $E \cong M \times \mathbb{R}$, and where the action of G on E is given by the trivial action, the differentiable cohomology will be denoted by $H^*(G)$ with the underlying complex $C^*(G)$. Hence, $C^k(G) = C^\infty(G^{(k)})$. That is, for $k > 0$, a k-cochain $f \in C^k(G)$ is a smooth function $f : G^{(k)} \longrightarrow \mathbb{R}$ with differential

$$\begin{aligned}(\delta f)(g_1, ..., g_{k+1}) := f(g_2, ..., g_{k+1}) + \sum_{i=1}^{k} (-1)^i f(g_1, ..., g_i g_{i+1}, ..., g_{k+1})& \\ + (-1)^{k+1} f(g_1, ..., g_k),&\end{aligned}$$

and 0-cochains are smooth functions $f : G^{(0)} = M \longrightarrow \mathbb{R}$ with $(\delta f)(g) := f(s(g)) - f(t(g))$.

There is a natural algebra structure on $C^*(G)$ given by the following bilinear product map:

$$C^*(G) \times C^*(G) \longrightarrow C^*(G), \quad (f, h) \mapsto f \cdot h \in C^{k+p}(G)$$

for $f \in C^k(G)$, $h \in C^p(G)$, defined by the cup product

$$(f \cdot h)(g_1, ..., g_k, g_{k+1}, ..., g_{k+p}) := \begin{cases} fh & k = p = 0 \\ f(t(g_1))h(g_1, ..., g_p) & k = 0, p > 0 \\ f(g_1, ..., g_k)h(s(g_k)) & p = 0, k > 0 \\ f(g_1, ..., g_k)h(g_{k+1}, ..., g_{k+p}) & k, p > 0 \end{cases}$$

for all $(g_1, ..., g_{k+p}) \in G^{(k+p)}$. Moreover, observe that the differential $\delta : C^k(G) \to C^{k+1}(G)$ satisfies the graded Leibniz rule

$$\delta(f \cdot h) = \delta(f) \cdot h + (-1)^k f \cdot \delta(h)$$

for all $f \in C^k(G)$, $h \in C^p(G)$ and hence makes the space $C^*(G)$ into a differential graded algebra (DGA).

Remark 2.2.3. *Given a vector bundle $E \to M$, one could still make sense of the graded vector space of differentiable cochains on G with values in E*

$$C^*(G, E) = \bigoplus_{k \in \mathbb{N}} C^k(G, E)$$

where each degree k part is defined as

$$C^k(G, E) := \Gamma(G^{(k)}, t^*E)$$

with t^*E being the pullback of the vector bundle $E \to M$ by the target map t and where $t(g_1, ..., g_k) = t(g_1)$.

$$\begin{tikzcd} t^*E \arrow[d] & E \arrow[d] \\ G^{(k)} \arrow[r, "t"] & M \end{tikzcd}$$

Given a vector bundle $E \to M$, we next note that the space $C^*(G, E)$ can be viewed as a right graded module over the algebra $C^*(G)$ via the map:

$$C^*(G, E) \times C^*(G) \longrightarrow C^*(G, E), \quad (w, f) \mapsto w \cdot f \in C^{k+p}(G, E)$$

for $w \in C^k(G, E)$, $f \in C^p(G)$, defined similarly by the cup product.

If E is additionally a representation of G, the cochain complex $(C^*(G, E), \delta)$ has the structure of a right differential graded module over $C^*(G)$ since the graded Leibniz rule

$$\delta(w \cdot f) = \delta(w) \cdot f + (-1)^k w \cdot \delta(f)$$

for $w \in C^*(G, E)$ and $f \in C^*(G)$ is satisfied.

2.3 Main example: The action groupoid

One of the major examples of Lie groupoids is that arising from an action of a Lie group on a smooth manifold. This groupoid also gives us a nice picture of how the deformation cohomology and differentiable cohomologies with values in representations can be related. Throughout this subsection, let G be a Lie group acting smoothly on a smooth manifold M.

A groupoid structure can be constructed in the following way. Let the product $G \times M$ be the space of arrows and let M be the space of objects. For all $g, h \in G$, $x \in M$, define the structure maps as:

- $s(g, x) = x$
- $t(g, x) = g \cdot x$
- $m((g, h \cdot x), (h, x)) = (gh, x)$
- $u(x) = (e, x)$
- $i(g, x) = (g^{-1}, g \cdot x)$.

The resulting groupoid is called the **action groupoid** (see Figure 4) and denoted by $G \ltimes M$.

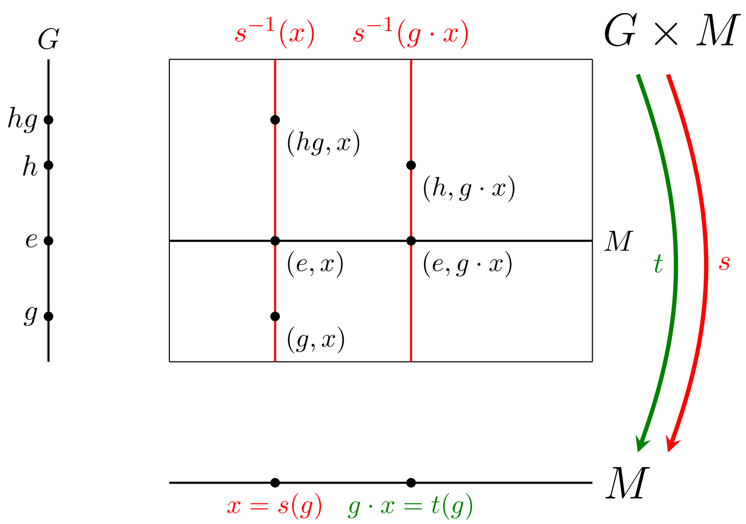

Figure 4: Action groupoid

Note that the set of all composable pairs of the action groupoid $G \ltimes M$ can be identified with $G \times G \times M$ since

$$\begin{aligned}(G \ltimes M)^{(2)} &= \{(g, x), (h, y) \in G \times M \mid s(g, x) = t(h, y)\} \\ &= \{(g, x), (h, y) \in G \times M \mid x = h \cdot y\} \\ &= \{(g, h \cdot y), (h, y) \in G \times M\} \\ &\cong \{(g, h, y) \in G \times G \times M\} \\ &= G \times G \times M. \end{aligned} \tag{4}$$

The associated Lie algebroid A is given by the trivial vector bundle $\mathfrak{g} \times M \longrightarrow M$ with typical fiber the Lie algebra $\mathfrak{g}$ of G. Note that the source fibers are exactly

$$s^{-1}(x) = G \times \{x\}$$

and hence the tangent spaces to the source fibers at the units are given by

$$A_x = T_{u(x)}s^{-1}(x) = T_eG \times T_x\{x\} \cong \mathfrak{g}$$

for all x in M. The anchor is defined as

$$\rho : \mathfrak{g} \times M \longrightarrow TM, \quad (X, x) \mapsto \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\exp(\varepsilon X) \cdot x)$$

where exp is the exponential map recalled in section 3.1. Moreover, the condition that

$$[\sigma_X, \sigma_Y] = \sigma_{[X,Y]}$$

for all $X, Y \in \mathfrak{g}$ and σ_X, σ_Y constant sections of $\mathfrak{g} \times M$, together with the Leibniz identity, uniquely determine the Lie bracket on the sections of A.

Furthermore, a representation E of the action groupoid $G \ltimes M$ is a vector bundle E over M together with an action of $G \ltimes M$ on E such that $(g, x) : E_x \to E_{g\cdot x}$ is a linear isomorphism $\forall\ g \in G,\ x \in M$. Hence, it is exactly an equivariant vector bundle over M by its very definition.

Given a representation E of $G \ltimes M$, the space $\Gamma(E)$ of smooth sections of E has the structure of a left module over G via

$$G \times \Gamma(E) \longrightarrow \Gamma(E), \quad (g \cdot \sigma)(x) := (g, x) \cdot \sigma(x) \in E_{g\cdot x}$$

for all $g \in G$, $\sigma \in \Gamma(E)$, $x \in M$. In light of this, one can identify the differentiable cohomology $H^*(G \ltimes M, E)$ with coefficients in E with the differentiable cohomology $H^*(G, \Gamma(E))$ with values in $\Gamma(E)$.

We now mention two natural representations of the action groupoid $G \ltimes M$, which are explained in [18, p. 148].

1. The Lie algebroid $A = \mathfrak{g} \times M$ together with the action

   $$(g, x) : A_x \cong \mathfrak{g} \longrightarrow A_{g\cdot x} \cong \mathfrak{g}, \quad (g, x) \cdot X := \mathrm{Ad}_g(X).$$

   By the above noted identification, we have $H^*(G \ltimes M, A) = H^*(G, \Gamma(A))$ where the smooth sections of the trivial bundle $A = \mathfrak{g} \times M$ are precisely the smooth functions from M to $\mathfrak{g}$ (see [14, Example 10.10(c)]).

2. The tangent bundle TM together with the action

   $$(g, x) : T_xM \longrightarrow T_{g\cdot x}M, \quad (g, x) \cdot V := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (g \cdot V(\varepsilon))$$

   where $V(\varepsilon)$ is a curve which represents the tangent vector $V \in T_xM$. Here, we get the identification $H^*(G \ltimes M, TM) = H^*(G, \mathfrak{X}(M))$.

It is worth to have a look at the relation between the deformation cohomology of the action groupoid $G \ltimes M$ and the above noted differentiable cohomologies. Indeed, we have the following long exact sequence, which is stated and proved in [18, Proposition 5.18].

$$\cdots \longrightarrow H^{k-1}(G\ltimes M, TM) \longrightarrow H^k_{\mathrm{def}}(G\ltimes M) \longrightarrow H^k(G\ltimes M, A) \xrightarrow{\rho_*} H^k(G\ltimes M, TM) \longrightarrow \cdots$$

where for a cocycle $c \in C^k(G \ltimes M, A)$, $\rho_* c(\gamma_1, ..., \gamma_k) := \rho_{g\cdot x}(c(\gamma_1, ..., \gamma_k)) \in T_{g\cdot x}M$ for all k-composable arrows $(\gamma_1, ..., \gamma_k)$ in $G \times M$ and $\gamma_1 = (g, x)$.

It is important to remark that one gets the following splitting as vector spaces

$$C^k_{\mathrm{def}}(G \ltimes M) \cong C^k(G \ltimes M, A) \oplus C^{k-1}(G \ltimes M, TM)$$

which does not respect the differentials. This is very much in the spirit of how the adjoint representation will be defined as a representation up to homotopy.

3 Deformations of group representations

Elements of a Lie group G can be represented via automorphisms of a finite-dimensional real vector space V given by a Lie group homomorphism $G \to \mathrm{Aut}(V)$. This, in turn, is equivalent to a smooth linear action of G on V as recalled in section 1.5. Thus, in our attempt to understand deformations of group representations, it is a natural approach to first study deformations of Lie group actions as well as deformations of Lie group homomorphisms. The discussion will be concluded by some rigidity results displayed in section 3.3.

3.1 Deformations of group actions

Throughout the subsection, let G be a Lie group with associated Lie algebra $\mathfrak{g}$, M a smooth manifold and I an open interval containing zero. This subsection is mainly based on the papers [23, 24] by Palais and Stewart. The definitions are stated in a slightly different manner to make the discussion parallel to deformations of groupoids in [7].

Definition 3.1.1 (Deformation of a manifold)**.** A **smooth deformation of** M is a family $\tilde{\psi} = \{\psi_\varepsilon\}_{\varepsilon \in I}$ of diffeomorphisms of M which is smoothly parametrized by $\varepsilon \in I$, i.e. $I \times M \to M$, $(\varepsilon, x) \mapsto \psi_\varepsilon(x)$ is smooth, and such that $\psi_0 = \mathrm{Id}_M$.

Definition 3.1.2 (Deformation of an action)**.** A **smooth deformation of a smooth action** φ of G on M is a family $\tilde{\varphi} = \{\varphi_\varepsilon\}_{\varepsilon \in I}$ of smooth actions of G on M which is smoothly parametrized by $\varepsilon \in I$, i.e. $G \times M \times I \to M$, $(g, x, \varepsilon) \mapsto \varphi_\varepsilon(g, x)$ is smooth, and such that $\varphi_0 = \varphi$.

The deformations and actions under study in this thesis will all be smooth unless specified otherwise.

Definition 3.1.3 (Constant deformation)**.** A deformation $\{\varphi_\varepsilon\}_{\varepsilon \in I}$ of an action φ of G on M will be called a **constant deformation** if $\varphi_\varepsilon = \varphi \ \forall \ \varepsilon \in I$.

Definition 3.1.4 (Equivalent deformations)**.** Let φ be an action of G on M. Two deformations $\tilde{\varphi} = \{\varphi_\varepsilon\}$ and $\tilde{\varphi}' = \{\varphi'_\varepsilon\}$ of φ are said to be **equivalent** if there exists a deformation $\tilde{\psi} = \{\psi_\varepsilon\}$ of M, such that each member ψ_ε is a G-equivariant map from M (with respect to φ_ε) to M (with respect to φ'_ε), in the sense that the following diagram commutes.

$$\begin{tikzcd}
G \times M \arrow[r, "\varphi_\varepsilon"] \arrow[d, "\mathrm{Id}_G \times \psi_\varepsilon"'] & M \arrow[d, "\exists\ \psi_\varepsilon"] \\
G \times M \arrow[r, "\varphi'_\varepsilon"] & M
\end{tikzcd}$$

i.e. $\varphi'_\varepsilon(g, x) = \psi_\varepsilon(\varphi_\varepsilon(g, \psi_\varepsilon^{-1}(x))) \ \ \forall \ g \in G$ and $x \in M$.

Definition 3.1.5 (Trivial deformation)**.** Let φ be a smooth action of G on M. A deformation $\tilde{\varphi}$ of φ is called a **trivial deformation** if $\tilde{\varphi}$ is equivalent to the constant deformation.

Definition 3.1.6 (Rigid action)**.** A smooth action φ of G on M is said to be **rigid** if every deformation of φ is trivial.

Next, let us take a glimpse at deformations of group actions from an infinitesimal perspective via the Lie algebra $\mathfrak{g}$ of the Lie group G as studied in [24]. Recall that the exponential map of G gives a natural smooth map from the Lie algebra to the Lie group, defined by

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \exp(X) := \gamma_X(1)$$

where $\gamma_X : \mathbb{R} \to G$ is the maximal integral curve of X starting at the identity e. One can show that $\exp(\varepsilon X) = \gamma_X(\varepsilon) \ \forall\, \varepsilon \in \mathbb{R}$. Consider a smooth action φ of G on M. Then, every left-invariant vector field $X \in \mathfrak{g}$ on G induces a smooth vector field X^* on M defined by

$$X^*_x := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\exp(\varepsilon X) \cdot x), \quad \forall\, x \in M.$$

Such a vector field is usually called the **action field** associated to X. The map

$$\mathfrak{g} \to \mathfrak{X}(M), \quad X \mapsto X^*$$

is a homomorphism of Lie algebras and is called the **infinitesimal generator** of φ. A linear map $D : \mathfrak{g} \to \mathfrak{X}(M)$ is called an **infinitesimal deformation** of φ if for all $X, Y \in \mathfrak{g}$,

$$D([X,Y]) = [D(X), Y^*] + [X^*, D(Y)].$$

Remark 3.1.7. *Given $Z \in \mathfrak{X}(M)$, the map $D : \mathfrak{g} \to \mathfrak{X}(M)$, $X \mapsto [Z, X^*]$ is an infinitesimal deformation of φ.*

Proof. By using the fact that $X \mapsto X^*$ is a Lie algebra map and the properties of the Lie bracket, we get for all X, Y in $\mathfrak{g}$, $D(X+Y) = [Z, (X+Y)^*] = [Z, X^* + Y^*] = [Z, X^*] + [Z, Y^*]$ which shows that D is linear. Moreover,

$$\begin{aligned} D([X,Y]) &= [Z, [X,Y]^*] \\ &= [Z, [X^*, Y^*]] \\ &= [[Z, X^*], Y^*] + [[Y^*, Z], X^*] \\ &= [D(X), Y^*] + [-D(Y), X^*] \\ &= [D(X), Y^*] + [X^*, D(Y)]. \end{aligned}$$

□

Infinitesimal deformations of φ of the form $D : \mathfrak{g} \to \mathfrak{X}(M)$, $X \mapsto [Z, X^*]$ for some $Z \in \mathfrak{X}(M)$ are called **trivial**.

In section 3.3 we will explore some conditions under which a Lie group action shows rigidity both on global and infinitesimal levels.

3.2 Deformations of group homomorphisms

Let G and H be connected Lie groups with $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras respectively. Let I denote an open interval containing zero and let the map $\phi : G \to H$ be a Lie group homomorphism.

Note that in this and the next subsection, Ψ will denote the group automorphism defined by conjugation

$$\Psi_g : G \to G, \quad r \mapsto grg^{-1} \quad \forall\, g, r \in G$$

and hence the differential of Ψ_g at the identity will just be Ad_g, where Ad is the adjoint representation of G.

Definition 3.2.1 (Deformation of a Lie group homomorphism)**.** A **smooth deformation** of $\phi : G \to H$ is a family $\tilde{\phi} = \{\phi_\varepsilon\}_{\varepsilon \in I}$ of Lie group homomorphisms from G to H which is smoothly parametrized by $\varepsilon \in I$, i.e. $G \times I \to H,\ (g, \varepsilon) \mapsto \phi_\varepsilon(g)$ is smooth, and such that $\phi_0 = \phi$.

Definition 3.2.2 (Constant deformation)**.** A deformation $\{\phi_\varepsilon\}_{\varepsilon \in I}$ of ϕ is called **constant** if $\phi_\varepsilon = \phi\ \forall\ \varepsilon \in I$.

Definition 3.2.3 (Equivalent deformations)**.** Two smooth deformations $\tilde{\phi} = \{\phi_\varepsilon\}_{\varepsilon \in I}$ and $\tilde{\phi}' = \{\phi'_\varepsilon\}_{\varepsilon \in I}$ of ϕ are said to be **equivalent** if there exists a smooth curve

$$h : I \to H, \quad \varepsilon \mapsto h_\varepsilon$$

starting at the identity e of H (i.e. $h_0 = e$) such that $\Psi_{h_\varepsilon} \circ \phi_\varepsilon = \phi'_\varepsilon$ for all $\varepsilon \in I$, in the sense that the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\phi_\varepsilon"] \arrow[dr, "\phi'_\varepsilon"'] & H \arrow[d, "\Psi_{h_\varepsilon}"] \\ & H \end{tikzcd}$$

Definition 3.2.4 (Trivial deformation)**.** A deformation $\tilde{\phi}$ of ϕ is called **trivial** if $\tilde{\phi}$ is equivalent to the constant deformation.

Definition 3.2.5 (Locally trivial deformation)**.** A deformation $\tilde{\phi} = \{\phi_\varepsilon\}_{\varepsilon \in I}$ of ϕ is called **locally trivial** if there exists a smooth curve $h : I \to H,\ \varepsilon \mapsto h_\varepsilon$ starting at the identity e of H such that $\Psi_{h_\varepsilon} \circ \phi = \phi_\varepsilon$ for ε small enough.

Definition 3.2.6 (Rigid homomorphism)**.** The Lie group homomorphism ϕ is said to be **rigid** if every deformation of ϕ is trivial.

Next, we examine how a deformation of a Lie group homomorphism gives rise to a differentiable 1-cocycle.

Let $\tilde{\phi} = \{\phi_\varepsilon\}$ be a deformation of the Lie group homomorphism $\phi : G \to H$. In order to study the behavior of ϕ under the deformation $\tilde{\phi}$, it is natural to look at the variation of ϕ_ε with respect to ε. Consider the expression

$$-\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon(g) \ \in \ T_{\phi(g)}H.$$

The choice of the minus sign will be clear in calculations. Now, using the isomorphism $T_hH \cong \mathfrak{h}$ for all h in H, given by right translation, we get that for $h = \phi(g)$

$$-dR_{\phi(g^{-1})}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon(g)\right) \ \in \ \mathfrak{h}. \tag{5}$$

Consider the representation of G on $\mathfrak{h}$ defined by the composition

$$G \xrightarrow{\ \phi\ } H \xrightarrow{\ \mathrm{Ad}\ } \mathrm{Aut}(\mathfrak{h})$$

where Ad is the adjoint representation of H.

Definition 3.2.7. Let $\tilde{\phi} = \{\phi_\varepsilon\}$ be a deformation of $\phi : G \to H$. The **differentiable cocycle** $w \in C^1(G, \mathfrak{h})$ associated to $\tilde{\phi}$ is defined by

$$w(g) := -dR_{\phi(g^{-1})} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g) \right) \in \mathfrak{h} \quad \forall\, g \in G$$

where $\mathfrak{h}$ is the representation of G defined as above.

Let us prove that w is indeed a cocycle.

Lemma 3.2.8. $w \in \ker(\delta : C^1(G, \mathfrak{h}) \to C^2(G, \mathfrak{h}))$.

Proof. Since each of ϕ_ε is a Lie group homomorphism from G to H, we have for all $g_1, g_2 \in G$

$$\phi_\varepsilon(g_1 g_2) = \phi_\varepsilon(g_1)\phi_\varepsilon(g_2).$$

Differentiating this identity with respect to ε at $\varepsilon = 0$, we get

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_1 g_2) = dR_{\phi(g_2)} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_1) \right) + dL_{\phi(g_1)} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_2) \right) \tag{6}$$

by using equation (3) for groups, that is, if m represents the multiplication of H

$$dm(X_{h_1}, X_{h_2}) = dR_{h_2}(X_{h_1}) + dL_{h_1}(X_{h_2}) \tag{7}$$

for all $h_1, h_2 \in H$, $X_{h_1} \in T_{h_1}H$, $X_{h_2} \in T_{h_2}H$.

Applying $dR_{\phi((g_1 g_2)^{-1})}$ on both sides of (6), we get

$$\begin{aligned}
dR_{\phi((g_1 g_2)^{-1})} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_1 g_2) \right) &= dR_{\phi((g_1 g_2)^{-1})} \circ dR_{\phi(g_2)} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_1) \right) \\
&\quad + dR_{\phi((g_1 g_2)^{-1})} \circ dL_{\phi(g_1)} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_2) \right) \\
&= dR_{\phi(g_1^{-1})} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_1) \right) \\
&\quad + d\Psi_{\phi(g_1)} \circ dR_{\phi(g_2^{-1})} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon(g_2) \right)
\end{aligned} \tag{8}$$

since

$$\begin{aligned}
dR_{\phi((g_1 g_2)^{-1})} \circ dL_{\phi(g_1)} &= dR_{\phi(g_1^{-1})} \circ dR_{\phi(g_2^{-1})} \circ dL_{\phi(g_1)} \\
&= dR_{\phi(g_1^{-1})} \circ dL_{\phi(g_1)} \circ dR_{\phi(g_2^{-1})} \\
&= d\Psi_{\phi(g_1)} \circ dR_{\phi(g_2^{-1})}.
\end{aligned}$$

But, (8) is exactly equivalent to the cocycle equation

$$\delta(w)(g_1 g_2) = Ad_{\phi(g_1)}(w(g_2)) - w(g_1 g_2) + w(g_1) = 0.$$

Therefore, w is indeed a cocycle. □

Recall the following well-known fact from Lie group theory.

