Abstract Lie algebroids of differentiable groupoids in tangent categories

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IMPRS Moduli Spaces

Differentiation of global geometric objects

Global geometric object $\overset{\text{differentiation}}{\rightsquigarrow}$ Infinitesimal counterpart

Lie group G $\rightsquigarrow$ Lie algebra $\mathfrak{g} := \mathfrak{X}(G)_{\text{r-inv}} \cong T_eG$

Lie groupoid $G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0$ $\rightsquigarrow$ Lie algebroid $A := \ker Ts|_{G_0}$, $\Gamma(G_0, A) \cong \mathfrak{X}(G_1)_{\text{r-inv}}$

Our goal: Generalize the well-understood differentiation procedure in the category of smooth manifolds to other less-understood settings, such as infinite-dimensional Lie groups, diffeomorphism groups, mapping spaces, situations with mild singularities (cusps, corners), moduli spaces, non-commutative tori, etc.

Our approach: We need a generalized differential calculus in the ambient category. In the 1980s, Rosický has axiomatized the natural structure of the tangent functor on euclidean spaces that is needed to construct the Lie bracket of vector fields. We have extended the framework and shown that a tangent structure on a category is sufficient for a generalized differentiation procedure.

Outlook: Geometric deformation theory: Deformation theory studies moduli spaces, that is, structures of a certain type (associative multiplications, complex structures, riemannian metrics, etc.) modulo isomorphisms. Deformations of a structure X are paths in the corresponding moduli space $\mathcal{M}$, which might admit singularities and does not come with a natural smooth structure. However, $\mathcal{M}$ can often be viewed as a stack presented by a diffeological groupoid $\mathcal{G}_{\mathcal{M}}$. The infinitesimal deformation theory of X should then be given by the fibers of $T\mathcal{M}$ presented by the abstract Lie algebroid of $\mathcal{G}_{\mathcal{M}}$.

The main theorem

The ambient framework in which our generalized construction and procedure of differentiation work will be a category equipped with a *cartesian tangent structure* with scalar multiplication. The main ingredients are:

Differentiable groupoid G $\overset{\text{generalized differentiation???}}{\rightsquigarrow}$ Abstract Lie algebroid $A := G_0 \times_{G_1} \ker Ts$

Theorem ([AB]). Let G be a differentiable groupoid in a tangent category $(\mathcal{C}, T)$. The bundle $A \to G_0$ defined by (4), together with the anchor $\rho : A \to TG_0$ defined in (5), and the Lie bracket on sections of A defined by (6) is an abstract Lie algebroid.

Tangent structure on a category

Let $\mathcal{E}\mathrm{ucl}$ denote the category of **euclidean spaces**, which has open subsets $U \subset \mathbb{R}^n$ ($n \geq 0$) as objects and smooth maps as morphisms. Let $T : \mathcal{E}\mathrm{ucl} \to \mathcal{E}\mathrm{ucl}$ be the usual tangent functor:

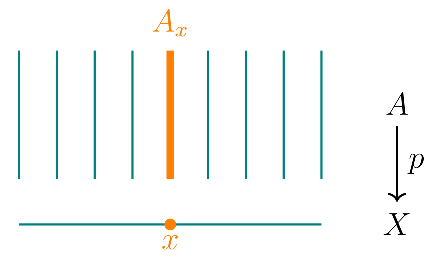
$$TU = U \times \mathbb{R}^n \qquad T^2U = U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \qquad TU \times_U TU = U \times \mathbb{R}^n \times \mathbb{R}^n\,,$$

$\pi_U : TU \longrightarrow U,$	$(u, u_1) \longmapsto u$	**bundle projection**
$0_U : U \longrightarrow TU,$	$u \longmapsto (u, 0)$	**zero section**
$+_U : TU \times_U TU \longrightarrow TU,$	$(u, u_1, v_1) \longmapsto (u, u_1 + v_1)$	**fiberwise addition**
$\lambda_U : TU \longrightarrow T^2U,$	$(u, u_1) \longmapsto (u, 0, 0, u_1)$	**vertical lift**
$\tau_U : T^2U \longrightarrow T^2U,$	$(u, u_1, u_2, u_{12}) \longmapsto (u, u_2, u_1, u_{12})$	**symmetric structure**

Definition ([Ros84]). A **tangent structure** on a category $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$, called **abstract tangent functor**, together with natural transformations $\pi : T \to 1$, $0 : 1 \to T$, $+ : T \times_1 T \to T$, $\lambda : T \to T^2$ and $\tau : T^2 \to T^2$, satisfying *certain axioms (many!)*.

