THE WEAK FORM OF HIRZEBRUCH'S PRIZE QUESTION VIA RATIONAL SURGERY

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ABSTRACT. We present a relatively elementary construction of a spin manifold with vanishing first rational Pontryagin class satisfying the conditions of Hirzebruch's prize question, using a modification of Sullivan's theorem for the realization of rational homotopy types by closed smooth manifolds. As such this is an alternative to the solutions of the problem given by Hopkins–Mahowald, though without the guarantee of the constructed manifold admitting a string structure. We present a particular solution which is rationally 7–connected with eighth Betti number equal to one; our approach yields many other solutions with complete knowledge of their rational homotopy type.

1. Introduction

We consider the following question of Hirzebruch [3, p.86]:

Question 1.1. Does there exist a 24–dimensional closed, oriented, smooth manifold M with $p_1(M) = 0$, $w_2(M) = 0$, $\hat{A}(M) = 1$ and $\hat{A}(M, TM \otimes \mathbb{C}) = 0$?

Here $\hat{A}(M)$ denotes the evaluation $\hat{A}(TM)[M]$ of the $\hat{A}$–genus of the tangent bundle on the fundamental class, and $\hat{A}(M, TM \otimes \mathbb{C})$ denotes $(\hat{A}(TM)\,\mathrm{ch}(TM \otimes \mathbb{C}))[M]$. The interest in such a manifold is the observation (loc.cit. p.86 f.) that one obtains the dimensions of irreducible representations of the Monster group from the $\hat{A}$–genus of certain linear combinations of symmetric powers of the complexified tangent bundle. For this observation to hold, which relies on the Witten genus being an integral modular form, one need only require that $p_1(M) = 0$ rationally (loc. cit. p.84), which is how we will interpret the condition $p_1(M) = 0$ in the above question.

Hopkins–Mahowald [4] point out that the existence of an answer to Question 1.1 follows directly from an understanding of the homotopy of $MString$, and they further construct and discuss explicit examples of such manifolds. In the present note we will construct an answer by different means, using rational surgery following Sullivan [6, Theorem 13.2]:

Theorem 1.2. (alternative solution to Question 1.1, solved in [4]) There is a 24–dimensional closed oriented simply connected smooth manifold M with $p_1(M) = 0$ rationally, $w_2(M) = 0$, $\hat{A}(M) = 1$, and $\hat{A}(M, TM \otimes \mathbb{C}) = 0$, which furthermore has $\dim H_i(M; \mathbb{Q}) = 0$ for $2 \leq i \leq 7$ and $\dim H_8(M; \mathbb{Q}) = 1$.

We emphasize that though our manifold will have vanishing first rational Pontryagin class, we are not able to detect whether it admits a string structure. However, our approach yields a large degree of flexibility in giving solutions to Question 1.1, though with the caveat that we know the resulting manifold only up to rational homotopy equivalence.

For any manifold satisfying the conditions of Question 1.1 that is furthermore string, one can of course perform normal surgery to the map to $BString$ classifying the stable normal bundle in order to make a 7–connected solution with eighth Betti number equal to one, string cobordant to the original. We remark that Hirzebruch also asked the question of whether there is a manifold as in Question 1.1 which furthermore admits a (faithful) action of the Monster group by diffeomorphisms; in this strong form the prize question is open.

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2. Rational realization for spin manifolds

Given a simply connected rational space X (i.e. one for which the natural maps $\tilde{H}_*(X;\mathbb{Z}) \to \tilde{H}_*(X;\mathbb{Q})$ are isomorphisms) satisfying Poincaré duality on its rational cohomology with respect to a class $[X] \in H_n(X;\mathbb{Q})$, and equipped with rational cohomology classes p_i in degrees $4i$, Sullivan described [6, Theorem 13.2] when there is a closed oriented manifold M with a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $p_i(TM) = f^*p_i$. A modification of the argument, replacing SO by $Spin$, yields the following realization result for spin manifolds; see Crowley–Nordström [1, §3.5]. For brevity we will state it only for dimension 24, though the appropriate analogous statement to [6, Theorem 13.2] holds for all dimensions > 4.

