L_∞-algebroids of Higher Groupoids in Tangent Categories

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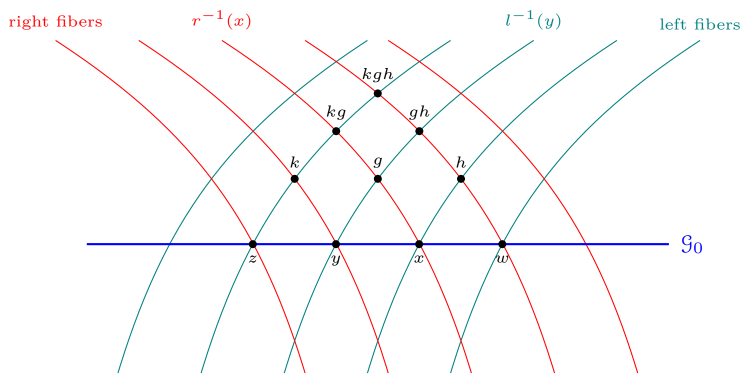
(Higher) Lie groupoids

A **groupoid** is a small category in which all morphisms are isomorphisms.

I will denote it by $\mathcal{G}_1 \overset{r}{\underset{l}{\rightrightarrows}} \mathcal{G}_0$, where $\mathcal{G}_1$ is the set of arrows, $\mathcal{G}_0$ the set of objects, and r, l the source and target maps.

Definition. A **Lie groupoid** is a groupoid internal to the category $\mathcal{M}\mathrm{fld}$ of smooth manifolds, where r and l are surjective submersions.



Example.

- G **Lie group** $\rightsquigarrow$ Lie groupoid with one object $G \rightrightarrows \{*\}$.
- M smooth manifold $\rightsquigarrow$ the **pair groupoid** $M \times M \rightrightarrows M$, where $r(x,y) = y$ and $l(x,y) = x$.

A **simplicial manifold** is a simplicial object in $\mathcal{M}\mathrm{fld}$, that is, a functor $\mathcal{G} : \Delta^{\mathrm{op}} \to \mathcal{M}\mathrm{fld}$, $[m] \mapsto \mathcal{G}_m$, where Δ denotes the category of finite ordinals. Explicitly, it is given by a family of smooth manifolds $\{\mathcal{G}_m\}_{m\geq 0}$ together with face and degeneracy maps satisfying the *simplicial relations* of Δ, as depicted below

$$\mathcal{G}_0 \leftrightarrows \mathcal{G}_1 \leftrightarrows \mathcal{G}_2 \quad \cdots$$

A simplicial manifold $\mathcal{G}$ is **Kan** if the natural restriction map

$$\mathrm{Hom}(\Delta^m, \mathcal{G}) \longrightarrow \mathrm{Hom}(\Lambda^m_j, \mathcal{G}) \tag{1}$$

is a surjective submersion for all $m \in \mathbb{N}$ and $0 \leq j \leq m$. Here, Δ^m denotes the standard simplicial m-simplex, and Λ^m_j denotes the j-th horn of Δ^m, which is obtained by removing the j-th face from Δ^m.

Definition ([Hen08], [Zhu09]). A **Lie ∞-groupoid** is a Kan simplicial manifold. For $n \in \mathbb{N}$, a **Lie n-groupoid** is a Kan simplicial manifold such that (1) is a diffeomorphism for all $m > n$ and $0 \leq j \leq m$.

Remark. A Lie 1-groupoid is (the nerve of) a usual Lie groupoid.

Similarly, one can define a higher groupoid in any category $\mathcal{C}$ equipped with a *Grothendieck pretopology* as Kan simplicial objects in $\mathcal{C}$.

