BONN INTERNATIONAL GRADUATE SCHOOL OF MATHEMATICS

Differentiation of higher groupoids in tangent categories

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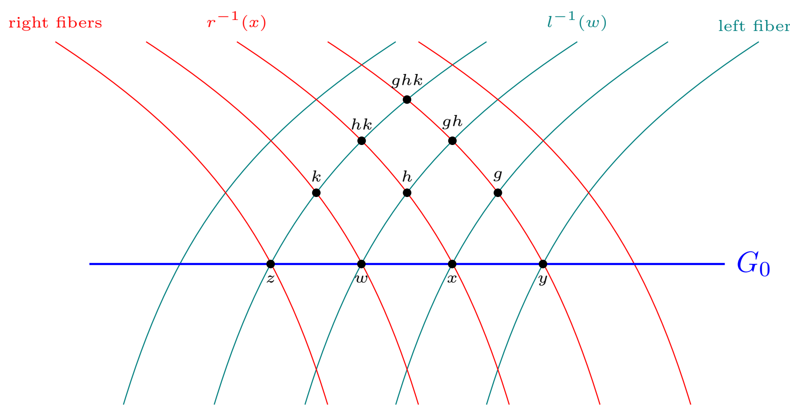
IMPRS Moduli Spaces

(Higher) Lie groupoids

A **groupoid** is a small category in which all morphisms are isomorphisms. I will denote it by $G_1 \overset{r}{\underset{l}{\rightrightarrows}} G_0$, where G_1 is the set of arrows, G_0 the set of objects, and r, l the right (target) and left (source) maps.

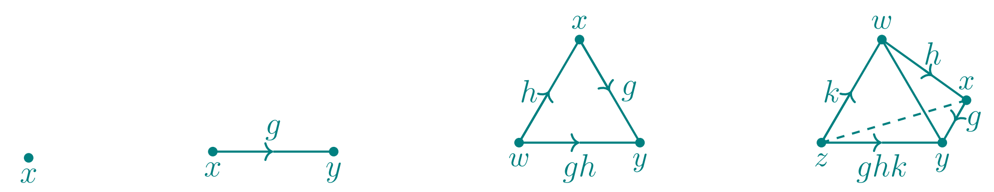
Definition. A **Lie groupoid** is a groupoid internal to the category $\mathcal{M}\mathrm{fld}$ of smooth manifolds, where r and l are surjective submersions.



Example.
- G **Lie group** ⇝ Lie groupoid with one object $G \rightrightarrows \{*\}$.
- M smooth manifold ⇝ the **pair groupoid** $M \times M \rightrightarrows M$, where $r(x, y) = y$ and $l(x, y) = x$.

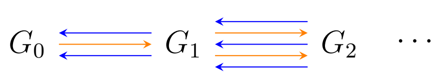
Remark. Given a Lie groupoid $G = G_1 \rightrightarrows G_0$, its nerve NG is a simplicial manifold, where

$$NG_n = \underbrace{G_1 \times_{G_0} \cdots \times_{G_0} G_1}_{n\text{-times}}.$$



One may observe that all horns of NG can be filled and all horns of degree > 1 can be filled uniquely.

A simplicial manifold $G : \Delta^{\mathrm{op}} \to \mathcal{M}\mathrm{fld}$, $[m] \mapsto G_m$, depicted by



is **Kan** if the horn projection

$$G_m \cong \mathrm{Hom}(\Delta^m, G) \longrightarrow \mathrm{Hom}(\Lambda^m_j, G) \tag{1}$$

is a surjective submersion for all $m \in \mathbb{N}$ and $0 \leq j \leq m$.

Definition ([Hen08], [Zhu09]). A **Lie ∞-groupoid** is a Kan simplicial manifold. For $n \in \mathbb{N}$, a **Lie n-groupoid** is a Kan simplicial manifold such that (1) is a diffeomorphism for all $m > n$ and $0 \leq j \leq m$.

Remark. A Lie n-groupoid is $(n+1)$-coskeletal.

More generally, one can define higher groupoid objects in any category $\mathcal{C}$ equipped with a *Grothendieck pretopology* as Kan simplicial objects in $\mathcal{C}$.

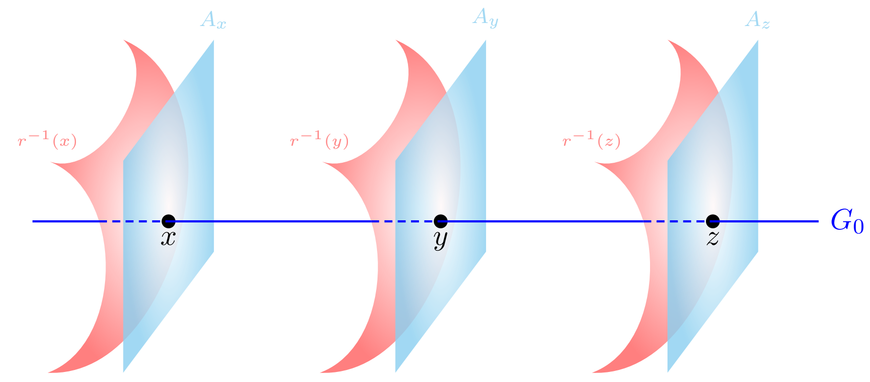
The Lie algebroid of a Lie groupoid

A Lie groupoid has an infinitesimal counterpart, its Lie algebroid. Abstractly, a **Lie algebroid** is a vector bundle $A \to M$ together with a Lie bracket on its space $\Gamma(A)$ of smooth sections with a vector bundle map $\rho : A \to TM$ called the *anchor*, satisfying a Leibniz rule.