Theorem 2.1. Let X be a simply connected rational space of finite type satisfying Poincaré duality on its rational cohomology with respect to a class $[X] \in H_{24}(X;\mathbb{Q})$. Furthermore, let $p_i \in H^{4i}(X;\mathbb{Q})$, $1 \le i \le 6$, be cohomology classes. Then there is a (simply connected) closed spin manifold M and a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $p_i(TM) = f^*(p_i)$ if (and only if)

- the rational numbers $(p_{i_1} p_{i_2} \cdots p_{i_r})[X]$ are integers that satisfy the Stong congruences of a spin manifold ([5, Theorem 1(c)], elaborated on below),
- the quadratic form on $H^{12}(X;\mathbb{Q})$ given by $q(\alpha,\beta) = (\alpha\beta)[X]$ is equivalent over $\mathbb{Q}$ to one of the form $\sum_i \pm y_i^2$,
- we have $L(p_1,\ldots,p_6)[X] = \tau(X)$, where L is Hirzebruch's L–genus, and τ is the signature of the quadratic form on $H^{12}(X;\mathbb{Q})$.

Remark 2.2. Two points are salient to the application of Sullivan's construction [6, Theorem 13.2] to $Spin$ instead of SO: that being spin is a stable property of a vector bundle, and that the homotopy fiber product of the map $X \xrightarrow{(p_1,p_2,\ldots)} BSO_{\mathbb{Q}} \simeq \prod_i K(\mathbb{Q},4i)$ and the map giving the universal dual rational Pontryagin classes $BSO \xrightarrow{(\bar{p}_1,\bar{p}_2,\ldots)} \prod_i K(\mathbb{Q},4i)$ (i.e. those determined by the equation $(1+p_1+p_2+\ldots)\cdot(1+\bar{p}_1+\bar{p}_2+\ldots) = 1$) is simply connected. The latter is a consequence of the homotopy fiber of the second map being simply connected. As this homotopy fiber product is the target space of the normal surgery performed in Sullivan's construction, the fact that it is simply connected allows for his argument to go through as in loc. cit.

Let us now discuss the Stong congruences for 24–dimensional closed spin manifolds [5, Theorem 1(c)]. These are obtained by understanding the image of the map from 24–dimensional spin bordism Ω_{24}^{Spin} to $H_{24}(BSpin;\mathbb{Q})$ which sends a bordism class to the pushforward of the fundamental class of any representative by the map classifying the stable tangent bundle with its spin structure.

In general, to describe the Stong congruences in any dimension, one considers the formal splitting of the universal rational Pontryagin class $1+p_1+p_2+\cdots = \prod_j (1+x_j^2)$, where x_j^2 are the Pontryagin roots (so $\deg(x_j^2) = 4$). Then, consider the set of variables given by $e^{x_j} + e^{-x_j} - 2$. Now form the elementary symmetric polynomials $\sigma_1, \sigma_2, \ldots$ in these variables $e^{x_j} + e^{-x_j} - 2$; note that these can be expressed as rational polynomials in the Pontryagin classes p_i (which are the elementary symmetric polynomials in x_j^2). For n divisible by 8, Stong describes the image of the map $\Omega_n^{Spin} \to H_n(BSpin;\mathbb{Q})$ as those $a \in H_n(BSpin;\mathbb{Q})$ such that

$$(z \cdot \hat{A})[a] \in \mathbb{Z} \quad \text{for all } z \in \mathbb{Z}[\sigma_1, \sigma_2, \ldots].$$

This corresponds to the pullback of any class of the form $z \cdot \hat{A}$ to a specified spin manifold M being an integer when integrated over the manifold, as guaranteed by the Atiyah–Singer index theorem.

3. Proof of Theorem 1.2

We now present a specific rational homotopy type that can be equipped with rational "Pontryagin classes" p_i such that the conditions of Theorem 2.1 and Question 1.1 are satisfied, thus producing the desired manifold.

