ERRATA AND COMMENTS ON PUBLICATIONS

ALEKSANDAR MILIVOJEVIĆ

The weak form of Hirzebruch's prize question via rational surgery, [13]**.**

- Inconsequential errata. Two silly typos that appeared during the process of extracting and reorganizing the argument as given in my thesis Section 3.5 (where these typos do not appear): First, on p.5, in the second iteration of writing the degree 24 part of the $\hat{A}$, the minus sign in front should be in the numerator alongside $769728p_2^3$, as it is a few lines above. The correct expression (the one given a few lines above) is the one used later on. Second, in the middle of p.6: "Hence, $\hat{A}(M, TM \otimes \mathbb{C}) = 1$" should of course say "Hence, $\hat{A}(M, TM \otimes \mathbb{C}) = 0$" (and $= 0$ is used in the next line and throughout).

On the minimal sum of Betti numbers on an almost complex manifold, [1]**.**

- Inconsequential erratum. The following statement at the beginning of Section 4 is incorrect: "By Adams' solution of the Hopf invariant one problem, any $2n$-dimensional manifold admitting a minimal cellular decomposition with three cells (that is, one 0-cell, one n-cell, and one $2n$-cell) has the homotopy type of $\mathbb{RP}^2, \mathbb{CP}^2, \mathbb{HP}^{\nvDash}$, or $\mathbb{OP}^{\nvDash}$". The claim is true for $2n = 2$ and $2n = 4$. There are six such homotopy types for $2n = 8$, and sixty homotopy types for $2n = 16$ by Eells–Kuiper, all realized by closed PL manifolds [6, p.1].

  As far as I can tell, whether or not there are such closed *smooth* manifolds with distinct homotopy types is not explicitly addressed in [6]. In any case, Kramer addresses this in [8]. Namely, he builds closed topological manifolds, called "models" and denoted $M_{a,b}$ in a given dimension (with dimensions 8 and 16 being of interest), which cover all the possible above enumerated homotopy types.

  In dimension 8, $M_{1+2t,s}$ is smoothable if and only if $s = 0$ and $t \equiv 0, 7, 48, 55 \bmod 56$. In dimension 16, $M_{7(1+2u),s}$ is smoothable if and only if $s = 0$ and $u \equiv 0, 127, 16128, 16255 \bmod 16256$ [8, Theorem 7.4].

  As for homotopy types: in dimension 8, $M_{r,s}$ is homotopy equivalent to $M_{r',s'}$ if and only if $r + 12s \equiv \pm(r' + 12s') \bmod 24$, and in dimension 16, $M_{r,s}$ is homotopy equivalent to $M_{r',s'}$ if and only if $r + 60s \equiv \pm(r' + 60s') \bmod 240$ [8, Theorem 8.4].

  In dimension 8 we can thus take $M_{1,0}$ and $M_{15,0}$ to obtain smooth projective plane-like manifolds with distinct homotopy types. In dimension 16 we can take $M_{7,0}$ and $M_{1785,0}$.

  These manifolds indeed have cell decompositions with three cells: by [8, Proposition 2.3] and [8, Corollary 2.6] they are simply connected with integral cohomology ring $\mathbb{Z}[x]/(x^3)$, and hence by a result of Smale (see e.g. [15, Proposition 4.1]) they have the desired cell decomposition.

Another proof of the persistence of Serre symmetry in the Frölicher spectral sequence.

- Inconsequential erratum. In [9, p.1], "or more generally, Moišezon" should be "or more generally, in Fujiki class $\mathcal{C}$".

Spin^h and further generalisations of spin, [2]**.**

- Consequential erratum, published corrigendum: `https://www.sciencedirect.com/science/article/pii/S0393044022002595`. In short, we do not have a proof that all orientable non-compact 6– and 7–manifolds admit spin^h structures. It is true that all compact orientable manifolds of dimension ≤ 7 are spin^h, and in fact the compactness assumption can be removed in dimension ≤ 5. We realized we could prove this without invoking [4] at all, and instead using earlier results of Atiyah, Dupont, and Hirsch. In dimensions 6 and 7 we can get that non-compact orientable 6– and 7–manifolds are spin^h under the additional assumption that all 4–torsion in $H^5(-;\mathbb{Z})$ is also 2–torsion.

The statement in the paragraph preceding [2, Remark 3.5], that every orientable n–manifold is spin$^{n-\alpha(n)}$ (where $\alpha(n)$ is the number of one's in the binary expansion of n) should at least be qualified with a compactness assumption. Removing the compactness assumption, we can of course appeal to Whitney's immersion theorem to conclude that every orientable n–manifold is spin^{n-1}; likewise for [2, Section 5].

On the topology of the space of almost complex structures on the six sphere, [7].

- Inconsequential erratum (the claim in question is a footnote not relevant to any argument in the paper). The argument given for the uniquess (up to homotopy) of the lift to $BSU(3)$ from $BU(3)$ in [7, footnote on p.1270] is incorrect (cf. the inclusion of the homotopy fiber $\mathbb{Z}/2$ into the total space BSO of the fibration $BSO \to BO$ being nullhomotopic as well). Of course, the lifts up to homotopy through lifts is a torsor over $H^1(M;\mathbb{Z})$. What the argument is showing is that there is a unique lift up to homotopy (not necessarily through lifts). Indeed, BU splits as $BSU \times BS^1$; a map to BU is thus a pair of maps (f_1, f_2) which then has a unique lift (up to homotopy) to BSU, namely f_1.

