# SETTING UP AND CALCULATING WITH THE FRÖLICHER SPECTRAL SEQUENCE

ALEKSANDAR MILIVOJEVIĆ

ABSTRACT. This is an expository note, based on the lectures and papers in the references.

The sections of the exterior bundle of the complexified cotangent bundle of a complex manifold naturally form a bigraded commutative algebra on which the de Rham differential splits into two anti-commuting bigraded pieces (one "holomorphic" and one "anti-holomorphic"). Filtering this algebra in a natural way yields a spectral sequence whose first page is the Dolbeault cohomology and whose last page is isomorphic to the (complexified) de Rham cohomology. We will discuss Frölicher's original treatment of this spectral sequence as presented in his 1955 paper "Relations between the cohomology groups of Dolbeault and topological invariants". The existence of this spectral sequence implies relations between the Betti numbers of a smooth manifold and the dimensions of the Dolbeault cohomology groups of any complex structure it may come equipped with.

The differentials on each page of the Frölicher spectral sequence can be explicitly described, and one can wonder whether these differentials uniformly vanish from some page onwards on a given complex manifold. If the manifold is compact and Kähler, or a compact complex curve or surface, then this phenomenon happens on the first page. We will look at some examples of non-Kähler compact complex threefolds for which the spectral sequence has non-trivial differentials past the first page. These examples will come from the easy-to-work-with family of nilmanifolds, and we will obtain explicit descriptions of each page of their spectral sequences.

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## 1. The spectral sequence of a filtered cochain complex

Before focusing on the Frölicher spectral sequence, let us consider the more general setup of a filtered cochain complex. This material comes from [Morgan]; we fill in some of the easier details.

Our data is the following:

- A bounded cochain complex $C^*$ of vector spaces, concentrated in non-negative degrees
$$0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots C^N \xrightarrow{d} 0,$$
- A decreasing filtration $F^* = F^*(C^*)$ of the cochain complex, i.e. a sequence of subspaces $\cdots F^n \subset F^{n+1} \subset F^{n+2} \subset \cdots$, which satisfies $F^0(C^*) = C^*$ and $F^{n+1}(C^n) = \{0\}$ for all $n$. We require that the differential $d$ in the cochain complex is compatible with the filtration in sense that $d(F^n) \subset F^n$ for all $n$. In expanded notation, this is the condition $d(F^n(C^k)) \subset F^n(C^{k+1})$, since $d(C^k) \subset C^{k+1}$. (By $F^n(C^k)$ we are denoting the intersection $F^n(C^*) \cap C^k$.)

A filtration compatible with the differential on a cochain complex is a powerful piece of data; it enables the following discussion, resulting in a trigraded sequence of vector spaces called the *spectral sequence* associated to the data of cochain complex and filtration. First, we note that the filtration on $C^*$ induces a filtration on the cocycles and the coboundaries. Denote the cycles in degree $n$ by $Z^n = (\ker d) \cap C^n$, and set

$$F^k Z^n = F^k(C^*) \cap Z^n = F^k(C^n) \cap Z^n.$$

Note that $F^{k+1}Z^n = F^{k+1}(C^n) \cap Z^n \subset F^k(C^n) \cap Z^n = F^k Z^n$, and we see that $F^*$ induces a filtration on the complex $Z^*$ with trivial differential, with the same properties as in the second point above. Similarly, denote the coboundaries in degree $n$ by $B^n = (\operatorname{image} d) \cap C^n$, and set

$$F^k B^n = F^k(C^*) \cap B^n = F^k(C^n) \cap B^n.$$

We also obtain a filtration on the cohomology $H^*(C^*)$. Namely, we have $H^n = H^n(C^*) = \frac{\ker d \cap C^n}{\operatorname{image} d \cap C^n}$, and we set

$$F^k H^n = \frac{\ker d \cap F^k C^n}{\operatorname{image} d \cap F^k C^n} = \frac{F^k Z^n}{F^k B^n}.$$

Since $F^{k+1}Z^n \subset F^k Z^n$ and $F^{k+1}B^n \subset F^k B^n$, we have $F^{k+1}H^n \subset F^k H^n$, along with $F^0 H^n = H^n$ and $F^{n+1}H^n = \{0\}$.

Now, consider the associated graded vector space $\bigoplus_{p=0}^n \frac{F^p(C^n)}{F^{p+1}(C^n)}$ of $C^n$ obtained from the filtration. Note that abstractly we have $C^n \cong \bigoplus_{p=0}^n \frac{F^p(C^n)}{F^{p+1}(C^n)}$. We can also consider the associated graded of $H^n(C^*)$, namely $\bigoplus_{p=0}^n \frac{F^p(H^n)}{F^{p+1}(H^n)}$. Let us obtain a more transparent description of $\frac{F^p(H^n)}{F^{p+1}(H^n)}$. We have $F^{p+1}H^n = \frac{F^{p+1}Z^n}{F^{p+1}B^n}$, and by the second isomorphism theorem applied to $F^{p+1}B^n =$

$F^{p+1}Z^n \cap F^pB^n$, we have

$$F^{p+1}H^n = \frac{F^{p+1}Z^n}{F^{p+1}B^n} \cong \frac{F^pB^n + F^{p+1}Z^n}{F^pB^n}.$$

Now, we have

$$\frac{F^pH^n}{F^{p+1}H^n} \cong \frac{\frac{F^pZ^n}{F^pB^n}}{\frac{F^pB^n+F^{p+1}Z^n}{F^pB^n}} \cong \frac{F^pZ^n}{F^{p+1}Z^n + F^pB^n}.$$

We would now like to form a sequence of approximations to the associated graded of $H^*$, starting with the associated graded of the cochain complex itself. At each stage of the approximation, at a given total degree $n$ and filtration level $F^pC^n$, we will take those cochains whose coboundary lives in some higher filtration level $F^{p+r}$ (i.e. the forms that are closed modulo that higher filtration level), modulo forms in the next filtration level $F^{p+1}$ satisfying this same property (that their coboundary is in $F^{p+r}$), and modulo coboundaries in $F^p$ of elements in a lower filtration, $F^{p-r+1}$. If we let $r$ grow very large, i.e. we are far along in this approximation procedure, then the "higher filtration level" is $\{0\}$ and the "lower filtration level" is all of $C^n$, and so the corresponding "approximation" will be all closed forms of total degree $n$ and filtration level $p$, modulo closed forms of total degree $n$ and filtration level $p+1$, and modulo coboundaries in $F^p$. That is, this "approximation" for large $r$ will be exactly $\frac{F^pZ^n}{F^{p+1}Z^n+F^pB^n}$, which is the piece of the associated graded of $H^n$ considered above.

For any $r$ (denoting how many steps of approximation we have done), we are considering the vector space

$$\frac{\{x \in F^pC^n \mid dx \in F^{p+r}C^{n+1}\}}{\{y \in F^{p+1}C^n \mid dy \in F^{p+r}C^{n+1}\} + d(F^{p-r+1}C^{n-1}) \cap F^pC^n}$$

as our $r$th approximation to the associated graded $\frac{F^pH^n}{F^{p+1}H^n}$ of cohomology. Before we continue let us verify that this approximation really is starting with the associated graded of the cochain complex itself, i.e. that for $r=0$ this vector space is $\frac{F^pC^n}{F^{p+1}C^n}$. Indeed, for $r=0$ we have

$$\frac{\{x \in F^pC^n \mid dx \in F^pC^{n+1}\}}{\{y \in F^{p+1}C^n \mid dy \in F^pC^{n+1}\} + d(F^{p+1}C^{n-1}) \cap F^pC^n}.$$

Since $d$ is compatible with the filtration, the numerator is just $F^pC^n$, and the denominator is $F^{p+1}(C^n)+d(F^{p+1}(C^{n-1}))\cap F^p$. Again since $d(F^{p+1}) \subset F^{p+1}$, we have that the denominator is just $F^{p+1}(C^n)$, and so this vector space for $r=0$ is $\frac{F^p(C^n)}{F^{p+1}(C^n)}$ as desired.

Now, we want a way to move from one approximating vector space to the next. First, note that in this sequence of approximating vector spaces indexed by $r$, we are keeping track of two indices: the filtration level $p$ and the total degree $n$. It will be easier to think about what follows if we write $p+q$ for the total degree instead of $n$. (Note that for $p > n$, the vector space above is trivial, due to $F^{n+1}C^k = \{0\}$. We relabel the associated graded of the cochain complex, and we define

$$E_0^{p,q} = \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}},$$

along with the approximating vector spaces for general $r$,

$$E_r^{p,q} = \frac{\{x \in F^pC^{p+q} \mid dx \in F^{p+r}C^{p+q+1}\}}{\{y \in F^{p+1}C^{p+q} \mid dy \in F^{p+r}C^{p+q+1}\} + d(F^{p-r+1}C^{p+q-1}) \cap F^pC^{p+q}}.$$

We saw that for large $r$, we get the associated graded of the cohomology, now denoted $\frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}$. We denote this by

$$E_\infty^{p,q} = \frac{\{x \in F^pC^{p+q} \mid dx = 0\}}{\{y \in F^{p+1}C^{p+q} \mid dy = 0\} + d(C^{p+q-1}) \cap F^pC^{p+q}} = \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}.$$

The $E_r^{p,q}$ form our trigraded sequence of vector spaces mentioned above. For each $r$, we call the bigraded sequence $E_r^{*,*}$ the $r$th *page* of the spectral sequence.