Lemma 3.2.9. *Let $h : I \to H$, $\varepsilon \mapsto h_\varepsilon$ be a smooth path in H starting at the identity $e = h_0$ of H. Then,*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_\varepsilon^{-1} = -\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_\varepsilon.$$

Proof. Let m denote the multiplication of H. Differentiating the identity $m(h_\varepsilon, h_\varepsilon^{-1}) = e$ with respect to ε at $\varepsilon = 0$, we get

$$
\begin{aligned}
& dm\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon^{-1}\right) = 0 \\
\Rightarrow \quad & dR_e\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon\right) + dL_e\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon^{-1}\right) = 0 \quad \text{by (7)} \\
\Rightarrow \quad & \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon^{-1} = 0
\end{aligned}
$$

and hence the result. $\square$

Remark 3.2.10. *Two equivalent deformations of ϕ give rise to two differentiable cocycles whose resulting cohomology classes are equal in $H^1(G, \mathfrak{h})$.*

Proof. Let $\tilde{\phi} = \{\phi_\varepsilon\}_{\varepsilon\in I}$ and $\tilde{\phi}' = \{\phi'_\varepsilon\}_{\varepsilon\in I}$ be two equivalent deformations of ϕ and let w and w' be their associated differentiable 1-cocycles respectively. Then, there exists a smooth curve $h : I \to H,\ \varepsilon \mapsto h_\varepsilon$ starting at the identity e of H such that $\Psi_{h_\varepsilon} \circ \phi_\varepsilon = \phi'_\varepsilon$. That is,

$$
\phi_\varepsilon(g)h_\varepsilon^{-1} = h_\varepsilon^{-1}\phi'_\varepsilon(g) \quad \forall\, g \in G. \tag{9}
$$

Now, let $X \in C^0(G, \mathfrak{h}) = \mathfrak{h}$ be a differentiable 0-cochain defined by

$$
X := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon \ \in T_eH \cong \mathfrak{h}.
$$

Let m denote the multiplication of H. Differentiating (9) with respect to ε at $\varepsilon = 0$, we get

$$
\begin{aligned}
& \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m(\phi_\varepsilon(g), h_\varepsilon^{-1}) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m(h_\varepsilon^{-1}, \phi'_\varepsilon(g)) \\
\Leftrightarrow \quad & dm\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon(g), \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon^{-1}\right) = dm\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon^{-1}, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi'_\varepsilon(g)\right) \\
\Leftrightarrow \quad & dR_e\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon(g)\right) + dL_{\phi(g)}(-X) = dR_{\phi(g)}(-X) + dL_e\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi'_\varepsilon(g)\right)
\end{aligned}
$$

by using the identity (7) and Lemma 3.2.9. Applying $dR_{\phi(g^{-1})}$ both sides, we get

$$
\begin{aligned}
dR_{\phi(g^{-1})}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon(g)\right) + dR_{\phi(g^{-1})} \circ dL_{\phi(g)}(-X) &= (-X) + dR_{\phi(g^{-1})}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi'_\varepsilon(g)\right) \\
\Leftrightarrow \quad -w(g) - \mathrm{Ad}_{\phi(g)}(X) &= -X - w'(g) \\
\Leftrightarrow \quad w'(g) - w(g) &= \mathrm{Ad}_{\phi(g)}(X) - X \\
&= \delta(X)(g).
\end{aligned}
$$

Thus,

$$
[w] = [w'] \ \in \ H^1(G, \mathfrak{h}).
$$

$\square$

3.3 Rigidity results

Geometric objects, structures or operations are called **rigid** if any attempt of deforming them leads to trivial deformations. The following theorem from [23] by Palais and Stewart shows the rigidity of actions of compact Lie groups on compact smooth manifolds. We prove the statement mainly by the steps done in [23], providing more details.

Theorem 3.3.1. *Let G be a compact Lie group acting by φ on a compact smooth manifold M. Then, any smooth deformation of φ is a trivial deformation.*

Proof. Let $\{\varphi_\varepsilon\}_{\varepsilon\in I}$ be a smooth deformation of φ. Our aim is to show that this is a trivial deformation, i.e. $\varphi_\varepsilon(g,x) = \psi_\varepsilon(\varphi(g,\psi_\varepsilon^{-1}(x)))$ holds $\forall\ g \in G,\ x \in M,\ \varepsilon \in I$ and for some deformation $\{\psi_\varepsilon\}_{\varepsilon\in I}$ of M.

As a first step, consider the following smooth action of G on $M \times I$ induced by the members φ_ε:

$$\phi : G \times (M \times I) \longrightarrow (M \times I), \quad (g,(x,\varepsilon)) \mapsto g \cdot (x,\varepsilon) := (\varphi_\varepsilon(g,x),\varepsilon).$$

This is indeed a smooth action since φ_ε are smooth actions $\forall\ \varepsilon \in I$.

Let $\phi_g : M \times I \to M \times I, (x,\varepsilon) \mapsto \phi(g,(x,\varepsilon))$ be the corresponding diffeomorphisms for all $g \in G$, and let $\pi : M \times I \to I$ be the projection on I. Then, $\pi \circ \phi_g = \pi$ for all $g \in G$ since $\pi(\phi_g(x,\varepsilon)) = \pi(\phi(g,(x,\varepsilon)) = \pi(\varphi_\varepsilon(g,x),\varepsilon) = \varepsilon = \pi(x,\varepsilon)$ for all $(x,\varepsilon) \in M \times I$, and hence

$$d\pi \circ d\phi_g = d\pi \quad \forall\ g \in G \tag{10}$$

as the differential commutes with composition.

As a second step, we aim to get a smooth time-dependent vector field on M, which will in turn generate a smooth deformation of M since M is compact. Let $V \in \mathfrak{X}(M \times I)$ be the vector field on $M \times I$ defined as $V_{(x,\varepsilon)} := (0_x, \frac{\partial}{\partial\varepsilon})$ for all $(x,\varepsilon) \in M \times I$. We will use the notation $\frac{\partial}{\partial\varepsilon} =: D_\varepsilon$.

Now, due to the compactness of G, we apply some averaging techniques by using the Haar measure μ on G (for details about averaging techniques and the Haar measure one may refer to any basic book about the topic, such as [31]). More precisely, to average V over G, we construct the vector field V^* on $M \times I$ defined by:

$$V^*_{(x,\varepsilon)} := \int_G d\phi_g\left(V_{\phi_g^{-1}(x,\varepsilon)}\right) d\mu(g).$$

Smoothness follows from that of ϕ. Moreover, V^* satisfies the invariance property, i.e.

$$d\phi_g\left(V^*_{(x,\varepsilon)}\right) = V^*_{\phi_g(x,\varepsilon)} \quad \forall\ g \in G,\ x \in M,\ \varepsilon \in I \tag{11}$$

since

$$d\phi_g \int_G d\phi_h\left(V_{\phi_h^{-1}(x,\varepsilon)}\right) d\mu(h) = d\phi_g \int_G d\phi_{h^{-1}h}\left(V_{\phi^{-1}_{h^{-1}h}(x,\varepsilon)}\right) d\mu(h) = \int_G d\phi_h\left(V_{(x,\varepsilon)}\right) d\mu(h)$$

by the property of invariance of Haar measure. Next, we show that V^* induces a smooth time-dependent vector field on M. To do that, we note:

$$d\pi\left(V_{\phi_g^{-1}(x,\varepsilon)}\right) = D_\varepsilon \quad \forall\ g \in G,\ x \in M,\ \varepsilon \in I \tag{12}$$

since $d\pi\left(V_{\phi_g^{-1}(x,\varepsilon)}\right) = d\pi\left(V_{(\varphi_\varepsilon(g^{-1},x),\varepsilon)}\right) = d\pi\left(0_{\varphi_\varepsilon(g^{-1},x)}, D_\varepsilon\right) = D_\varepsilon$, where π is the projection map. Putting (10) and (12), we get

$$d\pi\left(V^*_{(x,\varepsilon)}\right) = d\pi \int_G d\phi_g\left(V_{\phi_g^{-1}(x,\varepsilon)}\right) d\mu(g) = \int_G d\pi\left(V_{\phi_g^{-1}(x,\varepsilon)}\right) d\mu(g) = \int_G D_\varepsilon d\mu(g) = D_\varepsilon$$

and therefore

$$V^*_{(x,\varepsilon)} = (X(\varepsilon)_x, D_\varepsilon)$$

for some time-dependent vector field $X = \{X(\varepsilon)\}_{\varepsilon \in I}$ on M.

By Proposition 1.4.6, its corollaries and due to the compactness of M, X generates a smooth deformation $\{\psi_\varepsilon\}_{\varepsilon \in I}$ of M, which is given by $\psi_\varepsilon(x) = \gamma_x(\varepsilon)\ \forall\ \varepsilon \in I,\ x \in M$ for γ_x representing the unique maximal integral curve of X starting at x.

This is equivalent to saying that $\tau_{(x,0)} : I \to (M \times I),\ \varepsilon \mapsto (\gamma_x(\varepsilon), \varepsilon)$ is the integral curve of V^* starting at $(x, 0)$.

By the invariance property (11), $\varepsilon \mapsto \phi_g(\gamma_x(\varepsilon), \varepsilon) = \phi_g(\psi_\varepsilon(x), \varepsilon)$ is the integral curve of V^* starting at $\phi_g(x, 0) = \phi(g, (x, 0)) = (\varphi_0(g, x), 0) = (\varphi(g, x), 0)$.

But, by definition $\tau_{(\varphi(g,x),0)} : I \to (M \times I),\ \varepsilon \mapsto \big(\gamma_{\varphi(g,x)}(\varepsilon), \varepsilon\big) = (\psi_\varepsilon(\varphi(g, x)), \varepsilon)$ is the integral curve of V^* starting at $(\varphi(g, x), 0)$. Hence,

$$\phi_g(\psi_\varepsilon(x), \varepsilon) = (\psi_\varepsilon(\varphi(g, x)), \varepsilon) \quad \forall\ g \in G,\ x \in M,\ \varepsilon \in I$$

by uniqueness of integral curves. Now, $\phi_g(\psi_\varepsilon(x), \varepsilon) = \phi(g, (\psi_\varepsilon(x), \varepsilon)) = (\varphi_\varepsilon(g, \psi_\varepsilon(x)), \varepsilon)$ implies that

$$\varphi_\varepsilon(g, \psi_\varepsilon(x)) = \psi_\varepsilon(\varphi(g, x)) \quad \forall\ g \in G,\ x \in M,\ \varepsilon \in I.$$

In particular, for $\psi_\varepsilon^{-1}(x) \in M$, we get the desired result:

$$\varphi_\varepsilon(g, x) = \psi_\varepsilon(\varphi(g, \psi_\varepsilon^{-1}(x))) \quad \forall\ g \in G,\ x \in M,\ \varepsilon \in I.$$

□

It is natural to ask if this result still holds if the condition of compactness of G or M is dropped. There are counterexamples in the literature which clarify that if G or M is not compact, we would not necessarily get trivial deformations. Here is a counterexample from [9, Examples 5.1.2(b)], also mentioned in [23].

Example 3.3.2. Let the real line $\mathbb{R}$ act on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ by

$$\varphi : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2, \quad \varphi\left(x, \left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}\right)\right) := \left(e^{2\pi i(\theta_1 + x)}, e^{2\pi i\theta_2}\right).$$

Then, the deformation $\{\varphi_\varepsilon\}_{\varepsilon \in I}$ of φ given by

$$\varphi_\varepsilon\left(x, \left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}\right)\right) := \left(e^{2\pi i(\theta_1 + x)}, e^{2\pi i(\theta_2 + \varepsilon x)}\right)$$

is a non-trivial deformation since the topology of the orbits varies with respect to ε.

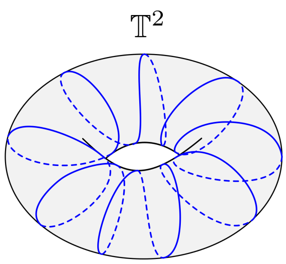

Figure 5: Orbits of the action φ_ε, for $\varepsilon \in \mathbb{Q}$

Observe that the orbit space of φ_ε has the indiscrete (trivial) topology if and only if ε is irrational. If ε is rational, the orbits look like the curve depicted in Figure 5.

The formulation of the next counterexample has been inspired by [18, Remark 73] and [9, Example 5.2.2].

Example 3.3.3. Firstly, it has been proved by McMillan [17] that there exist uncountably many topologically distinct open manifolds W_α of dimension 3 such that:

- $W_\alpha \times \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^{n+3}$ for all $n > 1$ and for each α
- W_α is not diffeomorphic to $\mathbb{R}^3$ for each α.

Next, let G be a compact Lie group and let $\psi : G \to \mathrm{GL}(\mathbb{R}^n)$ be a faithful representation (i.e. ψ injective) of G on $\mathbb{R}^n$, for $n > 1$. The existence of such representations is guaranteed by known results (see [31, p.70, Theorem 1]). Note that the associated smooth linear action given by $g \cdot v = \psi(g)(v)$ has only 0 as a fixed point.

We now construct an action of the compact Lie group G on the non-compact space $\mathbb{R}^{n+3} \cong W_\alpha \times \mathbb{R}^n$ referring to [22, Section 5] by Palais and Richardson. For each α, define the smooth action by

$$\varphi : G \times (W_\alpha \times \mathbb{R}^n) \longrightarrow W_\alpha \times \mathbb{R}^n, \quad \varphi(g, (w, v)) := (w, \psi(g)(v)).$$

Observe that the set of fixed points of φ is precisely $W_\alpha \times \{0\} \cong W_\alpha$.

By [25], any smooth action of a Lie group on a Euclidean space with at least one fixed point can be smoothly deformed into a linear action. Specifically, consider the above defined action $\varphi : G \times \mathbb{R}^{n+3} \to \mathbb{R}^{n+3}$ with $0 \in \mathbb{R}^{n+3}$ as a fixed point, and linearize it at 0 by the following deformation:

$$\varphi_\varepsilon : G \times \mathbb{R}^{n+3} \to \mathbb{R}^{n+3}, \quad \varphi_\varepsilon(g, v) := \frac{1}{\varepsilon}\varphi(g, \varepsilon v), \quad \varphi_0 := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\varphi(g, \varepsilon v).$$

Observe that the set of fixed points of φ_0 is $\mathbb{R}^3$ which is not diffeomorphic to W_α, which is the set of fixed points of each φ_ε, $\forall\ \varepsilon > 0$. Hence, we found a deformation of φ which is non-trivial.

Another desirable result yielded by compactness is the one established in [21]. Namely, there are at most countably many inequivalent smooth actions of a compact Lie group G on a compact smooth manifold M. The assumption of compactness of M is shown to be a necessary condition by the authors in [22], which actually follows from Example 3.3.3.

From an infinitesimal perspective, the authors in [24] have proved the following rigidity theorem [24, p. 638], where the compactness of M can be dropped, but the property of semi-simpleness of G is added. Recall that a Lie group is called semi-simple if its corresponding Lie algebra is semi-simple.

Theorem 3.3.4. *Let G be a compact semi-simple Lie group acting by φ on a smooth manifold M. Then, every infinitesimal deformation of φ is trivial.*

This theorem is a direct consequence of some results proved in [24] which make use of the cohomology theory of Lie algebras. Mainly, one can show that the triviality of every infinitesimal deformation of an action of G on M is equivalent to the vanishing of the first order cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{X}(M)$ viewed as a $\mathfrak{g}$-module.

The next rigidity theorem was mentioned in the paper [20, p. 178] by Nijenhuis and Richardson, where the proof is omitted and only referred to the paper [27, p. 152] by Weil for a similar proof in the discrete case. We prove the statement in this thesis by using our construction of the differentiable 1-cocycle associated to deformations of Lie group homomorphisms as explained in section 3.2.

Theorem 3.3.5. *Let G and H be connected Lie groups and let $\phi : G \longrightarrow H$ be a Lie group homomorphism. Consider the representation of G on the Lie algebra $\mathfrak{h}$ of H given by $G \to \mathrm{Aut}(\mathfrak{h}),\ g \mapsto \mathrm{Ad}_{\phi(g)}$.*

If $H^1(G, \mathfrak{h}) = 0$, then every smooth deformation of ϕ is locally trivial.

Proof. Let $\tilde{\phi} = \{\phi_\varepsilon\}_{\varepsilon \in I}$ be a smooth deformation of ϕ. Our aim is to show that there exists a smooth curve

$$h : I \to H, \quad \varepsilon \mapsto h_\varepsilon$$

starting at the identity e of H such that for ε small enough

$$h_\varepsilon \phi(g) h_\varepsilon^{-1} = \phi_\varepsilon(g), \quad \forall\, g \in G. \tag{13}$$

First of all, let us consider the differentiable 1-cocycle $w \in C^1(G, \mathfrak{h})$ associated to the deformation $\tilde{\phi}$, and the resulting cohomology class $[w] \in H^1(G, \mathfrak{h})$. Due to the vanishing of $H^1(G, \mathfrak{h})$, there exists an element $X \in C^0(G, \mathfrak{h}) = \mathfrak{h}$ with

$$w(g) = \delta(X)(g) \quad \forall\, g \in G$$

$$\Leftrightarrow \quad -dR_{\phi(g^{-1})}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon(g) \right) = \mathrm{Ad}_{\phi(g)}(X) - X, \quad \forall\, g \in G. \tag{14}$$

Next, we attempt to find the smooth curve $\varepsilon \mapsto h_\varepsilon$ in H such that (13) is satisfied. Now, since $X \in \mathfrak{h}$ is a left-invariant vector field on H, let us consider the unique maximal integral curve γ_X of X starting at the identity e of H:

$$\gamma_X : I \longrightarrow H, \quad \gamma_X(0) = e.$$

Note that $\gamma_X(\varepsilon)$ is exactly given by $\exp(\varepsilon X)$ for all $\varepsilon \in I$ (definition of the exponential map was recalled in section 3.1). Define

$$h_\varepsilon := \gamma_X(\varepsilon) = \exp(\varepsilon X), \quad \forall\, \varepsilon \in I$$

with the defining equation for integral curves

$$\frac{d}{d\varepsilon} h_\varepsilon = X_{h_\varepsilon}, \quad \forall\, \varepsilon \in I$$

and so at $\varepsilon = 0$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon = X_e = X.$$

Lastly, to show that (13) holds with our particular h_ε, we prove that the differential with respect to ε of

$$h_\varepsilon^{-1} \phi_\varepsilon(g) h_\varepsilon \phi(g^{-1})$$

at $\varepsilon = 0$ vanishes, as shown below.

Let m denote the multiplication of the group H. For all $g \in G$, we have

$$
\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m\left(m\left(h_\varepsilon^{-1},\phi_\varepsilon(g)\right),m\left(h_\varepsilon,\phi(g^{-1})\right)\right)\\
&= dm\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m\left(h_\varepsilon^{-1},\phi_\varepsilon(g)\right),\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m\left(h_\varepsilon,\phi(g^{-1})\right)\right)\\
&= dm\left(dm\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon^{-1},\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\phi_\varepsilon(g)\right),dm\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_\varepsilon,\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\phi(g^{-1})\right)\right)\\
&= dm\left(dm\left(-X,\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\phi_\varepsilon(g)\right),dm\left(X,0_{\phi(g^{-1})}\right)\right)\\
&= dm\left(dR_{\phi(g)}(-X)+dL_e\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\phi_\varepsilon(g)\right),dR_{\phi(g^{-1})}(X)\right)\\
&= dR_{\phi(g^{-1})}\circ dR_{\phi(g)}(-X)+dR_{\phi(g^{-1})}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\phi_\varepsilon(g)\right)+dL_{\phi(g)}\circ dR_{\phi(g^{-1})}(X)\\
&= -X-w(g)+\mathrm{Ad}_{\phi(g)}(X)\\
&= 0
\end{aligned}
$$

by using the identity (7), Lemma 3.2.9 and the cocycle equation (14). Therefore, for all g in G, we get that

$$h_\varepsilon^{-1}\phi_\varepsilon(g)h_\varepsilon\phi(g^{-1})=e,\quad \text{for } \varepsilon \text{ small enough}$$

and hence

$$h_\varepsilon\phi(g)h_\varepsilon^{-1}=\phi_\varepsilon(g),\quad \text{for } \varepsilon \text{ small enough}$$

which implies that the deformation $\tilde{\phi}$ of ϕ is locally trivial. □

Remark 3.3.6. *It seems likely that Theorem 3.3.5 still holds even if the condition of connectedness of G and H is dropped. We keep the condition, for the statement to be the same as the one mentioned in [20].*

The authors in [20] provide another approach to rigidity of Lie group homomorphisms. Here, we give an overview of these ideas.

Consider the set $\mathfrak{R}$ of all Lie group homomorphisms from G to H

$$\mathfrak{R}=\{\phi:G\to H\mid \phi \text{ is a Lie group homomorphism}\}$$

and equip it with the compact-open topology. In addition, let the group H act on the space $\mathfrak{R}$ by

$$H\times\mathfrak{R}\to\mathfrak{R},\quad h\cdot\phi:=\Psi_h\circ\phi$$

where Ψ_h is the group automorphism defined through conjugation. Then, the orbit $\mathrm{Orb}(\phi)$ of $\phi\in\mathfrak{R}$ is precisely composed of all Lie group homomorphisms which are equivalent to ϕ in the sense of Definition 3.2.3 since

$$
\begin{aligned}
\mathrm{Orb}(\phi)&=\{\phi'\in\mathfrak{R}\mid \exists\, h\in H \text{ with } h\cdot\phi=\phi'\}\\
&=\{\phi'\in\mathfrak{R}\mid \exists\, h\in H \text{ with } \Psi_h\circ\phi=\phi'\}.
\end{aligned}
$$

Having this in mind, the Lie group homomorphism ϕ is said to be rigid if its orbit $\mathrm{Orb}(\phi)$ is an open subset of the space $\mathfrak{R}$ as defined by the authors in [20]. The natural question would then be if this view on rigidity and our notion of rigidity in the sense of Definition 3.2.4 are equivalent, and if not, which one implies the other.

4 Representations of groupoids

Representation theory of Lie groupoids is a relatively recent theme which attempts to generalize the notion of representations of Lie groups. Due to their subtle nature, groupoids have only a few canonical representations. In sections 4.1.1 and 4.1.2, we will introduce two natural representations of a regular groupoid, in particular the isotropy and the normal representations, and study their effect on the deformation cohomology in low degrees. The results mentioned by Crainic, Mestre and Struchiner in [7] will be crucial in our study and hence mentioned with further details. In addition, sections 4.2.1-4.2.3 will mainly deal with the notion of representations up to homotopy, which were first introduced and studied by Arias Abad and Crainic in [2]. As a conclusion, we will explore the generalization of the adjoint representation to the case of Lie groupoids as a main example of representations up to homotopy and look into its relation to the deformation cohomology.

4.1 Representations of regular groupoids

First of all, recall that a representation of a Lie groupoid $G \rightrightarrows M$ is a vector bundle $\pi : E \to M$, together with a linear action of G on E. Moreover, a Lie groupoid is said to be regular if all its orbits have the same dimension. Sections 4.1.1 and 4.1.2 mainly refer to [7], which is further elaborated in [18].

4.1.1 Isotropy representation

In this subsection, the isotropy representation of a regular groupoid will be formally defined as the kernel of the anchor map of the Lie algebroid associated to the groupoid. This representation will lead to some important results of the deformation cohomology in degree zero. Firstly, recall from [14, Theorem 10.34] that the kernel of a smooth vector bundle map is a vector subbundle if and only if the given vector bundle map has constant rank.

Let $G \rightrightarrows M$ be a Lie groupoid with associated Lie algebroid $\mathrm{Lie}(G) = A$. Define the isotropy bundle as

$$\mathfrak{i} := \ker(\rho) = \bigcup_{x \in M} \ker(\rho_x) \subset A$$

where $\rho : A \longrightarrow TM$ is the anchor of A. If G is a regular groupoid, this is indeed a vector subbundle of A since ρ, as a vector bundle map, has constant rank. This is due to the fact that $\rho(A_x) = T_x \,\mathrm{Orb}(x) \;\forall\, x \in M$. If G is not a regular groupoid, then $\mathfrak{i}$ is not a subbundle of A. Nonetheless, one can still talk about its fibers and space of smooth sections. In the general case, one can show that the fibers $\mathfrak{i}_x$ of $\mathfrak{i}$ over x are precisely the Lie algebras of the isotropy groups G_x.