Differentiation of global objects

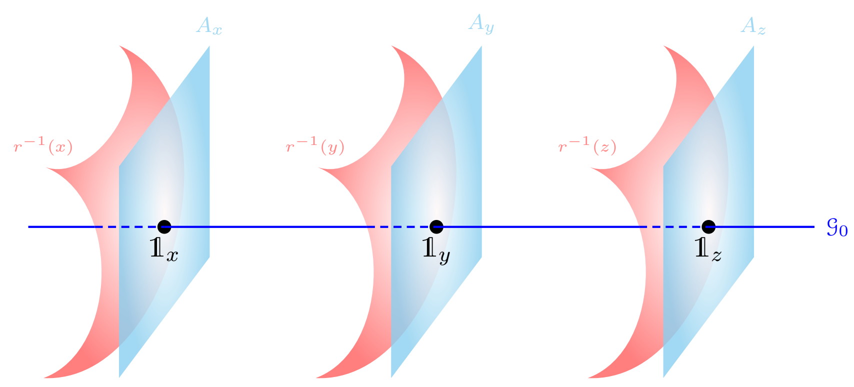
Global geometric object $\xrightarrow{\text{differentiation}}$ Infinitesimal object

- One of the main approaches: Notion of smooth paths.
- Leads to the notion of tangent vectors, tangent spaces, tangent maps, etc. in the category at hand.
- Need of a good differential calculus (e.g. Cartan calculus, tangent structure...).

The Lie algebroid of a Lie groupoid

A Lie groupoid has an infinitesimal counterpart, its Lie algebroid. Abstractly, a **Lie algebroid** is a vector bundle $A \to M$ together with a Lie bracket on its space $\Gamma(A)$ of smooth sections with a vector bundle map $\rho : A \to TM$ called the *anchor*, satisfying a Leibniz rule.

Just like the Lie algebra $\mathfrak{g}$ of a Lie group G consists of the right-invariant vector fields on G, the **Lie algebroid A associated to a Lie groupoid** $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ is defined by its right-invariant vector fields. However, we restrict the vector fields to be tangent to the right fibers, where right translation is well-defined. Then, A is defined by the pullback of the subbundle $\ker(Tr) \subset T\mathcal{G}_1$ by the unit map $\mathbb{1} : \mathcal{G}_0 \to \mathcal{G}_1$. Hence, the fibers of $A = \mathbb{1}^*(\ker(Tr))$ are precisely the tangent spaces to the right fibers at the units of the groupoid as depicted in the following picture.



The anchor of A is given by the restriction of Tl to A and the Lie bracket on $\Gamma(A)$ is defined using the identification of $\Gamma(A)$ with the space of right-invariant vector fields on $\mathcal{G}_1$.

Open question: What are the infinitesimal counterparts of higher Lie groupoids?

Ševera's construction

In [Šev06], Ševera has shown a method of differentiation:

$$\underbrace{\text{Higher Lie groupoids}}_{\text{Kan simplicial manifolds}} \xrightarrow{\text{differentiation}} \underbrace{\text{Higher Lie algebroids}}_{\text{dg-manifolds}}$$

using

- The odd line $\mathbb{R}^{0|1}$ as an infinitesimal model, where $\mathbb{R}^{0|1} = (*, \mathcal{O}_{\mathbb{R}^{0|1}})$ with $C^\infty(\mathbb{R}^{0|1}) := \mathcal{O}_{\mathbb{R}^{0|1}}(*) = S(\theta)$.
- $\underline{\mathrm{Hom}}(\mathbb{R}^{0|k}, M) \cong T[k]M$ for $k \geq 1$ for all supermanifolds M.
- $\exists$ $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M) \circlearrowleft \underline{\mathrm{End}}(\mathbb{R}^{0|1})$ $\iff$ structure of a complex on $\Omega(M)$ ($\mathbb{Z}$-grading and the de Rham differential).