To obtain some natural movement from $E_r^{*,*}$ to $E_{r+1}^{*,*}$, we first note that $d$ induces a map on every page of the spectral sequence. Let us focus on $r = 0$, i.e. $\frac{F^pC^{p+q}}{F^{p+1}C^{p+q}}$. Applying $d$ to an element in $F^pC^{p+q}$ gives us an element in $F^pC^{p+q+1}$, since $d$ is compatible with the filtration. Also, $d$ maps $F^{p+1}C^{p+q}$ into $F^{p+1}C^{p+q+1}$, and so $d$ induces a map

$$E_0^{p,q} = \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}} \xrightarrow{d_0} \frac{F^pC^{p+q+1}}{F^{p+1}C^{p+q+1}} = E_0^{p,q+1}.$$

In this new $(p,q)$ bigrading, the induced map $d_0$ is of bidegree $(0,1)$. Let us also briefly consider $E_1^{p,q} = \frac{\{x \in F^pC^{p+q} \mid dx \in F^{p+1}C^{p+q+1}\}}{F^{p+1}C^{p+q} + dF^pC^{p+q+1}}$. For an $x$ in the numerator, by definition we have $dx \in F^{p+1}C^{p+q+1}$, and $d(dx) = 0$. For $y + dz$ in the denominator, we have $d(y + dz) = dy$, where $y \in F^{p+1}$. Therefore $d$ induces a map

$$\frac{\{x \in F^pC^{p+q} \mid dx \in F^{p+1}C^{p+q+1}\}}{F^{p+1}C^{p+q} + dF^pC^{p+q+1}} \xrightarrow{d_1} \frac{\{x \in F^{p+1}C^{p+1+q} \mid dx \in F^{p+2}C^{p+q+2}\}}{F^{p+2}C^{p+1+q} + dF^{p+1}C^{p+q+2}},$$

i.e. a map $E_1^{p,q} \to E_1^{p+1,q}$ of bidegree $(1,0)$.

Before we make a general statement, let us observe that the cohomology of $E_0^{p,q}$ with respect to $d_0$ (which squares to zero since it is induced by $d$) is isomorphic to $E_1^{p,q}$. Indeed, consider

$$\frac{F^pC^{p+q-1}}{F^{p+1}C^{p+q-1}} \xrightarrow{d_0} \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}} \xrightarrow{d_0} \frac{F^pC^{p+q+1}}{F^{p+1}C^{p+q+1}},$$

and consider kernel modulo image at the middle slot. The kernel of $d_0$ is $\frac{\{x \in F^p \mid dx \in F^{p+1}\}}{F^{p+1}C^{p+q}}$, and the image of $d_0$ is $\frac{d(F^pC^{p+q-1})+F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}}$. (Here we are using that for $V' \subset V$, $W' \subset W$, and a map $V \xrightarrow{\phi} W$ that descends to a map $\frac{V}{V'} \to \frac{W}{W'}$, the image of the induced map is $\frac{\text{image}\,\phi + W'}{W'}$.) Taking the quotient we see that

$$H^{p,q}(E_0^{*,*}, d_0) = \frac{\frac{\{x \in F^p \mid dx \in F^{p+1}\}}{F^{p+1}C^{p+q}}}{\frac{d(F^pC^{p+q-1})+F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}}} \cong \frac{\{x \in F^p \mid dx \in F^{p+1}\}}{d(F^pC^{p+q-1}) + F^{p+1}C^{p+q}} = E_1^{p,q}.$$

In general, the $E_{r+1}$ page will be the cohomology of the $E_r$ page with respect to the differential $d_k$, though this is not so obvious as in the above case of $r = 0$. This is the content of the following theorem.

**Theorem 1.1.** *For any $r \geq 0$, the differential $d$ on the cochain complex $C^*$ induces an operator $E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}$. (The operator is of bidegree $(r, -r+1)$ and squares to zero.) The cohomology of $E_r^{*,*}$ with respect to $d_r$ is isomorphic to $E_{r+1}^{*,*}$.*

*Proof.* First let us note that $d$ induces a map $E_r^{p,q} \to E_r^{p+r,q-r+1}$, i.e. a map

$$\frac{\{x \in F^p | dx \in F^{p+r}\}}{\{y \in F^{p+1} | dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p} \xrightarrow{d_r} \frac{\{x \in F^{p+r} | dx \in F^{p+2r}\}}{\{y \in F^{p+r+1} | dy \in F^{p+2r}\} + dF^{p+1} \cap F^{p+r}}.$$

(We are omitting writing $C^{p+q}$ and $C^{p+q+1}$.) Note that for $x$ in the numerator of the left term, $dx$ satisfies the condition to be in the numerator of the right term, since $dx \in F^{p+r}$ and $d(dx) = 0 \in F^{p+2r}$. A general term in the denominator on the left is of the form $y + dz$, where $y \in F^{p+1}$, $dy \in F^{p+r}$, and $z \in F^{p+2r}$, $dz \in F^p$. Then $d(y + dz) = dy \in dF^{p+1} \cap F^{p+r}$. Therefore $d$ indeed induces a map on $E_r^{*,*}$. The map $d_r$ squares to zero since $d$ does.

Now, let us calculate the cohomology of $d_r$ at the term $E_r^{p,q}$. First we consider $\ker d_r$; a class $[x] \in E_r^{p,q}$ is in the kernel if for any representative $x \in \{x \in F^p | dx \in F^{p+r}\}$, we have $dx \in \{y \in F^{p+r+1} | dy \in F^{p+2r}\} + dF^{p+1} \cap F^{p+r}$. So, $dx = y + dz$ where $y$ and $z$ satisfy the listed properties. Since $z \in F^{p+1}$ and $dz \in F^{p+r}$, $z$ is in the first term of the denominator on the left, and so $[x] = [x - z]$. This new representative $x - z$ has the nice property that $d(x - z) = y \in F^{p+r+1}$. This is an "upgrade" from the previous $dx \in F^{p+r}$. So, every class $[x] \in \ker d_r$ has a representative in the space $V' = \{x \in F^p \mid dx \in F^{p+r+1}\}$. Denoting $\ker d_r = \frac{V}{\{y \in F^{p+1} | dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p} = \frac{V}{W}$, (which is just the general form of a subspace of a quotient vector space), and noting that our calculation above was carried out inside $V$ (which is the vector space of all representatives of elements in $\ker d_r$), we have $V' \subset V$ along with the fact that every element in $\frac{V}{W}$ is represented by an element in

$V'$. Now, the map

$$\frac{V'}{V' \cap W} \hookrightarrow \frac{V}{W}$$

induced by the inclusion is in fact an isomorphism. Indeed, a class is in the kernel if the image of a representative (i.e. the representative itself) is in $W$; since the representative is already in $V'$, it must be in $V' \cap W$, i.e. 0 in the quotient space. The map is surjective since for a class $[v] \in \frac{V}{W}$ we can choose the class in $\frac{V'}{V' \cap W}$ represented by an element representing $[v]$. Let us now find a simpler expression for $V' \cap W = \{x \in F^p \mid dx \in F^{p+r+1}\} \cap (\{y \in F^{p+1} | dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p)$. We claim

$$\{x \in F^p | dx \in F^{p+r+1}\} \cap (\{y \in F^{p+1} | dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p) = \{z \in F^{p+1} | dz \in F^{p+r+1}\} + dF^{p-r+1} \cap F^p.$$

Indeed, the right hand side is easily included into the left. An element on the left is of the form $x$ for $x \in F^p$ and $dx \in F^{p+r+1}$, but also of the form $y + dw$, for $y \in F^{p+1}$, $dy \in F^{p+r}$, $z \in F^{p-r+1}$, $dz \in F^p$. Since $dy = d(y + dw) = dx \in F^{p+r+1}$, we see that $y + dw$ fits the description of the right side of the equation. So, we have concluded

$$\ker d_r \cong \frac{\{x \in F^p \mid dx \in F^{p+r+1}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r+1}\} + dF^{p-r+1} \cap F^p}.$$

Now we consider the image $d_r(E_r^{p-r,q+r-1}) \subset E_r^{p,q}$, i.e. the image of the map

$$\frac{\{x \in F^{p-r} | dx \in F^p\}}{\{y \in F^{p-r+1} | dy \in F^p\} + dF^{p-2r+1} \cap F^{p-r}} \xrightarrow{d_r} \frac{\{x \in F^p | dx \in F^{p+r}\}}{\{y \in F^{p+1} | dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p}.$$

Since the image lands in the kernel (and we identify the kernel with the above quotient space under the inclusion map), we have

$$\begin{aligned} \operatorname{image} d_r &\cong \frac{\{dx \mid x \in F^{p-r}, dx \in F^p\} + (\{z \in F^{p+1} \mid dz \in F^{p+r+1}\} + dF^{p-r+1} \cap F^p)}{\{z \in F^{p+1} \mid dz \in F^{p+r+1}\} + dF^{p-r+1} \cap F^p} \\ &= \frac{dF^{p-r} \cap F^p + \{z \in F^{p+1} \mid dz \in F^{p+r+1}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r+1}\} + dF^{p-r+1} \cap F^p}, \end{aligned}$$

since in our usual notation $\{dx \mid x \in F^{p-r}, dx \in F^p\} = dF^{p-r} \cap F^p$, and $dF^{p-r+1} \cap F^p \subset dF^{p-r} \cap F^p$. Finally, we have

$$\begin{aligned} \frac{\ker d_r}{\operatorname{image} d_r} &\cong \frac{\frac{\{x \in F^p \mid dx \in F^{p+r+1}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r+1}\} + dF^{p-r+1} \cap F^p}}{\frac{dF^{p-r} \cap F^p + \{z \in F^{p+1} \mid dz \in F^{p+r+1}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r+1}\} + dF^{p-r+1} \cap F^p}} \\ &\cong \frac{\{x \in F^p \mid dx \in F^{p+r+1}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r+1}\} + dF^{p-r} \cap F^p}. \\ &= E_{r+1}^{p,q}. \end{aligned}$$

□

## 2. The Frölicher spectral sequence

Now we apply the story in the previous section to construct the Frölicher (also known as Hodge-to-de Rham) spectral sequence. Let $M$ be a compact complex manifold of complex dimension $n$. Denote by $A^{p,q}$ the space of smooth complex-valued sections of the bundle $\Lambda^p TM \oplus \Lambda^q \overline{TM}$, where $TM$ denotes the holomorphic tangent bundle. Equivalently, one can consider the complexification of the smooth tangent bundle, $TM \otimes \mathbb{C}$, which splits into eigenbundles for the complexification of the endomorphism $J$ of the tangent bundle that determines the complex structure on the real tangent bundle, i.e. in some coordinate chart $(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$ the operator defined by $J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$, $J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$. Denote the splitting by $T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}$ is the $i$-eigenbundle, and is in fact isomorphic as a complex bundle to the holomorphic tangent bundle, and $T^{0,1}$ is the $-i$-eigenbundle, isomorphic to the conjugate of the holomorphic tangent bundle. Then for any $k$ we can consider the exterior bundle $\Lambda^k TM \otimes \mathbb{C}$, which splits as $\Lambda^k TM \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^p T^{1,0}M \wedge \Lambda^q T^{0,1}M$; the space $A^{p,q}$ is then the space of smooth complex-valued sections of $\Lambda^p T^{1,0}M \otimes \Lambda^q T^{0,1}M$.

