GEOMETRICAL CONSTRUCTIONS USING ONLY A RULER

NJTEH MKHSIAN AND LORY AINTABLIAN

Course: Math 213

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Objective: We will prove that every construction that can be done with compass and straight-edge can be done with straight-edge alone given a fixed circle in the plane.

Outline:

1. Definition
2. Historical Background
3. Some Useful Theorems
4. Problems and Solutions
5. Conclusion
6. References

1. Definition

A point P in the Euclidean plane is said to be constructible if it is one of the following:

- The intersection point of two lines
- The intersection point of a line and a circle
- The intersection point of two circles

2. Historical Background

From the times of ancient Greece, mathematicians attempted constructions using a compass and straight edge only. In their constructions, the Greek got stuck on three famous problems:

- Squaring the circle
- Doubling the cube
- Trisecting an angle

It wasn't until the 19th century that these constructions were proven impossible using a compass and a ruler alone.

Lorenzo Mascheroni (1797) and Dane Georg Mohr (1672) gave a proof that every point constructible with a compass and a straight edge can be constructed using a compass alone.

Jean Victor Poncelet (1822) conjectured and Jacob Steiner (1833) proved that every point constructible with a compass and a straight edge can be constructed using a straight edge alone given a fixed circle and its center.

3. Some Useful Theorems

Theorem 1:
Given a trapezoid, the straight line joining the point of intersection of its diagonals and the point of intersection of its non-parallel sides bisects each of the parallel sides.

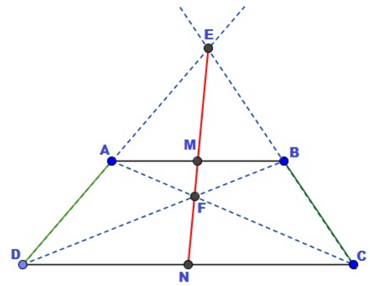

Figure 1

Theorem 2: (Converse of Thales' Theorem)
If a line d cuts the two sides AC and AB of a triangle ABC in points M and N respectively such that $\dfrac{AM}{MC} = \dfrac{AN}{NB}$, then d is parallel to side BC.

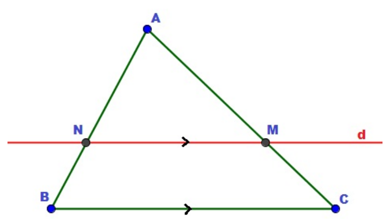

Figure 2

Theorem 3: (Ceva's Theorem)
If the cevians AA', BB' and CC' of a triangle ABC are concurrent, then $\dfrac{AC'}{C'B} \times \dfrac{BA'}{A'C} \times \dfrac{CB'}{B'A} = 1$

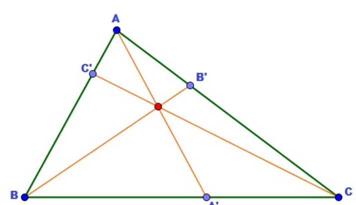

Figure 3

4. Problems and Solutions

Problem 1: Given a segment AB and its midpoint C. Through a given point D lying outside AB, draw a straight line parallel to AB.

Solution:
Join AD
Take any point E on AD
Join BD, BE and EC
EC cuts BD in F
Join AF
AF meets EB in G
DG is the required line

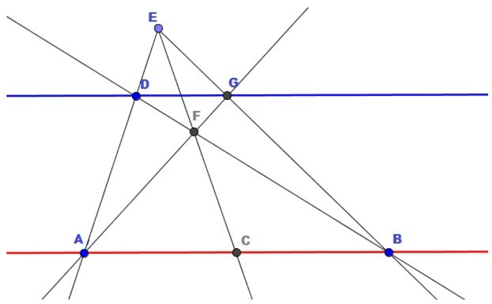

Figure 4

Reason: In triangle AEB: EC, AG and BD are concurrent

$\Rightarrow \dfrac{AC}{CB} \times \dfrac{BG}{GE} \times \dfrac{ED}{DA} = 1$

but, AC = CB (C is the midpoint of AB)

$\Rightarrow \dfrac{BG}{GE} = \dfrac{DA}{ED}$

⇒ DC // AB, by Theorem 3

Problem 2: Given two parallel straight lines a and b. Bisect the given segment AB on a.