Next, we define the action of G on $\mathfrak{i}$. Consider the conjugation, usually called the adjoint action, of $g : y \leftarrow x$ in G, on the isotropy group.

$$\Psi_g : G_x \longrightarrow G_y, \quad k \mapsto gkg^{-1}$$

which is equivalent to the composition of right and left translations

$$\begin{array}{ccccc} G_x & \xrightarrow{\;L_g\;} & s^{-1}(x) \cap t^{-1}(y) & \xrightarrow{\;\;R_{g^{-1}}\;\;} & G_y \\ k & \longmapsto & gk & \longmapsto & gkg^{-1} = (R_{g^{-1}} \circ L_g)(k). \end{array}$$

Note that the multiplication gkg^{-1} makes sense for $k \in G_x$ and that gkg^{-1} lies indeed in G_y. The action of G on $\mathfrak{i}$ is now defined to be the differentiation of the adjoint action at the units of G. That is, for all $g : y \curvearrowleft x$ in G, the action is given by

$$d(\Psi_g)|_{u(x)} =: \mathrm{ad}_g : T_{u(x)}G_x \longrightarrow T_{u(y)}G_y, \quad v \mapsto (dR_{g^{-1}} \circ dL_g)(v).$$

Using the fact that $T_{u(x)}G_x = \mathrm{Lie}(G_x) = \mathfrak{i}_x$, this is equivalent to

$$\mathrm{ad}_g : \mathfrak{i}_x \longrightarrow \mathfrak{i}_y, \quad v \mapsto g \cdot v = (dR_{g^{-1}} \circ dL_g)(v)$$

which are linear isomorphisms, since the right and left translations are local diffeomorphisms. If G is regular, then the isotropy bundle together with this action gives rise to a representation of G, called the **isotropy representation**.

More generally, when G is not necessarily regular and where $\mathfrak{i}$ is not necessarily a representation of G, define the sections of $\mathfrak{i}$ by $\Gamma(\mathfrak{i}) := \ker(\rho : \Gamma(A) \to \mathfrak{X}(M))$, where smoothness is defined by viewing them as smooth sections of A.

Remark 4.1.1. *If G is a regular groupoid over M, then $\Gamma(\mathfrak{i}) = \ker(\rho : \Gamma(A) \to \mathfrak{X}(M))$ holds.*

Proof. "$\subseteq$": Let $\sigma \in \Gamma(\mathfrak{i})$. Then, $\sigma \in \Gamma(A)$ with $\rho(\sigma)(x) = \rho(\sigma(x)) = 0\ \forall\ x \in M$ since $\sigma(x) \in \mathfrak{i} = ker(\rho : A \to TM)$. Thus, $\sigma \in \ker(\rho : \Gamma(A) \to \mathfrak{X}(M))$.

"$\supseteq$": Similarly, let $\sigma \in \ker(\rho : \Gamma(A) \to \mathfrak{X}(M))$. Then, $\sigma \in \Gamma(A)$ s.t. $\rho(\sigma) \equiv 0$ and so $\rho(\sigma(x)) = \rho(\sigma)(x) = 0\ \forall\ x \in M$. Hence $\sigma(x) \in \mathfrak{i} = \ker(\rho : A \to TM)$. □

Hence, for any Lie groupoid $G \rightrightarrows M$, talking about the space of smooth sections of $\mathfrak{i}$ makes sense. Recall that the differentiable cohomology of G with coefficients in a representation E of G in degree zero is exactly the space of smooth sections of E, which are invariant under the action of G. In the general case here, let

$$H^0(G, \mathfrak{i}) = \Gamma(\mathfrak{i})^{\mathrm{inv}}$$

where $\Gamma(\mathfrak{i})^{\mathrm{inv}} := \big\{\alpha \in \Gamma(\mathfrak{i}) \mid g \cdot \alpha_{s(g)} = \alpha_{t(g)} \quad \forall\ g \in G\big\}$.

Proposition 4.1.2. *$H^0_{\mathrm{def}}(G) \cong H^0(G, \mathfrak{i})$ for any Lie groupoid $G \rightrightarrows M$.*

Proof. Recall that $H^0_{\mathrm{def}}(G) = \{\alpha \in \Gamma(A) \mid \overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g = 0\ \forall\ g \in G\}$. Let $\alpha \in \Gamma(A)$. It is required to show that for all g in G, $\overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g = 0$ holds if and only if $\rho(\alpha) = 0$ and $g \cdot \alpha_{s(g)} = \alpha_{t(g)}$. First of all, we get that

$$\begin{aligned} dt(\overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g) &= 0 \\ \Rightarrow dt(\overrightarrow{\alpha}_g) + \cancel{dt(\overleftarrow{\alpha}_g)} &= 0 \\ \Rightarrow dt(\overrightarrow{\alpha}_g) &= 0 \end{aligned}$$

for all g in G, as the differential is linear and $\overleftarrow{\alpha}$ is left-invariant. Restricting dt to A, we get $dt|_A(\overrightarrow{\alpha}_{u(x)}) = dt|_A(\alpha_x) = 0\ \forall\ x \in M$ and since $\rho = dt|_A$, this implies that $\rho(\alpha) = 0$. Now, from the relations $\rho(\alpha_x) = \alpha_x + di(\alpha_x)$ and $\rho(\alpha_x) = 0\ \forall\ x \in M$, we get that $di(\alpha_x) = -\alpha_x$ $\forall\ x \in M$. Lastly, for all $g : y \curvearrowleft x \in G$:

$$\begin{aligned} \overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g &= 0 \\ \Leftrightarrow dR_g\alpha_y + dL_g di\alpha_x &= 0 \\ \Leftrightarrow dR_g\alpha_y - dL_g\alpha_x &= 0 \\ \Leftrightarrow \alpha_y &= dR_{g^{-1}}dL_g\alpha_x \\ \Leftrightarrow \alpha_y &= g \cdot \alpha_x. \end{aligned}$$

□

In higher degrees, the differentiable cohomology $H^*(G, \mathfrak{i})$ with coefficients in $\mathfrak{i}$ can still be understood for any Lie groupoid G. The k-cocycles $w \in C^k(G, \mathfrak{i})$ can still be defined as beforehand. Similar to the sections, smoothness is defined by viewing the cochains as having values in A. Consider the following inclusion:

$$r : C^k(G, \mathfrak{i}) \hookrightarrow C^k_{\text{def}}(G)$$

defined by $r(w)(g_1, ..., g_k) := dR_{g_1}(w(g_1, ..., g_k)) \in T_{g_1}G$ for all $w \in C^k(G, \mathfrak{i})$. This inclusion gives in fact an identification of $C^*(G, \mathfrak{i})$ to a subcomplex of $C^*_{\text{def}}(G)$ in the following way

$$C^k(G, \mathfrak{i}) \leftrightarrow \left\{ c \in C^k_{\text{def}}(G) \mid c(g_1, ..., g_k) \in \ker ds|_{g_1} \cap \ker dt|_{g_1} \ \forall \ (g_1, ..., g_k) \in G^{(k)} \right\}.$$

Remark 4.1.3. *The map r commutes with the differentials. That is, the following diagram commutes:*

$$\begin{tikzcd} C^k(G, \mathfrak{i}) \arrow[r, "\delta"] \arrow[d, "r"'] & C^{k+1}(G, \mathfrak{i}) \arrow[d, "r"] \\ C^k_{\text{def}}(G) \arrow[r, "\delta"] & C^{k+1}_{\text{def}}(G) \end{tikzcd}$$

The interchangeable usage of δ as the differential in the differentiable or deformation cases should be understood from the context.

Proposition 4.1.4. *The induced map $r : H^1(G, \mathfrak{i}) \longrightarrow H^1_{\text{def}}(G)$, $[w] \mapsto [r(w)]$, is injective, where $[w]$ is the cohomology class of the cocycle $w \in C^1(G, \mathfrak{i})$.*

Proof. Let $[w] \in \ker(r)$. That means, w is a differentiable 1-cocycle with $[r(w)] = 0$ in $H^1_{\text{def}}(G)$, and so

$$r(w) = \delta(\alpha) \tag{15}$$

for some $\alpha \in C^0_{\text{def}}(G) = \Gamma(A)$. Applying dt to (15), we get $0 = dt(\overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g) = dt(\overrightarrow{\alpha}_g)$ for all $g \in G$, since $r(w)$ is in the subcomplex of $C^1_{\text{def}}(G)$ where it is killed by ds and dt, and since $\overleftarrow{\alpha}$ is left-invariant. Restricting dt to A, we get that $\rho(\alpha) = 0$ and hence $\alpha \in \Gamma(\mathfrak{i})$. For any $g : y \curvearrowleft x$ in G, it holds:

$$\begin{aligned} r(w)(g) &= \delta(\alpha)(g) \\ dR_g(w(g)) &= \overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g \\ dR_g(w(g)) &= dR_g\alpha_y + dL_g di\alpha_x \\ w(g) &= \alpha_y + (dR_{g^{-1}} \circ dL_g)(di\alpha_x) \\ w(g) &= \alpha_y - (dR_{g^{-1}} \circ dL_g)(\alpha_x) \quad \text{using } di(\alpha_x) = -\alpha_x \\ w(g) &= \alpha_y - g \cdot \alpha_x \\ w(g) &= -\delta(\alpha)(g) \end{aligned}$$

for $\alpha \in \Gamma(\mathfrak{i}) = C^0(G, \mathfrak{i})$. As a conclusion, we get that w is in $\text{im}(\delta)$ and so its cohomology class $[w] = 0$ in $H^1(G, \mathfrak{i})$, and hence injectivity of r. $\square$

However, it is not in general true that the induced map $r : H^*(G, \mathfrak{i}) \rightarrow H^*_{\text{def}}(G)$ is injective for higher degrees.

4.1.2 Normal representation

Another significant representation of a regular groupoid is the normal representation, which will be defined as the cokernel of the anchor of the Lie algebroid associated to the Lie groupoid. Similar to the isotropy representation, the normal one will have important consequences on the deformation cohomology in lower degrees, especially in degree one. Recall from [14, Theorem 10.34] that the image of a smooth vector bundle map is a vector subbundle if and only if the given vector bundle map has constant rank.

Let $G \rightrightarrows M$ be a Lie groupoid and A be its Lie algebroid with anchor ρ. Define the normal bundle as

$$\mathfrak{v} := \mathrm{coker}(\rho) = TM/\rho(A).$$

If G is regular and hence ρ has constant rank, this is the quotient bundle of the tangent bundle mod the image of ρ. In the general case where G is not necessarily regular, we can still make sense of the fibers and space of smooth sections of $\mathfrak{v}$. However, this will be harder than the isotropy case because of the existence of a quotient.

In order to define the action of G on $\mathfrak{v}$, we let γ be a smooth path in G starting at $\gamma(0) = g$ for $g : y \curvearrowleft x \in G$. Now, $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} s(\gamma(\varepsilon))$ is a tangent vector to M at $s(g) = x$, and hence a representative of $[V] \in \mathfrak{v}_x$. We define the action as:

$$\mathrm{ad}_g : \mathfrak{v}_x \longrightarrow \mathfrak{v}_y,$$

$$[V] \mapsto g \cdot [V] = g \cdot \left[\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} s(\gamma(\varepsilon)) \right] := \left[\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} t(\gamma(\varepsilon)) \right].$$

Note that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} t(\gamma(\varepsilon))$ is a tangent vector to M at $t(g) = y$ and hence a representative of $g \cdot [V] \in \mathfrak{v}_y$.

Equivalently, for $g \in G$ and $[V] \in \mathfrak{v}_{s(g)}$, the action is defined as

$$g \cdot [V] := [dt(X)] \ \in \mathfrak{v}_{t(g)}$$

for $X \in \mathfrak{X}(G)$ being any s-lift of V.

Remark 4.1.5. *The action defined above is independent of the choices made and hence well-defined.*

If G is regular, then the normal bundle with the above defined action is a representation of G, called the **normal representation**. Similar to $\mathfrak{i}$, we consider the more general case, where G is not necessarily regular. Define the space of smooth sections of $\mathfrak{v}$ by

$$\Gamma(\mathfrak{v}) := \mathfrak{X}(M)/\mathrm{im}(\rho : \Gamma(A) \to \mathfrak{X}(M)).$$

To define the smooth invariant sections of $\mathfrak{v}$, let $[V] \in \Gamma(\mathfrak{v})$ such that $g \cdot [V_{s(g)}] = [V_{t(g)}]$ holds $\forall\ g \in G$. Note that this condition is equivalent to the fact that for any s-lift $X \in \mathfrak{X}(G)$ of V, the class of $V_{t(g)}$ is equal to the class of $dt(X_g)$ mod $\rho(A)$. This leads to the following definition of the space of smooth invariant sections of $\mathfrak{v}$:

$$\Gamma(\mathfrak{v})^{\mathrm{inv}} := \{[V] \in \Gamma(\mathfrak{v}) \mid \exists\ X \in \mathfrak{X}(G) \quad \text{which is } s\text{- and } t\text{-projectable to } V\}$$

and we let $H^0(G, \mathfrak{v}) = \Gamma(\mathfrak{v})^{\mathrm{inv}}$. Here, one can also check that in the case of a regular groupoid, these are just the usual definitions of smooth sections as well as smooth invariant sections.

Next, we show that there is a natural linear map between the deformation cohomology of G in degree one and the space of smooth invariant sections of the normal bundle. The main tool to do this is to describe the degree one deformation cohomology in terms of special types of multiplicative vector fields.

Given $\alpha \in \Gamma(A)$, vector fields of the form $\overrightarrow{\alpha} + \overleftarrow{\alpha} : G \longrightarrow TG,\ g \mapsto \overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g$ are called **inner multiplicative vector fields** on G over $\rho(\alpha)$. Denote the set of all inner multiplicative vector fields on G by $\mathfrak{X}_{\mathrm{inn}}(G)$.

Remark 4.1.6. *Inner multiplicative vector fields are indeed multiplicative.*

Proof. Firstly, note that the following diagram commutes

$$\begin{tikzcd}
G \arrow[r, "\overrightarrow{\alpha} + \overleftarrow{\alpha}"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & TG \arrow[d, shift right, "dt"'] \arrow[d, shift left, "ds"] \\
M \arrow[r, "\rho(\alpha)"] & TM
\end{tikzcd}$$

$$\begin{aligned}
ds(\overrightarrow{\alpha} + \overleftarrow{\alpha})(g) &= ds(\overrightarrow{\alpha}_g + \overleftarrow{\alpha}_g) \\
&= \cancel{ds(\overrightarrow{\alpha}_g)} + ds(\overleftarrow{\alpha}_g) \quad \text{since } \overrightarrow{\alpha} \text{ is right-invariant} \\
&= (ds \circ dL_g \circ di)(\alpha_{s(g)}) \\
&= dt(\alpha_{s(g)}) \\
&= \rho(\alpha_{s(g)}) \\
&= \rho(\alpha)(s(g))
\end{aligned}$$

for every g in G. Therefore,

$$ds \circ (\overrightarrow{\alpha} + \overleftarrow{\alpha}) = \rho(\alpha) \circ s.$$

By similar calculations and since $\overleftarrow{\alpha}$ is left-invariant, one gets

$$dt \circ (\overrightarrow{\alpha} + \overleftarrow{\alpha}) = \rho(\alpha) \circ t.$$

The commutativity of multiplication $(\overrightarrow{\alpha} + \overleftarrow{\alpha})_{gh} = dm((\overrightarrow{\alpha} + \overleftarrow{\alpha})_g, (\overrightarrow{\alpha} + \overleftarrow{\alpha})_h)$ follows from a step by step calculation by using for instance the formula of dm in terms of right and left translations corresponding to certain bisections (see [15, Theorem 1.4.14]). $\square$

Proposition 4.1.7. *The deformation cohomology of G in degree one is precisely the multiplicative vector fields on G modulo the inner multiplicative vector fields on G, i.e.*

$$H^1_{\mathrm{def}}(G) = \mathfrak{X}_{\mathrm{mult}}(G)/\mathfrak{X}_{\mathrm{inn}}(G).$$

Proof. Consider the coboundry maps of lower degrees:

$$C^0_{\mathrm{def}}(G) = \Gamma(A) \xrightarrow{\delta^0} C^1_{\mathrm{def}}(G) \xrightarrow{\delta^1} C^2_{\mathrm{def}}(G).$$

By definition, $H^1_{\mathrm{def}}(G) = \ker(\delta^1)/\operatorname{im}(\delta^0)$. Since

$$\operatorname{im}(\delta^0) = \left\{\delta^0(\alpha) \mid \alpha \in \Gamma(A)\right\} = \{\overrightarrow{\alpha} + \overleftarrow{\alpha} \mid \alpha \in \Gamma(A)\} = \mathfrak{X}_{\mathrm{inn}}(G),$$

it remains to show that a vector field $X \in \mathfrak{X}(G)$ being multiplicative is equivalent to X being a deformation 1-cocycle.

Let the pair (X, V) represent a multiplicative vector field of G. Hence, (X, V) is a morphism of groupoids between $G \rightrightarrows M$ and the induced tangent groupoid $TG \rightrightarrows TM$. This is equivalent to saying that $ds(X_g) = V_{s(g)}$ $\forall\ g \in G$ and $d\bar{m}(X_g, X_h) = X_{\bar{m}(g,h)}$ $\forall\ (g, h) \in G \times_s G$ by Proposition 1.2.8. Hence, X is s-projectable and using the second part with $g = g_1 g_2$ and $h = g_2$, we get

$$(\delta^1 X)(g_1, g_2) = -d\bar{m}(X_{g_1 g_2}, X_{g_2}) + (-1)^2 X_{g_1} = -d\bar{m}(X_{g_1 g_2}, X_{g_2}) + X_{\bar{m}(g_1 g_2, g_2)} = 0.$$

The reverse direction is easily obtained by similar steps. □

Now, we are ready to see the following result.

Proposition 4.1.8. *There exists a natural linear map*

$$\pi : H^1_{\text{def}}(G) \longrightarrow \Gamma(\mathfrak{v})^{\text{inv}}, \quad [X] \mapsto [V]$$

for $X \in \mathfrak{X}(G)$ a multiplicative vector field over the base field $V \in \mathfrak{X}(M)$.

Note that by the very definition of a multiplicative vector field, we know that $X \in \mathfrak{X}(G)$ is both s- and t-projectable to its base field V and hence $[V]$ is indeed an invariant section of $\mathfrak{v}$.

Proof. By straight forward calculation, we get for every (X, V) and (X', V') multiplicative vector fields

$$\begin{aligned} \pi([X] + [X']) &= \pi([X + X']) \\ &= [V + V'] \\ &= [V] + [V'] \\ &= \pi([X]) + \pi([X']) \end{aligned}$$

by using Remark 1.4.12. □

Next, we show that every invariant section of $\mathfrak{v}$ gives rise to a differentiable 2-cocycle with coefficients in $\mathfrak{i}$ by a linear map. First, consider the following lemma.

Lemma 4.1.9. *Let $[V] \in \Gamma(\mathfrak{v})^{\text{inv}}$ and let $X \in \mathfrak{X}(G)$ be an (s, t)-lift of V. Then:*

(i) $\delta(X) \in C^2(G, \mathfrak{i})$.

(ii) $[\delta(X)] \in H^2(G, \mathfrak{i})$ *does not depend on the choice of X.*

Proof. (i): $X \in \mathfrak{X}(G)$ is such that it is both s- and t-projectable and so $X \in C^1_{\text{def}}(G)$. So, $\delta(X) \in C^2_{\text{def}}(G)$. To prove that $\delta(X)$ is in $C^2(G, \mathfrak{i})$, recall the identification of $C^2(G, \mathfrak{i})$ with the subcomplex of $C^2_{\text{def}}(G)$ where the cochains are killed by ds and dt. For all $(g, h) \in G^{(2)}$

$$\begin{aligned} ds(\delta(X)(g, h)) &= ds(-d\bar{m}(X_{gh}, X_h) + X_g) \\ &= ds(-d\bar{m}(X_{gh}, X_h)) + ds(X_g) \\ &= -dt(X_h) + ds(X_g) \\ &= -V_{t(h)} + V_{s(g)} \quad (X \text{ is } s\text{- and } t\text{-projectable to } V) \\ &= 0 \quad (s(g) = t(h)). \end{aligned}$$

Similarly, we get that

$$\begin{aligned} dt(\delta(X)(g,h)) &= dt(-d\bar{m}(X_{gh}, X_h)) + dt(X_g) \\ &= -dt(X_g) + dt(X_g) \\ &= -V_{t(g)} + V_{t(g)} \\ &= 0 \end{aligned}$$

and hence $\delta(X)$ is killed both by ds and dt, which implies that it lies in $C^2(G, \mathfrak{i})$.

(ii): To show well-definedness of the cohomology class $[\delta(X)]$ induced by $[V]$, firstly let X' be another (s,t)-lift of V and let $c := X' - X$. Then, c lies in the subcomplex $C^1(G, \mathfrak{i})$ since $ds(c(g)) = ds(X'_g - X_g) = ds(X'_g) - ds(X_g) = V_{s(g)} - V_{s(g)} = 0$ and $dt(c(g)) = 0$ in the same way. Thus,

$$\delta(c) \in \operatorname{im}(\delta : C^1(G, \mathfrak{i}) \to C^2(G, \mathfrak{i}))$$

and so $[\delta(c) = 0]$ in $H^2(G, \mathfrak{i})$. Therefore, $[\delta(X')] = [\delta(X) + \delta(c)] = [\delta(X)]$ in $H^2(G, \mathfrak{i})$.

Secondly, let $V' \in \mathfrak{X}(M)$ be such that $[V] = [V'] \in \Gamma(\mathfrak{v}) = \mathfrak{X}(M)/\operatorname{im}(\rho)$. Then, $V' = V + \rho(\alpha)$ for some $\alpha \in \Gamma(A)$. Let X be an (s,t)-lift of V. Then, $X' := X + \overrightarrow{\alpha} + \overleftarrow{\alpha} = X + \delta(\alpha)$ is an (s,t)-lift of V'. Hence, $[\delta(X')] = [\delta(X) + \cancel{\delta(\delta(\alpha))}] = [\delta(X)]$. ☐

Proposition 4.1.10. *The map* $K : \Gamma(\mathfrak{v})^{\text{inv}} \to H^2(G, \mathfrak{i})$, $[V] \mapsto [\delta(X)]$ *as defined above is linear.*

Proof. For all $[V]$, $[V'] \in \Gamma(\mathfrak{v})^{\text{inv}}$ with X and X' some (s,t)-lifts of V and V' respectively,

$$\begin{aligned} K([V] + [V']) &= K([V + V']) \\ &= [\delta(X + X')] \quad \text{since } X + X' \text{ is an } (s,t)\text{-lift of } V + V' \\ &= [\delta(X) + \delta(X')] \\ &= [\delta(X)] + [\delta(X')] \\ &= K([V]) + K([V']). \end{aligned}$$

☐

Proposition 4.1.11. *There is an exact sequence:*

$$0 \to H^1(G, \mathfrak{i}) \xrightarrow{r} H^1_{\text{def}}(G) \xrightarrow{\pi} \Gamma(\mathfrak{v})^{\text{inv}} \xrightarrow{K} H^2(G, \mathfrak{i}) \xrightarrow{r} H^2_{\text{def}}(G)$$

Proof.