In local coordinates, an element of $A^{p,q}$ is simply a sum of terms of the form $f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, where $f$ is a smooth function with values in the complex numbers. On the bigraded algebra $\bigoplus_{i,j} A^{p,q}$ we can consider the operators $\partial$ and $\bar{\partial}$ of bidegree $(1,0)$ and $(0,1)$ respectively. The operator $\partial$ acts on a homogeneous term by

$$\partial f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} = \sum_k \frac{\partial f}{\partial z_k} dz_k \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

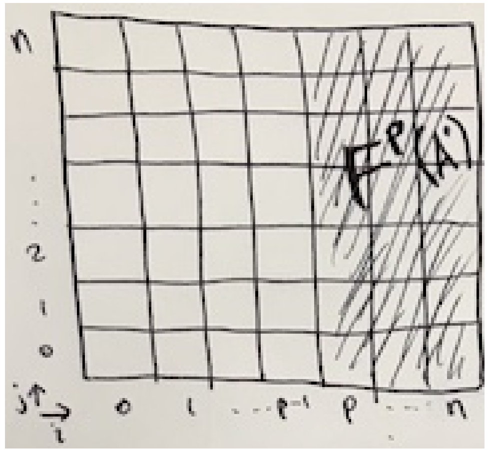
and similarly for $\bar{\partial}$. We can also consider the complexified de Rham exterior derivative $d$ on this bigraded algebra, and (as can be checked by noting that every form is locally a sum of products of functions and one-forms, and $d$, $\partial$, $\bar{\partial}$ satisfy the Leibniz rule) we have

$$d = \partial + \bar{\partial}.$$

Note that from $d^2 = 0$ it follows that $\partial^2 = 0$, $\bar{\partial}^2 = 0$, and $\partial\bar{\partial} = -\partial\bar{\partial}$, by looking at the pieces of the equation in each grading.

Let us now denote $A^k = \bigoplus_{p+q=k} A^{p,q}$. In this singly-graded algebra, $d$ is a degree $+1$ differential. The complex $(A^*, d)$ will be our cochain complex from the previous section. Our filtration will be

$$F^p A^k = \bigoplus_{i \geq p,\, i+j=k} A^{i,j}.$$




We have $F^0A^k = A^k$, and since the space $A^{p,q}$ is trivial as soon as $p$ or $q$ are $> n$, we have $F^{n+1}A^k = \{0\}$ for all $k$.

We now translate our conclusions from the previous section into this situation. The spaces $E_0^{p,q}$ are by definition

$$E_0^{p,q} = \frac{F^p(A^{p+q})}{F^{p+1}(A^{p+q})} = \frac{\bigoplus_{i\geq p,\, i+j=p+q} A^{i,j}}{\bigoplus_{i\geq p+1,\, i+j=p+q} A^{i,j}} \cong A^{p,q}.$$

The map $A^{p,q} \hookrightarrow E_0^{p,q}$ induced by inclusion is an isomorphism. Consider now the differential on the $E_0$ page, i.e. $\frac{F^p(A^{p+q})}{F^{p+1}(A^{p+q})} \xrightarrow{d_0} \frac{F^p(A^{p+q+1})}{F^{p+1}(A^{p+q+1})}$. An element in the domain is of the form $\alpha + F^{p+1}(A^{p+q})$, which under our identification of the $E_0$ page is the form $\alpha$. Applying $d$ yields $d\alpha + F^{p+1}$. Writing $d\alpha = \bar{\partial}\alpha + \partial\alpha$, and noting that $\partial\alpha \in F^{p+1}$ (since $\partial$ raises the $p$ degree by one), we have that $d\alpha + F^{p+1} = \bar{\partial}\alpha + F^{p+1}$, which we identify with $\bar{\partial}\alpha$. So under our above identification of the $E_0$ page, the $d_0$ differential is the $\bar{\partial}$ operator. That is, the class in $E_0$ given by $[x]$, where $x$ is the unique representative determined by the identification of $E_0$ above, is such that $d_0[x] = [\bar{\partial}x]$, since $\bar{\partial}$ is of bidegree $(0,1)$ and thus sends these unique representatives to other classes' unique representatives. Therefore we can calculate the cohomology of $(E_0^{*,*}, d_0)$ as the cohomology of $(A^{*,*}, \bar{\partial})$. So, we have (by our theorem in the previous section),

$$E_1^{p,q} \cong \frac{\ker \bar{\partial}(A^{p,q})}{\operatorname{image} \bar{\partial}(A^{p,q-1})} = H_{\bar{\partial}}^{p,q}(M),$$

where $H_{\bar{\partial}}^{p,q}(M)$ denotes the Dolbeault cohomology. These Dolbeault cohomology groups are known to be finite dimensional on a compact complex manifold (by Hodge theoretic arguments).

Now let us turn to the $E_\infty$ page. We know that abstractly $\oplus_{p+q=k} E_\infty^{p,q} \cong H^k(A^*, d)$, just by a dimension count. The cohomology $H^k(A^*, d)$ is simply the complexified de Rham cohomology, i.e. the cohomology of the complexified de Rham complex (since any form in $A^k$ is locally a sum of terms of the form $f dz_I d\bar{z}_J$ for some indexing sets $I$ and $J$, and complex valued function $f$; expanding $dz_i = dx_i + idy_i$ and $d\bar{z}_j = dx_j - idy_j$ and splitting $f$ into its real and imaginary parts, we have our conclusion). So, the sum of the complex dimensions of the spaces $E_\infty^{p,q}$ for $p+q=k$ is the $k$th Betti number $b_k$ (defined as the real dimension of the corresponding cohomology group in the real de Rham complex). On the other hand, note that for any $p, q, r$ we have $\dim E_r^{p,q} \geq \dim E_{r+1}^{p,q}$, since the $r+1$ page is the cohomology of the $r$ page. In particular,

$$\sum_{i+j=k} \dim E_1^{p,q} \geq \sum_{i+j=k} \dim E_2^{p,q} \geq \ldots \geq \sum_{i+j=k} \dim E_\infty^{p,q} = b_k.$$

Since $E_1^{p,q} \cong H_{\bar{\partial}}^{p,q}(M)$, denoting $h^{p,q} = \dim H_{\bar{\partial}}^{p,q}$, we thus have the following theorem:

**Theorem 2.1.** *On a compact complex manifold $M$, for any $k$ we have*

$$\sum_{p+q=k} h^{p,q} \geq b_k,$$

*where $h^{p,q}$ denotes the complex dimension of the Dolbeault cohomology group $H_{\bar{\partial}}^{p,q}(M)$, and $b_k$ denotes the real dimension of the $k$th real de Rham cohomology group $H_{dR}(M)$.*

**Remark 2.2.** It should be noted that $h^{p,q}$ in general *depends on the complex structure*, while $b_k$ is a homotopy invariant. See [Angella, Appendix A] for an example of two diffeomorphic 6–manifolds equipped with complex structures such that the $h^{p,q}$ numbers generally differ. In complex dimensions 1 and 2, it turns out that the numbers $h^{p,q}$ are in fact homotopy invariants.

**Example 2.3.** Suppose we have a complex manifold such that for some $k$, equality $\sum_{p+q=k} h^{p,q} = b_k$ is achieved. We can consider slightly deforming this complex structure, in which case $\ker \bar{\partial}$ can only "decrease in dimension", and image $\bar{\partial}$ can only "increase in dimension", and so a small deformation results in $H_{\bar{\partial}}^{p,q}$ decreasing in dimension (or staying the same). (The spaces involved are infinite-dimensional, so this is just a vague guiding idea.) Precisely, $h^{p,q}$ is an upper-semicontinuous function of the complex structure on the underlying smooth manifold. Since $\sum_{p+q=k} h^{p,q} = b_k$ before deforming the complex structure, and $b_k$ stays constant throughout the deformation (since the diffeomorphism type of the underlying manifold is preserved), and each $h^{p,q}$ stays the same or decreases, we conclude that $\sum_{p+q=k} h^{p,q} = b_k$ remains true in a neighborhood of our original complex structure.

We now turn to calculating the Euler characteristic of our complex manifold $M$ (a homotopy invariant) using only the numbers $h^{p,q}$. We use only the simple linear algebra fact that the alternating sum of the dimensions of the spaces in a finite-dimensional complex $(C^*, d)$ is equal to the alternating sum of the dimensions of its cohomology groups $H^*(C^*, d)$ (note the independence of the differential $d$), and the harder fact mentioned above that the first page $E_1^{*,*}$ of the Frölicher spectral sequence consists of finite-dimensional vector spaces. Consider the singly-graded complex $(\bigoplus_{p+q=*} E_1^{p,q}, d_0)$. The alternating sum of dimensions of its constituents is

$$\sum_k \sum_{p+q=k} (-1)^k h^{p,q}.$$

On the other hand, the cohomology of this complex is $\bigoplus_{p+q=*} E_2^{p,q}$. The alternating sums of the dimensions of the terms of this singly graded complex obtained by grouping the terms of the $E_2$ page with a common total degree is the Euler characteristic of its cohomology with respect to the $d_1$ differential. Inductively, we conclude that the Euler characteristic of $(\bigoplus_{p+q=*} E_1^{p,q}, d_0)$ is the Euler characteristic of $(\bigoplus_{p+q=*} E_\infty^{p,q}, 0)$, which is the alternating sum of the Betti numbers $b_*$, i.e. the topological Euler characteristic $\chi(M)$. We record this observation as our following theorem.