Solution:
Take a point D on b and join AD
Take a point E on AD and join BE
BE cuts b in F
Join BD and AF. They meet in G
Join EG
ABFD is a trapezoid (since a and b are parallel)
G is the intersection of its diagonals and E is the intersection of its non-parallel sides
⇒GE cuts AB at its midpoint, by Theorem 1

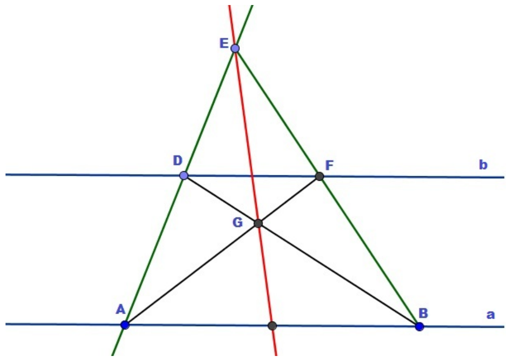

Figure 5

Problem 3: Given two parallel straight lines a and b. Through a point L lying outside a and b, draw a straight line parallel to a and b.

Solution:
Take an arbitrary segment DE on the line b
Bisect it using Problem 2
Construct the line parallel to b through L, by Problem 1

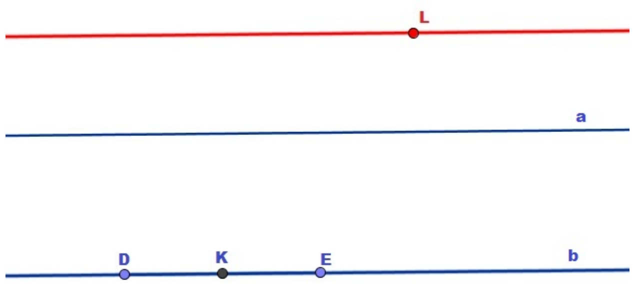

Figure 6

For the following problems, it's given a fixed circle c and its center A.

Problem 4: Given a line d. Through a point D not on d, draw a straight line parallel to d.

(i) If d passes through the center A of the given circle c.

Solution:
d intersects c in B and C
Since d passes through the center of the circle c, BC is a diameter
Hence, A is the midpoint of BC
Construct a line through D parallel to d, by Problem 1

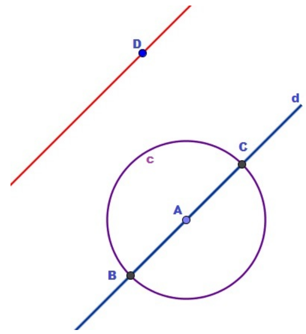

Figure 7

(ii) If d intersects the circle c, but does not pass through its center A.

Solution:
d intersects c in B and C
Join CA and BA
CA and BA cut c in D' and E respectively
Join ED'
Notice that ∠BCE, ∠CED', ∠ED'B and ∠D'BC are 90 (inscribed in a semi-circle)
⇒The quadrilateral BCED' is a rectangle and hence BC // D'E
Having two parallel lines d and D'E, and a given point D,
Construct a line through D parallel to d, by Problem 3

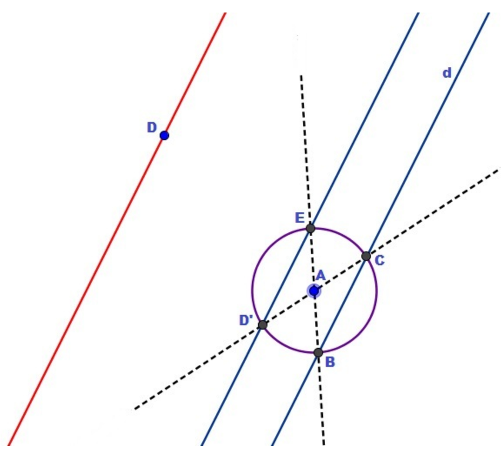

Figure 8

(iii) If d does not intersect the circle c.

Solution:
Take a point B on d
Join AB. It cuts the circle c in C and D'
Take a point E different from C and D' on c
Draw the line d' parallel to CD' through E

Case 1:
d' cuts c in another point F and d' cuts d in G
Join AE and AF. They meet c in I and H respectively
Join HI. It cuts d in J
HI // EF (HIFE is a rectangle)
⇒HFGJ is a trapezoid
A is the midpoint of FH
and AB is parallel to the sides FG and HJ of the trapezoid HFGJ
⇒B is the midpoint of GJ (By the midpoint theorem in a trapezoid)
Having a segment GJ on d with its midpoint B,
Construct the line parallel to d through D, by Problem 1.