- $r : H^1(G, \mathfrak{i}) \to H^1_{\text{def}}(G)$ is injective by Proposition 4.1.4.
- Let $[X] \in \operatorname{im}(r) \subset H^1_{\text{def}}(G)$. That is, $X \in \mathfrak{X}_{\text{mult}}(G)$, with $ds(X_g) = 0$ and $dt(X_g) = 0$ $\forall g \in G$. Thus, the base field of X is zero and so $\pi([X]) = 0$, hence $[X] \in \ker(\pi)$. Moreover,

$$\begin{aligned} &\ker(\pi) \\ &= \left\{[X] \in H^1_{\text{def}}(G) \mid X \in \mathfrak{X}_{\text{mult}}(G) \text{ with the class of its base field } [V] = 0 \text{ in } \Gamma(\mathfrak{v})\right\} \\ &= \left\{[X] \in H^1_{\text{def}}(G) \mid X \in \mathfrak{X}_{\text{mult}}(G) \text{ with base field } V = \rho(\alpha) \text{ for some } \alpha \in \Gamma(A)\right\} \\ &= \left\{[X - \delta(\alpha)] \in H^1_{\text{def}}(G) \mid X \in \mathfrak{X}_{\text{mult}}(G) \text{ s.t. } V - \rho(\alpha) = 0 \text{ for some } \alpha \in \Gamma(A)\right\}. \end{aligned}$$

  Therefore, $\ker(\pi)$ is composed of all classes $[X]$ of multiplicative vector fields with base field zero, and so of classes $[X]$, where X vanishes under ds and dt, hence lying in the image of r.

- Let $[X] \in H^1_{\text{def}}(G)$ with $\pi([X]) = [V] \in \text{im}(\pi)$, i.e. X is a multiplicative vector field of G with base field V. Then, $\delta(X)(g,h) = -d\bar{m}(X_{gh}, X_h) + X_g = -X_{\bar{m}(gh,h)} + X_g = -X_g + X_g = 0 \ \forall \ (g,h) \in G^{(2)}$. Hence, $K \circ \pi[X] = K[V] = [\delta(X)] = 0$ and so $\pi[X] \in \ker(K)$. On the other hand, let $[V] \in \ker(K)$. That is, $[V] \in \Gamma(\mathfrak{v})^{\text{inv}}$ such that $K([V]) = [\delta(X)] = 0$ in $H^2(G, \mathfrak{i})$ for some (s,t)-lift $X \in \mathfrak{X}(G)$ of V. Thus, $\delta(X) = \delta(c)$ for some c in the subcomplex of $C^1_{\text{def}}(G)$ where it vanishes under ds and dt. Hence, for all g in G we get that
$$ds(X - c)(g) = ds(X_g) - \cancel{ds(c(g))} = V_{s(g)}$$
$$dt(X - c)(g) = dt(X_g) - \cancel{dt(c(g))} = V_{t(g)}$$
which imply that $Y = X - c$ is another (s,t)-lift of V with the property that $\delta(Y) = \delta(X - c) = \delta(X) - \delta(c) = 0$. That is,
$$\begin{aligned} &-d\bar{m}(Y_{gh}, Y_h) + Y_g = 0 \\ \Rightarrow \quad & d\bar{m}(Y_{gh}, Y_h) = Y_g \\ \Rightarrow \quad & d\bar{m}(Y_{gh}, Y_h) = Y_{\bar{m}(gh,h)} \end{aligned}$$
$\forall \ (g,h) \in G^{(2)}$ and hence Y is a multiplicative vector field of G with base field V and with $[\delta(Y)] = [\delta(X)]$. Therefore, [V] comes from a multiplicative vector field.

- Let $[V] \in \Gamma(\mathfrak{v})^{\text{inv}}$ and so $K([V]) = [\delta(X)] \in \text{im}(K)$ for some (s,t)-lift $X \in \mathfrak{X}(G)$ of V. Then, $r \circ K([V]) = r([\delta(X)]) = 0$ in $H^2_{\text{def}}(G)$. Hence, $K[V] \in \ker(r)$. Furthermore, let $[c] \in \ker(r)$. That is, $[c]$ is an element of $H^2(G, \mathfrak{i})$ with $r([c]) = [r(c)] = 0$ in $H^2_{\text{def}}(G)$. So, there exists some $X \in C^1_{\text{def}}(G)$ s.t. $r(c) = \delta(X)$. But, we know that $r(c)$ is in the subcomplex of $C^2_{\text{def}}(G)$ where it vanishes under ds and dt. Hence, X is s- and t-projectable to some vector field $V \in \mathfrak{X}(M)$. Thus, $[c] = K([V]) \in \text{im}(K)$.

□

It is natural to ask if there is such an exact sequence for higher degrees of the cohomologies as well. As described in [18, Proposition 5.52], when G is a regular groupoid, there exists a long exact sequence
$$\cdots \to H^k(G, \mathfrak{i}) \to H^k_{\text{def}}(G) \to H^{k-1}(G, \mathfrak{v}) \to H^{k+1}(G, \mathfrak{i}) \to H^{k+1}_{\text{def}}(G) \to \cdots$$
To prove this statement as given in [18], it is useful to understand the adjoint representation of a groupoid and the deformation cohomology interpretation through it.

4.2 Representations up to homotopy

The key to proceed with representations up to homotopy is to represent a groupoid in a complex of vector bundles, instead of just a single vector bundle. Sections 4.2.1 to 4.2.3 refer mainly to [2].

4.2.1 Representations up to homotopy of groupoids

The following notations and definitions are needed for the discussion.

Definition 4.2.1 (Graded vector bundle)**.** A **graded vector bundle** over a smooth manifold M is a vector bundle $E \to M$ together with a direct sum decomposition
$$E = \bigoplus_{l \in \mathbb{Z}} E^l$$
which is called a grading of E.

An element $v \in E^l$ is said to have degree l. Note that the fibers of a graded vector bundle carry the structure of a graded space. Given two graded vector bundles E and F over M, we can construct the new graded space of morphisms, denoted by

$$\mathrm{Hom}(E,F) = \bigoplus \mathrm{Hom}^l(E,F)$$

where each $T \in \mathrm{Hom}^l(E,F)$ is a vector bundle map $T : E \to F$ of degree l, that is $T(E^m) \subseteq F^{m+l}\ \forall\ m \in \mathbb{Z}$. For $F = E$, we denote $\mathrm{Hom}(E,E) = \mathrm{End}(E)$ and call it the space of endomorphisms of E.

Given a Lie groupoid $G \rightrightarrows M$ and a vector bundle E over M, recall from Remark 2.2.3, that the space $C^*(G,E)$ of differentiable cochains on G with values in E makes sense as a graded vector space. If E is a graded vector bundle over M, $C(G,E)^*$ has the following total grading

$$C(G,E)^n = \bigoplus_{k+l=n} C^k(G,E^l)$$

where elements $w \in C^k(G,E^l)$ are said to have bidegree (k,l) and total degree $k+l$. The space $C(G,E)^*$ has also the structure of a right graded $C^*(G)$-module similar to the ungraded case.

Definition 4.2.2 (Complex of vector bundles)**.** A **cochain complex of vector bundles** over M is a graded vector bundle $E = \bigoplus_{l\in\mathbb{Z}} E^l$ over M equipped with an endomorphism $\partial \in \mathrm{End}(E)$ of degree 1

$$\cdots \xrightarrow{\partial} E^0 \xrightarrow{\partial} E^1 \xrightarrow{\partial} E^2 \xrightarrow{\partial} \cdots$$

and such that $\partial^2 = 0$.

We are now ready to define a representation up to homotopy of a Lie groupoid according to [2]. Firstly, we unpack the shortest but less intuitive definition. Later, through a bijective correspondence between representations up to homotopy and sequences of particular differentiable cochains, the structure of a representation up to homotopy will be more explicitly revealed. In what follows, let $G \rightrightarrows M$ be a Lie groupoid.

Definition 4.2.3 (Representation up to homotopy)**.** A **representation up to homotopy** of G is a bounded graded vector bundle $E = \bigoplus E^l$ over M together with a linear operator on $C(G,E)$ of degree one

$$D : C(G,E)^n \longrightarrow C(G,E)^{n+1}$$

such that $D^2 = 0$, and for all $w \in C(G,E)^k$, $f \in C^*(G)$, the graded Leibniz identity

$$D(w \cdot f) = D(w) \cdot f + (-1)^k w \cdot \delta(f)$$

is satisfied.

$$\begin{tikzcd} G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & E = \bigoplus E^l \arrow[dl] \\ M & \end{tikzcd}$$

Equivalently, a representation up to homotopy of G is a bounded graded vector bundle $E = \bigoplus E^l$ over M together with a differential D on $C(G,E)$ which gives $C(G,E)$ the structure of a right differential graded module over $C(G)$. D as in the above definition is usually called the **structure operator** of the representation up to homotopy E of G.

Clearly, a usual representation E of G can be viewed as a representation up to homotopy of G concentrated in degree zero, since the space $C(G,E)$ of E-valued cochains on G is endowed with the usual differential δ which makes $C(G,E)$ into a right differential graded $C(G)$-module as mentioned in section 2.2.

Definition 4.2.4. Let E and F be two representations up to homotopy of G with structure operators D_E and D_F respectively. A **morphism** $\Phi : E \to F$ between E and F is a $C^*(G)$-linear map of degree zero

$$\Phi : C(G,E)^* \longrightarrow C(G,F)^*$$

such that $D_F \circ \Phi = \Phi \circ D_E$.

The following proposition is an essential one stated and proved in [2, Proposition 3.2]. Here, we only mention the statement and look into its interpretation. At first note that, given a graded vector bundle E over M, the bigraded vector space $C(G, \mathrm{End}(E))$ in bidegree (k,l) is precisely

$$C^k(G, \mathrm{End}^l(E)) = \Gamma(G^{(k)}, \mathrm{Hom}(s^*E^\bullet, t^*E^{\bullet+l}))$$

where $s(g_1,...,g_k) = s(g_k)$ and $t(g_1,...,g_k) = t(g_1)$.

Proposition 4.2.5. *There exists a bijective correspondence between representations up to homotopy (E,D) of G and graded vector bundles E over M together with sequences $\{R_k\}_{k\geq 0}$ composed of differentiable cochains $R_k \in C^k(G, \mathrm{End}^{1-k}(E))$ satisfying*

$$\sum_{j=1}^{k-1}(-1)^j R_{k-1}(g_1,...,g_j g_{j+1},...,g_k) = \sum_{j=0}^{k}(-1)^j R_j(g_1,...,g_j) \circ R_{k-j}(g_{j+1},...,g_k) \qquad (16)$$

for all $k \geq 0$.

The elements R_i of the sequence are also called the structure operators of the associated representation up to homotopy. Through the first elements of the sequence, one can get an intuitive idea of the notion of a representation up to homotopy.

For k=0: $R_0 : G^{(0)} = M \longrightarrow \mathrm{End}^1(E)$ induces an endomorphism of E of degree 1 denoted by

$$\partial := R_0 : E^* \longrightarrow E^*.$$

In this case, (16) becomes $0 = R_0 \circ R_0 = \partial \circ \partial$ which means that E, as a graded vector bundle over M, together with ∂ becomes a cochain complex of vector bundles over M.

For k=1: $R_1 : G^{(1)} = G \longrightarrow \mathrm{End}^0(E) = \mathrm{Hom}^0(s^*E, t^*E)$ induces a (graded) quasi-action of G on E denoted by

$$\lambda_g := R_1(g) : E_{s(g)} \longrightarrow E_{t(g)}.$$

Moreover, (16) implies that the differential ∂ of the cochain complex commutes with the quasi-action since

$$\begin{aligned} 0 &= R_0 \circ R_1 - R_1 \circ R_0 \\ \Rightarrow R_1 \circ R_0 &= R_0 \circ R_1 \\ \Rightarrow \lambda_g \circ \partial &= \partial \circ \lambda_g \end{aligned}$$

and hence an arrow $g \in G$ acts via a map λ_g of cochain complexes.

For k=2: Here, (16) becomes

$$\begin{aligned} -R_1(g_1g_2) &= R_0 \circ R_2(g_1, g_2) - R_1(g_1) \circ R_1(g_2) + R_2(g_1, g_2) \circ R_0 \\ \Rightarrow R_1(g_1) \circ R_1(g_2) - R_1(g_1g_2) &= R_0 \circ R_2(g_1, g_2) + R_2(g_1, g_2) \circ R_0 \\ \Rightarrow \lambda_{g_1} \circ \lambda_{g_2} - \lambda_{g_1g_2} &= \partial \circ R_2(g_1, g_2) + R_2(g_1, g_2) \circ \partial \end{aligned}$$

for all composable pairs $(g_1, g_2) \in G^{(2)}$. This means that the quasi-actions as induced by R_1 are not necessarily associative. That is, for $(g_1, g_2) \in G^{(2)}$, $\lambda_{g_1} \circ \lambda_{g_2}$ and $\lambda_{g_1g_2}$ are not necessarily the same maps of complexes, but they are homotopic and the homotopies are controlled by the map R_2.

Definition 4.2.6. A representation up to homotopy E of G is said to be **unital** if $\forall\, x \in M$, the map $R_1(u(x)) : E_x \to E_x$ is the identity map, and if $R_k(g_1, ..., g_k) = 0$ for $k \geq 2$ with one of the g_i's being a unit.

Similar to the case of usual representations of Lie groupoids, one can define the differentiable cohomology of G with coefficients in a representation up to homotopy of G.

Definition 4.2.7. Let (E, D) be a representation up to homotopy of the Lie groupoid G. The **differentiable cohomology** of G with coefficients in E, denoted by $H(G, E)^*$, is the cohomology of the complex $(C(G, E)^*, D)$.

Remark 4.2.8 (Related to VB-groupoids)**.** *In relationship to* VB-*groupoids, which are Lie groupoids endowed with some compatible linear structure, there is a bijective correspondence between* VB-*groupoids and unital 2-term representations $E = E^0 \oplus E^1$ up to homotopy of Lie groupoids up to isomorphism. This result, which is stated and studied in [11] by Gracia-Saz and Mehta, provides a geometrical model for unital 2-term representations up to homotopy. In addition, there is a cochain complex associated to any* VB-*groupoid which is isomorphic to the complex of Lie groupoid cochains with coefficients in the corresponding representation up to homotopy according to [11]. This in turn gives a canonical model of the differentiable cohomologies of Lie groupoids with values in a representation up to homotopy.*

Remark 4.2.9 (Related to weak representations)**.** *In a very recent paper by Wolbert [30], the notion of a weak representation of a Lie groupoid is introduced as a consequence of some issues and restrictions with representations up to homotopy of Lie groupoids. Among the issues is the lack of geometrical interpretation of the higher order equations between the homotopies associated to the representations up to homotopy. Also, there is no general well-defined notion of a "right" representation up to homotopy. Moreover, the linear structure of the graded vector bundle E plays a huge role in the definition of a representation up to homotopy. In contrast, weak representations, which are defined in [30] via so-called weak actions, do not possess these issues. Furthermore, it can be shown that weak representations of a Lie groupoid G are in bijective correspondence with 2-term representations up to homotopy of G.*

As a first and important example of a 2-term representation up to homotopy, we will study the adjoint representation Ad generalized in the setting of groupoids in the coming subsections. Its significance in the deformation theory of a groupoid G will be clear through the isomorphism between the differentiable cohomology $H(G, \mathrm{Ad})^*$ of G with coefficients in its adjoint representation and the deformation cohomology $H^*_{\mathrm{def}}(G)$ of G.

4.2.2 Connections and basic curvatures on groupoids

This subsection will provide the necessary background for the construction of the adjoint representation of a groupoid as a representation up to homotopy. More specifically, we introduce the notion of a certain type of connection on groupoids, namely Ehresmann connections, as well as the induced basic curvature. Although the notion of connections on groupoids has appeared in the literature earlier, we will mainly follow the definition given in [2].

Let $G \rightrightarrows M$ be a Lie groupoid with its associated Lie algebroid A. Observe that

$$0 \longrightarrow t^*A \xrightarrow{dR} TG \xrightarrow{ds} s^*TM \longrightarrow 0$$

is a short exact sequence of vector bundles, where $s^*TM = TM|_{s(G)}$ and $t^*A = A|_{t(G)}$ are the pullback bundles of TM and A by the source and target maps respectively. For each arrow $g \in G$, the maps are precisely given by $dR_g : A_{t(g)} \to T_gG$ and $ds_g : T_gG \to T_{s(g)}M$. The injectivity of dR_g follows by similar arguments as before and the surjectivity of ds_g is due to the fact that s is a submersion. Moreover, $\ker(ds_g)$ and $\mathrm{im}(dR_g)$ coincide since they are composed of all the vectors tangent to the source fiber at g. It is a known result on vector bundles, also mentioned in details in [15, Proposition 5.2.6], that the short exact sequence above induces a right inverse $\sigma : s^*TM \to TG$ to ds (i.e. $ds \circ \sigma = \mathrm{Id}$) and a left inverse $w : TG \to t^*A$ to dR (i.e. $w \circ dR = \mathrm{Id}$), usually called a right splitting and a left splitting of the sequence respectively.

Definition 4.2.10 (Ehresmann connection)**.** An **Ehresmann connection** on G is defined to be one of the following equivalent structures:

(i) A subbundle H of the tangent bundle TG of G such that

- $H_g \oplus \ker(ds_g) = T_gG \quad \forall\, g \in G$
- $H_{u(x)} = T_xM \quad \forall\, x \in M$.

(ii) A right splitting $\sigma : s^*TM \to TG$ of the above sequence, such that it restricts to the canonical splitting at the units, i.e. $\sigma_{u(x)} = du_x \;\forall\, x \in M$.

(iii) A left splitting $w : TG \to t^*A$ of the above sequence, such that it restricts to the canonical splitting at the units.

The existence of such a right or left splitting (see [15, Proposition 5.2.6]) proves the fact that every Lie groupoid has an Ehresmann connection. Moreover, the following crucial relation holds between the maps:

$$dR \circ w + \sigma \circ ds = \mathrm{Id}\,. \tag{17}$$

In what follows, we will usually use the right splitting σ as the chosen connection on the groupoid.

Let σ be a connection on G. Our aim in the remaining part of the subsection is to study a particular type of curvature associated to the connection σ. In order to do that, we first note that σ induces quasi-actions λ of G on A and TM, defined in the following way:

$$\lambda_g : A_x \longrightarrow A_y, \quad v \longmapsto \lambda_g(v) := -w_g(dL_g di(v))$$
$$\lambda_g : T_xM \longrightarrow T_yM, \quad X \longmapsto \lambda_g(X) := dt_g(\sigma_g(X))$$

for all $g : y \leftarrow x$ in G, $v \in A_x$, $X \in T_xM$.

For a given composable pair $(g,h) \in G^{(2)}$ and a tangent vector $X \in T_{s(h)}M$, consider the following expression lying in $T_{gh}G$

$$L(g,h)X := \sigma_{gh}(X) - dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)).$$

Lemma 4.2.11. *The expression $L(g,h)X \in \ker(ds_{gh})$ $\forall$ $(g,h) \in G^{(2)}$, $X \in T_{s(h)}M$.*

Proof.

$$\begin{aligned} ds(L(g,h)X) &= ds(\sigma_{gh}(X) - dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X))) \\ &= ds(\sigma_{gh}(X)) - ds(dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X))) \\ &= X - ds(\sigma_h(X)) \\ &= X - X \\ &= 0 \end{aligned}$$

using the fact that σ is a right-inverse to ds and hence $ds \circ \sigma = \mathrm{Id}$. ☐

The fact that the expression $L(g,h)X$ is killed by ds_{gh} further implies that $L(g,h)X$ lies in the image of $dR_{gh} : A_{t(g)} \to T_{gh}G$. Denote by

$$K_\sigma^{\mathrm{bas}}(g,h)X \in A_{t(g)}$$

the preimage of $L(g,h)X$ under dR_{gh}. That is

$$dR_{gh}(K_\sigma^{\mathrm{bas}}(g,h)X) = \sigma_{gh}(X) - dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)). \tag{18}$$

Definition 4.2.12. Let σ be an Ehresmann connection on G. The **basic curvature** associated to σ is defined to be the resulting section

$$K_\sigma^{\mathrm{bas}} \in \Gamma(G^{(2)}, \mathrm{Hom}(s^*TM, t^*A))$$

satisfying (18).

In other words, the basic curvature is an operator K_σ^{bas} which associates to each composable pair $(g,h) \in G^{(2)}$ and tangent vector $X \in T_{s(h)}M$, the vector

$$K_\sigma^{\mathrm{bas}}(g,h)X \in A_{t(g)}$$

characterized by (18).

Remark 4.2.13. *The basic curvature K_σ^{bas} is precisely given by the formula:*

$$K_\sigma^{\mathrm{bas}}(g,h)X = -w_{gh}(dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)))$$

$\forall$ $(g,h) \in G^{(2)}$, $X \in T_{s(h)}M$.

Proof. By using equation (17), we get that for all $(g,h) \in G^{(2)}$, $X \in T_{s(h)}M$,

$$\begin{aligned} dR_{gh} \circ w_{gh}(dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X))) &= (\mathrm{Id} - \sigma_{gh} \circ ds_{gh})((dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)))) \\ &= (dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X))) - \sigma_{gh} \circ ds_{gh}(\sigma_h(X)) \\ &= (dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X))) - \sigma_{gh}(X) \\ &= -dR_{gh}(K_\sigma^{\mathrm{bas}}(g,h)X). \end{aligned}$$

Therefore,

$$K_\sigma^{\mathrm{bas}}(g,h)X = -w_{gh}(dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X))).$$

☐

4.2.3 Adjoint representation and the deformation cohomology

The attempt of generalizing the well-defined adjoint representation of Lie groups to the case of Lie groupoids via the usual definition of representations of groupoids has not been completely successful. For instance, there is no natural action of a Lie groupoid on its algebroid which may be thought of a generalization of the adjoint action. In this respect, the concept of representations up to homotopy as introduced in [2] has played a crucial role since the adjoint representation has a well-defined generalization to Lie groupoids as an isomorphism class of representations up to homotopy. As mentioned in the introduction, the construction of the adjoint representation of a Lie groupoid is completely parallel to the infinitesimal case of constructing the adjoint representation of a Lie algebroid also introduced and studied by Arias Abad and Crainic in [1].

Let $G \rightrightarrows M$ be a Lie groupoid and let A be its associated Lie algebroid throughout the subsection. In order to construct the adjoint representation of G, we first have to choose an Ehresmann connection σ on the groupoid G. The anchor ρ associated to A, the quasi-actions λ of G on A and TM as well as the basic curvature K_σ^{bas} associated to σ will be essential in defining the structure operators of the adjoint representation.

Definition 4.2.14 (Adjoint representation of a groupoid)**.** Let σ be an Ehresmann connection on G with K_σ^{bas} its associated basic curvature. The **adjoint representation** of G is a 2-term representation up to homotopy (E, R_0, R_1, R_2) of G where:

(i) $E = \mathrm{Ad} := A \oplus TM$ is the cochain complex of vector bundles over M, where A has degree zero and TM has degree one, and with differential given by the anchor map ρ. The complex Ad

$$0 \longrightarrow A \xrightarrow{\rho} TM \longrightarrow 0$$

is usually called the **adjoint complex** of G.

(ii) The structure operators are defined by

- $R_0 := \rho$ (anchor map)
- $R_1 := \lambda$ (quasi-actions of G on A and TM)
- $R_2 := K_\sigma^{\mathrm{bas}}$ (basic curvature associated to σ).

For a given connection σ on G, the adjoint representation of G will be denoted by Ad_σ.

Proposition 4.2.15. *Let σ be an Ehresmann connection on G. Then, the adjoint representation Ad_σ of G is indeed a unital representation up to homotopy of G.*

The proof is based on [2, Proposition 2.15], providing more details.

Proof. We need to show that the structure operators satisfy a set of equations:

(i) $\rho \circ \lambda_g = \lambda_g \circ \rho$

(ii) $\lambda_g \lambda_h(X) - \lambda_{gh}(X) = -\rho(K_\sigma^{\mathrm{bas}}(g,h)X)$

(iii) $\lambda_g \lambda_h(v) - \lambda_{gh}(v) = -K_\sigma^{\mathrm{bas}}(g,h)(\rho(v))$

(iv) $\lambda_g K_\sigma^{\mathrm{bas}}(h,k) - K_\sigma^{\mathrm{bas}}(gh,k) + K_\sigma^{\mathrm{bas}}(g,hk) - K_\sigma^{\mathrm{bas}}(g,h)\lambda_k = 0$

$\forall\ (g,h,k) \in G^{(3)}$, $X \in T_{s(h)}M$, $v \in A_{s(h)}$.

