Rational homotopy type of low-dimensional homogeneous spaces $SO(2n)/U(n)$

When considering the problem of finding an almost-complex structure on a real $2n$–manifold M, we can consider the obstructions to lifting the classifying map $M \to BSO(2n)$ of the tangent bundle to a map to $BU(n)$ (through the map $BU(n) \to BSO(2n)$ induced by inclusion $U(n) \hookrightarrow SO(2n)$. The homotopy fiber of the map $BU(n) \to BSO(2n)$ is the homogeneous space $SO(2n)/U(n)$, so our obstructions lie in $H^*(M, \pi_{*-1}(SO(2n)/U(n))$. For ease of reference when considering almost-complex structures in dimensions up to twelve, we compute the rational minimal models of these spaces. Of course, these do not tell us the full information about the homotopy fibers, but they do lend some guidance.

Our procedure for finding the minimal model of $SO(2n)/U(n)$ will be to consider the map

$$\mathrm{Model}(BSO(2n)) \to \mathrm{Model}(BU(n))$$

on minimal models induced by the map $BU(n) \to BSO(2n)$. These classifying spaces are formal, and their cohomology rings are free, so their minimal models are isomorphic to their cohomology (as algebras over our ground field). The map we are considering between the classifying spaces is formal as well, so the induced map between models is given by the map on cohomology. As $SO(2n)/U(n)$ is the fiber of the map $BU(n) \to BSO(2n)$, a model of $SO(2n)/U(n)$ will be obtained as the cofiber of the map $\mathrm{Model}(BSO(2n)) \to \mathrm{Model}(BU(n))$. Explicitly, we build a differential graded algebra E containing $\mathrm{Model}(BSO(2n))$ along with a quasi-isomorphism to $\mathrm{Model}(BU(n))$ such that the diagram

$$\begin{array}{ccc} & E & \\ \nearrow\!\!\!\!\hookrightarrow & & \searrow \sim \\ \mathrm{Model}(BSO(2n)) & \longrightarrow & \mathrm{Model}(BU(n)) \end{array}$$

commutes. The model of $SO(2n)/U(n)$ will be E modulo the ideal generated by positive degree elements in $\mathrm{Model}(BSO(2n))$, with the induced differential.

Recall that $H^*(BU(n), \mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c_n]$, the polynomial algebra in the universal Chern classes, while $H^*(BSO(2n), \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_{n-1}, e]$, where e is the universal degree $2n$ Euler class, whose square is the top Pontryagin class p_n.

- $SO(4)/U(2)$. The map on cohomology $H^*(BSO(4)) \to H^*(BU(2))$, i.e. $\mathbb{Q}[p_1, e] \to \mathbb{Q}[c_1, c_2]$ is given by the universal relations $p_1 \mapsto c_1^2 - 2c_2$ and $e \mapsto c_2$. To build the larger differential graded algebra E, we put in the variables p_1 and e, as necessary, and introduce a variable $\bar{c_1}$ which we will map to c_1. So, for now we have $E = \Lambda(p_1, e, \bar{c_1})$ with trivial differential. Our map from E to $H^*(BU(2))$ is given by $p_1 \mapsto c_1^2 - 2c_2$, $e \mapsto c_2$, and $\bar{c_1} \mapsto c_1$. Note that this is not a quasi-isomorphism, since $\bar{c_1}^2 - p_1 + 2e$ represents a non-trivial class in cohomology in E, while it gets mapped to the zero class in $H^*(BU(2))$. For this reason, we introduce into E a new variable η_3 (of degree 3) and set $d\eta = p_1 - \bar{c_1}^2 - 2e$. Further, we map η to zero in $\mathrm{Model}(BU(2))$. This map now induces an isomorphism on cohomology, and we are ready to take the cofiber.

The model of $SO(4)/U(2)$ is given by

$$\frac{\Lambda(p_1, e, \bar{c_1}, \eta,\ dp_1 = de = d\bar{c_1} = 0, d\eta = \bar{c_1}^2 - p_1 + 2e)}{\text{ideal}(p_1, e)} = \Lambda(\bar{c_1}, \eta_3,\ d\bar{c_1} = 0, d\eta = \bar{c_1}^2),$$

which is the minimal model of S^2. Therefore $SO(4)/U(2) \cong_{\mathbb{Q}} S^2$. (In fact, there is a diffeomorphism to the two-sphere.)