**Theorem 2.4.** *For a compact complex manifold $M$, we have*

$$\sum_{p,q} (-1)^{p+q} h^{p,q} = \chi(M).$$

**Remark 2.5.** There is a symmetry among the numbers $h^{p,q}$ on any compact complex manifold, namely $h^{p,q} = h^{n-p,n-q}$, where $n$ is the complex dimension of $M$. This follows from the classical *Serre duality*. We also note that if $M$ is connected, then $h^{0,0} = 1$ (since $H_{\bar\partial}^{0,0}$ is the space of holomorphic functions on $M$), and so by Serre duality $h^{n,n} = 1$ as well.

## 3. A visual description of the terms and differentials

In the bicomplex $(A^{*,*}, \partial, \bar\partial)$ of forms (which we identify with the $E_0$ page) on a compact complex manifold, we can ask for a picture of what the elements on a later page $E_r$ look like, and how the induced differential $d_r$ is determined from this picture.

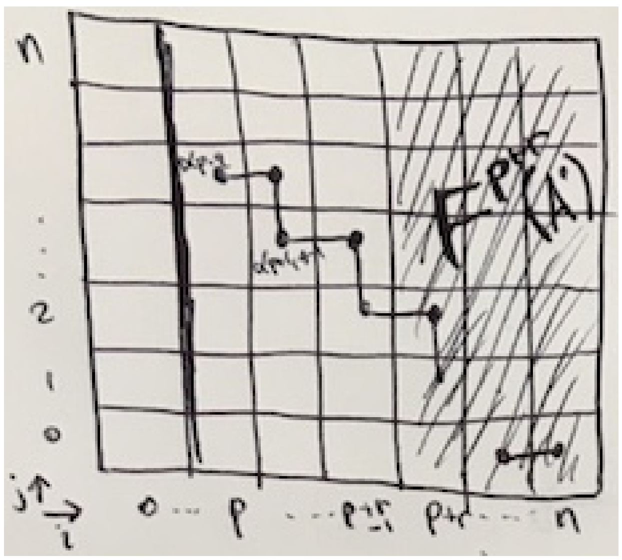
As a somewhat special case, let us first make a picture for the $E_1$ page and the $d_1$ differential. Recall, $E_1^{p,q} = \frac{\{x \in F^p \mid dx \in F^{p+1}\}}{F^{p+1} + dF^p}$. A representative of a class in $E_1^{p,q}$ is a class $\alpha_{p,q} + \alpha_{p+1,q-1} + \ldots + \alpha_{p+q,0}$, where $d(\alpha_{p,q} + \alpha_{p+1,q-1} + \ldots + \alpha_{p+q,0}) \in F^{p+1}$. (The subscripts denote the bidegree of each form.)

From here, as we saw, it follows that $\bar{\partial}\alpha_{p,q} = 0$. Now, $d_1(\alpha_{p,q} + \alpha_{p+1,q-1} + \ldots + \alpha_{p+q,0})$ is represented by

$$d(\alpha_{p,q} + \alpha_{p+1,q-1} + \ldots + \alpha_{p+q,0}) = \partial\alpha_{p,q} + d(\alpha_{p+1,q-1} + \ldots + \alpha_{p+q,0}),$$

which in $E_1^{p+1,q}$ is the class of $\partial\alpha_{p,q}$. Therefore, under our identification of the $E_0$ page with the bicomplex of forms, a term is on the $E_1$ page if it is $\bar{\partial}$-closed, and its differential on the $E_1$ page is represented by $\partial$ of the representative used under the identification of $E_0$.

Let us now consider an element in $E_r^{p,q} = \frac{\{x \in F^p | dx \in F^{p+r}\}}{\{y \in F^{p+1} | dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p}$. A representative is of the form $x = \alpha_{p,q} + \alpha_{p+1,q-1} + \ldots + \alpha_{p+q,0}$, with the property that $d(\alpha_{p,q} + \alpha_{p+1,q-1} + \ldots + \alpha_{p+q,0}) \in F^{p+r}$. This means that some cancellation has to occur in order for the differential to land in the higher filtration level $F^{p+r}$.



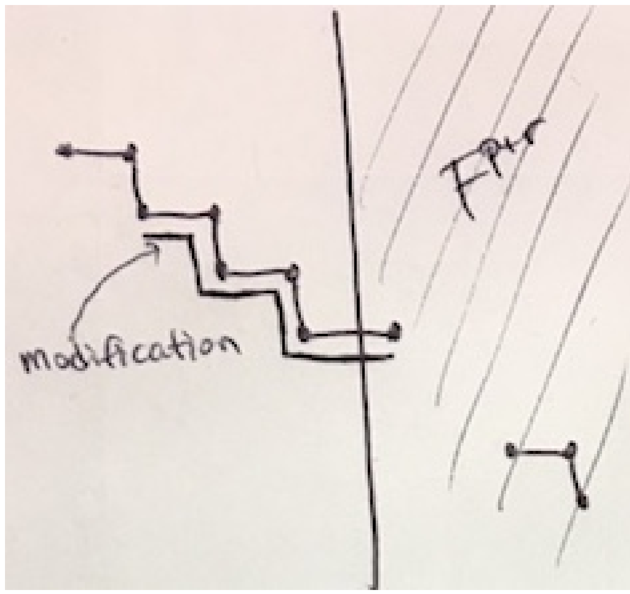

Specifically, the following has to happen:

- $\bar{\partial}\alpha_{p,q} = 0$,
- $\partial\alpha_{p,q} + \bar{\partial}\alpha_{p+1,q-1} = 0$,
- $\partial\alpha_{p+1,q-1} + \bar{\partial}\alpha_{p+2,q-2} = 0$,

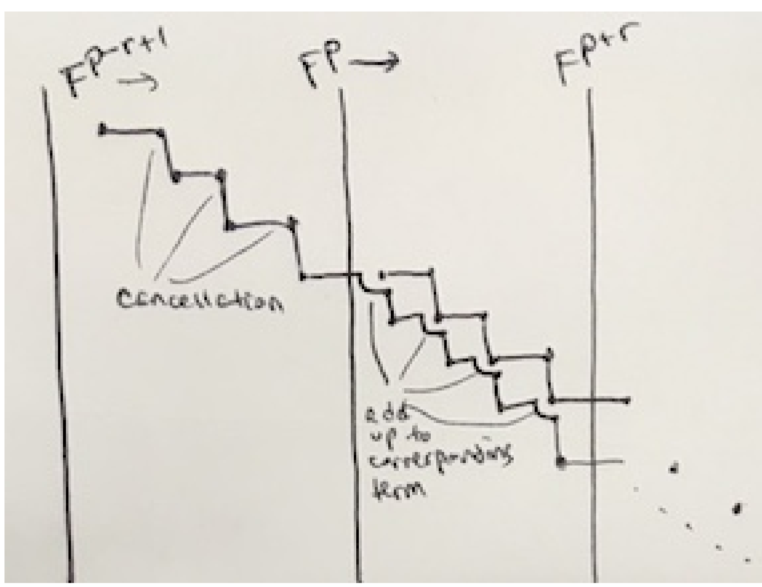
  $\vdots$
- $\partial\alpha_{p+r-2,q-r+2} + \bar{\partial}\alpha_{p+r-1,q-r+1} = 0$.

Think of $x$ therefore as a sum of homogeneous elements in $F^p$, such that the differentials applied to the terms outside of the filtration level $F^{p+r}$ link together (i.e. cancel each other out) to form a zig-zag that makes its way into $F^{p+r}$.

Now let us look at the denominator of $E_r^{p,q}$. An element in $\{y \in F^{p+1} | dy \in F^{p+r}\}$ has the same description as our representative $x$ above, except $y \in F^{p+1}$ instead of $F^p$. So, in $E_r^{p,q}$ we can modify our zig-zag $x$ by any such "lower" zig-zag $y$ without changing the corresponding class.




We also mod out by elements in $dF^{p-r+1} \cap F^p$, i.e. the differential applied to elements in $F^{p-r+1}$ with a zig-zag that makes it way into $F^p$ causing a cancellation of all the terms in the differential that are outside of $F^p$.



The following theorem from [CFUG] cleans up and makes precise the above description.