Realization for almost complex manifolds.

- Consequential erratum (fixed in published version). The realizability claim in dimensions $4k$ when the signature and Euler characteristic are zero in [10, p.1678, (1)] and [12, Corollary 3.1.2] needs the additional assumption that the quadratic form is in the image of the integral Witt ring; this is fixed in the published version [11, Corollary 6.4].
- Improvement (included in published version). The restriction to dimensions not being congruent to four mod eight can be removed in [10, p.1677, Corollary] and [12, Corollary 3.1.1]; see the published version [11, Corollary 6.1].
- Comment. This is a comment on the following lemma used in [11], [12]: a homotopy commutative square of simply connected spaces

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where the vertical maps are rationalizations, is a homotopy pullback if and only if it is a homotopy pushout. This is a special case of [14, Lemma 6.1], used as [11, Proposition 5.5] (where the simply-connected assumption should have been made explicit) and [12, Lemma 2.3.4] (where it is properly stated). There is a sketch of proof of this statement in [14, Lemma 6.1], but I would like to point out that the "pullback implies pushout" direction is covered in detail in [3, Proposition 3.1].

It was a general question of Milgram on when fibrations are also cofibrations. See [3, p.1] for further discussion and references to a proof that (in particular), for a rationalization map $X \to X_{\mathbb{Q}}$ where X is simply connected, the fibration $F \to X \to X_{\mathbb{Q}}$, where F is the homotopy fiber, is also a cofibration.

To show the "pushout implies pullback" direction of [14, Lemma 6.1], we apply the above to the fibrations $F \to A \to C$ and $F' \to B \to D$. Then we extend both cofibration sequences to the right and consider the induced map on suspensions $\Sigma F \to \Sigma F'$. By assumption this map is an isomorphism on homotopy groups. Since F and F' are both connected, these spaces are simply connected and hence the map is an isomorphism on homology. This lets us conclude that the map $F \to F'$ is a homology isomorphism. Note that F and F' are connected, and are nilpotent, being the homotopy fibers of maps of nilpotent spaces [5, Corollary 7.2] (in fact, only having nilpotent domain suffices). By Dror's generalized Whitehead theorem [5, §4.3], this map is a weak homotopy equivalence and hence the diagram is a homotopy pullback.

References

[1] Albanese, M. and Milivojević, A., 2019. *On the minimal sum of Betti numbers of an almost complex manifold.* Differential Geometry and its Applications, 62, pp.101-108.

[2] Albanese, M. and Milivojević, A., 2021. *$Spin^h$ and further generalisations of spin.* Journal of Geometry and Physics, 164, p.104174.

[3] Alonso, J.M., 1983. *Fibrations that are cofibrations.* Proceedings of the American Mathematical Society, 87(4), pp.749-753.

[4] Cohen, R.L., 1985. *The immersion conjecture for differentiable manifolds.* Annals of Mathematics, 122(2), pp.237-328.

[5] Dror, E., 1971. *A generalization of the Whitehead theorem.* In *Symposium on Algebraic topology*(pp. 13-22). Springer, Berlin, Heidelberg.

[6] Eells, J. and Kuiper, N.H., 1962. *Manifolds which are like projective planes.* Publications Mathématiques de l'IHÉS, 14, pp.5-46.

[7] Ferlengez, B., Granja, G., Milivojević, A. 2021. *On the topology of the space of almost complex structures on the six sphere.* New York J. Math. 27, 1258–1273

[8] Kramer, L., 2003. *Projective planes and their look-alikes.* Journal of Differential Geometry, 64(1), pp.1-55.

[9] Milivojević, A., 2020. *Another proof of the persistence of Serre symmetry in the Frölicher spectral sequence.* Complex Manifolds, 7(1), pp.141-144.

[10] Milivojević, A., 2021. *Realization of rational spaces by almost complex manifolds*, in Angella, D., Cirici, J., Demailly, J.P. and Wilson, S., 2021. Mini-Workshop: Almost Complex Geometry. Oberwolfach Reports, 17(4), pp.1657-1691.

[11] Milivojević, A., 2022. *On the characterization of rational homotopy types and Chern classes of closed almost complex manifolds.* Complex Manifolds, 9(1), pp.138-169.

[12] Milivojević, A., 2022. *On the characterization of rational homotopy types and Chern classes of closed almost complex manifolds*, doctoral dissertation, Stony Brook University.

[13] Milivojević, A., 2022. *The weak form of Hirzebruch's prize question via rational surgery.* Journal of Topology and Analysis, pp.1-8.

[14] Taylor, L. and Williams, B., 1979. *Local surgery: foundations and applications.* In *Algebraic Topology Aarhus 1978* (pp. 673-695). Springer, Berlin, Heidelberg.

[15] Wall, C.T.C., 1965. *Finiteness conditions for CW-complexes.* Annals of Mathematics, pp.56-69.

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn
Email address: milivojevic@mpim-bonn.mpg.de