- $SO(6)/U(3)$ The map on cohomology $H^*(BSO(6)) \to H^*(BU(3))$, i.e.

$$\mathbb{Q}[p_1, p_2, e] \to \mathbb{Q}[c_1, c_2, c_3]$$

is given by $p_1 \mapsto c_1^2 - 2c_2$, $p_2 \mapsto c_2^2 - 2c_1c_3$, $e \mapsto c_3$. Build E by putting in p_1, p_2, e, along with a $\bar{c_1}$ to map to c_1. Note that now $\bar{c_1}^2 - p_1$ in E is mapped to c_2 in $H^*(BU(3))$. (The variables coming from $H^*(BSO(6))$ have to get mapped to what their images under the map $H^*(BSO(6)) \to H^*(BU(3))$ for the diagram to commute.) Now observe that we have a non-trivial cohomology class $(\bar{c_1}^2 - p_1)^2 - p_2 - 2\bar{c_1}e$ mapping to zero in $H^*(BU(3))$. We thus introduce a variable η_7 into E such that $d\eta = (\bar{c_1}^2 - p_1)^2 - p_2 - 2\bar{c_1}e$, and map η to zero in $H^*(BU(3))$. This map is now induces an isomorphism on cohomology, and we have that a model of $SO(6)/U(3)$ is

$$\frac{\Lambda(p_1, p_2, e, \bar{c_1}, \eta_7,\ d\eta = (\bar{c_1}^2 - p_1)^2 - p_2 - 2\bar{c_1}e)}{\text{ideal}(p_1, p_2, e)} = \Lambda(\bar{c_1}, \eta_7,\ d\eta = \bar{c_1}^4),$$

which is the minimal model of $\mathbb{CP}^3$. So, $SO(6)/U(3) \cong_{\mathbb{Q}} \mathbb{CP}^3$. These two manifolds are in fact diffeomorphic.

- $SO(8)/U(4)$. The map on cohomology $H^*(BSO(8)) \to H^*(BU(4))$, i.e.

$$\mathbb{Q}[p_1, p_2, p_3, e] \to \mathbb{Q}[c_1, c_2, c_3, c_4]$$

is given by

$$\begin{aligned} p_1 &\mapsto c_1^2 - 2c_2, \\ p_2 &\mapsto c_2^2 + 2c_4 - 2c_1c_3, \\ p_3 &\mapsto c_3^2 - 2c_2c_4, \\ e &\mapsto c_4. \end{aligned}$$

Build E by placing p_1, p_2, p_3, e in it and introducing a $\bar{c_1}$ and a $\bar{c_3}$ to map c_1 and c_3 respectively. Note that $\frac{1}{2}\bar{c_1}^2 - p_1$ maps to zero, and so we have non-trivial cohomology classes in E given by $\frac{1}{4}(\bar{c_1} - p_1)^2 + 2e - 2\bar{c_1}\bar{c_3} - p_2$ and $\bar{c_3}^2 - \frac{1}{4}(\bar{c_1}^2 - p_1)e$ that should not be there. So, we introduce variables η_7 and η_{11} to kill these classes, and map them to zero. Now our map is a quasi-isomorphism, and we have that a model of $SO(8)/U(4)$ is given by

$$\frac{\Lambda(p_1, p_2, p_3, e, \bar{c_1}, \bar{c_3}, \eta_7, \eta_{11},\ d\eta_7 = \frac{1}{4}(\bar{c_1} - p_1)^2 + 2e - 2\bar{c_1}\bar{c_3} - p_2, d\eta_{11} = \bar{c_3}^2 - \frac{1}{4}(\bar{c_1}^2 - p_1)e)}{\text{ideal}(p_1, p_2, p_3, e)}$$
$$= \Lambda(\bar{c_1}, \bar{c_3}, \eta_7, \eta_{11},\ d\eta_7 = \tfrac{1}{4}\bar{c_1}^4 - 2\bar{c_1}\bar{c_3}, d\eta_{11} = \bar{c_3}^2).$$

- $SO(10)/U(5)$. We have the map $H^*(BSO(10)) \to H^*(BU(5))$, i.e.

$$\mathbb{Q}[p_1, p_2, p_3, p_4, e] \to \mathbb{Q}[c_1, c_2, c_3, c_4, c_5]$$

given by

$$\begin{aligned} p_1 &\mapsto c_1^2 - 2c_2, \\ p_2 &\mapsto c_2^2 + 2c_4 - 2c_1c_3, \\ p_3 &\mapsto c_3^2 - 2c_2c_4 + 2c_1c_5, \\ p_4 &\mapsto c_4^2 - 2c_3c_5, \\ e &\mapsto c_5. \end{aligned}$$

We place p_1, p_2, p_3, p_4, e in E, map them accordingly to $H^*(BU(5))$, and introduce $\bar{c_1}$ and $\bar{c_3}$ to map to c_1 and c_3 respectively. We see that $\frac{1}{2}(\bar{c_1}^2 - p_1)$ is mapped to c_2, and that $p_2 - \frac{1}{4}(\bar{c_1}^2 - p_1)^2 + 2\bar{c_1}\bar{c_3}$ is mapped to c_4, and so we introduce generators η_{11} and η_{15} to set

$$\begin{aligned} d\eta_{11} &= \bar{c_3}^2 - (\bar{c_1}^2 - p_1)(p_2 - \tfrac{1}{4}(\bar{c_1}^2 - p_1)^2 + 2\bar{c_1}\bar{c_3}) + 2\bar{c_1}e - p_3, \\ d\eta_{15} &= (p_2 - \tfrac{1}{4}(\bar{c_1}^2 - p_1)^2 + 2\bar{c_1}\bar{c_3})^2 - 2\bar{c_3}e - p_4. \end{aligned}$$