Case 2:
d' in tangent to c
By similar steps, construction follows

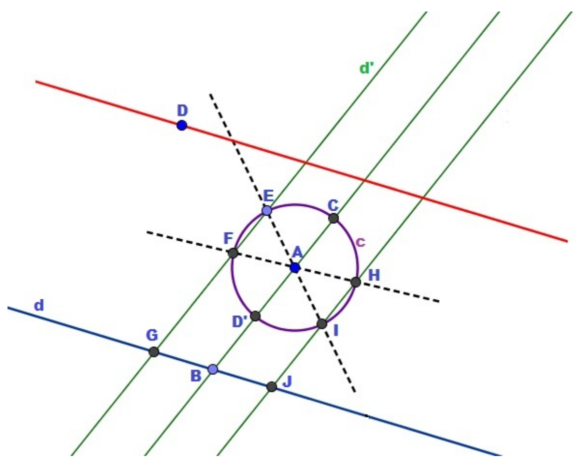

Figure 9

Problem 5: Given a straight line a. Through a given point B, draw a straight line perpendicular to a.

(i) If a intersects c in two distinct points C and D, and a doesn't pass through A.

Solution:
Join CA. It cuts c in another point E
Join ED
Then, $\angle EDC = 90$ (inscribed in a semi-circle)
$\Rightarrow$ ED $\perp$ a
Construct a line through B parallel to ED, by Problem 4(ii)

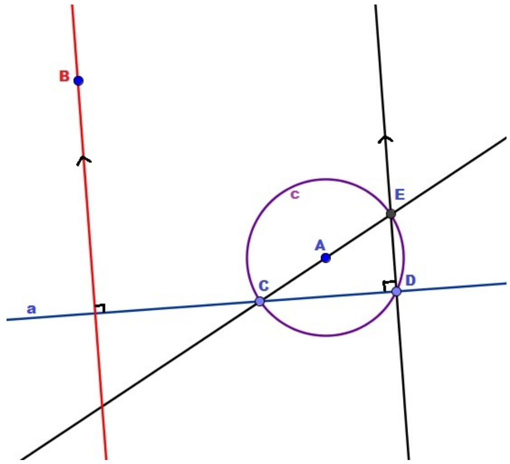

Figure 10

(ii) If a does not intersect c.

Solution:
Take a point C inside the circle c
Draw through C the parallel a' to the given line a, by Problem 4(iii)
a' certainly cuts c (since C is inside c)
Let D and E be the points of intersection of a' and c
Draw through D the perpendicular d to a', by Problem 5(i)
d ⊥ a' and a // a' ⇒ d ⊥ a
Construct through B the parallel e to d, by Problem 4(iii)
d ⊥ a and e // d ⇒ e ⊥ a
So, e is the required line

Note: If a passes through A, by similar steps the required line can be constructed

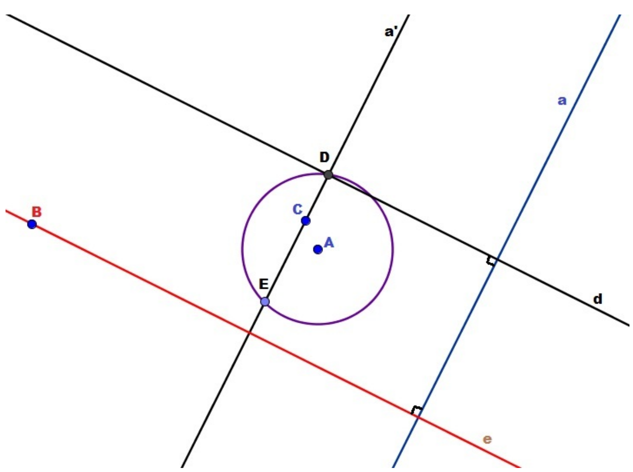

Figure 11

Problem 6: A circle a is given by its center C and radius CB (but not sketched) and given a straight line b. Construct the points of intersection of the circle a and the line b.

Solution:
Draw the radius AD of c parallel to BC
Join BD and AC. They meet in E
Take a point F on b
Join CF and draw through A the parallel h to CF
Join FE. It cuts h in G
Draw through G the parallel g to b. It cuts c in H and I
Join EH and EI. They cut b in L and M respectively
L and M are the points of intersection of the circle a and line b

(The proof of this can be found in the resources mentioned at the end)