Let Wibble be an algebraic theory, such as a monoid, an abelian group, an R-module (if $R \in \mathcal{C}$ is a ring object), an $\mathbb{R}$-vector space (if $\mathbb{R} \in \mathcal{C}$), etc.

Given an object $X \in \mathcal{C}$ such that the overcategory $\mathcal{C} \downarrow X$ has finite products, a **bundle of Wibbles** over X is a Wibble object $A \to X$ in $\mathcal{C} \downarrow X$. For all *points* $x : * \to X$ in $\mathcal{C}$ ($* \in \mathcal{C}$ terminal object), the fibers $A_x := * \times_X A$ have the structure of a Wibble.



Caution. A bundle of vector spaces makes no assumptions on local trivialization and hence is considerably more general than a vector bundle.

The axioms of a tangent structure capture some of the properties of the tangent functor on $\mathcal{E}\mathrm{ucl}$. For instance, $\pi_X : TX \to X$, together with 0_X and $+_X$ is required to be a bundle of abelian groups for all $X \in \mathcal{C}$.

$v, w \in \Gamma(X, TX)$ vector fields $\overset{\text{tangent structure on } \mathcal{C}}{\rightsquigarrow}$ Lie bracket $[v, w]$ of vector fields

Example. The category of *elastic* diffeological spaces has a tangent structure, given by the left Kan extension of the tangent structure on euclidean spaces [Blo24].

Our tangent categories are assumed to have a ring object R with some additional *nice* properties.

Differentiable groupoids

Definition. A **groupoid object** in $\mathcal{C}$ is a simplicial object $G : \Delta^{\text{op}} \to \mathcal{C}$ such that the limits

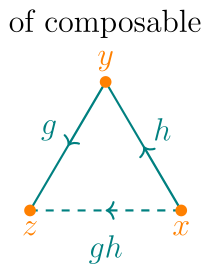
(1) $$G[\Lambda^n_i] := \lim_{\Delta^k \to \Lambda^n_i} G_k$$

exist and the natural morphisms

$$G_n \longrightarrow G[\Lambda^n_i]$$

are isomorphisms for all $n \geq 2$ and $0 \leq i \leq n$.

A pair of composable arrows



$$G_2 \cong G[\Lambda^2_1] \cong G_1 \times^{s,t}_{G_0} G_1$$

Definition. A groupoid object G in a tangent category $(\mathcal{C}, T)$ is called **differentiable** if

$$TG : \Delta^{\text{op}} \longrightarrow \mathcal{C}, \;\; [n] \longmapsto (TG)_n := TG_n$$

is a groupoid object in $\mathcal{C}$ and if the face and degeneracy maps satisfy *additional properties.*

If G is a differentiable groupoid, then for all $k \geq 2$ the natural morphism

(2) $$T(\underbrace{G_1 \times^{s,t}_{G_0} \ldots \times^{s,t}_{G_0} G_1}_{k\text{-factors}}) \longrightarrow \underbrace{TG_1 \times^{Ts,Tt}_{TG_0} \ldots \times^{Ts,Tt}_{TG_0} TG_1}_{k\text{-factors}}$$

is an isomorphism, where s and t are the source and target maps given by the first face maps of G.

Remark. In the literature [MZ15], groupoids are often defined in categories equipped with a pretopology. In smooth manifolds, the surjective submersions are the only morphisms which ensure the existence of the limits (1) and the commutativity of T with the nerve of G (2). However, in many categories (such as diffeological spaces) these universal conditions may be too strong.

Example. Every Lie groupoid is differentiable.

