BONN INTERNATIONAL GRADUATE SCHOOL OF MATHEMATICS

Deformation theory of Lie groupoids and differentiable stacks

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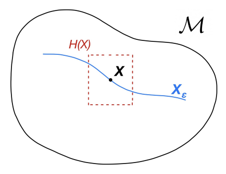
Deformations via differential graded Lie algebras

Deformation theory is useful in many areas of Mathematics. It starts from the observation that often a structure X, like the multiplication map of an algebra, a complex structure, a riemannian metric, etc. depends smoothly on real parameters. This means that the set $\mathcal{S}$ of all such structures is equipped with a notion of smooth paths. A **deformation** of X is a smooth path $\mathbb{R} \to \mathcal{S}$, $\varepsilon \mapsto X_\varepsilon$ through $X = X_0$. If all structures X_ε are isomorphic, we call it a **trivial deformation**. Non-trivial deformations, i.e. deformations up to isomorphism can be viewed as paths in the moduli space $\mathcal{M}$ of structures of type X modulo isomorphisms.



Differentiating a smooth deformation at $\varepsilon = 0$ yields an **infinitesimal deformation**. Owing to the structure of $\mathcal{M}$, which typically has singular points, strata, etc. the differentiation requires a derived approach in which the tangent space is replaced with a differential complex in which deformations in $\mathcal{S}$ are given by cochains and trivial deformations by coboundaries. Finding the right complex can be quite involved.
The prototypical example for this example is Gerstenhaber's work [4], who has shown that the infinitesimal deformations m_ε of an associative algebra (A, m) over a field k are controlled by the second order Hochschild cohomology group $HH^2(A)$ of A with coefficients in itself. The idea behind this is encoded in the associativity of m_ε, i.e. $m_\varepsilon(m_\varepsilon(a,b),c) = m_\varepsilon(a, m_\varepsilon(b,c))$ for all a, b, c in A, which gives rise to the cocycle condition of $m_1 \in H^2(A)$. Moreover, viewing the Hochschild complex $H^{\bullet+1}(A)$ together with the Gerstenhaber bracket $[.,.]$ and the Hochschild differential δ as a differential graded Lie algebra (dgLa), it has been shown that the set of infinitesimal deformations of A is precisely described via the set $\mathcal{MC}(H^{\bullet+1}(A))$ of Maurer-Cartan elements of the Hochschild dgLa.

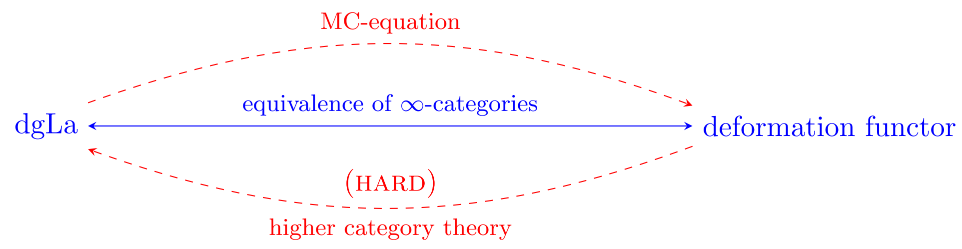
Definition. *Let $(\mathfrak{g}, [.,.], \partial)$ be a differential graded Lie algebra. A* ***Maurer-Cartan element*** *of $\mathfrak{g}$ is a homogeneous degree 1 element $\alpha \in \mathfrak{g}^1$ which satisfies the* ***Maurer-Cartan equation***

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

In many more examples, the deformation theory of structures is described in this way by a dgLa, such as commutative algebras (Harrison cohomology), Lie algebras (Chevalley-Eilenberg cohomology), complex structures (Kodaira-Spencer cohomology), etc. This has led to the conjecture that *every deformation theory is governed by a dgLa*. Using the modern language of higher category theory, Lurie [9], and independently Pridham [12] have proved this.

Theorem (Lurie, Pridham)**.** *Over a field of characteristic zero, there is an equivalence of ∞-categories between the ∞-category of formal moduli problems and the ∞-category of differential graded Lie algebras.*

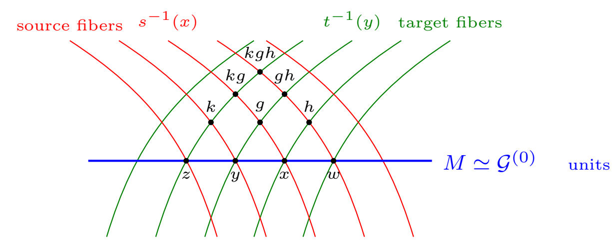
However, as promising as this theorem sounds, constructing (a computable model of) the dgLa controlling a specific deformation problem turns out to be very difficult. Kontsevich has called this an "art" [5].