Just like the Lie algebra $\mathfrak{g}$ of a Lie group G consists of the right-invariant vector fields on G, the **Lie algebroid A associated to a Lie groupoid** $G_1 \rightrightarrows G_0$ is defined by the right-invariant vector fields on G_1. However, we restrict the vector fields to be tangent to the right fibers, where right translation is well-defined. Then,

$$A := \ker(Tr)|_{G_0} \longrightarrow G_0\,,$$

with fibers the tangent spaces to the right fibers at the units of the groupoid as depicted below



The anchor of A is given by the restriction of Tl to A and the Lie bracket on $\Gamma(A)$ is defined using the identification of $\Gamma(A)$ with the space of right-invariant vector fields on G_1.

Ševera's construction: revisited

In [Šev06], Ševera has shown a method of differentiation from higher Lie groupoids to higher Lie algebroids using the supermanifold $\mathbb{R}^{0|1}$ as an infinitesimal model and the fact that $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$ is represented by the odd tangent bundle ΠTM for any (super)manifold M. Ševera's main idea suggests replacing $\mathbb{R}^{0|1}$ by the nerve $(\mathbb{R}^{0|1})^{\bullet+1}$ of the pair groupoid of $\mathbb{R}^{0|1}$. Let G be a simplicial manifold.

$$\Pi TM \;=\; \underline{\mathrm{Hom}}\;(\mathbb{R}^{0|1}\;,\;M)$$

enrichment of functor categories ([Kel05]) · Ševera · simplicial manifold

$$LG \;:=\; \int_{[m]\in\Delta^{\mathrm{op}}} \underline{\mathrm{Hom}}\;((\mathbb{R}^{0|1})^{m+1},\;G_m)$$

The integral sign above denotes the *categorical end*, which is a universal construction and in particular a special limit of some diagram. Moreover, (co)ends support a calculus [Lor21].

Categorical generalization of Ševera's construction

Let $n \in \mathbb{N}$ and let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology and an abstract tangent functor $T : \mathcal{C} \to \mathcal{C}$ in the sense of Rosický [Ros84] (cf. [Blo23, section 2]). Let G be an n-groupoid in $\mathcal{C}$.

Definition. The **Lie n-algebroid** $\mathrm{Lie}(G)$ **of** G is defined by

$$\mathrm{Lie}(G) := \int_{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} T^{m+1}G_m\,. \tag{2}$$

Remark. The iterated tangent bundle $T^{\bullet+1}$ has an augmented cosimplicial structure, using the fact that T is a monad. Hence, the end in equation (2) is well-defined.

Note that the definition of $\mathrm{Lie}(G)$ is compatible with Ševera's approach (up to degree shift of the fibers) using the fact that $(\Pi T)^k M \cong \underline{\mathrm{Hom}}((\mathbb{R}^{0|1})^k, M)$ for all (super)manifolds M and $k \geq 1$.

Computing the end

Theorem (Blohmann, K.). *The Lie n-algebroid of G is isomorphic to the following fiber product*

$$\mathrm{Lie}(G) \cong T^1G_0 \times_{R_1} T^2G_1 \times_{R_2} \cdots \times_{R_{n+1}} T^{n+2}G_{n+1}\,, \tag{3}$$

where

$$R_i = (T^iG_i)^i \times (T^{i+1}G_{i-1})^{i+1}.$$

for $i \in \{1, \ldots, n+1\}$.

Proof. Since ends are special limits, one can compute the end $\mathrm{Lie}(G)$ (cf. [ML98, prop. IX.5.1]) explicitly and obtain the limit of the following diagram

□

Note that the formula given in (3) and its proof assert that $\mathrm{Lie}(G)$ is obtained via powers of the tangent bundles of the nerve of G under some relations imposed by the (co)face and (co)degeneracy maps.

Remark. When $n = 1$ and $\mathcal{C} = \mathcal{M}\mathrm{fld}$, we recover the *symmetric version* of the usual Lie algebroid of G. That is, we recover $\mathrm{Lie}(G)$ without making a choice between the left and right maps.

Proposition (K.). *The (co)face-relations in the highest box together with the (co)degeneracy-relations in the lower boxes imply all the other relations.*

The cohomological vector field

Given a vector bundle $A \to M$, it is due to Vaintrob [Vai97] that the Lie algebroid structures on A are in one-to-one correspondence with cohomological vector fields on the degree one graded manifold $A[1]$, which has core M and sheaf of functions $\mathcal{O}(A[1]) = \Gamma(\Lambda^\bullet A^*)$. In this way, **Lie n-algebroids** are defined as $\mathbb{N}$-graded manifolds of degree n together with a cohomological vector field.

Given a Lie n-groupoid, in order to fully capture the structure of $\mathrm{Lie}(G)$, our aim is to understand the sheaf of functions on $\mathrm{Lie}(G)$ and the cohomological vector field. Using the fact that $\mathcal{O}(\Pi TM)$ is the cochain complex of differential forms on M, we define

$$\mathcal{O}(\mathrm{Lie}(G)) := \int^{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} \underline{\mathrm{Hom}}((\Pi T)^{m+1}G_m, \mathbb{R}) \cong \int^{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} \mathcal{O}((\Pi T)^{m+1}G_m)\,.$$

By construction, $\mathcal{O}(\mathrm{Lie}(G))$ is a differential graded algebra.

It is still an open question and work-in-progress how this generalizes to categories with a Grothendieck pretopology and an abstract tangent functor.

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