Abstract Lie algebroids

Definition. An **abstract Lie algebroid** in $(\mathcal{C}, T)$ consists of a bundle of R-modules $A \to X$, a morphism $\rho : A \to TX$ of bundles of R-modules, called the **anchor**, and a Lie bracket on the $\mathcal{C}(X, R)$-module of sections $\Gamma(X, A)$, such that for all sections a, b of A and all morphisms $f \in \mathcal{C}(X, R)$

$$[a, fb] = f[a, b] + ((\rho \circ a) \cdot f)b\,,$$

called the **Leibniz rule**, and

(3) $$\rho \circ [a, b] = [\rho \circ a, \rho \circ b]\,.$$

Remark. In the definition of a usual Lie algebroid, the Jacobi identity and the Leibniz rule imply that ρ is a Lie algebra homomorphism, so that condition (3) is not needed. However, the proof relies on the identification of vector fields with derivations of $C^\infty(X)$, which is not the case here.

The abstract Lie algebroid of a differentiable groupoid

Let G be a differentiable groupoid in a tangent category $(\mathcal{C}, T)$. Consider the s-vertical tangent bundle:

$$\begin{array}{ccc} VG_1 := \ker Ts = TG_1 \times_{TG_0} G_0 & \xrightarrow{i_{VG_1}} & TG_1 \\ \downarrow & & \downarrow Ts \\ G_0 & \xrightarrow[0_{G_0}]{} & TG_0 \end{array}$$

Proposition ([AB]). The map $\pi'_{G_1} : VG_1 \xrightarrow{i_{VG_1}} TG_1 \xrightarrow{\pi_{G_1}} G_1$ is a bundle of R-modules.

Define the abstract Lie algebroid of G by restricting the s-vertical bundle to the identity bisection:

(4) $$\begin{array}{ccc} A := G_0 \times_{G_1} VG_1 & \xrightarrow{i_A} & VG_1 \\ \downarrow & & \downarrow \pi'_{G_1} \\ G_0 & \xrightarrow[1]{} & G_1 \end{array}$$

The anchor ρ is defined as the restriction of Tt to A:

(5) $$\rho : A \xrightarrow{i_A} VG_1 \xrightarrow{i_{VG_1}} TG_1 \xrightarrow{Tt} TG_0\,.$$

Proposition ([AB]). There is a right groupoid action $R : VG_1 \times_{G_0} G_1 \longrightarrow VG_1$, called **right translation**.

Definition ([AB]). A vector field $v \in \Gamma(G_1, TG_1)$ is called **right invariant** if the diagrams

$$\begin{array}{ccc} G_1 & \xrightarrow{v} & TG_1 \\ & {\scriptstyle \widetilde{v}} \searrow \quad \nearrow {\scriptstyle i_{VG_1}} & \\ & VG_1 & \end{array} \qquad \begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{\widetilde{v} \times_{G_0} \mathrm{id}_{G_1}} & VG_1 \times_{G_0} G_1 \\ m \downarrow & & \downarrow R \\ G_1 & \xrightarrow[\widetilde{v}]{} & VG_1 \end{array}$$

commute for some morphism $\widetilde{v} : G_1 \to VG_1$.

Proposition ([AB]).

(i) The set $\mathfrak{X}(G_1)_{\text{r-inv}}$ of right invariant vector fields is closed under the Lie bracket of vector fields on G_1.

(ii) There is an isomorphism of $\mathcal{C}(G_0, R)$-modules $\Gamma(G_0, A) \cong \mathfrak{X}(G_1)_{\text{r-inv}}$. (6)

References

[AB] Lory Aintablian and Christian Blohmann. Differentiable groupoids and their abstract Lie algebroids. *Work in progress.*

[Blo24] Christian Blohmann. Elastic diffeological spaces. *Contemp. Math.*, 794:49–86, 2024.

[MZ15] Ralf Meyer and Chenchang Zhu. Groupoids in categories with pretopology. *Theory and Applications of Categories*, 30(55):1906–1998, 2015.

[Ros84] J. Rosický. Abstract tangent functors. *Diagrammes*, 12:JR1–JR11, 1984.

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