For instance, the deformation theory of Lie groups is yet not fully described. It is known (cf. [10]) that infinitesimal deformations of a Lie group G are described by the group cohomology with values in the adjoint representation $\mathfrak{g}$. Surprisingly, the question of the dgLa that governs these deformations remains open.

Lie groupoids and Lie algebroids

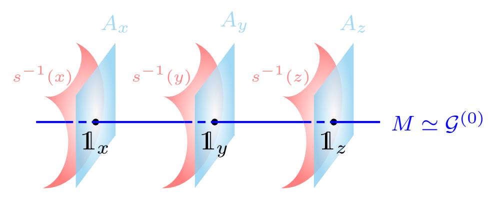
We are primarily interested in the deformation theory of so-called *Lie groupoids*, which are geometric objects that model a large class of geometric structures, such as representations, foliations, orbifolds, differentiable stacks, convolution C^*-algebras, etc.
A **groupoid** is a small category in which all morphisms are isomorphisms. As such, a groupoid can be viewed as a generalization of the notion of a group, where there are many identities, namely the set of objects M of the category, and where the set of groupoid elements $\mathcal{G}$ is given by the arrows between these objects. The groupoid multiplication is then simply the composition of the (composable) arrows. It is customary to denote a groupoid by $\mathcal{G} \underset{t}{\overset{s}{\rightrightarrows}} M$, where the parallel maps s and t associate the source and target objects to each arrow $g \in \mathcal{G}$ respectively.
A **Lie groupoid** is a groupoid internal to the category $\mathcal{M}$fld of smooth manifolds, where the source and target maps are surjective submersions. The following figure depicts a Lie groupoid and its structure maps (inspired from [3]).



Some of the geometric objects that can be described via Lie groupoids are shown in the following examples.

- A **Lie group** G can be viewed as a Lie groupoid with a single object $G \rightrightarrows \{*\}$.
- A **smooth manifold** M is a Lie groupoid $M \rightrightarrows M$ over itself where s and t are the identity maps.
- Given a smooth manifold M, one can define the **pair groupoid** $M \times M \rightrightarrows M$, where $s(x, y) = y$ and $t(x, y) = x$.
- To every smooth action of a Lie group G on a smooth manifold M one can associate the **action groupoid** $G \ltimes M \rightrightarrows M$, where the space of objects is M and the space of arrows is $G \times M$ with $s(g, x) = x$ and $t(g, x) = g \cdot x$.
- Let $P \to M$ be a principal G-bundle. Then, one can construct the **gauge groupoid** Gauge$(P) \rightrightarrows M$, where M is the space of objects and the quotient $(P \times P)/G$ with respect to the diagonal action of G on $P \times P$ is the space of arrows.

A Lie groupoid has an infinitesimal counterpart, its Lie algebroid, which generalizes the relation between a Lie group and its Lie algebra. Abstractly, a **Lie algebroid** is a vector bundle $A \to M$ together with a Lie bracket on its space of smooth sections $\Gamma(A)$ with a vector bundle map $\rho : A \to TM$ called the *anchor*, satisfying a Leibniz rule.
Just like the Lie algebra $\mathfrak{g}$ of a Lie group G consists of the right-invariant vector fields on G, the **Lie algebroid** A **associated to the Lie groupoid** $\mathcal{G} \rightrightarrows M$ is defined by its right-invariant vector fields. However, we restrict the vector fields to be tangent to the source fibers, where right translation is well-defined. Then, A is defined by the pullback of the subbundle $\ker(Ts) \subset T\mathcal{G}$ by the unit map $\mathbb{1} : M \to \mathcal{G}$ (which associates to each object its identity arrow). Hence, the fibers of $A = \mathbb{1}^*(\ker(Ts))$ are precisely the tangent spaces to the source fibers at the units of the groupoid as depicted in the following figure.



The anchor of A is given by the restriction of Tt to A and the Lie bracket on $\Gamma(A)$ is defined using the identification of $\Gamma(A)$ with the space of right-invariant vector fields on $\mathcal{G}$.