(i) For $g : y \curvearrowleft x \in G$ and $v \in A_x$,

$$\begin{aligned}
\rho_y(\lambda_g(v)) &= dt_g \circ dR_g(\lambda_g(v)) \quad \text{as } \rho = dt|_A \\
&= -dt_g \circ dR_g \circ w_g \circ dL_g \circ di(v) \\
&= -dt_g \circ (Id - \sigma_g \circ ds_g) \circ dL_g \circ di(v) \quad \text{using (17)} \\
&= \cancel{-dt_g \circ dL_g \circ di(v)} + dt_g \circ \sigma_g \circ ds_g \circ dL_g \circ di(v) \\
&= dt_g \circ \sigma_g(\rho(v)) \quad \text{since } ds_g \circ dL_g \circ di = dt_{u(x)} = \rho_x \\
&= \lambda_g(\rho_x(v)).
\end{aligned}$$

Thus, $\rho \circ \lambda_g = \lambda_g \circ \rho$.

(ii) For $(g,h) \in G^{(2)}$ and $X \in T_{s(h)}M$,

$$\begin{aligned}
\rho(K_\sigma^{\text{bas}}(g,h)X) &= dt_{gh} \circ dR_{gh}(K_\sigma^{\text{bas}}(g,h)X) \\
&= -dt_{gh} \circ dR_{gh} \circ w_{gh} \circ dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)) \\
&= -dt_{gh} \circ (Id - \sigma_{gh} \circ ds_{gh}) \circ dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)) \\
&= -dt_{gh} \circ dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)) \\
&\quad + dt_{gh} \circ \sigma_{gh} \circ ds_{gh} \circ dm|_{(g,h)}(\sigma_g(\lambda_h(X)), \sigma_h(X)) \\
&= -dt_{gh}(\sigma_g(\lambda_h(X))) + dt_{gh} \circ \sigma_{gh} \circ ds_{gh}(\sigma_h(X)) \\
&= -dt_{gh}(\sigma_g(\lambda_h(X))) + dt_{gh}(\sigma_{gh}(X)) \\
&= -\lambda_g\lambda_h(X) + \lambda_{gh}(X).
\end{aligned}$$

Hence, $\lambda_g\lambda_h(X) - \lambda_{gh}(X) = -\rho(K_\sigma^{\text{bas}}(g,h)X)$.

(iii) For the next equation, we need the following relation:

$$\sigma_g \circ \rho = dL_g di + dR_g \lambda_g.$$

This indeed holds since $\forall\ g \in G,\ v \in A_{s(g)}$,

$$\begin{aligned}
dL_g di(v) + dR_g\lambda_g(v) &= dL_g di(v) - dR_g w_g dL_g di(v) \\
&= dL_g di(v) - (Id - \sigma_g ds_g) dL_g di(v) \\
&= \cancel{dL_g di(v)} - \cancel{dL_g di(v)} + \sigma_g ds_g dL_g di(v) \\
&= \sigma_g(\rho(v)).
\end{aligned}$$

Now, for $(g,h) \in G^{(2)}$ and $v \in A_{s(h)}$,

$$\begin{aligned}
K_\sigma^{\text{bas}}(g,h)(\rho(v)) &= -w_{gh} \circ dm|_{(g,h)}(\sigma_g\lambda_h(\rho(v)), \sigma_h(\rho(v))) \\
&= -w_{gh} \circ dm|_{(g,h)}(\sigma_g\rho(\lambda_h(v)), \sigma_h(\rho(v))) \quad \text{by (i)} \\
&= -w_{gh} \circ dm|_{(g,h)}(dL_g di\lambda_h(v) + dR_g\lambda_g\lambda_h(v), dL_h di(v) + dR_h\lambda_h(v)) \\
&= -w_{gh} \circ dm|_{(g,h)}(dR_g\lambda_g\lambda_h(v), 0_h) \\
&\quad - w_{gh} \circ dm|_{(g,h)}(0_g, dL_h di(v)) \\
&\quad \cancel{- w_{gh} \circ dm|_{(g,h)}(dL_g di\lambda_h(v), dR_h\lambda_h(v))} \\
&= -w_{gh} \circ dR_{gh}(\lambda_g\lambda_h(v)) - w_{gh} \circ dL_{gh} di(v) \\
&= -\lambda_g\lambda_h(v) + \lambda_{gh}(v).
\end{aligned}$$

Hence, $\lambda_g\lambda_h(v) - \lambda_{gh}(v) = -K_\sigma^{bas}(g,h)(\rho(v))$.

Note that in the above calculations we used the fact that for all $(g,h) \in G^{(2)}$,

$$dm|_{(g,h)}(X_g, 0_h) = dR_h(X_g), \tag{19}$$

$$dm|_{(g,h)}(0_g, X_h) = dL_g(X_h), \tag{20}$$

which are special cases of (3).

(iv) For simplicity, let $K = K^{\mathrm{bas}}_\sigma$. For all $(g,h,k) \in G^{(3)}$, $X \in T_{s(k)}M$,

$$\begin{aligned}
K(g,hk)X &= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_{hk}(X), \sigma_{hk}(X)) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X) + \sigma_g\rho(K(h,k)X), \sigma_{hk}(X)) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X) + dL_g di(K(h,k)X) \\
&\quad + dR_g\lambda_g(K(h,k)X), \sigma_{hk}(X)) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X) + dL_g di(K(h,k)X), \sigma_{hk}(X)) \\
&\quad - w_{ghk} \circ dm|_{(g,hk)}(dR_g\lambda_g(K(h,k)X), 0_{hk}) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X) + dL_g di(K(h,k)X), \sigma_{hk}(X)) \\
&\quad - w_{ghk} \circ dR_{ghk}\lambda_g(K(h,k)X) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X) + dL_g di(K(h,k)X), \sigma_{hk}(X)) - \lambda_g(K(h,k)X).
\end{aligned}$$

This implies that

$$\begin{aligned}
K(g,hk)X + \lambda_g K(h,k)X &= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X) + dL_g di(K(h,k)X), \sigma_{hk}(X)) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X) + dL_g di(K(h,k)X), \sigma_{hk}(X)) \\
&\quad - w_{ghk} \circ dm|_{(g,hk)}(-dL_g di(K(h,k)X), -dR_{hk}(K(h,k)X)) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X), \sigma_{hk}(X) - dR_{hk}(K(h,k)X)) \\
&= -w_{ghk} \circ dm|_{(g,hk)}(\sigma_g\lambda_h\lambda_k(X), dm|_{(h,k)}(\sigma_h\lambda_k(X), \sigma_k(X))).
\end{aligned}$$

By the associativity of the multiplication we get further that

$$\begin{aligned}
K(g,hk)X + \lambda_g K(h,k)X &= -w_{ghk} \circ dm|_{(gh,k)}(dm|_{(g,h)}(\sigma_g\lambda_h\lambda_k(X), \sigma_h\lambda_k(X)), \sigma_k(X)) \\
&= -w_{ghk} \circ dm|_{(gh,k)}(dm|_{(g,h)}(\sigma_g\lambda_h\lambda_k(X), \sigma_h\lambda_k(X)) \\
&\quad + dR_{gh}K(g,h)\lambda_k(X), \sigma_k(X)) \\
&\quad + K(g,h)\lambda_k(X) \\
&= -w_{ghk} \circ dm|_{(gh,k)}(\sigma_{gh}\lambda_k(X), \sigma_k(X)) + K(g,h)\lambda_k(X) \\
&= K(gh,k)X + K(g,h)\lambda_k(X).
\end{aligned}$$

Therefore, $\lambda_g K(h,k) - K(gh,k) + K(g,hk) - K(g,h)\lambda_k = 0$.

Finally, to show unitality, observe that the quasi-actions $\lambda_{u(x)}$ $(x \in M)$ on A and TM at units are the identity maps. Also, for all $g : y \curvearrowleft x$, $X \in T_xM$,

$$\begin{aligned}
K^{\mathrm{bas}}_\sigma(g, u(x))X &= -w_g \circ dm|_{(g,u(x))}(\sigma_g\lambda_{u(x)}(X), \sigma_{u(x)}(X)) \\
&= -w_g \circ dm|_{(g,u(x))}(\sigma_g(X), du(X)) \\
&= -w_g \circ \sigma_g(X) \\
&= 0.
\end{aligned}$$

Similarly, $K^{\mathrm{bas}}_\sigma(u(y), g)X = 0$ for all $g : y \curvearrowleft x$, $X \in T_xM$. □

The issue with this definition of the adjoint representation is that it relies primarily on the choice of an Ehresmann connection on G. It is natural to ask how the adjoint representations defined upon different choices of connections are related. The following crucial result from [2, Proposition 3.16] solves this issue.

Theorem 4.2.16. *Let σ and σ' be two Ehresmann connections on G. Then, we get the following isomorphism of representations*

$$\mathrm{Ad}_\sigma \cong \mathrm{Ad}_{\sigma'} .$$

It can be therefore concluded that the adjoint representation Ad of G is a well-defined unital representation up to homotopy, which, up to isomorphism, is independent of the choice of the connection.

The following significant result is stated and proved in [18, Lemma 5.53]. It generalizes a similar result in the case of Lie groups (Theorem 2.1.4) and shows the relation between the deformation cohomology and adjoint representation of a groupoid.

Recall that by the total grading, $C(G, \mathrm{Ad}_\sigma)^k = C^k(G, A) \oplus C^{k-1}(G, TM)$.

Theorem 4.2.17. *Let σ be an Ehresmann connection on G. Then, the map*

$$I^k_\sigma : C(G, \mathrm{Ad}_\sigma)^k \longrightarrow C^k_{\mathrm{def}}(G), \quad (u, v) \longmapsto I^k_\sigma(u, v)$$

defined by

$$I^k_\sigma(u, v)(g_1, ..., g_k) := dR_{g_1}(u(g_1, ..., g_k)) - \sigma_{g_1}(v(g_2, ..., g_k)) \in T_{g_1}G$$

is an isomorphism of cochain complexes.

Corollary 4.2.18. *Given an Ehresmann connection σ on G, $H^*_{\mathrm{def}}(G) \cong H(G, \mathrm{Ad}_\sigma)^*$.*

Remark 4.2.19 (Related to VB-groupoids)**.** *Given a Lie groupoid $G \rightrightarrows M$, the associated tangent groupoid $TG \rightrightarrows TM$ has the structure of a* VB-*groupoid which corresponds precisely to the adjoint representation of G as shown in [11]. Moreover, the following isomorphism*

$$H^*_{\mathrm{VB}}(TG) \cong H(G, \mathrm{Ad})^* \cong H^*_{\mathrm{def}}(G)$$

where $H^*_{\mathrm{VB}}(TG)$ is the cohomology of the cochain complex of the tangent groupoid viewed as a VB-*groupoid, gives a canonical model of the deformation cohomology.*

5 Deformations of groupoids

The goal of this section is to familiarize ourselves with deformations of Lie groupoids. At first, specific kinds of deformations will be considered and studied, which will pave the way to understanding general deformations. Furthermore, the relation between deformations and the deformation cohomology will be clearer. Namely, we will see how deformations of groupoids give rise to 2-cocycles. Lastly, the vanishing of the deformation cohomology $H^2_{\text{def}}(G)$ in degree 2 will lead to some important rigidity results. The upcoming discussion is based on the paper [7] by Crainic, Mestre and Struchiner with further illustration given in the PhD thesis [18] by Mestre. In this thesis, we provide more details and explanations.

5.1 Basic definitions

In this subsection, $G \rightrightarrows M$ denotes a Lie groupoid and I an open set containing zero as before.

Definition 5.1.1 (Deformation of Lie groupoids)**.** A **smooth deformation** of G is a family

$$\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}_{\varepsilon \in I}$$

of Lie groupoids which is smoothly parametrized by $\varepsilon \in I$ and such that $G_0 = G$ as groupoids.

The structure maps of G_ε will be naturally denoted by $s_\varepsilon, t_\varepsilon, m_\varepsilon, \bar{m}_\varepsilon, u_\varepsilon, i_\varepsilon$, whereas the structure maps of the original groupoid G are given by $s, t, m, \bar{m}, u, i$ as before.

In other words, a deformation of G can be understood as a Lie groupoid $\tilde{G} \rightrightarrows \tilde{M}$, with structure maps $\tilde{s}, \tilde{t}, \tilde{m}, \tilde{\bar{m}}, \tilde{u}, \tilde{i}$, together with a surjective submersion $\pi : \tilde{M} \longrightarrow I$ with the property that $\pi \circ \tilde{s} = \pi \circ \tilde{t}$ and so that for each $\varepsilon \in I$, $G_\varepsilon := \tilde{s}^{-1}(M_\varepsilon) \rightrightarrows M_\varepsilon := \pi^{-1}(\varepsilon)$ denotes the groupoid over the fiber $\pi^{-1}(\varepsilon)$ such that $G_0 = G$ as groupoids.

Definition 5.1.2 (Equivalent deformations)**.** Two smooth deformations $\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}$ and $\tilde{G}' = \{G_\varepsilon{}' \rightrightarrows M_\varepsilon{}'\}$ of G are said to be **equivalent** if $\exists$ a family

$$\{(F^\varepsilon : G_\varepsilon \to G'_\varepsilon,\ f^\varepsilon : M_\varepsilon \to M'_\varepsilon)\}$$

of isomorphisms of Lie groupoids which is smoothly parametrized by an open interval containing zero and such that $(F^0, f^0) = \text{Id}$.

In light of the reinterpretation of the definition of deformations above, two deformations $\tilde{G} \rightrightarrows \tilde{M} \xrightarrow{\pi} I$ and $\tilde{G}' \rightrightarrows \tilde{M}' \xrightarrow{\pi'} I$ are called equivalent if $\exists$ a Lie groupoid isomorphism $\tilde{F} : \tilde{G} \to \tilde{G}',\ \tilde{f} : \tilde{M} \to \tilde{M}'$ with $\pi' \circ \tilde{f} = \pi$ and such that at $\varepsilon = 0$, it is the identity.

As it is difficult to study such general deformations in a straightforward manner, we will first consider different types of deformations with additional properties. Understanding such specific deformations will pave the way to the analysis as well as rigidity theorems for general deformations.

Definition 5.1.3. Let $\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}_{\varepsilon \in I}$, equivalently $\tilde{G} \rightrightarrows \tilde{M} \xrightarrow{\pi} I$ be a deformation of G. Then, the deformation $\tilde{G}$ is called:

- **strict** if $G_\varepsilon = G$ as manifolds for all $\varepsilon \in I$
- ***s*-constant** if it is strict and s_ε is ε-independent
- ***t*-constant** if it is strict and t_ε is ε-independent

- **(s,t)-constant** if it is strict, s-constant and t-constant
- **constant** if $G_\varepsilon = G$ as groupoids for all $\varepsilon \in I$
- **proper** if $\tilde{G} \rightrightarrows \tilde{M}$ is a proper groupoid
- **trivial** if $\tilde{G}$ is equivalent to the constant deformation.

Definition 5.1.4 (Rigid groupoid)**.** The Lie groupoid $G \rightrightarrows M$ is said to be **rigid** if every deformation of G is trivial.

5.2 Deformations

For our exploration of deformations of Lie groupoids, let $G \rightrightarrows M$ be a Lie groupoid with associated Lie algebroid A, and let I be an open interval containing zero throughout the coming subsections.

5.2.1 (s,t)-constant deformations

Recall that an (s,t)-constant deformation of G is a deformation $\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}_{\varepsilon \in I}$ such that $G_\varepsilon = G$ as manifolds and where the source s_ε and the target t_ε of G_ε do not depend on ε. The aim of this section is to investigate the behavior of the groupoid G which undergoes an (s,t)-constant deformation. Naturally, the behavior of the groupoid under deformations is primarily understood via the variation of its structure maps.

Fix an (s,t)-constant deformation $\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}_{\varepsilon \in I}$ of G. Under the restriction of ε-independent source and target maps of G_ε, we get the advantage that any composable pair $(g,h) \in G^{(2)}$ in the original groupoid G is still composable in each G_ε. That is, $m_\varepsilon(g,h)$ would still make sense for any $(g,h) \in G^{(2)}$. We will typically stick to the notation $m_0(g,h) = m(g,h) = gh$ to denote the multiplication of a composable pair (g,h) in the original groupoid.

In light of these thoughts, we study the variation of the multiplication of the groupoid G (see Figure 6) under the deformation $\tilde{G}$. Consider the expression

$$-\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,h) \tag{21}$$

which is a well-defined element in $T_{gh}G$ for any $(g,h) \in G^{(2)}$.

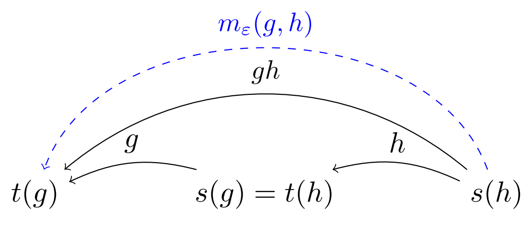

Figure 6: Variation of the multiplication of the groupoid

Remark 5.2.1. *The expression* (21) *lies in* $\ker(ds)|_{gh}$ *and* $\ker(dt)|_{gh}$.

Proof. First of all,

$$\begin{aligned}
-ds\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,h)\right) &= -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (s\circ m_\varepsilon(g,h))\\
&= -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} s(h)\\
&= 0.
\end{aligned}$$

Similarly, we get that

$$-dt\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,h)\right) = -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} t(g) = 0.$$

□

The fact that the expression (21) is killed by ds really means that it is tangent to the source fiber $s^{-1}(s(h))$ at gh, and hence comes by right translation from an element which is tangent to the source fiber $s^{-1}(t(g))$ at the unit $u(t(g))$. That is,

$$-\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,h) \;\in\; \mathrm{im}(dR_{gh} : A_{t(g)} \to T_{gh}G).$$

Note that Remark 5.2.1 points out that (21) is additionally killed by dt which further implies that it comes from an element in $\mathfrak{i}_{t(g)} \subset A_{t(g)}$, where $\mathfrak{i}$ is the isotropy bundle defined in section 4.1.1. Therefore, we get that

$$-\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,h) = dR_{gh}(u_0(g,h)) \;\in\; T_{gh}G$$

for some differentiable 2-cochain $u_0 \in C^2(G,\mathfrak{i})$ with values in $\mathfrak{i}$. By definition, this means that u_0 is a smooth map

$$u_0 : G^{(2)} \longrightarrow \mathfrak{i}, \quad (g,h) \longmapsto u_0(g,h) \in \mathfrak{i}_{t(g)}.$$

Recall from section 4.1.1 that $C^*(G,\mathfrak{i})$ still makes sense for any groupoid G, which is not necessarily regular. What we aim to do next is to show that u_0 is actually a cocycle.

Lemma 5.2.2. $u_0 \in \ker(\delta : C^2(G,\mathfrak{i}) \to C^3(G,\mathfrak{i}))$.

Proof. The main hint in proving this is the fact that the multiplication m_ε of the groupoid G_ε is associative. That is, for all $(g,h,k) \in G^{(3)}$

$$m_\varepsilon(m_\varepsilon(g,h),k) = m_\varepsilon(g,m_\varepsilon(h,k)).$$

Differentiating this associativity equation with respect to ε at $\varepsilon = 0$, we get

$$\begin{gathered}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(m_\varepsilon(g,h),k) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,m_\varepsilon(h,k))\\
dm\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,h), 0_k\right) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(gh,k) = dm\left(0_g, \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(h,k)\right) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,hk)\\
dm(-dR_{gh}(u_0(g,h)),0_k) - dR_{ghk}(u_0(gh,k)) = dm(0_g, -dR_{hk}(u_0(h,k))) - dR_{ghk}(u_0(g,hk))\\
-dR_{ghk}(u_0(g,h)) - dR_{ghk}(u_0(gh,k)) = -dL_g dR_{hk}(u_0(h,k)) - dR_{ghk}(u_0(g,hk)).
\end{gathered}$$

Applying $dR_{(ghk)^{-1}}$ to both sides, we get

$$u_0(g,h) + u_0(gh,k) = dR_{g^{-1}}dL_g(u_0(h,k)) + u_0(g,hk) \tag{22}$$

where $dR_{g^{-1}}dL_g(u_0(h,k)) = \mathrm{ad}_g(u_0(h,k)) = g \cdot u_0(h,k)$, ad being the action of G on $\mathfrak{i}$. Hence equation (22) really means that

$$\delta(u_0)(g,h,k) = g \cdot u_0(h,k) - u_0(gh,k) + u_0(g,hk) - u_0(g,h) = 0$$

for all $(g,h,k) \in G^{(3)}$ and thus u_0 is indeed a cocycle. Note that we have used equations (19) and (20) from section 4.2.3 in our caculations. □

Remark 5.2.3. *Two equivalent (s,t)-constant deformations of G give rise to two differentiable cocycles whose resulting cohomology classes are equal in $H^2(G, \mathfrak{i})$.*

Proof. Let $\tilde{G} = \{G_\varepsilon\}$ and $\tilde{G}' = \{G'_\varepsilon\}$ be two equivalent (s,t)-constant deformations of G and let u_0 and u'_0 be their associated differentiable 2-cocycles respectively. Then, there is a family $\{(F^\varepsilon : G_\varepsilon \to G'_\varepsilon, f^\varepsilon : M_\varepsilon \to M'_\varepsilon)\}$ of groupoid isomorphisms which is the identity at $\varepsilon = 0$. Without loss of generality, assume here that $f_\varepsilon = \mathrm{Id}$ on the units. Then,

$$ds\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(g)\right) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (s \circ F^\varepsilon(g)) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f^\varepsilon(s(g)) = 0$$

since groupoid morphisms commute with the source map. Similarly,

$$dt\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(g)\right) = 0$$

and hence $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(g)$ is tangent to both the source and target fibers. For simplicity, denote

$$\eta(g) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(g).$$

Now, let $X \in C^1(G, \mathfrak{i})$ be a differentiable 1-cochain defined as

$$X(g) := dR_{g^{-1}}\eta(g) \ \in \ \mathfrak{i}_{t(g)} \quad \forall\, g \in G.$$

Next, we use the fact that a groupoid morphism commutes with multiplication and hence $\forall\ (g,h) \in G^{(2)}$

$$m'_\varepsilon(F^\varepsilon(g), F^\varepsilon(h)) = F^\varepsilon(m_\varepsilon(g,h)).$$

Differentiating with respect to ε at $\varepsilon = 0$, we get

$$dm(\eta(g), \eta(h)) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m'_\varepsilon(g,h) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g,h) + \eta(gh).$$

Applying $dR_{(gh)^{-1}}$ to both sides, we get

$$\begin{aligned}
u'_0(g,h) - u_0(g,h) &= dR_{(gh)^{-1}}dm(\eta(g),\eta(h)) - dR_{(gh)^{-1}}\eta(gh) \\
&= dR_{(gh)^{-1}}dR_h(\eta(g)) + dR_{(gh)^{-1}}dL_g(\eta(h)) - dR_{(gh)^{-1}}\eta(gh) \\
&= dR_{g^{-1}}\eta(g) + dR_{g^{-1}}dL_g dR_{h^{-1}}\eta(h) - dR_{(gh)^{-1}}\eta(gh) \\
&= X(g) + g \cdot X(h) - X(gh) \\
&= \delta(X)(g,h)
\end{aligned}$$

by using (3) and the definition of δ. Thus

$$[u_0] = [u'_0] \ \in \ H^2(G, \mathfrak{i}).$$

□

Let us recall from section 4.1.1 that the complex $C^*(G, \mathfrak{i})$ can be identified with the subcomplex of the deformation complex $C^*_{\text{def}}(G)$ where the deformation cochains take values in $\ker(ds)$ and $\ker(dt)$ by the following inclusion (here, specifically in degree 2):

$$r : C^2(G, \mathfrak{i}) \hookrightarrow C^2_{\text{def}}(G).$$

Denote by ξ_0 the image of u_0 under the inclusion r. That is, $\xi_0 \in C^2_{\text{def}}(G)$ such that for all $(g, h) \in G^{(2)}$, $ds(\xi_0(g, h)) = dt(\xi_0(g, h)) = 0$ and

$$\xi_0(g, h) = dR_g(u_0(g, h)) \in T_gG.$$

We conclude this section by establishing a formula for ξ_0 in terms of the variation of the division map $\bar{m}_\varepsilon$ of G_ε which will lead to understanding s-constant deformations for reasons explained in the next section (5.2.2).