**Theorem 3.1** (CFUG Theorem 1)**.** *The space $E_r^{p,q}$, for $r \geq 2$, is isomorphic to the quotient*

$$\frac{\begin{array}{l}\{\alpha_{p,q} \mid \bar{\partial}\alpha_{p,q} = 0 \text{ and there exist } \alpha_{p+i,q-i} \text{ such that} \\ \quad \partial\alpha_{p,q} + \bar{\partial}\alpha_{p+1,q-1} = 0, \dots, \partial\alpha_{p+r-2,q-r+2} + \bar{\partial}\alpha_{p+r-1,q-r+1} = 0\}\end{array}}{\begin{array}{l}\{\partial\beta_{p-1,q} + \bar{\partial}\beta_{p,q-1} \mid \text{ there exist } \beta_{p-2,q+1}, \dots, \beta_{p-r+1,q+r-2} \\ \quad \text{such that } \partial\beta_{p-i,q+i-1} + \bar{\partial}\beta_{p-i+1,q+i-2} = 0 \text{ and } \bar{\partial}\beta_{p-r+1,q+r-2} = 0\}.\end{array}}$$

*Proof.* Recall $E_r^{p,q} = \frac{\{x \in F^p \mid dx \in F^{p+r}\}}{\{y \in F^{p+1} \mid dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p}$. By the third isomorphism theorem for vector spaces, we have

$$E_r^{p,q} \cong \frac{\frac{\{x \in F^p \mid dx \in F^{p+r}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r}\}}}{\frac{\{y \in F^{p+1} \mid dy \in F^{p+r}\} + dF^{p-r+1} \cap F^p}{\{z \in F^{p+1} \mid dz \in F^{p+r}\}}}.$$

Now applying the second isomorphism theorem for vector spaces (i.e. $\frac{V+W}{V} \cong \frac{W}{V \cap W}$) to the expression in the denominator, we have

$$E_r^{p,q} \cong \frac{\frac{\{x \in F^p \mid dx \in F^{p+r}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r}\}}}{\frac{dF^{p-r+1} \cap F^p}{\{y \in F^{p+1} \mid dy \in F^{p+r}\} \cap dF^{p-r+1} \cap F^p}}.$$

We see that $\{y \in F^{p+1} \mid dy \in F^{p+r}\} \cap dF^{p-r+1} \cap F^p = dF^{p-r+1} \cap F^{p+1}$, since $F^{p+1} \subset F^p$ and $dy = 0$ for any such $y \in dF^{p-r+1}$. So, we have

$$E_r^{p,q} \cong \frac{\frac{\{x \in F^p \mid dx \in F^{p+r}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r}\}}}{\frac{dF^{p-r+1} \cap F^p}{dF^{p-r+1} \cap F^{p+1}}}.$$

We focus now on the numerator, and note

$$\begin{aligned}\frac{\{x \in F^p \mid dx \in F^{p+r}\}}{\{z \in F^{p+1} \mid dz \in F^{p+r}\}} &= \frac{\{\alpha_{p,q} + \alpha_{p+1,q-1} + \cdots + \alpha_{p+q,0} \mid d(\alpha_{p,q} + \alpha_{p+1,q-1} + \cdots + \alpha_{p+q,0}) \in F^{p+r}\}}{\{\alpha_{p+1,q-1} + \cdots + \alpha_{p+q,0} \mid d(\alpha_{p+1,q-1} + \cdots + \alpha_{p+q,0}) \in F^{p+r}\}} \\ &\cong \{\alpha_{p,q} \mid \text{ there exist } \alpha_{p+1,q-1}, \dots, \alpha_{p+q,0} \\ &\qquad \text{such that } d(\alpha_{p,q} + \alpha_{p+1,q-1} + \cdots + \alpha_{p+q,0}) \in F^{p+r}\}.\end{aligned}$$

Indeed, the map from the last expression into the second-to-last expression obtained by taking any $\alpha_{p+1,q-1}, \dots, \alpha_{p+q,0}$ for a given $\alpha_{p,q}$ to its class in the quotient is an isomorphism. This is exactly the description of the numerator we want from the statement of the theorem.

As for the denominator term, we have

$$\frac{dF^{p-r+1} \cap F^p}{dF^{p-r+1} \cap F^{p+1}} = \frac{\{d\beta \in dF^{p-r+1} | (d\beta)_{p-r+1,q+r-1}=0, \ldots, (d\beta)_{p-1,q-1}=0\}}{\{d\beta \in dF^{p-r+1} | (d\beta)_{p-r+1,q+r-1}=0, \ldots, (d\beta)_{p-1,q-1}=0, (d\beta)_{p,q}=0\}}.$$

As in the case of the numerator, we conclude that this is isomorphic to the desired denominator in the statement of the theorem. (The assumption $r \geq 2$ is used here; otherwise the indexing of the terms in the denominator of the statement of the theorem isn't meaningful.) □

Using this description we obtain a simple description of the differential on the second page onwards. (Recall we have identified the differentials on $E_0$ and $E_1$.) Note that $d$ is compatible with the description of $E_r^{*,*}$ given in the theorem, in the sense that it induces a map from the numerator of the new description of $E_r^{p,q}$ to the numerator of the new description of $E_r^{p+r,q-r+1}$ that sends the denominator to the denominator. (This follows immediately from $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$.)

**Theorem 3.2.** *For $[\alpha] \in E_r^{p,q}$ (suppose $r \geq 2$), we have*

$$d_r[\alpha] = [\partial \alpha_{p+r-1,q-r+1}] \in E_r^{p+r,q-r+1},$$

*where $\alpha = \alpha_{p,q} + \alpha_{p+1,q-1} + \cdots + \alpha_{p+r-1,q-r+1} + \cdots + \alpha_{p+q,0}$ is any representative for $[\alpha]$.*

*Proof.* Consider

$$d(\alpha_{p,q} + \cdots + \alpha_{p+r-1,q-r+1} + \alpha_{p+r,q-r} + \cdots + \alpha_{p+q,0})$$
$$= \partial \alpha_{p+r-1,q-r+1} + d(\alpha_{p+r,q-r} + \cdots + \alpha_{p+q,0}).$$

Now, $d_r[\alpha]$ is represented by $(d\alpha)_{p,q} = \partial \alpha_{p+r-1,q-r+1}$ (since $\bar{\partial}\alpha_{p,q-1}$ can be absorbed into the denominator as $\bar{\partial}\alpha_{p,q-1} + \partial 0$), which satisfies the new description of the numerator of $E_r^{p+r-1,q-r+1}$ with all other forms $\alpha_{i,j} = 0$ (since $\partial\partial\alpha_{p+r-1,q-r+1} = 0$ and so we can take all the other forms to be trivial). □

## 4. Degeneration of the spectral sequence

For a complex $n$–dimensional manifold $X$, we note that the space $A^{p,q}$ is trivial if $p$ or $q$ is at least $n+1$. So, we have $E_{n+1}^{p,q} = E_\infty^{p,q}$, i.e. the terms on the $n+1$-st page of the Frölicher spectral sequence are the associated graded pieces of the complexified de Rham cohomology. We can ask whether we have $E_r^{p,q} \cong E_\infty^{p,q}$ for all $p, q$ for some $r < n+1$. Note that since $E_r^{p,q}$ decreases in dimension as $r$ increases, $E_r^{p,q} \cong E_\infty^{p,q}$ implies $E_r^{p,q} \cong E_{r+1} \cong \cdots \cong E_\infty^{p,q}$. Since $E_{r+1}^{p,q}$ is the cohomology of $E_r^{p,q}$ with respect to $d_r$, this implies that $d_r, d_{r+1}, \ldots$ are all identically 0. Conversely, $0 = d_r = d_{r+1} = \cdots$ at all $(p, q)$

implies that $E_r \cong E_\infty$. We say the spectral sequence *has degenerated* at page $r$ if $0 = d_r = d_{r+1} = \cdots$. For a complex manifold $X$ let us denote by $r(X)$ the minimum such number, and we say the spectral sequence *degenerates on* $E_r^{p,q}$ if $r$ is the smallest number such that $E_r^{p,q} \cong E_\infty^{p,q}$. As we saw, $r(X)$ exists and satisfies $r(X) \leq \dim_{\mathbb{C}}(X)+1$. Since $E_0^{p,q}$ is an infinite-dimensional vector space, and $E_1^{p,q}$ is finite-dimensional, we also have $r(X) \geq 1$.

**Theorem 4.1.** *If the complex manifold $X$ is Kähler (i.e. there exists a Kähler metric for the complex structure), then $r(X) = 1$.*

*Proof.* First let us see why $d_1 = 0$. The only property of Kähler manifolds that we will use is the $\partial\bar{\partial}$-lemma, i.e. the property that if a form $\alpha_{p,q}$ is $d$-closed (which is equivalent to being $\partial$- and $\bar{\partial}$-closed for homogeneous forms) and $\partial$- or $\bar{\partial}$-exact, then $\alpha_{p,q}$ is $\partial\bar{\partial}$-exact, i.e. there is a form $\beta_{p-1,q-1}$ such that $\partial\bar{\partial}\beta_{p-1,q-1} = \alpha_{p,q}$. (Note that then we also have $\bar{\partial}\partial(-\beta) = \alpha$.) Now, take a representative $\alpha_{p,q}$ of a class in $E_1^{p,q}$; note that this means $\bar{\partial}\alpha_{p,q} = 0$. Then, as we saw, $\partial\alpha_{p,q}$ represents $d_1[\alpha_{p,q}]$ in $E_1^{p+1,q}$. However, $\partial\alpha_{p,q}$ satisfies $\partial\partial\alpha_{p,q} = 0$ and $\bar{\partial}\partial\alpha_{p,q} = -\partial\bar{\partial}\alpha_{p,q} = -\partial 0 = 0$, and so there is a $\beta_{p,q-1}$ such that $\bar{\partial}\partial\beta = \partial\alpha_{p,q}$. In particular, $\partial\alpha_{p,q}$ is $\bar{\partial}$-exact, and so represents the 0 class in $E_1^{p+1,q}$. We conclude $d_1 = 0$.

Now let us see that $d_r = 0$ for $r \geq 2$ (the argument will be virtually the same). We will use Theorems 3.1 and 3.2 for the description of elements in $E_r^{p,q}$ and their differentials. Let $\alpha_{p,q}$ represent a class in $E_r^{p,q}$; this means there exist $\alpha_{p+1,q-1}, \ldots, \alpha_{p+q,0}$ such that $\partial\alpha_{p,q} + \bar{\partial}\alpha_{p+1,q-1} = 0, \ldots, \partial\alpha_{p+r-2} + \bar{\partial}\alpha_{p+r-1,q-r+1} = 0$, and $d_r[\alpha_{p,q}]$ is represented by $\partial\alpha_{p+r-1,q-r+1}$. Note that by the same argument we used above, $\partial\alpha_{p,q} = \partial\bar{\partial}\beta_{p,q-1}$ for some $\beta_{p,q-1}$. Then we can replace $\alpha_{p+1,q-1}$ by $\partial\beta_{p,q-1}$, since $\partial\alpha_{p,q} + \bar{\partial}\partial\beta_{p,q-1} = \partial\alpha_{p,q} - \partial\alpha_{p,q} = 0$, and we can replace $\alpha_{p+2,q-2}, \ldots, \alpha_{p+q,0}$ by 0's. Now $d_r[\alpha_{p,q}]$ is represented by $\partial 0 = 0$ and so $d_r[\alpha_{p,q}] = 0$.