Ševera's main idea suggests replacing $\mathbb{R}^{0|1}$ by the nerve $(\mathbb{R}^{0|1})^{\bullet+1}$ of the pair groupoid of $\mathbb{R}^{0|1}$. Let $\mathcal{G}$ be a simplicial manifold.

$$T[1]M \;=\; \underline{\mathrm{Hom}}\;(\mathbb{R}^{0|1}\;,\; M)$$

(enrichment of functor categories ([Kel05]); Ševera; simplicial manifold)

$$T\mathcal{G} \;:=\; \int_{[m]\in\Delta^{\mathrm{op}}} \underline{\mathrm{Hom}}\;((\mathbb{R}^{0|1})^{m+1},\; \mathcal{G}_m)$$

The integral sign above denotes the *categorical end*, which is a special limit of some diagram.

Theorem?. *Let $\mathcal{G}$ be a Lie n-groupoid. Then, $T\mathcal{G} = \int_{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} \underline{\mathrm{Hom}}((\mathbb{R}^{0|1})^{m+1}, \mathcal{G}_m)$ exists in the category of graded manifolds.*

Categorical generalization of Ševera's construction

Let $n \in \mathbb{N}$ and let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology and an abstract tangent functor (in the sense of Rosický). Let $\mathcal{G}$ be an n-groupoid in $\mathcal{C}$.

Definition. The **Lie n-algebroid** $\mathcal{L}(\mathcal{G})$ **of** $\mathcal{G}$ is

$$\mathcal{L}(\mathcal{G}) := \int_{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} T^{m+1}\mathcal{G}_m\,. \tag{2}$$

Remark. For a fixed object $C \in \mathcal{C}$, the iterated tangent bundle $T^{\bullet+1}C$ has a cosimplicial structure, and hence the end in equation (2) is well-defined.

Remark. The definition of $\mathcal{L}(\mathcal{G})$ given in (2) is compatible with Ševera's approach. Namely,

$$T\mathcal{G} = \int_{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} \underline{\mathrm{Hom}}((\mathbb{R}^{0|1})^{m+1}, \mathcal{G}_m) \cong \int_{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} T[m+1]\mathcal{G}_m \cong \mathcal{L}(\mathcal{G})\,.$$

Open question: How are the (higher) Lie brackets encoded in $\mathcal{L}(\mathcal{G})$?

Computing the end

Since ends are special limits, using Mac Lane's formula [ML98, prop. 1, p. 220], one can compute the end $\mathcal{L}(\mathcal{G})$.

Theorem. *The Lie n-algebroid of $\mathcal{G}$ is precisely given by the following fiber product*

$$\mathcal{L}(\mathcal{G}) \cong T^1\mathcal{G}_0 \times_{R_{0,1}} T^2\mathcal{G}_1 \times_{R_{1,2}} \cdots \times_{R_{n,n+1}} T^{n+2}\mathcal{G}_{n+1}\,, \tag{3}$$

where

$$R_{i-1,i} = (T^i\mathcal{G}_i)^i \times (T^{i+1}\mathcal{G}_{i-1})^{i+1}.$$

for $i \in \{1, \ldots, n+1\}$.

Proof. The formula given in (3) is the limit of the following diagram:

□

Note that the formula given in (3) and its proof assert that $\mathcal{L}(\mathcal{G})$ is obtained via powers of the tangent bundles of the nerve of $\mathcal{G}$ under some relations imposed by the (co)face and (co)degeneracy maps.

Remark. When $n = 1$ and $\mathcal{C} = \mathcal{M}\mathrm{fld}$ (and hence $\mathcal{G}$ is the nerve of a Lie groupoid), we recover the *symmetric version* of the usual Lie algebroid of $\mathcal{G}$. That is, we recover $\mathcal{L}(\mathcal{G})$ without making a choice between the left and right maps.

Conjecture 1. The (co)face-relations in the highest box together with the (co)degeneracy-relations in the lower boxes imply all the other relations.

Conjecture 2. There are natural actions of the symmetric groups S_i on $T^i\mathcal{G}_{i-1}$ for all $i \in \{2, \ldots, n+2\}$ and $\mathcal{L}(\mathcal{G})$ is invariant under these actions.

Conjectures 1 and 2 hold in the case of usual Lie groupoids.

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