A trichotomy of consequences of the existence of holomorphic charts on the six sphere

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Abstract

We investigate the topology of the space of almost complex structures (inducing a fixed orientation) on the six–sphere; we determine its rational homotopy type to be that of $\mathbb{RP}^7$, with the inclusion of the subspace $SO(7)/G_2$ of almost complex structures orthogonal with respect to the round metric, provided by the octonions, inducing an isomorphism on rational homotopy groups. Though additionally both of these spaces have fundamental groups of order two, they are not homotopy equivalent. We then go on to consider the naive and homotopy quotients of the space of almost complex structures and its subspace of integrable structures by the action of the diffeomorphism group of the sphere, obtaining three putative statements involving these spaces and the stabilizers of integrable structures, one of which must be true if there exist integrable structures.

1 The rational homotopy type of the space of almost complex structures

An almost complex structure on $\mathbb{R}^{2n}$ is a linear endomorphism $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $J^2 = -\mathrm{id}$, and such a J endows $\mathbb{R}^{2n}$ with the structure of a complex vector space, where the complex scalar multiplication can be defined as $(x + iy)\vec{v} := x\vec{v} + yJ(\vec{v})$.

$GL(2n, \mathbb{R})$ acts on almost complex structures on $\mathbb{R}^{2n}$ by conjugation. This action is transitive and its stabilizer is $GL(n, \mathbb{C})$. Therefore, the space of almost complex structures on $\mathbb{R}^{2n}$ can be identified with $GL(2n, \mathbb{R})/GL(2n, \mathbb{C})$ (see page 116 of [KN96]).

An almost complex structure on a manifold is a cross-section of the bundle associated to the space of almost complex structures on each tangent space, i.e. of the bundle $GL(2n, \mathbb{R})/GL(n, \mathbb{C}) \to \mathfrak{J}(M) \to M$, where $\mathfrak{J}(M)$ denotes the bundle of linear frames over M modulo $GL(n, \mathbb{C})$ (see page 113 in [KN96]).

We consider the special case of the six–sphere,

$$\begin{array}{ccc} GL(2n,\mathbb{R})\big/GL(n,\mathbb{C}) & \longrightarrow & \mathfrak{J}(S^6) \\ & & \big\downarrow \\ & & S^6 \end{array}$$

Firstly, from elementary obstruction theory, we know that the space of cross-sections is non-empty and has two components (given orientation, the standard almost complex structures given by octonion multiplication provides a family of such a cross-section).

We replace $GL(6,\mathbb{R})/GL(3,\mathbb{C})$ by $SO(6)/U(3)$, as they are homotopy equivalent (see Remark 1.3). Since the space of almost complex structures on $\mathbb{R}^8$ fibers over S^6 with fibers being the space of almost complex structures on $\mathbb{R}^6$, we can replace $\mathfrak{J}(S^6)$ by $SO(8)/U(4)$.

The Γ-construction (suggested by Sullivan on page 314 of [Sull77]), providing a rational homotopy model of the space of sections Γ, works as follows:

Given a fibration $F \to E \xrightarrow{p} B$, we consider the pairs $b^* \otimes f$, where b^* is a linear basis vector of the differential graded coalgebra that is the dual of the (finite type) model $\mathcal{B}$ of the base and f is a cdga generator of a (finite rank) model $\Lambda(V)$ of the fiber. The degree of the pair $b^* \otimes f$ is defined to be $|f| - |b|$. We mod out the free algebra generated by those $b^* \otimes f$ pairs by the ideal generated by pairs that have a negative degree and cocycles in degree zero. That way, we obtain a connected free cdga denoted by Γ.