□

**Remark 4.2.** The above theorem tells us that for any compact complex manifold $X$ satisfying the $\partial\bar{\partial}$-lemma property, we have

$$\bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X) \cong H^k_{dR}(X) \otimes \mathbb{C}$$

for all $k$. On a Kähler manifold, it is well-known that this splitting has further structure, where it is known as the Hodge decomposition.

**Remark 4.3.** Note that any complex structure on a complex curve ($n = 1$) is Kähler (in fact any metric compatible with the complex structure will be Kähler), and so the spectral sequence degenerates on the first page. Also, even though not every complex surface ($n = 2$) is Kähler (for example the Hopf surface, diffeomorphic to $S^1 \times S^3$), it is true that $r(X) = 1$ for all complex surfaces $X$ (see [BHPV, IV.2.8]).

Let us now see that there are complex manifolds with $r(X) \geq 2$. Both examples will belong to the class of manifolds called *nilmanifolds*, which we will see are easy to work with.

**Example 4.4.** (Iwasawa manifold) Consider the complex Lie group $G$ consisting of upper-triangular matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix},$$

where $x, y, z$ are complex coordinates. Consider the subgroup $\Gamma$ of (Gaussian) integer points in this group, i.e. matrices as above with integer entries $x, y, z$. Note that left multiplication by elements in $\Gamma$ gives biholomorphic maps $G \to G$, and so the quotient $X = \Gamma\backslash G$ will be a complex compact threefold ($n = 3$), known as the *Iwasawa manifold*.

Let us see what the left invariant holomorphic one-forms on $G$ are. For $A \in G$, consider the expression $A^{-1}dA$; the left-invariant holomorphic forms will be the entries of the resulting matrix. We have

$$\begin{aligned} A^{-1}dA &= \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & dx & dz \\ 0 & 1 & dy \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & dx & dz \\ 0 & 1 & dy \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & dx & dz - xdy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and so a basis of the space of left-invariant holomorphic one-forms on $G$ is given by $\{dx, dy, dz - xdy\}$.

Considering $G$ as a real Lie group equipped with a left-invariant complex structure (namely, the one sending the vector field corresponding to the real part of one of these complex coordinates to the vector field corresponding to the imaginary part of that same complex coordinate), we have that the real dual of its real Lie algebra $\mathfrak{g}$ satisfies

$$\begin{aligned} \mathfrak{g}^* \otimes \mathbb{C} &\cong \text{span}(dx, dy, dz - xdy) \oplus \overline{\text{span}(dx, dy, dz - xdy)} \\ &= \text{span}(dx, dy, dz - xdy) \oplus \text{span}(d\bar{x}, d\bar{y}, d\bar{z} - \bar{x}d\bar{y}) \end{aligned}$$

on the level of vector spaces. In fact, this splitting is also true on the level of (dual) Lie algebras, due to the integrability of the complex structure. We can now form the exterior product $\Lambda\mathfrak{g}^*\otimes\mathbb{C}$ of $\mathfrak{g}^*\otimes\mathbb{C}$ (in analogy with forming the exterior product of $TM \otimes \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$), and we have

$$\Lambda^k\mathfrak{g}^* \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \Lambda^p\text{span}(dx, dy, dz - xdy) \wedge \Lambda^q\text{span}(d\bar{x}, d\bar{y}, d\bar{z} - \bar{x}d\bar{y})$$

as vector spaces. Let us denote $\alpha = dx$, $\beta = dy$, $\gamma = dz - xdy$, and note that we have $d\alpha = d\beta = 0$ and $d\gamma = -d\alpha d\beta$ (and similarly for the conjugates). The algebra $\Lambda\mathfrak{g}^*\otimes\mathbb{C}$ naturally has the structure of a differential graded algebra where $d$ is determined on the generators in degree one by the above relations, and is extended over all higher-degree forms according to the (graded) Leibniz rule.

Note that left-invariant smooth forms on $G$ descend to forms on $X = \Gamma\backslash G$. This map is multiplicative, commutes with the exterior derivative $d$, and sends elements of bidegree $(p, q)$ to $(p, q)$ forms. So, $p$ is a map of cochain complexes that respects the filtration used in setting up the Frölicher spectral sequence. It follows that $p$ induces a map on every page of the ensuing spectral sequences. By a classical theorem of Nomizu, the following is known about this situation focusing only on the underlying smooth manifolds:

**Theorem 4.5.** *(Nomizu) Suppose $G$ is a simply-connected nilpotent (real) Lie group, and suppose $\Gamma$ is a subgroup such that $\Gamma\backslash G$ is a closed manifold. Then the map of differential graded algebras $(\Lambda\mathfrak{g}^*, d) \xrightarrow{p} A^*_{dR}(\Gamma\backslash G, d)$ sending a left-invariant smooth form on $G$ to its descended form on $\Gamma\backslash G$ induces an isomorphism on cohomology. (Here $A_{dR}$ denotes the space of smooth forms.)*

We note that our group $G$ of upper-triangular matrices is indeed nilpotent (all threefold commutators will be the identity matrix) and simply-connected (since $G$ is diffeomorphic to $\mathbb{C}^3$ via the coordinates $x, y, z$). In general, manifolds obtained as quotients of simply connected nilpotent Lie groups by a discrete subgroup are known as *nilmanifolds*.

In our scenario of a Lie group with left-invariant complex structure (in fact, we have more, namely a complex Lie group), Nomizu's theorem guarantees that the map of filtered cochain complexes $(\Lambda^*\mathfrak{g}^* \otimes \mathbb{C}, d, F^p) \xrightarrow{p} (A^k_{dR}(\Gamma\backslash G) \otimes \mathbb{C}, d, F^p)$ induces an isomorphism on the $E_\infty$ page (i.e. on the associated graded of the cohomology). In order to do calculations with the Frölicher spectral sequence on $\Gamma\backslash G$, we would like to know that $p$ induces an isomorphism on the $E_1$ page. From here, it would follow that $p$ induces an isomorphism on all subsequent pages as well (since one page is the cohomology of the previous one, and the map is compatible with the filtrations). Note that $p$ cannot induce an isomorphism on the $E_0$ page simply for dimension reasons.

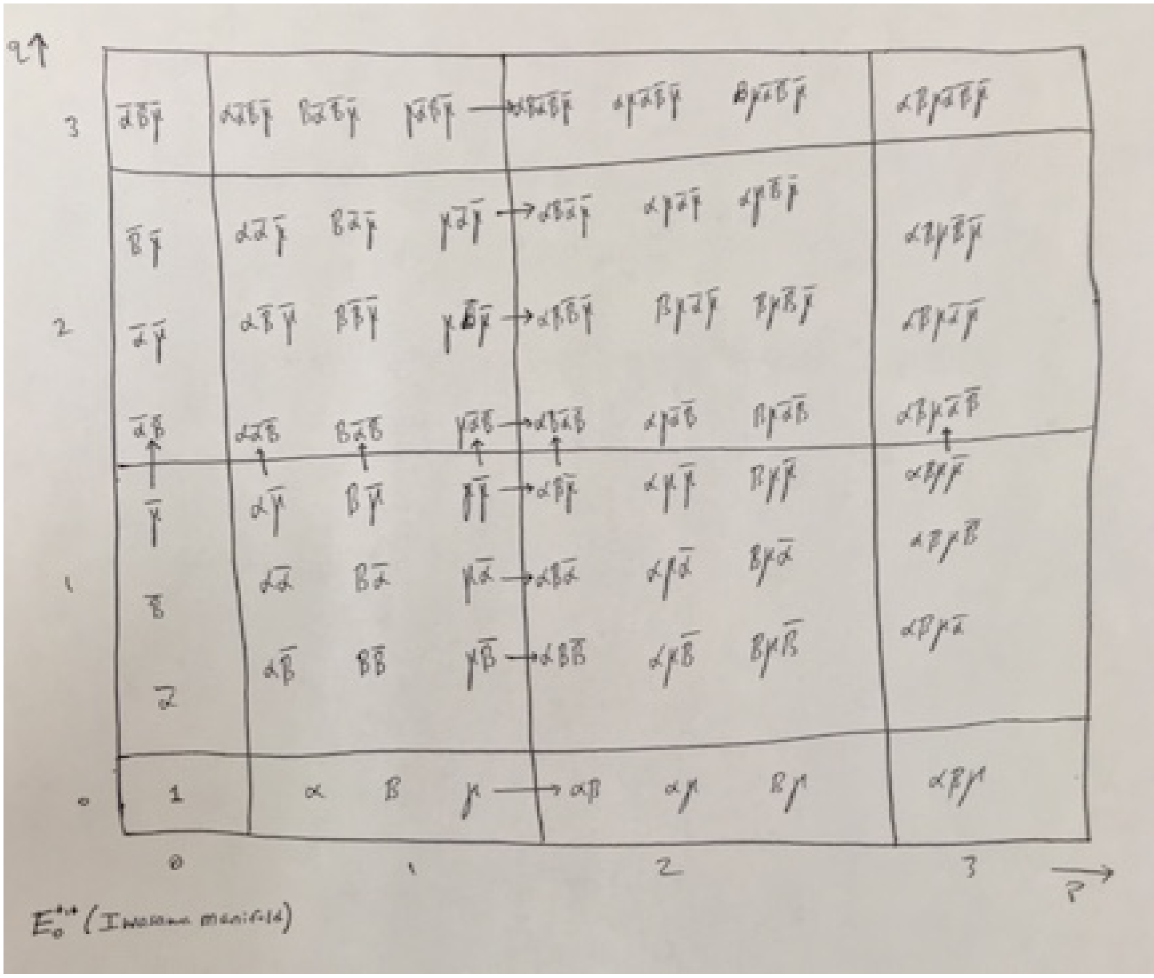
It is not known in general that such a $p$, mapping from a Lie group with left-invariant complex structure to its quotient, induces an isomorphism on $E_1$ (such maps are known as $E_1$-*quasi-isomorphisms*), but in this example and the next, it is known to satisfy this property. An account on the state of this problem can be found in [Rollenske].