Lemma 5.2.4. $\xi_0(g, h) = -dR_{h^{-1}} \left(\frac{d}{d\varepsilon}\big|_{\varepsilon=0} m_\varepsilon(g, h) \right)$

Proof. For any composable pair (g, h), it holds that

$$\begin{aligned}
\xi_0(g, h) &= dR_g(u_0(g, h)) \\
&= dR_{ghh^{-1}}(u_0(g, h)) \\
&= dR_{h^{-1}} \circ dR_{gh}(u_0(g, h)) \\
&= -dR_{h^{-1}} \left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g, h) \right).
\end{aligned}$$

∎

Proposition 5.2.5. *The image ξ_0 of u_0 under r is exactly given by:*

$$\xi_0(g, h) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \bar{m}_\varepsilon(gh, h).$$

Proof. The trick here is to consider the following identity

$$m_\varepsilon(\bar{m}_\varepsilon(gh, h), h) = gh.$$

Differentiating with respect to ε at $\varepsilon = 0$, we get

$$\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(\bar{m}_\varepsilon(gh, h), h) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} gh \\
dm \left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\bar{m}_\varepsilon(gh, h), 0_h \right) &= -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(\bar{m}(gh, h), h) \\
dR_h \left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\bar{m}_\varepsilon(gh, h) \right) &= -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g, h) \quad \text{using equation (19).}
\end{aligned}$$

Applying $dR_{h^{-1}}$ to both sides, we get

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\bar{m}_\varepsilon(gh, h) = -dR_{h^{-1}} \left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} m_\varepsilon(g, h) \right).$$

Therefore, by using Lemma 5.2.4, we arrive to the desired result

$$\xi_0(g, h) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \bar{m}_\varepsilon(gh, h).$$

∎

5.2.2 s-constant deformations

Next, we slightly generalize the previous case by dropping the condition of ε-independence of the target map t_ε of G_ε. That is, we will consider s-constant deformations $\tilde{G} = \{G_\varepsilon\}_{\varepsilon \in I}$ where the target map t_ε of G_ε could depend on ε and study the behavior of the groupoid accordingly. The discussion of (s,t)-constant deformations in the previous section (5.2.1) will be very useful in defining the anticipated deformation 2-cocycle ξ_0 associated to s-constant deformations. Also, the formula for ξ_0 in terms of the division map $\bar{m}_\varepsilon$ given by Proposition 5.2.5 is advantageous in this case since division in a groupoid is defined for elements which have the same source.

It is worth to mention here that looking at the variation of the division map (see Figure 7) instead of the multiplication map to study deformations is a crucial approach, since one can as well recover all the structure maps of the groupoid G from the division and source maps. This is well-explained in the Appendix of [7]. One could for instance also sense this in Proposition 1.2.8.

Definition 5.2.6. Let $\tilde{G} = \{G_\varepsilon\}_{\varepsilon \in I}$ be an s-constant deformation of G. The **deformation cocycle** $\xi_0 \in C^2_{\mathrm{def}}(G)$ associated to $\tilde{G}$ is defined by

$$\xi_0(g,h) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(gh,h) \ \in \ T_gG \quad \forall\ (g,h) \in G^{(2)}.$$

$$\begin{tikzcd}
t_\varepsilon(g) & t(g) & s(g) = t(h) = s(h^{-1}) \arrow[l, bend right, "g"'] \arrow[ll, dashed, bend right=40, "\bar{m}_\varepsilon(gh,h)"'] \arrow[r, bend right, "h^{-1}"'] & s(h) = t(h^{-1}) \arrow[l, bend right, "h"']
\end{tikzcd}$$

Figure 7: Variation of the division of the groupoid

Similar to before, let us first show that ξ_0 is indeed a cocycle.

Lemma 5.2.7. $\xi_0 \in \ker(\delta : C^2_{\mathrm{def}}(G) \to C^3_{\mathrm{def}}(G))$.

Proof. Consider the associativity equation of the division map

$$\bar{m}_\varepsilon(\bar{m}_\varepsilon(u,w), \bar{m}_\varepsilon(v,w)) = \bar{m}_\varepsilon(u,v)$$

which makes sense for arrows u, v, w having the same source. Using the same trick as before, let us differentiate this equation with respect to ε at $\varepsilon = 0$. We get

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(\bar{m}_\varepsilon(u,w), \bar{m}_\varepsilon(v,w)) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(u,v)$$

$$\Rightarrow d\bar{m}\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(u,w), \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(v,w)\right) + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(uw^{-1}, vw^{-1}) - \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(u,v) = 0.$$

Now, by letting $u = ghk$, $v = hk$, $w = k$ for some $(g,h,k) \in G^{(3)}$, we get

$$d\bar{m}\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(ghk,k), \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(hk,k)\right) + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(gh,h) - \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{m}_\varepsilon(ghk,hk) = 0$$

which implies by using the definition of ξ_0 and the differential δ

$$\delta(\xi_0)(g,h,k) = -d\bar{m}(\xi_0(gh,k),\xi_0(h,k)) - \xi_0(g,h) + \xi_0(g,hk) = 0.$$

□

Remark 5.2.8. *The cohomology classes of the deformation cocycles associated to two equivalent s-constant deformations of G are equal in $H^2_{\mathrm{def}}(G)$.*

Proof. Let $\tilde{G} = \{G_\varepsilon\}$ and $\tilde{G}' = \{G'_\varepsilon\}$ be two equivalent s-constant deformations of G and let ξ_0 and ξ'_0 be their associated deformation 2-cocycles respectively. Then, there exists a family $\{F^\varepsilon : G_\varepsilon \to G'_\varepsilon\}$ of groupoid isomorphisms which is the identity at $\varepsilon = 0$.

Let $X \in C^1_{\mathrm{def}}(G)$ be a deformation 1-cochain defined by

$$X(g) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(g) \;\in\; T_gG \quad \forall\, g \in G.$$

Using Proposition 1.2.8, we know that F^ε, being a groupoid isomorphism, commutes with the division map. That is,

$$\bar{m}'_\varepsilon(F^\varepsilon(gh), F^\varepsilon(h)) = F^\varepsilon(\bar{m}_\varepsilon(gh,h))$$

for all $(g,h) \in G^{(2)}$, which gives us after differentiating with respect to ε at $\varepsilon = 0$

$$d\bar{m}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(gh), \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(h)\right) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \bar{m}'_\varepsilon(gh,h) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \bar{m}_\varepsilon(gh,h) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F^\varepsilon(g)$$

which implies that

$$\begin{aligned} d\bar{m}(X(gh),X(h)) + \xi'_0(g,h) &= \xi_0(g,h) + X(g) \\ \Rightarrow \quad \xi'_0(g,h) - \xi_0(g,h) &= \delta(X)(g,h) \end{aligned}$$

by using the definition of the differential δ and thus

$$[\xi_0] = [\xi'_0] \;\in\; H^2_{\mathrm{def}}(G).$$

□

Our goal in the remaining part of this subsection is to show that if the induced deformation cocycles ξ_ε of an s-constant deformation $\{G_\varepsilon\}$ of G come from a smooth time-dependent vector field $X = \{X_\varepsilon\}$ on G, then the flow ψ_X of X is locally a groupoid morphism between the members of the deformation. In order to achieve this goal, we will first reinterpret s-constant deformations and their associated deformation cocycles.

Let $\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}_{\varepsilon\in I}$ be a strict deformation of the Lie groupoid $G \rightrightarrows M$. That is, each $G_\varepsilon = G$ as manifolds. Then, $\tilde{G}$ can be identified with the groupoid $\tilde{G} \rightrightarrows \tilde{M}$ where $\tilde{G} := G \times I$ and $\tilde{M} := M \times I$, with structure maps given by:

- $\tilde{s}(g,\varepsilon) = (s_\varepsilon(g),\varepsilon)$
- $\tilde{t}(g,\varepsilon) = (t_\varepsilon(g),\varepsilon)$
- $\tilde{m}((g,\varepsilon),(h,\varepsilon)) = (m_\varepsilon(g,h),\varepsilon)$
- $\tilde{\bar{m}}((g_1,\varepsilon),(g_2,\varepsilon)) = (\bar{m}_\varepsilon(g_1,g_2),\varepsilon)$

- $\tilde{u}(x,\varepsilon) = (u_\varepsilon(x), \varepsilon)$
- $\tilde{i}(g,\varepsilon) = (i_\varepsilon(g), \varepsilon)$

for $(g,h) \in G^{(2)}$ and $(g_1, g_2) \in G \times_s G$.

This is very much in the spirit of the reinterpretation of the definition of deformations of groupoids as demonstrated in Definition 5.1.1, where the surjective submersion π here is simply the projection on I.

Note that if $\tilde{G}$ is specifically s-constant, one can view the element $\frac{\partial}{\partial\varepsilon}$ as one lying in $C^1_{\text{def}}(\tilde{G})$, using the fact that it is $\tilde{s}$-projectable, and where as a vector field on $\tilde{G}$

$$\frac{\partial}{\partial\varepsilon}(g,\varepsilon) = \left.\frac{d}{d\varepsilon_0}\right|_{\varepsilon_0=0} (g, \varepsilon+\varepsilon_0) \;\in\; T_{(g,\varepsilon)}\tilde{G}. \tag{23}$$

Viewing s-constant deformations from this perspective, one can interpret the associated deformation cocycles in the following way.

Lemma 5.2.9. *Given an s-constant deformation $\tilde{G} = \{G_\varepsilon\}_{\varepsilon\in I}$ of G, define*

$$\xi := -\delta\left(\frac{\partial}{\partial\varepsilon}\right) \;\in\; C^2_{\text{def}}(\tilde{G})$$

where $\frac{\partial}{\partial\varepsilon} \in C^1_{\text{def}}(\tilde{G})$ as described above. Then,

$$\xi_0 = \xi|_{G_0} \;\in\; C^2_{\text{def}}(G_0).$$

Proof. For a composable pair $(g,0),(h,0) \in G_0 = G \times \{0\}$,

$$\begin{aligned}
\xi((g,0),(h,0)) &= -\delta\left(\frac{\partial}{\partial\varepsilon}\right)((g,0),(h,0)) \\
&= d\bar{\tilde{m}}\left(\frac{\partial}{\partial\varepsilon}\tilde{m}((g,0),(h,0)), \frac{\partial}{\partial\varepsilon}(h,0)\right) - \frac{\partial}{\partial\varepsilon}(g,0) \\
&= d\bar{\tilde{m}}\left(\frac{\partial}{\partial\varepsilon}(gh,0), \frac{\partial}{\partial\varepsilon}(h,0)\right) - \frac{\partial}{\partial\varepsilon}(g,0) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \bar{\tilde{m}}((gh,\varepsilon),(h,\varepsilon)) - \frac{\partial}{\partial\varepsilon}(g,0) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\bar{m}_\varepsilon(gh,h),\varepsilon) - \frac{\partial}{\partial\varepsilon}(g,0) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\bar{m}_\varepsilon(gh,h)) \\
&= \xi_0(g,h)
\end{aligned}$$

Note that in the proof we have used equation (23) in the special case where $\varepsilon = 0$ and hence

$$\frac{\partial}{\partial\varepsilon}(g,0) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (g,\varepsilon).$$

□

The proof of Lemma 5.2.9 also implies that $\xi((g,0),(h,0))$ lies indeed in $T_{(g,0)}G_0 \subset T_{(g,0)}\tilde{G}$ for any composable pair $((g,0),(h,0)) \in G_0^{(2)}$.

The following proposition from [18] is a crucial one which will lead to the main result of this subsection. Before stating it, recall from section 1.4 that a time-dependent vector field $V = \{V(\varepsilon)\}$ on a manifold M can be identified with a vector field V^* on $M \times I$ defined as $V^*_{(x,\varepsilon)} = \left(V(\varepsilon)_x, \frac{\partial}{\partial\varepsilon}\right) \;\in\; T_{(x,\varepsilon)}(M \times I)$.

Proposition 5.2.10. *Let $\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}_{\varepsilon\in I}$ be an s-constant deformation of the Lie groupoid $G \rightrightarrows M$ with induced deformation cocycles $\xi_\varepsilon \in C^2_{\text{def}}(G_\varepsilon)$. That is, for all $(g,h) \in G^{(2)}_\varepsilon$*

$$\xi_\varepsilon(g,h) = \left.\frac{d}{d\varepsilon_0}\right|_{\varepsilon_0=0} \bar{m}_{\varepsilon+\varepsilon_0}(m_\varepsilon(g,h),h) \ \in \ T_g G_\varepsilon.$$

Also, let $\tilde{G} \rightrightarrows \tilde{M}$ be the Lie groupoid associated to the deformation $\tilde{G}$. Consider a smooth time-dependent vector field $X = \{X(\varepsilon)\}_{\varepsilon\in I}$ on G. Then, the following are equivalent:

(i) $\delta(X(\varepsilon)) = \xi_\varepsilon \quad \forall\, \varepsilon \in I$

(ii) *the induced vector field $X^* \in \mathfrak{X}(\tilde{G})$ is multiplicative.*

Proof. Firstly, note that $X^* \in \mathfrak{X}(\tilde{G})$ being a multiplicative vector field means that it is $\tilde{s}$-projectable and commutes with the division map by Proposition 1.2.8. In case of s-constant deformations, the $\tilde{s}$-projectability of X^* follows naturally. The commutativity with the division map means that $\forall\ (g,h) \in G^{(2)}$

$$\begin{aligned}
d\bar{\tilde{m}}\left(X^*_{\tilde{m}((g,\varepsilon),(h,\varepsilon))}, X^*_{(h,\varepsilon)}\right) &= X^*_{\bar{\tilde{m}}(\tilde{m}((g,\varepsilon),(h,\varepsilon)),(h,\varepsilon))}\\
\Leftrightarrow \quad d\bar{\tilde{m}}\left(X^*_{(m_\varepsilon(g,h),\varepsilon)}, X^*_{(h,\varepsilon)}\right) &= X^*_{(g,\varepsilon)}\\
\Leftrightarrow \quad d\bar{\tilde{m}}\left(X(\varepsilon)_{m_\varepsilon(g,h)} + \frac{\partial}{\partial\varepsilon}, X(\varepsilon)_h + \frac{\partial}{\partial\varepsilon}\right) &= X(\varepsilon)_g + \frac{\partial}{\partial\varepsilon}\\
\Leftrightarrow \quad d\bar{\tilde{m}}\left(X(\varepsilon)_{m_\varepsilon(g,h)}, X(\varepsilon)_h\right) + d\bar{\tilde{m}}\left(\frac{\partial}{\partial\varepsilon}, \frac{\partial}{\partial\varepsilon}\right) &= X(\varepsilon)_g + \frac{\partial}{\partial\varepsilon}
\end{aligned}$$

where the first expression on the left-hand side is precisely

$$d\bar{\tilde{m}}\left(X(\varepsilon)_{m_\varepsilon(g,h)}, X(\varepsilon)_h\right) = d\bar{m}_\varepsilon\left(X(\varepsilon)_{m_\varepsilon(g,h)}, X(\varepsilon)_h\right)$$

since $X(\varepsilon)$ is tangent to G_ε. Whereas, using equation (23), the second expression is given by

$$\begin{aligned}
d\bar{\tilde{m}}\left(\frac{\partial}{\partial\varepsilon}((m_\varepsilon(g,h)),\varepsilon), \frac{\partial}{\partial\varepsilon}(h,\varepsilon)\right) &= d\bar{\tilde{m}}\left(\left.\frac{d}{d\varepsilon_0}\right|_{\varepsilon_0=0}(m_\varepsilon(g,h),\varepsilon+\varepsilon_0), \left.\frac{d}{d\varepsilon_0}\right|_{\varepsilon_0=0}(h,\varepsilon+\varepsilon_0)\right)\\
&= \left.\frac{d}{d\varepsilon_0}\right|_{\varepsilon_0=0} \bar{\tilde{m}}((m_\varepsilon(g,h),\varepsilon+\varepsilon_0),(h,\varepsilon+\varepsilon_0))\\
&= \left.\frac{d}{d\varepsilon_0}\right|_{\varepsilon_0=0} (\bar{m}_{\varepsilon+\varepsilon_0}(m_\varepsilon(g,h),h),\varepsilon+\varepsilon_0)\\
&= \xi_\varepsilon(g,h) + \frac{\partial}{\partial\varepsilon}.
\end{aligned}$$

Therefore, we get that the multiplicativity of X^* means exactly that

$$d\bar{m}_\varepsilon(X(\varepsilon)_{m_\varepsilon(g,h)}, X(\varepsilon)_h) + \xi_\varepsilon(g,h) + \cancel{\frac{\partial}{\partial\varepsilon}} = X(\varepsilon)_g + \cancel{\frac{\partial}{\partial\varepsilon}}$$

holds $\forall (g,h) \in G^{(2)}_\varepsilon$, which is equivalent to

$$\begin{aligned}
\xi_\varepsilon(g,h) &= -d\bar{m}_\varepsilon(X(\varepsilon)_{m_\varepsilon(g,h)}, X(\varepsilon)_h) + X(\varepsilon)_g\\
&= \delta(X(\varepsilon))(g,h).
\end{aligned}$$

□

We now state the main result of this subsection, which reflects the importance of the 2-parameter dependence of time-dependent flows of time-dependent vector fields in relating members of s-constant deformations.

Proposition 5.2.11. *Let $\tilde{G} = \{G_\varepsilon \rightrightarrows M_\varepsilon\}_{\varepsilon\in I}$ be an s-constant deformation of the Lie groupoid $G \rightrightarrows M$ with induced deformation cocycles $\xi_\varepsilon \in C^2_{\text{def}}(G_\varepsilon)$. Assume that there exists a smooth time-dependent vector field $X = \{X(\varepsilon)\}_{\varepsilon\in I}$ on G such that*

$$\delta(X(\varepsilon)) = \xi_\varepsilon \quad \textit{for } \varepsilon \textit{ small enough.}$$

Denote by $\psi_X^{(\varepsilon_1,\varepsilon_2)}$ and $\psi_V^{(\varepsilon_1,\varepsilon_2)}$ the respective time-dependent flows of X and V where $V = \{V(\varepsilon)\}_{\varepsilon\in I}$ is the time-dependent vector field on M given by $V(\varepsilon) = ds(X(\varepsilon))$.

Then, $\left(\psi_X^{(\varepsilon_1,\varepsilon_2)}, \psi_V^{(\varepsilon_1,\varepsilon_2)}\right)$ is locally a morphism from G_{ε_2} to G_{ε_1} for ε_1 and ε_2 small enough.

Proof. Firstly, Proposition 5.2.10 implies that the induced vector field $X^* \in \mathfrak{X}(\tilde{G})$ of X is multiplicative. And thus, by Proposition 1.4.13, the flow $\psi^\varepsilon_{X^*}$ of X^* is locally a groupoid morphism over the flow $\psi^\varepsilon_{V^*}$ of V^* for ε small enough and where $V^* \in \mathfrak{X}(\tilde{M})$ is the induced vector field of V.

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\psi^\varepsilon_{X^*}"] \arrow[d, shift right, "\tilde{t}"'] \arrow[d, shift left, "\tilde{s}"] & \tilde{G} \arrow[d, shift right, "\tilde{t}"'] \arrow[d, shift left, "\tilde{s}"] \\
\tilde{M} \arrow[r, "\psi^\varepsilon_{V^*}"] & \tilde{M}
\end{tikzcd}$$

But, recall from section 1.4, equation (2), that for all $g \in G,\ x \in M,\ \varepsilon_1, \varepsilon_2$ small enough

$$\psi^{\varepsilon_1}_{X^*}(g, \varepsilon_2) = \left(\psi_X^{(\varepsilon_1+\varepsilon_2,\varepsilon_2)}(g), \varepsilon_1 + \varepsilon_2\right),$$

$$\psi^{\varepsilon_1}_{V^*}(x, \varepsilon_2) = \left(\psi_V^{(\varepsilon_1+\varepsilon_2,\varepsilon_2)}(x), \varepsilon_1 + \varepsilon_2\right).$$

Thus, $\left(\psi_X^{(\varepsilon_1,\varepsilon_2)}, \psi_V^{(\varepsilon_1,\varepsilon_2)}\right)$ is locally a groupoid morphism from $G_{\varepsilon_2} \rightrightarrows M_{\varepsilon_2}$ to $G_{\varepsilon_1} \rightrightarrows M_{\varepsilon_1}$ for ε_1 and ε_2 small enough. $\square$

Observe that, Remark 1.4.14 would further imply the following.

Remark 5.2.12. *If G is additionally proper, then the flow $\psi_X^{(\varepsilon_1,\varepsilon_2)}(g)$ is defined whenever the flows $\psi_V^{(\varepsilon_1,\varepsilon_2)}(s(g))$ and $\psi_V^{(\varepsilon_1,\varepsilon_2)}(t(g))$ are defined.*

5.2.3 General deformations

The main objective of this subsection is to generalize the previously studied particular types of deformations and understand deformations in their general context. Contrary to (s,t)-constant and s-constant deformations, the choice of 2-cocycles for general deformations is not canonical. However, the resulting cohomology classes will be the same. The reinterpretation of the definition of deformations of groupoids from Definition 5.1.1 will be very useful for the discussion.

Recall that a deformation of G may be viewed as a Lie groupoid $\tilde{G} \rightrightarrows \tilde{M}$ together with a surjective submersion $\pi : \tilde{M} \longrightarrow I$ with the property that $\pi \circ \tilde{s} = \pi \circ \tilde{t}$, and so that for each $\varepsilon \in I$, $G_\varepsilon := \tilde{s}^{-1}(M_\varepsilon) \rightrightarrows M_\varepsilon := \pi^{-1}(\varepsilon)$ denotes the groupoid over the fiber $\pi^{-1}(\varepsilon)$ such that $G_0 = G$ as groupoids.

Let $\tilde{G} \rightrightarrows \tilde{M} \to I$ be a general deformation of the Lie groupoid $G \rightrightarrows M$ throughout the subsection. The primary goal is to define and show the existence of the deformation class $[\xi_0] \in H^2_{\mathrm{def}}(G)$ associated to $\tilde{G}$. In order to achieve that, we will proceed in a similar way as the reinterpretation of strict, and specifically s-constant, deformations. Namely, the reinterpretation given in Lemma 5.2.9 will serve as a model of how to construct the deformation class of $\tilde{G}$. The so-called transverse vector fields will be introduced such that they are well-defined elements lying in $C^1_{\mathrm{def}}(\tilde{G})$ and hence they will give us a way to generalize $\frac{\partial}{\partial\varepsilon}$.

Definition 5.2.13 (Transverse vector field)**.** A vector field $\tilde{X} \in \mathfrak{X}(\tilde{G})$ is said to be a **transverse vector field** for the deformation $\tilde{G}$ if it is $\tilde{s}$-projectable to some vector field $\tilde{V} \in \mathfrak{X}(\tilde{M})$ such that $\tilde{V}$ is π-projectable to $\frac{\partial}{\partial\varepsilon}$.