Haefliger shows in [Haef82] that there is a unique differential we can put on this free cdga that makes the evaluation map $\mathrm{ev} : \mathcal{B} \otimes \Lambda(V) \to \mathcal{B} \otimes \Gamma$, a map of cdga's:

$$d_\Gamma(b^* \otimes f) := \pm \partial b^* \otimes f \pm b^* \otimes \mathrm{ev}(df),$$

where $\mathcal{B} \otimes \Lambda(V)$ models the total space E of the fibration, ∂b^* is the transpose of $d_{\mathcal{B}}$ and the $b^* \otimes \mathrm{ev}(df)$ term is obtained by taking the differential of f in the the total space and then using the coalgebra structure of $\mathcal{B}^*$ to evaluate them at B to reduce the term to a product of the generators of Γ.

The generators of Γ (with positive degrees) in the S^6 case are $1^*_{\mathcal{B}} \otimes x_2$ (with degree 2), $1^*_{\mathcal{B}} \otimes y_7$ (with degree 7) and $w_6^* \otimes y_7$ (with degree 1). (The indices indicate the degrees of the generators in their respective algebras.)

$\Lambda(1_{\mathcal{B}}, w_6, z_{11})$ is a model of S^6 of finite type with $\mathrm{d}_{\mathcal{B}}(1) = \mathrm{d}_{\mathcal{B}}(w_6) = 0$ and $\mathrm{d}(z_{11}) = w_6^2$.

$\Lambda(1_F, x_2, y_7)$ is a model of $\mathbb{C}P^3$, which is diffeomorphic to $SO(6)/U(3)$ and it has finite rank linearized cohomology.

The differential of $1^*_{\mathcal{B}} \otimes x_2 = 0$. To find the differential of $1^*_{\mathcal{B}} \otimes y_7$ and $w_6^* \otimes y_7$, we need to figure out what $\mathrm{d}y_7$ is. The crucial part of the computation is to show that the fibration

$$\begin{array}{ccc} \mathbb{C}P^3 & \longrightarrow & SO(8)/U(4) \\ & & \downarrow \\ & & S^6 \end{array}$$

is not a product fibration and that follows from the fact that $SO(8)/U(4)$ is Hermitian symmetric (see page 518 in [Hel76]) and hence a Kähler manifold.

Therefore, it should have a closed 2-form whose sixth power is nonzero. The only closed 2-form in our model is x_2. That means $x_2^4 \neq 0$ and therefore $0 \neq c \in \mathbb{R}$ in $\mathrm{d}y_7 = x_2^4 + c x_2 w_6$. Then $\mathrm{d}_\Gamma(w_6^* \otimes y_7) = (1^*_{\mathcal{B}} \otimes x_2)$ and we end up with one cohomology class represented by $(1^*_{\mathcal{B}}, y_7)$ as claimed.

This shows that the space of almost complex structures has the de Rham homotopy type of S^7 (or $\mathbb{R}P^7$, which is $\mathbb{Q}$-equivalent to S^7).

Remark 1.1. Our considered space of sections Γ is nilpotent since the fiber is simply connected and the base is a finite complex [M87], and hence we are in the setting of [Sull77] where information obtained from (minimal) models directly corresponds to geometric information.

Hence we have the following:

Theorem 1.2. *The space* Γ *of almost complex structures on* S^6 *(inducing a given orientation) is nilpotent and a minimal model is given by* $(\Lambda(y_7), dy_7 = 0)$*.*

Remark 1.3. One can consider the spaces of smooth or continuous sections of the relevant $GL(6,\mathbb{R})/GL(3,\mathbb{C})$ or $SO(6)/U(3)$ bundles (corresponding to smooth or continuous almost complex structures); we note that all of these spaces are weakly homotopy equivalent via the natural inclusions. First of all, that the inclusion of the space of smooth sections of the $SO(6)/U(3)$ bundle into the space of its continuous sections is a weak homotopy equivalence follows from classical smooth approximation arguments (see [St51, I.6.7], and [Ki20, Theorem 1.2] for a definitive modern treatment). To observe that the inclusion of continuous sections of the $SO(6)/U(3)$ bundle into the continuous sections of the $GL(6,\mathbb{R})/GL(3,\mathbb{C})$ bundle is a weak homotopy equivalence, we work simplicially and use the naturality of the Moore–Postnikov decomposition of fibrations and the description of the weak homotopy type of the space of sections given in [Haef82, p.611].