So, let us take this result that $p$ in our situation $G \to \Gamma\backslash G$ is an $E_1$-quasi-isomorphism (i.e. it induces an isomorphism on $H_{\bar\partial}$). This means that we can calculate the terms in the Frölicher spectral sequence on $\Gamma\backslash G$ simply by working with the left-invariant complex forms $\alpha, \beta, \gamma, \bar\alpha, \bar\beta, \bar\gamma$ which span the degree 1 part of $\Lambda\mathfrak{g}^* \otimes \mathbb{C}$.

The following picture shows the bigraded algebra $\Lambda\mathfrak{g}^* \otimes \mathbb{C}$ with all differentials $d = \partial + \bar\partial$. For example, for the $(1,1)$ form $\gamma\bar\gamma$, we have

$$d(\gamma\bar\gamma) = (d\gamma)\bar\gamma - \gamma(d\bar\gamma) = \alpha\beta\bar\gamma - \gamma\bar\alpha\bar\beta,$$

and so $\bar{\partial}(\gamma\bar{\gamma}) = -\gamma\bar{\alpha}\bar{\beta}$ and $\partial(\gamma\bar{\gamma}) = \alpha\beta\bar{\gamma}$. In the picture, below, the absence of an arrow indicates that the corresponding differential vanishes. An arrow indicates that the destination hits the target (up to sign).



The $E_1$ page is the cohomology with respect to $\bar{\partial}$, i.e. the vertical arrow, of this picture. Consider the $(1,0)$ form $\gamma$. Since $\bar{\partial}\gamma = 0$, it represents a form in $E_1^{1,0}$. Now, $d_1[\gamma]$ is represented by $[\alpha\beta] \in E_1^{1,0}$, which is a non-zero class, since $\alpha\beta$ is not $\bar{\partial}$-exact. Therefore $d_1[\gamma] \neq 0$. Therefore $r(\Gamma\backslash G) \geq 2$. Since there are no non-trivial "zig-zags" of differentials, by Theorem 3.2 we conclude that $d_k$ for $k \geq 2$ are all trivial, and so $r(\Gamma\backslash G) = 2$.

**Example 4.6.** Now we consider an example of a real 6–dimensional nil-manifold with complex structure $X$ such that $r(X) = 3$. Take $G$ to be the simply connected nilpotent Lie group consisting of matrices of the form

$$\begin{pmatrix} 1 & -\bar{x} & -\frac{1}{2}\bar{x}^2 & z \\ 0 & 1 & \bar{x} & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\Gamma$ to be the subgroup of such matrices with Gaussian integer entries. Note that $G$ is indeed a complex manifold, despite the peculiar coordinates; namely it is covered by a single chart homeomorphic to $\mathbb{C}^3$ via the coordinates $(x, y, z)$. However, $G$ is *not* a complex Lie group, as the multiplication is not holomorphic. Indeed, in terms of coordinates, the map $\mathbb{C}^3 \to \mathbb{C}^3$ given by left multiplication by $(x, y, z)$ is given by

$$\begin{aligned} (x,y,z)\cdot(x',y',z') &= \begin{pmatrix} 1 & -\bar{x} & -\frac{1}{2}\bar{x}^2 & z \\ 0 & 1 & \bar{x} & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\bar{x}' & -\frac{1}{2}\bar{x}'^2 & z' \\ 0 & 1 & \bar{x}' & y' \\ 0 & 0 & 1 & x' \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\bar{x}' - \bar{x} & -\frac{1}{2}\bar{x}'^2 - \bar{x}\bar{x}' - \frac{1}{2}\bar{x}^2 & z' - \bar{x}y' - \frac{1}{2}\bar{x}^2 x' + z \\ 0 & 1 & \bar{x}' + \bar{x} & y' + \bar{x}x' + y \\ 0 & 0 & 1 & x' + x \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= (x' + x, y' + \bar{x}x' + y, z' - \bar{x}y' - \tfrac{1}{2}\bar{x}^2 x' + z), \end{aligned}$$

which is not complex-linear.

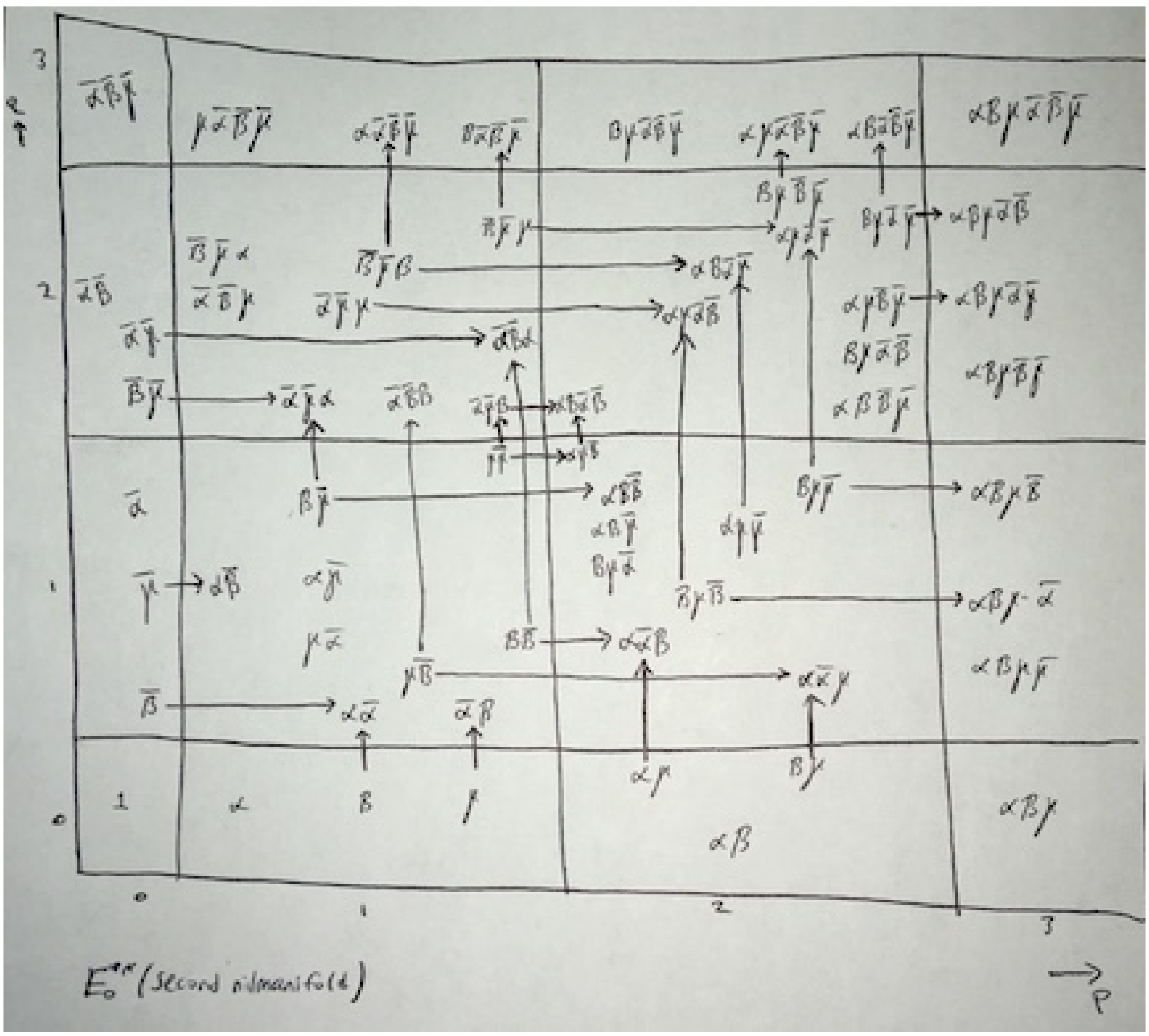
As in the previous example, we calculate $A^{-1}dA$ for a general element $A \in G$;

$$\begin{aligned} A^{-1}dA &= \begin{pmatrix} 1 & \bar{x} & -\frac{1}{2}\bar{x}^2 & \frac{1}{2}x\bar{x}^2 - \bar{x}y - z \\ 0 & 1 & -\bar{x} & x\bar{x} - y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -d\bar{x} & -\bar{x}d\bar{x} & dz \\ 0 & 0 & d\bar{x} & dy \\ 0 & 0 & 0 & dx \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -d\bar{x} & 0 & dz + \bar{x}dy - \frac{1}{2}\bar{x}^2 dx \\ 0 & 0 & d\bar{x} & dy - \bar{x}dx \\ 0 & 0 & 0 & dx \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The multiplication is not holomorphic, so we can only conclude that $dx$, $dy - \bar{x}dx$, $dz + \bar{x}dy - \frac{1}{2}\bar{x}^2 dx$ span the space of left-invariant smooth complex-valued $(1,0)$-forms, and their conjugates span the space of $(0,1)$ forms. (Even though not all the conjugates appear in the above expression $A^{-1}dA$, we can conclude that they are left invariant by writing each coordinate as a sum of a real and an imaginary coordinate.) Denote $\alpha = dx$,

$\beta = dy - \bar{x}dx$, $\gamma = dz + \bar{x}dy$. We have $d\alpha = 0$, $d\beta = -d\bar{x}dx = -\bar{\alpha}\alpha$, $d\gamma = d\bar{x}dy - \bar{x}d\bar{x}dx = d\bar{x}(dy - \bar{x}dx) = \bar{\alpha}\beta$, with the differential on the conjugate forms $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ determined by $d\bar{\eta} = \overline{d\eta}$. The map sending a left-invariant $(p, q)$ form on $G$ to its descended form on $\Gamma\backslash G$ turns out to be an $E_1$-quasi-isomorphism as in the previous example. We can therefore calculate the spectral sequence by studying only the complex $(\Lambda^*\mathrm{span}(\alpha, \beta, \gamma, \bar{\alpha}, \bar{\beta}, \bar{\gamma}), d)$ as before.