Lemma 5.2.14. *There exist transverse vector fields for the deformation $\tilde{G}$ of G.*

Proof. The idea of proving this is to use the nature of the maps $\tilde{s}$ and π as surjective submersions, normal forms and partitions of unity. □

Proposition 5.2.15. *Let $\tilde{X} \in \mathfrak{X}(\tilde{G})$ be a transverse vector field for the deformation $\tilde{G}$ of G and let $\xi := -\delta(\tilde{X}) \in C^2_{\mathrm{def}}(\tilde{G})$. Then,*

(i) *the restriction of ξ to G_0 induces a cocycle*

$$\xi_0 := \xi|_{G_0} \in C^2_{\mathrm{def}}(G_0),$$

(ii) *the cohomology class $[\xi_0]$ of ξ_0 is independent of the choice of the transverse vector field $\tilde{X}$.*

Proof.

(i) Recall that $G_0 = G$ as groupoids. So, for all composable arrows $(g,h) \in G_0^{(2)} = G^{(2)}$, we need to show that $\xi_0(g,h)$ is tangent to the fiber groupoid G_0 and hence lies in $T_g G_0$. That is, we will show that it vanishes under $d(\pi \circ \tilde{s})$.

$$\begin{aligned} d\pi \circ d\tilde{s}(\xi_0(g,h)) &= d\pi \circ d\tilde{s}(-\delta(\tilde{X})(g,h)) \\ &= d\pi \circ d\tilde{s}\left(d\bar{\tilde{m}}\left(\tilde{X}_{\tilde{m}(g,h)}, \tilde{X}_h\right) - \tilde{X}_g\right) \\ &= d\pi \circ d\tilde{s}\left(d\bar{\tilde{m}}\left(\tilde{X}_{\tilde{m}(g,h)}, \tilde{X}_h\right)\right) - d\pi \circ d\tilde{s}(\tilde{X}_g) \\ &= d\pi \circ d\tilde{t}(\tilde{X}_h) - d\pi \circ d\tilde{s}(\tilde{X}_g) \\ &= d\pi \circ d\tilde{s}(\tilde{X}_h - \tilde{X}_g) \end{aligned}$$

using the fact that $\pi \circ \tilde{s} = \pi \circ \tilde{t}$. Moreover, due to the transversality of $\tilde{X}$, it is $\tilde{s}$-projectable to some vector field $\tilde{V} \in \mathfrak{X}(\tilde{M})$, which is in turn π-projectable to $\frac{\partial}{\partial\varepsilon}$ and thus we get

$$\begin{aligned} d\pi \circ d\tilde{s}(\tilde{X}_h - \tilde{X}_g) &= d\pi(\tilde{V}_{\tilde{s}(h)} - \tilde{V}_{\tilde{s}(g)}) \\ &= \frac{\partial}{\partial\varepsilon}\pi(\tilde{s}(h)) - \frac{\partial}{\partial\varepsilon}\pi(\tilde{s}(g)). \end{aligned}$$

But, $\pi(\tilde{s}(h)) = \pi(\tilde{s}(g)) = 0$ since g and h are arrows in G_0 which is the groupoid over $M_0 = \pi^{-1}\{0\}$, and therefore $\xi_0(g,h)$ vanishes under $d\pi \circ d\tilde{s}$.

(ii) Let $\tilde{X}'$ be another transverse vector field. Define $Y := \tilde{X}' - \tilde{X}$ and let $Y_0 := Y|_{G_0}$, the restriction of Y to G_0. Then, for all g in G_0

$$\begin{aligned} d(\pi \circ \tilde{s})(Y_0(g)) &= d\pi \circ d\tilde{s}(Y_0(g)) \\ &= d\pi \circ d\tilde{s}\left(\tilde{X}'_g - \tilde{X}_g\right) \\ &= 0 \end{aligned}$$

by similar calculations as in (i) since $\tilde{X}'$ and $\tilde{X}$ are transverse vector fields. Hence, $Y_0(g) \in T_g G_0$ for all $g \in G_0$ and so Y_0 lives in $C^1_{\text{def}}(G_0)$.

As a final step, observe that $\forall\ (g,h) \in G_0^{(2)} = G^{(2)}$,

$$\begin{aligned} \xi_0(g,h) - \xi'_0(g,h) &= -\delta(\tilde{X})(g,h) + \delta(\tilde{X}')(g,h) \\ &= \delta(Y_0)(g,h) \end{aligned}$$

where ξ_0 and ξ'_0 are the cocycles associated to $\tilde{X}$ and $\tilde{X}$ respectively and thus

$$[\xi_0] = [\xi'_0] \in H^2_{\text{def}}(G).$$

$\square$

Definition 5.2.16. Choose a transverse vector field $\tilde{X} \in \mathfrak{X}(\tilde{G})$ for the deformation $\tilde{G}$ of G. The **deformation class** associated to $\tilde{G}$ is defined to be the cohomology class

$$[\xi_0] \in H^2_{\text{def}}(G)$$

where $\xi_0 = -\delta(\tilde{X})|_{G_0} \in C^2_{\text{def}}(G_0)$.

This definition is well-defined by Proposition 5.2.15.

In light of the discussion about (s,t)-constant, s-constant and general deformations, one gets the intuition behind proving rigidity results. Essentially, we will try to find multiplicative vector fields $\tilde{X} \in \mathfrak{X}(\tilde{G})$ which are also transverse, and consider their flows to get isomorphisms between the members of the deformation and the groupoid G.

Naturally, the question of the existence of multiplicative transverse vector fields arises. In fact, by the exact sequence given in Proposition 4.1.11, this is equivalent to the existence of elements $[V]$ in $\Gamma(\mathfrak{v})^{\text{inv}}$ which are killed by K, and where V is π-projectable to $\frac{\partial}{\partial \varepsilon}$. More precisely, having an element $[V]$ in $\Gamma(\mathfrak{v})^{\text{inv}}$ means that there exists a vector field $X \in \mathfrak{X}(G)$ which is (s,t)-projectable to $V \in \mathfrak{X}(M)$. Also, $[V] \in \ker(K)$ means that it comes from a multiplicative vector field. The vanishing of the cohomology $H^2_{\text{def}}(G)$ in degree 2 will be useful in that sense as described in the next subsection. It is also important to have in mind the issue of the domains of definition of the flows generated by such vector fields.

5.3 Rigidity results

This subsection will highlight some rigidity results for deformations of Lie groupoids. Bearing in mind results from [21, 22, 23, 24] as described in section 3.3, where the compactness of Lie groups played a significant role in proving rigidity results, a key approach for obtaining rigidity is to consider compact and proper groupoids, as well as proper deformations. Recall that the notion of properness of groupoids generalizes the notion of compactness of groups.

First of all, recall that a Lie groupoid $G \rightrightarrows M$ is said to be proper if G is Hausdorff and if the map $G \to M \times M$, $g \mapsto (s(g), t(g))$ is a proper map. Recall as well that a deformation $\tilde{G} \rightrightarrows \tilde{M} \to I$ of G is called proper if the groupoid $\tilde{G} \rightrightarrows \tilde{M}$ is proper.

We now look at the deformation cohomology in higher degrees for proper groupoids. Note that in degree zero, $H^0_{\text{def}}(G) \cong \Gamma(\mathfrak{i})^{\text{inv}}$ for any Lie groupoid G by Proposition 4.1.2.

Fix a Lie groupoid $G \rightrightarrows M$ with its corresponding Lie algebroid A throughout the subsection and let I be an open interval containing zero as before.

Proposition 5.3.1. *Let the groupoid $G \rightrightarrows M$ be proper. Then,*

(i) $H^k(G, \mathfrak{i}) = 0 \quad \forall\, k \geq 0$,

(ii) $H^k_{\text{def}}(G) = 0 \quad \forall\, k \geq 2$.

For a detailed proof of Proposition 5.3.1, one may refer to [18, Theorem 5.41], which follows the same method as in [5, Proposition 1] for the vanishing of the differentiable cohomology in the proper case. The main idea for part (ii) is to show that for every deformation k-cocycle $c \in C^k_{\text{def}}(G)$, $k \geq 2$, one can construct a deformation $(k-1)$-cochain $X \in C^{k-1}_{\text{def}}(G)$ by using a Haar system and a cut-off function on the proper groupoid G, such that $\delta(X) = c$. The vanishing of the differentiable cohomology $H^k(G, \mathfrak{i})$, $k \geq 0$, of G with coefficients in $\mathfrak{i}$ in part (i) follows directly from [5, Proposition 1].

Corollary 5.3.2. *If the groupoid $G \rightrightarrows M$ is proper, then $H^1_{\text{def}}(G) \cong \Gamma(\mathfrak{v})^{\text{inv}}$.*

Proof. Consider the exact sequence from Proposition 4.1.11:

$$0 \to H^1(G, \mathfrak{i}) \to H^1_{\text{def}}(G) \to \Gamma(\mathfrak{v})^{\text{inv}} \to H^2(G, \mathfrak{i}) \to H^2_{\text{def}}(G)$$

By Proposition 5.3.1, we get that $H^1(G, \mathfrak{i}) = 0$, $H^2(G, \mathfrak{i}) = 0$ and $H^2_{\text{def}}(G) = 0$ and hence $H^1_{\text{def}}(G) \cong \Gamma(\mathfrak{v})^{\text{inv}}$. □

Another direct consequence of Proposition 5.3.1 is the following.

Corollary 5.3.3. *Let $\tilde{G} \rightrightarrows \tilde{M} \to I$ be a proper deformation of the groupoid $G \rightrightarrows M$. For $k \geq 2$, consider a smooth family $\{u^\varepsilon\}_{\varepsilon \in I}$ of deformation cocycles $u^\varepsilon \in C^k_{\text{def}}(G_\varepsilon)$.*

Then, there exists a smooth family $\{X(\varepsilon)\}_{\varepsilon \in I}$ of deformation cochains $X(\varepsilon) \in C^{k-1}_{\text{def}}(G_\varepsilon)$ satisfying

$$\delta(X(\varepsilon)) = u^\varepsilon \quad \forall\, \varepsilon \in I.$$

Proof. The deformation $\tilde{G}$ being proper means that the groupoid $\tilde{G} \rightrightarrows \tilde{M}$ is a proper groupoid. Hence, using the fact that the deformation cohomology vanishes in degree $k \geq 2$, we get the result. □

The next lemma is important and solves the issue of the existence of multiplicative transverse vector fields for proper deformations of the groupoid G where one makes use of the isomorphism given in Corollary 5.3.2.

Lemma 5.3.4. *For any proper deformation $\tilde{G} \rightrightarrows \tilde{M} \xrightarrow{\pi} I$ of the Lie groupoid $G \rightrightarrows M$, there exists a multiplicative transverse vector field $\tilde{X} \in \mathfrak{X}(\tilde{G})$.*

Proof. By the explanation at the end of section 5.2.3 and by corollary 5.3.2, the existence of a multiplicative transverse vector field for $\tilde{G}$ amounts to finding a vector field $\tilde{V} \in \mathfrak{X}(\tilde{M})$ such that it is π-projectable to $\frac{\partial}{\partial \varepsilon}$ and the class $[\tilde{V}] \in \Gamma(\mathfrak{v})^{\text{inv}}$.

Let first $\tilde{X} \in \mathfrak{X}(\tilde{G})$ be a transverse vector field for $\tilde{G}$. (Existence is due to lemma 5.2.14). Then, $\tilde{X}$ is $\tilde{s}$-projectable to some vector field $\tilde{W} \in \mathfrak{X}(\tilde{M})$, which in turn is π-projectable to $\frac{\partial}{\partial \varepsilon}$.

Now, using a Haar system and a cut-off function on the proper groupoid $\tilde{G}$, let $\tilde{V}$ be the vector field on $\tilde{M}$ defined by

$$\tilde{V}_p := \int_p d\tilde{t}(\tilde{X}_a)da \quad \in \; T_p\tilde{M} \quad \text{for each } p \in \tilde{M}$$

where $\int_p d\tilde{t}(\tilde{X}_a)da$ simply denotes $\int_{t^{-1}(p)} d\tilde{t}(\tilde{X})$ as described in [18]. We now verify that our candidate $\tilde{V}$ is π-projectable to $\frac{\partial}{\partial \varepsilon}$.

$$\begin{aligned} d\pi(\tilde{V}_p) &= \int_p d\pi d\tilde{t}(\tilde{X}_a)da \\ &= \int_p d\pi d\tilde{s}(\tilde{X}_a)da \quad \text{since } \pi \circ \tilde{t} = \pi \circ \tilde{s} \\ &= \int_p d\pi(\tilde{W}_{\tilde{s}(a)})da \\ &= \int_p \frac{\partial}{\partial \varepsilon}(\pi(\tilde{s}(a)))da \\ &= \frac{\partial}{\partial \varepsilon}\pi(p). \end{aligned}$$

Next, the aim is to show that $[\tilde{V}] \in \Gamma(\mathfrak{v})$ is actually invariant in the sense that there exists a vector field on $\tilde{G}$ which is $(\tilde{s}, \tilde{t})$-projectable to $\tilde{V}$. Define the vector field $\tilde{X}' \in \mathfrak{X}(\tilde{G})$ by

$$\tilde{X}'_b := \int_{\tilde{s}(b)} d\bar{\tilde{m}} \left(\tilde{X}_{\tilde{m}(b,a)}, \tilde{X}_a \right) da \quad \in \; T_b\tilde{G} \quad \text{for each } b \in \tilde{G}$$

and observe that

$$\begin{aligned} d\tilde{s}(\tilde{X}'_b) &= \int_{\tilde{s}(b)} d\tilde{s} d\bar{\tilde{m}} \left(\tilde{X}_{\tilde{m}(b,a)}, \tilde{X}_a \right) da \\ &= \int_{\tilde{s}(b)} d\tilde{t}(\tilde{X}_a)da \\ &= \tilde{V}_{\tilde{s}(b)} \end{aligned}$$

and by using the left-invariance of the integral

$$\begin{aligned} d\tilde{t}(\tilde{X}'_b) &= \int_{\tilde{s}(b)} d\tilde{t} d\bar{\tilde{m}} \left(\tilde{X}_{\tilde{m}(b,a)}, \tilde{X}_a \right) da \\ &= \int_{\tilde{s}(b)} d\tilde{t} \left(\tilde{X}_{\tilde{m}(b,a)} \right) da \\ &= \int_{\tilde{t}(b)} d\tilde{t}(\tilde{X}_a)da \\ &= \tilde{V}_{\tilde{t}(b)}. \end{aligned}$$

Therefore, $\tilde{X}'$ is indeed $(\tilde{s}, \tilde{t})$-projectable to $\tilde{V}$. □

Lastly, we put together the techniques, ideas and statements developed so far to state and prove some rigidity results of deformations of Lie groupoids. As mentioned earlier, this whole section about deformations of Lie groupoids and in particular the coming rigidity theorems refer to the ones from [18, Section 5.6].

Theorem 5.3.5. *Let the Lie groupoid $G \rightrightarrows M$ be proper. Then, any (s,t)-constant deformation of G is trivial.*

Proof. Let $\tilde{G} = \{G_\varepsilon\}_{\varepsilon \in I}$ be an (s,t)-constant deformation of G. The hidden statement that this deformation is strict implies that the deformation is given by $\tilde{G} \rightrightarrows \tilde{M} \xrightarrow{\pi} I$ with $\tilde{G} = G \times I$ and $\tilde{M} = M \times I$ where π is the projection on the second component. Also, the properness of the Lie groupoid $G \rightrightarrows M$ implies the properness of the Lie groupoid $G \times I \rightrightarrows M \times I$ and hence the properness of the deformation $\tilde{G}$.

Consider the induced deformation cocycles $\xi_\varepsilon \in C^2_{\mathrm{def}}(G_\varepsilon)$. By Corollary 5.3.3, there exists a smooth family $X = \{X(\varepsilon)\}_{\varepsilon \in I}$ of deformation cochains $X(\varepsilon) \in C^1_{\mathrm{def}}(G_\varepsilon)$ satisfying $\delta(X(\varepsilon)) = \xi_\varepsilon$.

Moreover, Proposition 5.2.11 tells us that for ε_1 and ε_2 small enough, the pair of flows $\left(\psi_X^{(\varepsilon_1,\varepsilon_2)}, \psi_V^{(\varepsilon_1,\varepsilon_2)}\right)$ is a local groupoid morphism from G_{ε_2} to G_{ε_1}, where $V = \{V(\varepsilon)\}$ is the time-dependent vector field on M given by $ds(X(\varepsilon)) = V(\varepsilon)$.

Next, we need to make sure that $\psi_X^{(\varepsilon_1,\varepsilon_2)} : G_{\varepsilon_2} \longrightarrow G_{\varepsilon_1}$ is defined on the whole G_{ε_2}. But, the properness of G and Remark 5.2.12 imply that $\psi_X^{(\varepsilon_1,\varepsilon_2)}(g)$ is defined precisely when $\psi_V^{(\varepsilon_1,\varepsilon_2)}(s(g))$ and $\psi_V^{(\varepsilon_1,\varepsilon_2)}(t(g))$ are. And so it is enough to prove that $\psi_V^{(\varepsilon_1,\varepsilon_2)}$ is defined on the whole M.

Since we are dealing with (s,t)-constant deformations, we know from section 5.2.1 that the induced deformation cocycles ξ_ε actually live in the subcomplex $C^2(G, \mathfrak{i}) \hookrightarrow C^2_{\mathrm{def}}(G)$ with values in the isotropy bundle $\mathfrak{i}$. Hence, solving the equation $\delta(X(\varepsilon)) = \xi_\varepsilon$ amounts to finding differentiable cochains $X(\varepsilon) \in C^1(G, \mathfrak{i})$ with values in $\mathfrak{i}$ satisfying the equation. But, elements in the subcomplex $C^*(G, \mathfrak{i})$ of $C^*_{\mathrm{def}}(G)$ are the deformation cochains which are killed by ds and dt, and thus it can be arranged that $V(\varepsilon) = 0$ and the issue with the domains of definition of the flows would be solved.

As a conclusion, we get a family of groupoid isomorphisms $G_\varepsilon \xrightarrow{\psi_X^{(0,\varepsilon)}} G_0$, which is smoothly parametrized by ε such that it is the identity at $\varepsilon = 0$ by using the properties of a time-dependent flow of a time-dependent vector field given in Theorem 1.4.5. Therefore, the deformation $\tilde{G}$ of G is a trivial deformation. $\square$

In [28, Theorem 7.1], Weinstein has already stated and proved that any (s,t)-constant deformation of a proper regular Lie groupoid is trivial, where he makes use of the isotropy bundle $\mathfrak{i}$ as a well-defined bundle over the base of the groupoid. Recall from section 4.1.1, that $\mathfrak{i}$ is indeed a vector bundle in the regular case. Here, we see that the rigidity result still holds even when the condition of regularness has been dropped, keeping in mind again from section 4.1.1 that $C^*(G, \mathfrak{i})$ still makes sense as a vector space.

Theorem 5.3.6. *Let the Lie groupoid $G \rightrightarrows M$ be compact. Then, any s-constant deformation of G is trivial.*

Before proving the statement, recall that the groupoid $G \rightrightarrows M$ is said to be compact if G is Hausdorff and compact as a manifold. Note that this in particular implies that the groupoid is proper. It follows from this definition that the base M of the groupoid is also compact, since it can be viewed as a closed embedded submanifold of G.

Proof. Such a deformation is also proper and all the statements about proper deformations and proper groupoids are valid here. Thus, this theorem can be proved by basically following the lines of the proof of the previous theorem until a local groupoid morphism $\left(\psi_X^{(\varepsilon_1,\varepsilon_2)}, \psi_V^{(\varepsilon_1,\varepsilon_2)}\right)$ is obtained from G_{ε_2} to G_{ε_1} for ε_1 and ε_2 small enough. In order to prove that $\psi_V^{(\varepsilon_1,\varepsilon_2)}$ is defined on the whole M in this case, one uses the compactness of M and Corollary 1.4.8. $\square$

Note that s-constant deformations of proper groupoids are not necessarily trivial (see Example 3.3.3 or [18, Remark 73]). In general, we have the following results from [18], which we only state here.

Theorem 5.3.7. *Let the Lie groupoid $G \rightrightarrows M$ be compact. Then,*

(i) *any strict deformation of G is trivial.*

(ii) *any proper deformation of G is locally trivial.*

Recently, the authors in [9] have provided another method of proving such rigidity results in the compact and proper cases. Instead of a cohomological point of view as in [7, 18, 28], they approach from a more geometrical perspective to understand rigidity by using so-called groupoid fibrations and associated linearization results.

For further exploration of rigidity results, one may refer to [18], where general proper deformations have been thoroughly studied from a semi-local perspective, as well as some applications to linearization problems have been presented.

6 Deformations of group representations revisited

In light of the discussion on deformations of Lie group actions and homomorphisms in section 3, we proceed by giving precise definitions of deformations of Lie group representations. Furthermore, for a given representation φ of a group G on a vector space V in the sense of a smooth linear action φ of G on V, we will consider the associated action groupoid $G \ltimes V$. It will be clear that a deformation of φ would naturally give rise to an s-constant deformation of $G \ltimes V$. We conclude the section with open questions about how the rigidity of representations and of the corresponding action groupoids could be related.

Throughout the coming subsections, let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let V be a finite-dimensional real vector space. I denotes an open interval containing zero as before. Let

$$\psi : G \longrightarrow \mathrm{GL}(V) = \mathrm{Aut}(V)$$

be a given representation of G on V. Recall from Definition 1.5.5 that the representation ψ is equivalent to the smooth linear action

$$\varphi : G \times V \longrightarrow V, \quad \varphi(g, v) = g \cdot v = \psi(g)(v).$$

6.1 Basic definitions

Definition 6.1.1 (Deformation of a representation)**.** A **smooth deformation** of the representation ψ of G on V is a smooth deformation $\tilde{\psi} = \{\psi_\varepsilon\}_{\varepsilon \in I}$ of the Lie group homomorphism ψ. That is, it is a family

$$\{\psi_\varepsilon : G \longrightarrow \mathrm{GL}(V)\}_{\varepsilon \in I}$$

of representations of G on V which is smoothly parametrized by ε and such that $\psi_0 = \psi$.

Note that a smooth deformation $\tilde{\psi} = \{\psi_\varepsilon\}_{\varepsilon \in I}$ of ψ is corresponds to the smooth deformation $\tilde{\varphi} = \{\varphi_\varepsilon\}_{\varepsilon \in I}$ of the associated action φ of ψ, where each φ_ε is given by

$$\varphi_\varepsilon(g, v) := \psi_\varepsilon(g)(v) \quad \forall\, g \in G,\ v \in V,\ \varepsilon \in I.$$

Definition 6.1.2 (Constant deformation)**.** A deformation $\{\psi_\varepsilon\}_{\varepsilon \in I}$ of ψ is called **constant** if $\psi_\varepsilon = \psi\ \forall\ \varepsilon \in I$.

Definition 6.1.3 (Equivalent deformations)**.** Two deformations $\{\psi_\varepsilon\}_{\varepsilon \in I}$ and $\{\psi'_\varepsilon\}_{\varepsilon \in I}$ of ψ are said to be **equivalent** if $\forall\ \varepsilon \in I,\ \exists\ f_\varepsilon \in \mathrm{Aut}(V)$ such that $f_\varepsilon \circ \psi_\varepsilon(g) \circ f_\varepsilon^{-1} = \psi'_\varepsilon(g)$ for all $g \in G$ and where $f_0 = \mathrm{Id}_V$.

Definition 6.1.4 (Trivial deformation)**.** A deformation $\tilde{\psi}$ of ψ is called **trivial** if $\tilde{\psi}$ is equivalent to the constant deformation.

Definition 6.1.5 (Locally trivial deformation)**.** A deformation $\tilde{\psi} = \{\psi_\varepsilon\}_{\varepsilon \in I}$ of ψ is called **locally trivial** if for ε small enough, $\exists\ f_\varepsilon \in \mathrm{Aut}(V)$ such that $f_\varepsilon \circ \psi(g) \circ f_\varepsilon^{-1} = \psi_\varepsilon(g)$ for all $g \in G$ and where $f_0 = \mathrm{Id}_V$.

Definition 6.1.6 (Rigid representation)**.** The representation ψ of G on V is said to be **rigid** if every deformation of ψ is trivial.