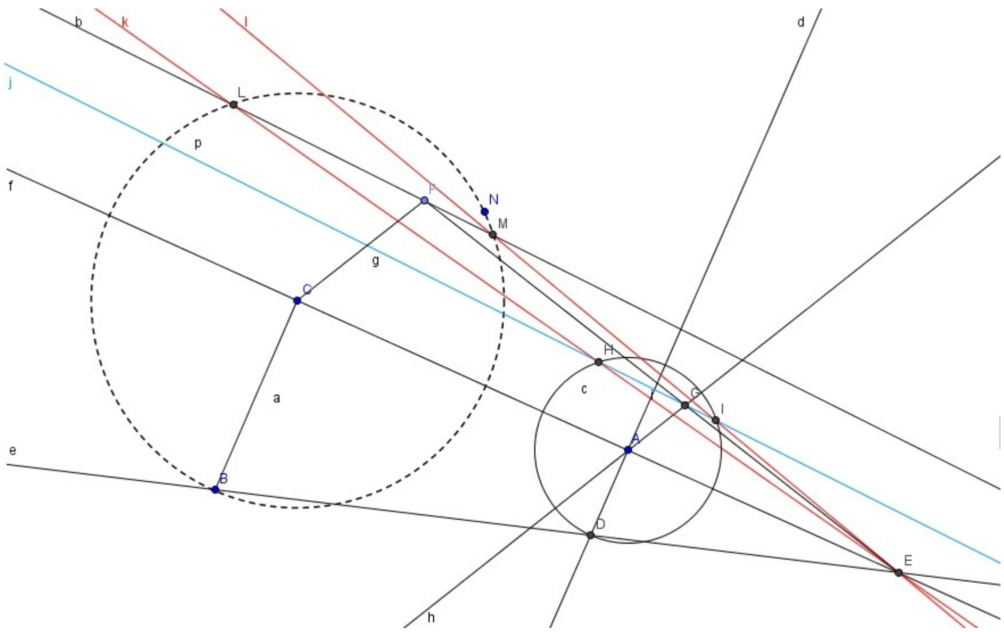

Figure 12

Problem 7: Given the centers B and E of circles a and b and their radii BC and ED respectively. Construct the points of intersection of the circles a and b.

Solution:
Join CD and BE
Take the midpoint F of CD
Join EF and BF
Draw h, the perpendicular to BF through C
Draw i, the perpendicular to EF through D
h and i meet in G
G is the radical center of the circles a, b, and c
Draw j, the perpendicular to BE through G
Then, j is the radical axis of a and b

We know that the points of intersection lie on the radical axis of a and b

Thus, the problem is reduced to finding the points of intersection of j and one of the circles a or b, which can be handled by Problem 6.

(The proof of this can be found in the resources mentioned at the end)

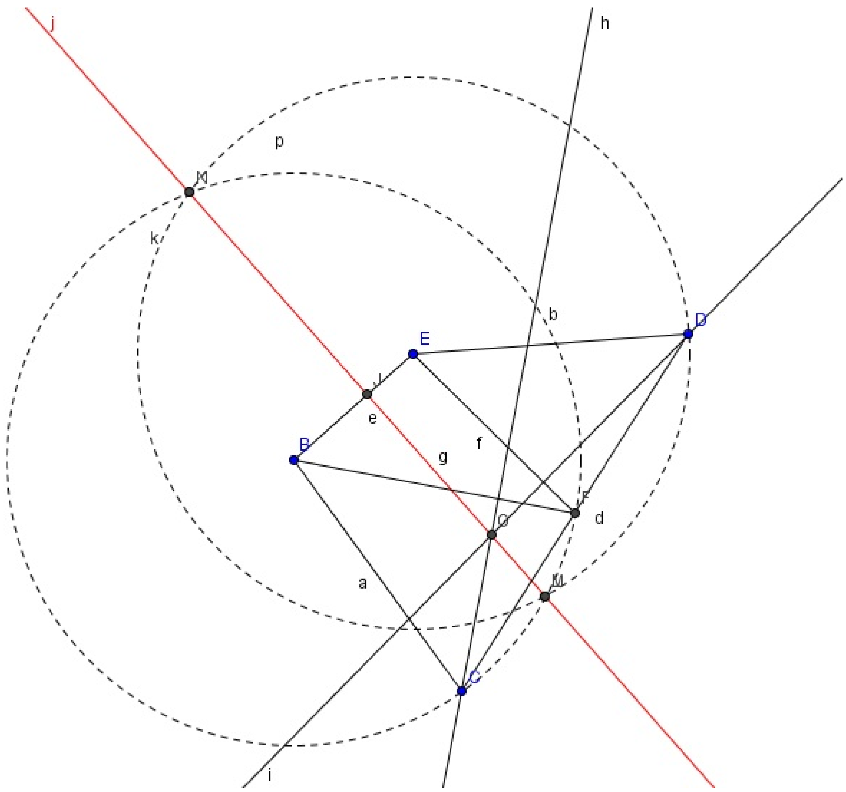

Figure 13

5. Conclusion

Every construction that can be done with compass and straight-edge can be done with straight-edge alone given a fixed circle in the plane.

6. References

(i) A. S. Smogorzhevskii "Ruler in Geometrical Constructions" (Volume 5)

(ii) J. Steiner "Geometrical constructions carried out with straight lines and a fixed circle" (Berlin, 1833)