We have the following picture of the bicomplex as before:

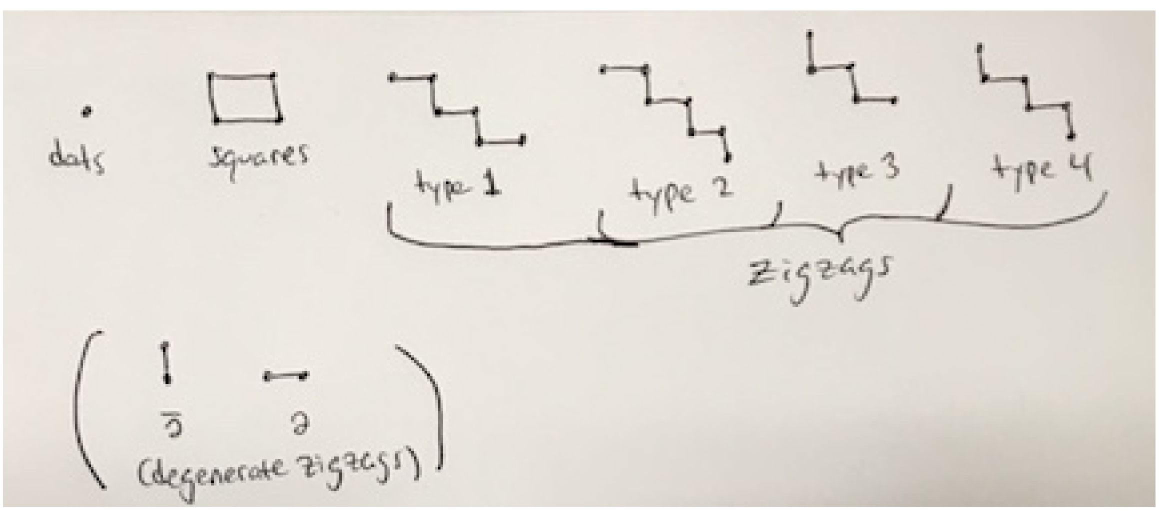


Note the zig-zag involving the terms $\bar{\beta}\bar{\gamma}$, $\bar{\alpha}\bar{\gamma}\alpha$, $\beta\bar{\gamma}$, $\alpha\beta\bar{\beta}$, and that $\bar{\partial}(\bar{\beta}\bar{\gamma}) = 0$. By our description of the terms and differentials in the spectral sequence from the previous section, we see that $\bar{\beta}\bar{\gamma}$ induces a non-zero class in $E_2^{0,2}$, and that $d_2[\bar\beta\bar\gamma] = [\alpha\beta\bar\beta] \neq 0$. Therefore $r(\Gamma\backslash G) \geq 3$. All the differentials on the subsequent pages $E_3$ and $E_4$ are trivial; indeed, any zig-zag corresponding to a differential on those pages terminates in a trivial class. For example, there is the long zig-zag involving the terms $\bar\alpha\bar\gamma$, $\bar\alpha\bar\beta\alpha$, $\beta\bar\beta$, $\alpha\bar\alpha\beta$, $\alpha\gamma$, though $\partial(\alpha\gamma)$ is the zero class and so $d_3[\bar\alpha\bar\gamma] = 0$. So, $r(\Gamma\backslash G)$. We will see in the next section that this is the maximum possible value of $r(X)$ for a compact complex threefold (namely, $r(X) \leq n$ for a compact complex manifold $n$ of dimension $n$).

## 5. A static viewpoint of the spectral sequence

Instead of considering the spectral sequence as a "moving" iteration of pages, to address questions of degeneration it is useful to also have a "static" picture in mind. For any bounded double complex $(A^{p,q}, \partial, \partial')$, i.e. a bi-graded vector space $\bigoplus_{p,q} A^{p,q}$ (such that $A^{p,q} = \{0\}$ for all but finitely many $(p,q)$) with maps $\partial$ and $\partial'$ of bidegree $(1,0)$ and $(0,1)$ respectively, satisfying $\partial^2 = 0$, $\partial'^2 = 0$, $\partial\partial' + \partial'\partial = 0$, we have the following structure theorem (see [Stelzig]):

**Theorem 5.1.** *A bounded double complex $(A^{p,q}, \partial, \partial')$ can be uniquely written/displayed as a direct sum of "shapes" of the following forms:*




*Here, an arrow denotes an isomorphism between one-dimensional spaces. The absence of an arrow denotes that the corresponding differential is trivial.*

*Modulo squares, only finitely many such shapes appear in this decomposition. Note that there is no assumption of finite-dimensionality on the vector spaces $A^{p,q}$.*

**Remark 5.2.** Zigzags of type 1 correspond to non-trivial differentials in the spectral sequence. Zigzags of type 2 correspond to cohomology classes (i.e. terms in the $E_\infty$ page) represented by the top-left corner in the associated graded of cohomology. Zigzags of type 3 and 4 are not detected by the spectral sequence.

We now consider the implications of this theorem for the bounded double complex of forms $(A^{p,q}, \partial, \bar{\partial})$ on a compact complex manifold $X$.

First of all, we note that the pairing $A^{p,q} \otimes A^{n-p,n-q} \to \mathbb{C}$ given by

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

is non-degenerate. Indeed, for any $\alpha$, its (conjugate) Hodge star $*\alpha$ will pair with $\alpha$ non-trivially. We can define a dual complex $DA$ to $(A^{*,*}, \partial, \bar{\partial})$ by setting $(DA)^{p,q} = (A^{n-p,n-q})^*$, and setting the $(1,0)$ and $(0,1)$ differentials to be the graded duals of $\partial$ and $\bar{\partial}$ respectively (namely, for $\phi \in (DA)^{p,q}$, set $\partial^* \phi = (-1)^{p+q+1} \phi \circ \partial$). It follows that the map $A^{p,q} \to (DA)^{p,q}$ which sends a form $\alpha$ to the functional $\int_X \alpha \wedge -$ is a map of complexes $(A^{*,*}, \partial, \bar{\partial}) \to ((DA)^{*,*}, \partial^*, \bar{\partial}^*)$ and so induces a map of the associated Frölicher spectral sequences. Serre duality then tells us that this map induces an isomorphism on Dolbeault cohomology, i.e. it is an $E_1$-quasi-isomorphism. By [Stelzig, Lemma 1.24] it follows that every zigzag that appears in the spectral sequence for $A^{*,*}$ also appears in the same place in the spectral sequence for $(DA)^{*,*}$. Since every zigzag in $A^{*,*}$ certainly appears in $(DA)^{*,*}$ after a central reflection through the middle of the bicomplex $(\lfloor \frac{n}{2} \rfloor), (\lfloor \frac{n}{2} \rfloor)$, we have the following conclusion.

**Proposition 5.3.** *The direct sum decomposition of the double complex of forms on $X$, modulo squares, has a central symmetry.*

Similarly we can associate to $A^{*,*}$ a conjugate double complex $\mathrm{Conj}(A)^{p,q} = A^{q,p}$ with differentials $\bar{\partial}$ (of degree $(1,0)$ here) and $\partial$ (of degree $(0,1)$). Then conjugation gives us an $E_0$-quasi-isomorphism of double complexes $(A^{*,*}, \partial, \bar{\partial}) \to (\mathrm{Conj}(A)^{*,*}, \bar{\partial}, \partial)$, i.e. an isomorphism of vector spaces that evidently preserves the grading and filtration. Therefore, we also have the following:

**Proposition 5.4.** *The direct sum decomposition of the double complex has a conjugation symmetry.*

As an easy corollary, we will now see that for a compact complex $n$-fold, $r(X) \leq n$ (improving on the upper bound $r(X) \leq n+1$).

**Proposition 5.5.** *For a compact complex manifold $X$ of complex dimension $n$, we have $r(X) \leq n$.*

*Proof.* We note that there are at most two possibly non-trivial differentials on $E_n^{*,*}$. Namely, the differential $E_n^{0,n} \to E_n^{n,1}$ and the differential $E_n^{0,n-1} \to E_n^{n,0}$. By the conjugation symmetry (or, in this case equivalently by the central symmetry) we see that all differentials of one type are trivial if and only if all differentials of the other type are trivial. So let us show that every differential $E_n^{0,n-1} \to E_n^{n,0}$ is trivial. Indeed, suppose there are form $\alpha_{0,n-1} + \alpha_{1,n-2} + \ldots + \alpha_{n-1,0}$ such that $d(\alpha_{0,n-1} + \alpha_{1,n-2} + \ldots + \alpha_{n-1,0}) = \alpha \in A^{n,0}$. In particular, $\alpha$ is $d$-exact; let us write $\alpha = d\beta$ for brevity. If we locally write $\alpha = f dz_I$, for some smooth complex-valued function $f$, we have $\alpha\bar{\alpha} = |f|^2 dz_I d\bar{z}_I$. Integrating over the manifold, noting that $d\bar{\alpha} = \overline{d\alpha} = 0$, and applying Stokes' theorem we have

$$0 \leq \int \alpha\bar{\alpha} = \int d(\beta\bar{\alpha}) = 0,$$

from where we conclude $f \equiv 0$ and so $\alpha = 0$. □

**Remark 5.6.** It is known that for any $n$, there is some compact complex manifold whose Frölicher spectral sequence degenerates after page $n$ (see [BR]); the examples provided in [BR] are of dimension $4n-2$. Currently there are no known compact complex $n$-folds, $n \geq 4$, for which $r(X) = n$.

If one knows the decomposition of the double complex from the above theorem, it is immediate to read off at which page the corresponding spectral sequence degenerates; namely, one looks for a zig-zag of type 1 of longest possible length. Given a compact complex manifold, the problem is determining what the decomposition is; even though one can draw zig-zags corresponding to some basis of complex forms, this basis is not necessarily the *correct* one for this decomposition.

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