For the rest of the discussion, fix a deformation $\tilde{\psi} = \{\psi_\varepsilon\}_{\varepsilon\in I}$ (equivalently $\tilde{\varphi} = \{\varphi_\varepsilon\}_{\varepsilon\in I}$) of the representation ψ (equivalently φ) of G on V.

Let $\mathfrak{gl}(V) = \mathrm{Lie}(\mathrm{GL}(V))$. We get from section 3.2, that the deformation $\tilde{\psi}$ of ψ gives rise to a differentiable 1-cocycle $w \in C^1(G, \mathfrak{gl}(V))$, defined as

$$w(g) := -dR_{\psi(g^{-1})}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \psi_\varepsilon(g)\right) \in \mathfrak{gl}(V) \quad \forall\, g \in G$$

(see Definition 3.2.7) whose resulting cohomology class depends only on the equivalence class of the deformation $\tilde{\psi}$ (see Remark 3.2.10).

6.2 Deformation of the associated action groupoid

Next, let us consider the action groupoid $G \ltimes V$ associated to the action φ of G on V, as defined and explained in section 2.3.

$$\begin{array}{ccc} G \ltimes V & = & G \times V \\ & & t \Big\downarrow\Big\downarrow s \\ & & V \end{array}$$

The deformation $\tilde{\varphi} = \{\varphi_\varepsilon\}_{\varepsilon\in I}$ of φ will lead to a natural s-constant deformation of $G \ltimes V$, denoted by Γ and defined in the following way:

- $s_\varepsilon(g, v) = v$
- $t_\varepsilon(g, v) = \varphi_\varepsilon(g, v)$
- $m_\varepsilon((g, \varphi_\varepsilon(h, v)), (h, v)) = (gh, v)$
- $u_\varepsilon(v) = (e, v)$
- $i_\varepsilon(g, v) = (g^{-1}, \varphi_\varepsilon(g, v))$

for all $g, h \in G$, $v \in V$.

Lemma 6.2.1. *The s-constant deformations of $G \ltimes V$ coming from two equivalent deformations of $\varphi : G \curvearrowright V$ are equivalent.*

Proof. Let $\tilde{\varphi} = \{\varphi_\varepsilon\}_{\varepsilon\in I}$ and $\tilde{\varphi}' = \{\varphi'_\varepsilon\}_{\varepsilon\in I}$ be equivalent deformations of φ. Denote by Γ and Γ' the resulting s-constant deformations of $G \ltimes V$ respectively. Then, there exists a family $\{f_\varepsilon\}_{\varepsilon\in I}$ of automorphisms of V, which is smoothly parametrized by $\varepsilon \in I$ with $f_0 = \mathrm{Id}_V$, and such that each f_ε is a G-equivariant map in the sense that the following diagram commutes:

$$\begin{array}{ccc} G \times V & \xrightarrow{\varphi'_\varepsilon} & V \\ {\scriptstyle \mathrm{Id}_G \times f_\varepsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle \exists\, f_\varepsilon} \\ G \times V & \xrightarrow{\varphi_\varepsilon} & V \end{array}$$

This implies further that the following diagram commutes:

$$
\begin{array}{ccc}
G \times V & \xrightarrow{\mathrm{Id}_G \times f_\varepsilon} & G \times V \\
t'_\varepsilon \Big\downarrow\Big\downarrow s & & t_\varepsilon \Big\downarrow\Big\downarrow s \\
V & \xrightarrow{\quad f_\varepsilon \quad} & V
\end{array}
$$

since for all $g \in G$, $v \in V$

- $s(g, f_\varepsilon(v)) = f_\varepsilon(v) = f_\varepsilon(s(g, v))$,
- $t_\varepsilon(g, f_\varepsilon(v)) = \varphi_\varepsilon(g, f_\varepsilon(v)) = f_\varepsilon(\varphi'_\varepsilon(g, v)) = f_\varepsilon(t'_\varepsilon(g, v))$.

Moreover, $\forall\ g, h \in G$, $v \in V$

- $m_\varepsilon((g, f_\varepsilon(\varphi'_\varepsilon(h, v))), (h, f_\varepsilon(v))) = (gh, f_\varepsilon(v))$
- $(\mathrm{Id}_G \times f_\varepsilon)(m'_\varepsilon((g, \varphi'_\varepsilon(h, v)), (h, v))) = (\mathrm{Id}_G \times f_\varepsilon)(gh, v) = (gh, f_\varepsilon(v))$.

Therefore, $\{(\mathrm{Id}_G \times f_\varepsilon, f_\varepsilon)\}$ gives a smooth family of groupoid isomorphisms between the members of the deformations Γ and Γ' of $G \ltimes V$ such that at $\varepsilon = 0$, it is the identity. □

Recall from section 5.2.2 that to any s-constant deformation of a groupoid, there is an associated deformation 2-cocycle (see Definition 5.2.6). We now describe explicitly what this cocycle ξ_0 is in our specific case. For a composable pair $(g, h \cdot v)$, $(h, v) \in G \times V$,

$$
\begin{aligned}
\xi_0((g, h \cdot v), (h, v)) &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \bar{m}_\varepsilon(m((g, h \cdot v), (h, v)), (h, v)) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \bar{m}_\varepsilon((gh, v), (h, v)) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} m_\varepsilon((gh, v), i_\varepsilon(h, v)) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} m_\varepsilon((gh, v), (h^{-1}, \varphi_\varepsilon(h, v))) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (g, \varphi_\varepsilon(h, v)) \\
&= \left(0_g, \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \varphi_\varepsilon(h, v)\right).
\end{aligned}
$$

Hence, using the identification (4), the deformation 2-cocycle $\xi_0 \in C^2_{\mathrm{def}}(G \ltimes V)$ associated to the s-constant deformation Γ of $G \ltimes V$ is given by

$$
\xi_0 : G \times G \times V \longrightarrow TG \times TV, \quad \xi_0(g, h, v) = \left(0_g, \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \varphi_\varepsilon(h, v)\right) \in T_gG \times T_{h \cdot v}V.
$$

Remark 6.2.2. *Two equivalent deformations of φ give rise to two deformation 2-cocycles in $C^2_{\mathrm{def}}(G \ltimes V)$ whose cohomology classes are equal in $H^2_{\mathrm{def}}(G \ltimes V)$.*

Proof. Result follows directly from Lemma 6.2.1 and Remark 5.2.8. □

6.3 Rigidity results

We state here a rigidity theorem which is an immediate consequence of results from sections 3.2-3.3.

Theorem 6.3.1. *If G is compact and connected, then every smooth deformation of the representation $\psi : G \longrightarrow \mathrm{GL}(V)$ of G on V is locally trivial.*

Proof. Note that the differentiable cohomology in degree $k \geq 1$ of a compact group with coefficients in any representation vanishes (see [5, Proposition 1]). In particular, $H^1(G, \mathfrak{gl}(V)) = 0$. Moreover, as G is connected and ψ is smooth, we get that $\psi(G)$ is connected. The fact that the Lie group homomorphism ψ maps the identity e of G to the identity Id_V of $\mathrm{GL}(V)$ implies that $\psi(G)$ lies in the connected component of Id_V. Hence, result follows from Theorem 3.3.5. □

In consideration of possible relations between rigidity of representations and of the corresponding action groupoids, two natural questions can be asked:

Q1 : Does rigidity of group representations imply rigidity of the associated action groupoid?

Q2 : Does rigidity of the action groupoid imply rigidity of the underlying group representation?

The answer of Question 1 is negative and can be shown by the following counterexample.

Example 6.3.2. Let G be a compact and connected Lie group. Consider the action $\varphi : G \times \mathbb{R}^{n+3} \to \mathbb{R}^{n+3}$, the deformation $\{\varphi_\varepsilon\}$ and φ_0 as defined in Example 3.3.3. Note that φ_0 is a representation of G, which is (locally) rigid because of Theorem 6.3.1. However, as an action, it is not rigid since $\{\varphi_\varepsilon\}$ is a non-trivial deformation of it as shown in Example 3.3.3. Thus, the corresponding action groupoid will also not be rigid.

Further exploration on rigidity results of group representations, as well as the answer of Question 2 is beyond the scope of this thesis and is left open for future research.

7 Deformations of groupoid representations

This section will provide one approach to deformations of groupoid representations. Due to their subtle nature, this proves to be more challenging and involved than the case of groups. The discussion is built in a way that is parallel to the study of deformations of group representations as in previous sections.

7.1 Representations of groupoids

Let $G \rightrightarrows M$ be a Lie groupoid. First of all, recall from Definition 1.5.9 that a representation of G is a vector bundle $E \to M$ together with a smooth linear action of G on E, that is, for each arrow $g : y \curvearrowleft x$ in G, $g : E_x \to E_y$ is a linear isomorphism. In this subsection, we show that this definition is equivalent to having some Lie groupoid morphism from G to the so-called general linear groupoid of the vector bundle E, similar to the case of group representations.

Definition 7.1.1 (General linear groupoid of a vector bundle)**.** Let E be a vector bundle over M. The **general linear groupoid** of E is denoted by $\mathrm{GL}(E)$ and defined to be the set of all linear isomorphisms from E_x to E_y for all $x, y \in M$.

Let $x, y, z \in M$ and let $\varphi : E_x \to E_y$ and $\varphi' : E_z \to E_x$ be elements of $\mathrm{GL}(E)$. Construct a groupoid structure on $\mathrm{GL}(E)$ with base M by the following structure maps:

- $s(\varphi) = x$
- $t(\varphi) = y$
- $m(\varphi, \varphi') = \varphi \circ \varphi'$
- $u(x) : E_x \to E_x$ is the identity map on E_x
- $i(\varphi) = \varphi^{-1} : E_y \to E_x$ is the inverse map of φ.

Note that $\mathrm{GL}(E)$ has the structure of a Lie groupoid as shown in [15, Example 1.1.12].

Now, let $E \to M$ be a given vector bundle. Consider the representation (E, φ) of $G \rightrightarrows M$, where φ is the smooth action of G on E and denote by $\varphi_g : E_{s(g)} \to E_{t(g)}$ the linear isomorphism induced by each arrow g. Define the Lie groupoid morphism (ψ, Id_M) by $\psi(g) := \varphi_g$, as shown in the following diagram.

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathrm{GL}(E) \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M \arrow[r, "\mathrm{Id}_M"] & M \end{tikzcd}$$

The usage of the same letters for the structure maps of G and $\mathrm{GL}(E)$ should be clear from the context. Now, this is indeed a goupoid morphism since for all $g, h \in G^{(2)}$

- $s(\psi(g)) = s(g) = \mathrm{Id}(s(g))$
- $t(\psi(g)) = t(g) = \mathrm{Id}(t(g))$
- $\psi(m(g, h)) = \psi(gh) = \varphi_{gh} = \varphi_g \circ \varphi_h = m(\varphi_g, \varphi_h) = m(\psi(g), \psi(h))$

using the axiom of associativity of actions of groupoids.

The reverse direction of obtaining a representation of G from a given Lie groupoid morphism (ψ, Id_M) from G to $\mathrm{GL}(E)$ follows by a similar manner.

Bearing the above discussion in mind, the coming subsections will mainly deal with deformations of Lie groupoid morphisms. As our aim is to understand deformations of groupoid representations, it will be enough to look at groupoid morphisms between groupoids over the same base such that they are the identity over the base.

7.2 Cohomology of groupoid morphisms

Similar to before, deformations of Lie groupoid morphisms will give rise to deformation cocycles living in the so-called deformation complex of groupoid morphisms. This complex is very similar to the deformation complex of Lie groupoids and was first introduced by Crainic, Mestre and Struchiner in [7, Remark 8.2]. In this subsection, we give the definition of this complex and its associated cohomology. We restrict to the case of groupoids over the same base.

Let $G \rightrightarrows M$ and $H \rightrightarrows M$ be two Lie groupoids and let (F, Id_M) be a Lie groupoid morphism from G to H.

Definition 7.2.1 (Deformation cohomology of a groupoid morphism)**.** The **deformation cohomology** $H^*_{\mathrm{def}}(F)$ of the morphism F is the cohomology of the **deformation complex** $(C^*_{\mathrm{def}}(F), \delta)$ of F which is defined as:

- $k \geq 1$: the k-cochains $c \in C^k_{\mathrm{def}}(F)$ are the smooth maps

$$c : G^{(k)} \longrightarrow TH, \quad (g_1, ..., g_k) \mapsto c(g_1, ..., g_k) \in T_{F(g_1)}H$$

  such that $ds_H \circ c(g_1, ..., g_k)$ does not depend on g_1, and where the differential is defined as

$$\delta : C^k_{\mathrm{def}}(F) \longrightarrow C^{k+1}_{\mathrm{def}}(F)$$
$$\begin{aligned}(\delta c)(g_1, ..., g_{k+1}) := & - d\bar{m}_H(c(g_1 g_2, ..., g_{k+1}), c(g_2, ..., g_{k+1})) \\ & + \sum_{i=2}^{k} (-1)^i c(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\ & + (-1)^{k+1} c(g_1, ..., g_k).\end{aligned}$$

- $k = 0$: the 0-cochains $c \in C^0_{\mathrm{def}}(F)$ are the smooth sections of the Lie algebroid A_H of H. Hence, $C^0_{\mathrm{def}}(F) = \Gamma(A_H)$. The differential is defined as $\delta : C^0_{\mathrm{def}}(F) \to C^1_{\mathrm{def}}(F)$ with

$$(\delta c)(g) := \overrightarrow{c}_{F(g)} + \overleftarrow{c}_{F(g)}$$

  where $\overrightarrow{c}$ and $\overleftarrow{c}$ are the induced right- and left-invariant vector fields of c respectively.

Remark 7.2.2. $H^*_{\mathrm{def}}(G) = H^*_{\mathrm{def}}(\mathrm{Id}_G)$.

For further details of this complex, especially about its relation to the deformation complexes of G and H, one may refer to [7, Remark 8.2].

7.3 Deformations of groupoid morphisms

The goal of this subsection is to define deformations of groupoid morphisms and show how they give rise to deformation 1-cocycles. The discussion will be parallel to that of deformations of group homomorphisms as in section 3.2. Throughout the subsection, let $G \rightrightarrows M$ and $H \rightrightarrows M$ be two Lie groupoids over the same base M and let (F, Id_M) be a Lie groupoid morphism from G to H. I denotes an open interval containing zero.

Definition 7.3.1 (Deformation of a Lie groupoid morphism)**.** A **smooth deformation** of (F, Id_M) is a family $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ of Lie groupoid morphisms from G to H which is smoothly parametrized by $\varepsilon \in I$ and such that $F_0 = F$.

Definition 7.3.2 (Constant deformation)**.** A deformation $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ of (F, Id_M) is called **constant** if $F_\varepsilon = F \ \forall\ \varepsilon \in I$.

To define equivalent deformations of groupoid morphisms through conjugation, the subtlety in the case of groupoids lies in the fact that right and left translations are only defined on the source and target fibers respectively. This issue is solved via the notion of bisections, as defined below.

Definition 7.3.3 (Bisections)**.** A **bisection** of the groupoid G is a smooth section σ of the source map s with $t \circ \sigma$ a diffeomorphism. That is, it is a smooth map $\sigma : M \to G$ such that $s \circ \sigma = \mathrm{Id}_M$ and where $t \circ \sigma$ is a diffeomorphism on M. Denote by $\mathrm{Bis}(G)$ the set of all bisections of G.

It is easy to prove that $\mathrm{Bis}(G)$ is actually a group under the operation

$$(\sigma\tau)(x) := \sigma(t \circ \tau(x))\tau(x) \quad \forall\ \sigma,\ \tau \in \mathrm{Bis}(G),\ x \in M$$

and with identity the unit map $u_G : M \to G$ of G as shown in [15, Proposition 1.4.2].

Note that, right and left translations corresponding to bisections can be now defined on the whole G (see [15, p. 22, 24]). We now define the automorphism Ψ via conjugation using bisections. Let $\sigma \in \mathrm{Bis}(G)$. Define

$$\Psi_\sigma : G \longrightarrow G, \quad g \longmapsto \sigma(t(g))g\sigma(s(g))^{-1}.$$

Definition 7.3.4 (Equivalent deformations)**.** Two smooth deformations $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ and $\{(F'_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ of (F, Id_M) are said to be **equivalent** if $\forall\ \varepsilon \in I,\ \exists\ \sigma_\varepsilon \in \mathrm{Bis}(H)$ such that $\Psi_{\sigma_\varepsilon} \circ F_\varepsilon = F'_\varepsilon$ in the sense that the following diagram commutes,

$$\begin{tikzcd} G \arrow[r, "F_\varepsilon"] \arrow[dr, "F'_\varepsilon"'] & H \arrow[d, "\Psi_{\sigma_\varepsilon}"] \\ & H \end{tikzcd}$$

and such that σ_ε varies smoothly with respect to ε and where $\sigma_0 = u_H : M \to H$ is the unit map of H (i.e. the identity of the group $\mathrm{Bis}(H)$).

Definition 7.3.5 (Trivial deformation)**.** A deformation $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ of (F, Id_M) is called **trivial** if it is equivalent to the constant deformation.

Definition 7.3.6 (Rigid morphism)**.** The Lie groupoid morphism (F, Id_M) is said to be **rigid** if every deformation of (F, Id_M) is trivial.

Next, we examine how a deformation of a Lie groupoid morphism gives rise to a deformation 1-cocycle.

Let $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ be a given deformation of the Lie groupoid morphism (F, Id_M).

Definition 7.3.7. The **deformation cocycle** $c \in C^1_{\mathrm{def}}(F)$ associated to the deformation $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ of F is defined by

$$c(g) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(g) \;\in\; T_{F(g)}H \quad \forall\, g \in G.$$

It is important to realize that c lies in $C^1_{\mathrm{def}}(F)$. Indeed, note that

$$\begin{aligned} ds(c(g)) &= ds\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(g)\right) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} s\circ F_\varepsilon(g) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Id}(s(g)) \\ &= 0. \end{aligned}$$

Let us now prove that c is a cocycle.

Lemma 7.3.8. *$c \in \ker(\delta : C^1_{\mathrm{def}}(F) \to C^2_{\mathrm{def}}(F))$.*

Proof. Since each of F_ε is a Lie groupoid morphism from G to H and by Proposition 1.2.8, we have for all $(u_1, u_2) \in G \times_{s_G} G$

$$F_\varepsilon(\bar{m}_G(u_1, u_2)) = \bar{m}_H(F_\varepsilon(u_1), F_\varepsilon(u_2))$$

Differentiating this identity with respect to ε at $\varepsilon = 0$, we get

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(\bar{m}_G(u_1, u_2)) = d\bar{m}_H\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(u_1), \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(u_2)\right)$$

and by letting $u_1 = g_1g_2$, $u_2 = g_2$, we get

$$\begin{aligned} -d\bar{m}_H\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(g_1g_2), \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(g_2)\right) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(\bar{m}_G(g_1g_2, g_2)) = 0 \\ \Rightarrow \quad -d\bar{m}_H(c(g_1g_2), c(g_2)) + c(g_1) = 0 \\ \Rightarrow \quad \delta(c)(g_1, g_2) = 0. \end{aligned}$$

□

7.4 Rigidity results

We state here a generalization of Theorem 3.3.5 to the case of Lie groupoids.

Theorem 7.4.1. *Let $G \rightrightarrows M$ and $H \rightrightarrows M$ be Lie groupoids over a compact base M and let (F, Id_M) be a Lie groupoid morphism from G to H. If $H^1_{\mathrm{def}}(F) = 0$, then every smooth deformation of (F, Id_M) is locally trivial.*

Proof. Let $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ be a smooth deformation of (F, Id_M). Our aim is to show that for ε small enough, there are bisections $\sigma_\varepsilon \in \mathrm{Bis}(H)$ that vary smoothly with respect to ε, $\sigma_0 = u_H$ and such that

$$\sigma_\varepsilon(t(g))F(g)\sigma_\varepsilon(s(g))^{-1} = F_\varepsilon(g), \quad \forall\, g \in G. \tag{24}$$

First of all, let us consider the deformation 1-cocycle $c \in C^1_{\mathrm{def}}(F)$ associated to the deformation $\{(F_\varepsilon, \mathrm{Id}_M)\}$, and the resulting cohomology class $[c] \in H^1_{\mathrm{def}}(F)$. Due to the vanishing of $H^1_{\mathrm{def}}(F)$, there exists an element $\alpha \in C^0_{\mathrm{def}}(F) = \Gamma(A_H)$ with

$$\begin{aligned} & c(g) = \delta(\alpha)(g) \quad \forall\, g \in G \\ \Leftrightarrow \quad & \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(g) = \overrightarrow{\alpha}_{F(g)} + \overleftarrow{\alpha}_{F(g)}, \quad \forall\, g \in G. \end{aligned} \tag{25}$$

Next, we attempt to find the bisections σ_ε such that (24) is satisfied. From [15, Proposition 3.6.1] and due to the compactness of M, we get that for $\alpha \in \Gamma(A_H)$ and for ε small enough, there exists a smooth family of bisections on H, denoted by $\exp(\varepsilon\alpha)$ satisfying similar properties as the usual exponential map in Lie groups. Define

$$\sigma_\varepsilon := \exp(\varepsilon\alpha).$$

Lastly, to show that (24) holds with our particular σ_ε, we differentiate

$$\sigma_\varepsilon(t(g))F(g)\sigma_\varepsilon(s(g))^{-1}$$

with respect to ε at $\varepsilon = 0$. For $g \in G$, let $s(F(g)) = s(g) = x$ and $t(F(g)) = t(g) = y$.

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \sigma_\varepsilon(y)F(g)\sigma_\varepsilon(x)^{-1} \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\alpha)(y)F(g)\exp(\varepsilon\alpha)(x)^{-1} \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\alpha)(y)F(g)\exp(0\alpha)(x)^{-1} \\
&\quad + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(0\alpha)(y)F(g)\exp(\varepsilon\alpha)(x)^{-1} \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} R_{F(g)}(\exp(\varepsilon\alpha)(y)) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_{F(g)}\exp(\varepsilon\alpha)(x)^{-1} \\
&= dR_{F(g)}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\exp(\varepsilon\alpha)(y))\right) + dL_{F(g)}di\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\alpha)(x)\right) \\
&= dR_{F(g)}\alpha_y + dL_{F(g)}di\alpha_x \\
&= \overrightarrow{\alpha}_{F(g)} + \overleftarrow{\alpha}_{F(g)} \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F_\varepsilon(g)
\end{aligned}$$

by using properties (i), (ii) and (iv) of [15, Proposition 3.6.1] and (25). Therefore, for all g in G, we get that

$$\sigma_\varepsilon(t(g))F(g)\sigma_\varepsilon(s(g))^{-1} = F_\varepsilon(g), \quad \text{for } \varepsilon \text{ small enough}$$

which implies that the deformation $\{(F_\varepsilon, \mathrm{Id}_M)\}_{\varepsilon\in I}$ of (F, Id_M) is locally trivial. $\square$

Corollary 7.4.2. *Let $G \rightrightarrows M$ be a Lie groupoid over a compact base M and let (E, φ) be a representation of G. Denote by (ψ, Id_M) the corresponding Lie groupoid morphism from G to $\mathrm{GL}(E)$. If $H^1_{\mathrm{def}}(\psi) = 0$, then every smooth deformation of (ψ, Id_M) is locally trivial.*

Remark 7.4.3. *In contrast to Lie groups, the condition of properness, or even compactness of the Lie groupoid $G \rightrightarrows M$ does not guarantee the vanishing of $H^1_{\mathrm{def}}(\psi)$ in degree 1. However, a natural question would be if the compactness of the Lie groupoid $G \rightrightarrows M$ would imply that the cocycle associated to the deformation of the representation ψ is actually a coboundry.*

This ends our discussion on deformations of Lie groupoid morphisms and of Lie groupoid representations. As mentioned earlier, this is one approach to this topic and is open for further research and investigation.

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