Take the algebra over $\mathbb{Q}$ generated by α in degree 8 with $\alpha^4 = 0$, and 22720000 variables β_i in degree 12 such that $\beta_i\alpha = 0$, $\beta_i\beta_j = 0$ for $i \neq j$, and $\beta_i^2 + \alpha^3 = 0$ (which implies $\beta_i^3 = 0$); that is, take

$$\mathbb{Q}[\alpha, \beta_i]/(\alpha^4, \beta_i\alpha, \beta_i\beta_j \text{ for } i \neq j, \beta_i^2 + \alpha^3),\ 1 \leq i, j \leq 22720000.$$

Realize this algebra as the rational cohomology of a rational space X, and take the fundamental class $[X] \in H_{24}(X; \mathbb{Q})$ to be such that $\alpha^3[X] = 1$; notice that Poincaré duality is satisfied. The nondegenerate pairing in middle degree is given by $22720000(-1)$. Prescribe the rational classes p_i as $p_1 = 0$, $p_2 = 144\alpha$, $p_3 = 0$, $p_4 = -4583424\alpha^2$, $p_5 = 0$, $p_6 = -5165220096\alpha^3$. Then we have

$$p_2^3[X] = 2985984, \quad p_2p_4[X] = -660013056, \quad p_6[X] = -5165220096.$$

We now check that the signature of the middle degree pairing is calculated correctly from evaluating Hirzebruch's L–genus on these "Pontryagin numbers", and that they satisfy the Stong congruences.

For the signature, we indeed have

$$L_{24} = \tfrac{1}{638512875}(2828954p_6 - 159287p_2p_4 + 8718p_2^3) = -22720000.$$

Now we describe the Stong congruences. Considering terms only up to degree 24, we have

$$e^{x_j} + e^{-x_j} - 2 = x_j^2 + \frac{x_j^4}{12} + \frac{x_j^6}{360} + \frac{x_j^8}{20160} + \frac{x_j^{10}}{1814400} + \frac{x_j^{12}}{239500800}$$

where $1 \leq j \leq 6$. Modulo odd-index Pontryagin classes, we have the following expressions for the elementary symmetric polynomials σ_i in the variables $e^{x_j} + e^{-x_j} - 2$:

$$\begin{aligned}
\sigma_1 &= -\tfrac{1}{119750400}p_2^3 + \tfrac{1}{39916800}p_2p_4 - \tfrac{1}{39916800}p_6 + \tfrac{1}{10080}p_2^2 - \tfrac{1}{5040}p_4 - \tfrac{1}{6}p_2 \\
\sigma_2 &= \tfrac{1}{1814400}p_2^3 - \tfrac{11}{604800}p_2p_4 + \tfrac{31}{604800}p_6 + \tfrac{1}{720}p_2^2 + \tfrac{1}{40}p_4 + p_2 \\
\sigma_3 &= -\tfrac{1}{7560}p_2p_4 - \tfrac{4}{945}p_6 - \tfrac{1}{3}p_4 \\
\sigma_4 &= \tfrac{1}{720}p_2p_4 + \tfrac{19}{240}p_6 + p_4 \\
\sigma_5 &= -\tfrac{1}{2}p_6 \\
\sigma_6 &= p_6.
\end{aligned}$$

Note that the lowest order term of $\sigma_{2i-1}, \sigma_{2i}$ is of degree $8i$. Now, any integer polynomial in the above σ_i multiplied by the $\hat{A}$–genus must evaluate to an integer on our desired manifold. Again modulo p_1, p_3, p_5, the $\hat{A}$–genus up to degree 24 is given by:

$$\begin{aligned}
\hat{A}_0 &= 1, \\
\hat{A}_8 &= -\tfrac{4}{5760}p_2, \\
\hat{A}_{16} &= \tfrac{1}{464486400}(208p_2^2 - 192p_4), \\
\hat{A}_{24} &= \tfrac{1}{2678117105664000}(-769728p_2^3 + 1476352p_2p_4 - 707584p_6),
\end{aligned}$$

with $\hat{A}_4 = \hat{A}_{12} = \hat{A}_{20} = 0$. From now on we will implicitly assume all degree 24 classes are paired with the fundamental class. Each of the following expressions must be an integer:

$$\hat{A}_{24} = -\tfrac{769728p_2^3+719872p_3^2+1476352p_2p_4-707584p_6}{2678117105664000}$$
$$(\sigma_1 \cdot \hat{A})_{24} = -\tfrac{97}{638668800}p_2^3 + \tfrac{37}{159667200}p_2p_4 - \tfrac{1}{39916800}p_6$$
$$(\sigma_2 \cdot \hat{A})_{24} = \tfrac{1}{29030400}p_2^3 - \tfrac{29}{806400}p_2p_4 + \tfrac{31}{604800}p_6$$
$$(\sigma_3 \cdot \hat{A})_{24} = \tfrac{1}{10080}p_2p_4 - \tfrac{4}{945}p_6$$
$$(\sigma_4 \cdot \hat{A})_{24} = \tfrac{1}{1440}p_2p_4 + \tfrac{19}{240}p_6$$
$$(\sigma_5 \cdot \hat{A})_{24} = -\tfrac{1}{2}p_6$$
$$(\sigma_6 \cdot \hat{A})_{24} = p_6$$
$$(\sigma_1^2 \cdot \hat{A})_{24} = -\tfrac{19}{362880}p_2^3 + \tfrac{1}{15120}p_2p_4$$
$$(\sigma_1^3 \cdot \hat{A})_{24} = -\tfrac{1}{216}p_2^3$$
$$(\sigma_1\sigma_2 \cdot \hat{A})_{24} = -\tfrac{1}{60480}p_2^3 - \tfrac{11}{2520}p_2p_4$$
$$(\sigma_1^2\sigma_2 \cdot \hat{A})_{24} = \tfrac{1}{36}p_2^3$$
$$(\sigma_1\sigma_2^2 \cdot \hat{A})_{24} = -\tfrac{1}{6}p_2^3$$
$$(\sigma_1\sigma_3 \cdot \hat{A})_{24} = \tfrac{1}{18}p_2p_4$$
$$(\sigma_1\sigma_4 \cdot \hat{A})_{24} = -\tfrac{1}{6}p_2p_4$$
$$(\sigma_2^2 \cdot \hat{A})_{24} = \tfrac{1}{480}p_2^3 + \tfrac{1}{20}p_2p_4$$
$$(\sigma_2^3 \cdot \hat{A})_{24} = p_2^3$$
$$(\sigma_2\sigma_3 \cdot \hat{A})_{24} = -\tfrac{1}{3}p_2p_4$$
$$(\sigma_2\sigma_4 \cdot \hat{A})_{24} = p_2p_4$$

The above is the full set of required congruences. Furthermore, recall that we require $\hat{A}(M) = 1$ and $\hat{A}(M, TM \otimes \mathbb{C}) = 0$. Using Newton's identities we calculate the top degree of the latter to be

$$\begin{aligned}\hat{A}(M, TM \otimes \mathbb{C})_{24} &= (\hat{A}(M) \cdot \mathrm{ch}(TM \otimes \mathbb{C}))_{24} \\ &= -\tfrac{8389}{52835328000}p_2^3 + \tfrac{9707}{39626496000}p_2p_4 - \tfrac{311}{9906624000}p_6.\end{aligned}$$

Hence $\hat{A}(M, TM \otimes \mathbb{C}) = 1$ is, given that p_1, p_3, p_5 vanish, equivalent to

$$p_6 = -\tfrac{25167}{4976}p_2^3 + \tfrac{9707}{1244}p_2p_4.$$

Assuming this, the first requirement $\hat{A}(M) = 1$ is then equivalent to

$$p_2p_4 = \frac{873600000p_2^3 - 832894419861504000}{1257984000}.$$

We check directly that these requirements and all the above congruences are satisfied for our choice of p_i and fundamental class on X. This completes the proof of Theorem 1.2.

Remark 3.1. Simplifying the above, one can see that all the requirements will be satisfied if the odd-index p_i vanish and if

$$\begin{aligned}p_2^3 &= 155520k + 31104, \\ p_2p_4 &= 108000k - 662065056, \\ p_6 &= 56160k - 5166287136,\end{aligned}$$