Deformation cohomology of a Lie groupoid

Deformations of Lie groupoids have first been investigated by Crainic, Mestre and Struchiner in [2] through a careful analysis of the cohomology theory controlling such deformations. The authors have constructed the so-called *deformation complex* $C^*_{\mathrm{def}}(\mathcal{G})$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$, whose cohomology group $H^*_{\mathrm{def}}(\mathcal{G})$ controls the deformations of $\mathcal{G}$. In particular, they have shown that deformations of a Lie groupoid $\mathcal{G}$ give rise to 2-cocycles and, hence, to cohomology classes in $H^2_{\mathrm{def}}(\mathcal{G})$. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $A = \mathrm{Lie}(\mathcal{G})$ its Lie algebroid; denote by $\mathcal{G}^{(k)}$ the set of k composable arrows.

Definition. *The* ***deformation complex*** *$(C^*_{\mathrm{def}}(\mathcal{G}), \delta)$ of $\mathcal{G}$ is defined as follows:*

- *For $k \geq 1$ the k-cochains $c \in C^k_{\mathrm{def}}(\mathcal{G})$ are the smooth maps*
$$c : \mathcal{G}^{(k)} \longrightarrow T\mathcal{G}, \quad (g_1, \dots, g_k) \mapsto c(g_1, \dots, g_k) \in T_{g_1}\mathcal{G}$$
which are s-projectable, i.e. $Ts \circ c(g_1, \dots, g_k)$ does not depend on g_1. The differential is defined by
$$(\delta c)(g_1, \dots, g_{k+1}) := -T\bar{m}(c(g_1 g_2, \dots, g_{k+1}), c(g_2, \dots, g_{k+1})) + \sum_{i=2}^{k}(-1)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}) + (-1)^{k+1} c(g_1, \dots, g_k).$$
where $\bar{m}(g_1, g_2) = g_1 g_2^{-1}$ denotes the division map of $\mathcal{G}$, defined on arrows with the same source.
- *For $k = 0$ the cochains are given by $C^0_{\mathrm{def}}(\mathcal{G}) := \Gamma(A)$. The differential of $c \in \Gamma(A)$ is defined by*
$$\delta(c) := \overrightarrow{c} + \overleftarrow{c}$$
where $\overrightarrow{c}$ and $\overleftarrow{c}$ are the induced right- and left-invariant vector fields of c respectively.

Similar to rigidity results of compact Lie groups (cf. [11]), one obtains rigidity of proper and compact Lie groupoids. Recall that $\mathcal{G}$ is a **proper groupoid** if the map $g \mapsto (s(g), t(g))$ is a proper map.

Theorem (Crainic, Mestre, Struchiner)**.**

- *Proper Lie groupoids are rigid with respect to (s, t)-constant deformations.*
- *Compact Lie groupoids are rigid with respect to s-constant deformations.*

One application is deformations of Lie group representations. A representation of a Lie group G on a finite-dimensional vector space V is a linear G-action on V. It can be encoded by the action groupoid $G \ltimes V \rightrightarrows V$. This relationship has been investigated in [1]. Another application is deformations of differentiable stacks, which are by definition presented by Lie groupoids. One can think of differentiable stacks as Lie groupoids up to *Morita equivalence*.

Open question: What is the dg**La** controlling the deformation theory of a Lie group(oid)?

Approaches of Lurie-Pridham and Kontsevich-Soibelman

In our attempt to understand the deformation theory of Lie groupoids, two approaches to be explored in this PhD project are:

LURIE-PRIDHAM: Applying their theorem to Lie group(oid)s

As mentioned above, Lurie and Pridham have formalized the equivalence of deformation problems and dgLa's using higher category theory. Because of its high level of abstraction, we first must spell out the details of Lurie's theorem (and its proof) in order to find the dgLa controlling deformations of Lie groups and Lie groupoids.

KONTSEVICH-SOIBELMAN: A geometric version of the deformation theory of linear operads

An operad consists of sets of operations satisfying relations, such as the multiplication and unit of an algebra satisfying associativity and unitality. A large class of algebraic structures can be encoded by operads.

Operadic methods have been extensively used in deformation theory, such as in the deformation quantization of Poisson manifolds by Kontsevich [6]. Furthermore, the deformation theory of algebras over linear operads is well understood (cf. [7], [8]). By this method, we recover the expected Hochschild dgLa in the case of associative algebras.

We are interested in a geometric realization of the Kontsevich-Soibelman approach, where we can go from algebras of operads in smooth spaces to linear operads by a procedure of differentiation. In a first step, we will try to describe the deformation theory of Lie groups in this way and then generalize this to Lie groupoids.

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