Now we note that all of these spaces have the homotopy type of CW complexes, so they are in fact all homotopy equivalent. Indeed, from the proof of Proposition 2.3 and the statement of [Haef82, Proposition p.611] we see that the spaces of continuous sections has the homotopy type of a CW complex. For the spaces of smooth sections we can appeal to results of Palais (see for ex. [duP76, p.301]).

Let us now consider the subspace of almost complex structures on S^6 provided by the octonions, and its relation to the space Γ.

Thinking of $S^6 = \{p^2+1=0\}$ as the unit sphere in the space of imaginary octonions, we have the standard almost complex structure given by $\hat{J}_p(v) = pv$, where on the right-hand side we have multiplication of octonions. Indeed, since any $\mathbb{R}$–subalgebra of the octonions generated by two elements is associative, we have $\hat{J}_p(\hat{J}_p(v)) = p(pv) = (p^2)v = -v$. Furthermore, with respect to the standard inner product on $\mathbb{O} \cong \mathbb{R}^8$ we have $\langle p, pv\rangle = \langle 1, v\rangle = 0$ since left multiplication by p is an orthogonal transformation, and v is imaginary. We also have $\langle pv, 1\rangle = -\langle v, p\rangle = 0$, and hence pv is imaginary and orthogonal to p, i.e. it lies in the tangent space T_pS^6.

Now notice that $SO(7)$ acts on the space of almost complex structures by $(AJ)_p(v) = A^{-1}J_{Ap}(Av)$ for an $A \in SO(7)$ and J almost complex structure; AJ induces the same orientation on S^6 as J does. The stabilizer of $\hat{J}$ under this action consists of those $A \in SO(7)$ such that $A^{-1}((Ap)(Av)) = pv$, i.e. $A(pv) = (Ap)(Av)$ for all orthogonal pairs of unit octonions p, v. This is enough to conclude that A is an $\mathbb{R}$–algebra automorphism of the octonions, i.e. an element of G_2. So, the orbit of $\hat{J}$ under the action of $SO(7)$ is homeomorphic to $\mathbb{RP}^7$.

On the other hand, for each unit octonion $u \in S^7$, we have the conjugation $x \mapsto ux\bar{u}$. Conjugation preserves the space of imaginary octonions; indeed, for an imaginary p we have $\langle up\bar{u}, 1\rangle = \langle p\bar{u}, \bar{u}\rangle = \langle p, 1\rangle = 0$, and so we have a map $S^7 \to SO(7)$ which factors through the action of ± 1 to produce a map $\mathbb{RP}^7 \to SO(7)$. The composite map $\mathbb{RP}^7 \to SO(7)/G_2 \cong \mathbb{RP}^7$ is not a homeomorphism. Indeed, by a result of Brandt (see [CoSm03, Theorem 8.7.14 on page 98]), conjugation by a unit octonion u is an algebra automorphism if and only if u is a sixth root of unity. The solution to $u^6 = 1$ consists of $u = \pm 1$ and the two 6-spheres parallel to the imaginary 7-plane at heights $\pm\frac{1}{2}$ on the real axis. The upper 6-sphere is the solution to $1 - u + u^2 = 0$, and the lower 6-sphere the solution to $1 + u + u^2 = 0$; the translation $u \mapsto u - 1$ maps the upper sphere to the lower one. This whole solution set gets sent to a point in the orbit $SO(7)/G_2 \cong \mathbb{RP}^7$.