for some integer k. For $k = 19$, we have the particularly nice solutions we started with, since then $p_2^3 = 144^3$. (This is the smallest k such that $155520k + 31104$ is a cube; there are infinitely many others.) This made it straightforward to construct a rational homotopy type with a class p_2 giving the desired Pontryagin number p_2^3. In fact, in the process of constructing the desired manifold using Theorem 2.1, for simplicity one would want the rational cohomology of the realizing manifold to be $\mathbb{Q}[\alpha]/(\alpha^4)$. The nonzero value of the L–genus then informs us to place an appropriate number of new variables β_i in middle degree. Of course, one could replace the cohomology algebra we realized with any simply connected rational Poincaré duality algebra with the same signature into which it includes, and still obtain an answer to Question 1.1 (indeed, we can just make the same choice of Pontryagin classes). The same holds on the level of rational homotopy types: if we can realize a (simply connected) rational homotopy type X, then any rational homotopy type whose cohomology contains $H^*(X;\mathbb{Q})$ can be realized. Even for a fixed $H^*(X;\mathbb{Q})$, there are potentially many corresponding rational homotopy types (though not for our rationally 7–connected example, as every such 24–dimensional Poincaré duality algebra is formal).

If one wished to avoid the search for a k such that $155520k + 31104$ is a cube as before, one could also, for example, recall that every integer is a sum of five cubes and settle for building a rationally 7–connected manifold with eighth Betti number possibly larger than one. For example, for $k = 0$ we have $31104 = 21^3 + 20^3 + 19^3 + 19^3 + 5^3$. We then start with the algebra

$$\mathbb{Q}[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]/(\alpha_i\alpha_j \text{ for } i \neq j, \alpha_i^4, \alpha_1^3 - \alpha_i^3 \text{ for } 2 \leq i \leq 5),$$

where the α_i are in degree 8, with fundamental class dual to α_1^3, and choose $p_2 = 21\alpha_1 + 20\alpha_2 + 19\alpha_3 + 19\alpha_4 + 5\alpha_5$, giving $p_2^3 = 31104$. We can then choose $p_4 = 4\alpha_1^2 - 33103257\alpha_2^2$ to achieve $p_2p_4 = -662065056$. With $p_6 = -5166287136\alpha_1^3$, the L–genus then evaluates to -22724256, so we add degree 12 elements β_j, $1 \leq j \leq 22724256$, to our algebra to correct for this; i.e. we take

$$\mathbb{Q}[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_j]/(\alpha_i\alpha_j \text{ for } i \neq j, \alpha_j^4, \alpha_1^3 - \alpha_i^3 \text{ for } 2 \leq i \leq 5, \beta_j\alpha, \beta_i\beta_j \text{ for } i \neq j, \beta_j^2 + \alpha_1^3).$$

Remark 3.2. Using the results of Hopkins–Mahowald [4], Han–Huang have computed the Pontryagin numbers for an integral basis M_1, M_2, M_3, M_4 of Ω_{24}^{String} [2, Lemma 2.3, Lemma 3.2, Lemma 4.12]. One then obtains a realization theorem for 24–dimensional string manifolds in analogy to Theorem 2.1 (again see [1, §3.5]; note $BSpin \simeq BO\langle 4\rangle$ and $BString \simeq BO\langle 8\rangle$). One need only additionally require that $p_1 = 0$ rationally and that the Pontryagin numbers $(p_2^3, p_3^2, p_2p_4, p_6)$ furthermore lie in the lattice in $\mathbb{Q}^4$ spanned by the Pontryagin numbers of M_1, M_2, M_3, M_4, i.e. in the lattice spanned by

$$(2^{13} \cdot 3^5 \cdot 5^3,\ 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2,\ 2^{12} \cdot 3^5 \cdot 5^3,\ 2^9 \cdot 3^4 \cdot 5^2 \cdot 89),$$
$$(-2^{13} \cdot 3^5 \cdot 5^3 \cdot 41,\ 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 31,\ -2^{12} \cdot 3^5 \cdot 5^3 \cdot 41,\ -2^9 \cdot 3^4 \cdot 5^2 \cdot 11^2),$$
$$(2^7 \cdot 3^5 \cdot 5,\ 0,\ 2^5 \cdot 3^3 \cdot 5^3,\ 2^5 \cdot 3^3 \cdot 5 \cdot 13),$$
$$(2^4 \cdot 3^5,\ 2^3 \cdot 5^2,\ 2^2 \cdot 3 \cdot 239,\ 2 \cdot 11 \cdot 89).$$

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