From here, as usual, we see that the minimal model of $SO(10)/U(5)$ is given by

$$\Lambda(\bar{c_1}, \bar{c_3}, \eta_{11}, \eta_{15},\ d\eta_{11} = \bar{c_3}^2 + \tfrac{1}{4}\bar{c_1}^6 - 2\bar{c_1}^3\bar{c_3},\ d\eta_{15} = (2\bar{c_1}\bar{c_3} - \tfrac{1}{4}\bar{c_1}^4)^2.$$

- $SO(12)/U(6)$. We have the map $H^*(BSO(12)) \to H^*(BU(6))$, i.e.

$$\mathbb{Q}[p_1, p_2, p_3, p_4, p_5, e] \to \mathbb{Q}[c_1, c_2, c_3, c_4, c_5, c_6]$$

given by

$$\begin{aligned} p_1 &\mapsto c_1^2 - 2c_2, \\ p_2 &\mapsto c_2^2 + 2c_4 - 2c_1c_3, \\ p_3 &\mapsto c_3^2 + 2c_1c_5 - 2c_2c_4 - 2c_6, \\ p_4 &\mapsto c_4^2 + 2c_2c_6 - 2c_3c_5, \\ p_5 &\mapsto c_5^2 - 2c_4c_6, \\ e &\mapsto c_6. \end{aligned}$$

We build E by introducing variables $\bar{c_1}$, $\bar{c_3}$, $\bar{c_5}$ (mapping to c_1, c_2, c_3) alongside $p_1, \ldots, p_5, e$. We see that $\frac{1}{2}(\bar{c_1}^2 - p_1)$ maps to c_2 and that and that $p_2 - \frac{1}{4}(\bar{c_1}^2 - p_1)^2 + 2\bar{c_1}\bar{c_3}$ is mapped to c_4, so we introduce generators η_{11}, η_{15}, η_{19} such that

$$\begin{aligned} d\eta_{11} &= \bar{c_3}^2 + 2\bar{c_1}\bar{c_5} - (\bar{c_1}^2 - p_1)(p_2 - \tfrac{1}{4}(\bar{c_1}^2 - p_1)^2 + 2\bar{c_1}\bar{c_3}) - 2e - p_3, \\ d\eta_{15} &= (\bar{c_1}^2 - p_1)e - 2\bar{c_3}\bar{c_5} + (p_2 - \tfrac{1}{4}(\bar{c_1}^2 - p_1)^2 + 2\bar{c_1}\bar{c_3})^2 - p_4, \\ d\eta_{19} &= \bar{c_5}^2 - 2(p_2 - \tfrac{1}{4}(\bar{c_1}^2 - p_1)^2 + 2\bar{c_1}\bar{c_3})e - p_5, \end{aligned}$$

whence we obtain that the minimal model of $SO(12)/U(6)$ is

$$\Lambda(\bar{c_1}, \bar{c_3}, \bar{c_5}, \eta_{11}, \eta_{15}, \eta_{19},\ d\eta_{11} = \bar{c_3}^2 + 2\bar{c_1}\bar{c_5} + \tfrac{1}{4}\bar{c_1}^6 - 2\bar{c_1}^3\bar{c_3},\ d\eta_{15} = \tfrac{1}{16}\bar{c_1}^8 + 4\bar{c_1}^2\bar{c_3}^2 - 2\bar{c_3}\bar{c_5},\ d\eta_{19} = \bar{c_5}^2).$$

• We can observe that the first $2n-2$ rational homotopy groups (starting with π_1) of these homogeneous spaces have been computed to be

$$\begin{aligned}
\pi_*(SO(4)/U(2))0, \mathbb{Q}, \\
\pi_*(SO(6)/U(3)) &= 0, \mathbb{Q}, 0, 0, \\
\pi_*(SO(8)/U(4)) &= 0, \mathbb{Q}, 0, 0, 0, \mathbb{Q}, \\
\pi_*(SO(10)/U(5)) &= 0, \mathbb{Q}, 0, 0, 0, \mathbb{Q}, 0, 0, \\
\pi_*(SO(12)/U(6)) &= 0, \mathbb{Q}, 0, 0, 0, \mathbb{Q}, 0, 0, 0, \mathbb{Q},
\end{aligned}$$

consistent with the result of Bott that the limiting space SO/U has the homotopy type of a connected component of ΩSO, which has integral homotopy groups

$$\pi_* \Omega SO = 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots.$$

For $k \leq 2n-2$, the homotopy groups $\pi_k(SO(2n)/U(n))$ are those of the limiting space $\pi_k(SO/U)$.

We can further observe that in the unstable range $2k \geq 2n-1$, we have

$$\pi_{2k}(SO(2n)/U(n)) \otimes \mathbb{Q} = 0,$$

i.e. the even homotopy groups of $SO(2n)/U(n)$ in the unstable range are all torsion.