Proposition 1.4. *The inclusion $SO(7)/G_2 \hookrightarrow \Gamma$ of the $SO(7)$–orbit of $J_p(v) = pv$ into the space of all almost complex structures, is an isomorphism on rational homotopy groups.*

Proof. Consider the evaluation map $(SO(7)/G_2)\times S^6 \to SO(8)/U(4)$. In [CalGlu93], Calabi and Gluck describe this map upon isometrically identifying $SO(8)/U(4)$ with the Grassmannian $Gr^+(2,8)$ of oriented real 2–planes in $\mathbb{R}^8$. A fixed $J \in SO(7)/G_2 = \mathbb{RP}^7$ sends S^6 to the sub-Grassmannian of two–planes containing a fixed line in $\mathbb{R}^8$; the space of such lines is precisely parametrized by the $\mathbb{RP}^7$ of considered almost complex structures. From this description, using Ehresmann's lemma and $\pi_1 Gr^+(2,8) = 0$, we see that the evaluation map is a locally trivial fibration with fiber S^1 (parametrized by the space of lines with a choice of orientation in a given oriented 2–plane).

We will now describe this fibration in terms of rational homotopy minimal models. To do so, first we record the minimal model of $Gr^+(2,8) = SO(8)/U(4)$. By Λ we will denote the free graded–commutative algebra on a set of generators, or, as will be clear from context, a minimal model of a given space.

Lemma 1.5. *The minimal model $\Lambda SO(8)/U(4)$ is given by*

$$(\Lambda(x_2, x_6, y_7, y_{11}),\ dy_7 = x_2^4 - x_2x_6, dy_{11} = x_6^2).$$

The degrees of the generators are denoted by their subscripts.

Proof. This follows from our earlier discussion in the proof of Theorem 1.2, and a rescaling of the generators. □

Now we see that in the fibration $S^1 \to SO(7)/G_2 \times S^6 \to Gr^+(2,8)$, the degree 1 generator of ΛS^1 must map to x_2 under the differential, giving the model $(\Lambda(x_6, y_7, y_{11}), dx_6 = 0, dy_7 = 0, dy_{11} = x_6^2)$ for $SO(7)/G_2 \times S^6 = \mathbb{RP}^7 \times S^6$ (which is rationally equivalent to $S^7 \times S^6$). In particular, the degree 7 generator y_7 of $SO(8)/U(4)$ pulls back to the degree 7 generator of $SO(7)/G_2 \times S^6$ by the evaluation. Since this evaluation map factors through the evaluation $\Gamma \times S^6 \to SO(8)/U(4)$ via the map $SO(7)/G_2 \times S^6 \hookrightarrow \Gamma \times S^6$ given by the inclusion and identity, we see that the degree 7 generator of $\Lambda\Gamma$ must pull back to the degree 7 generator of $SO(7)/G_2$. Therefore $SO(7)/G_2 \hookrightarrow \Gamma$ is an isomorphism on rational homotopy groups.

□

2 On the integral homotopy type

We will now record some topological information on the space of (orientation–compatible) almost complex structures on S^6; recall that the inclusion of metric–compatible almost complex structures into all almost complex structures is a weak homotopy equivalence (from here on let us fix the standard round metric on S^6 inherited from its ambient Euclidean space $\mathbb{O} \cong \mathbb{R}^8$).

Proposition 2.1. *The fundamental group of the space of almost complex structures on S^6 has order two.*

Proof. First of all, the space of orientation–compatible orthogonal almost complex structures on S^6 can be identified with the space of sections Γ of the projectivized bundle $P\mathbb{S}^+$ of positive spinors (see [LawMic16] Proposition IV.9.8 and Remark IV.9.12); the complex bundle $\mathbb{S}^+$ is of rank four, and so the fiber of the projectivized bundle is $\mathbb{CP}^3$.

We will use the following result of Crabb and Sutherland:

Theorem 2.2 (CS84, Proposition 2.7 and Theorem 2.12)**.** *Let X be an oriented closed connected $2n$–manifold and ξ a complex rank $n+1$ bundle over X. Denote by $N\xi$ the space of sections of the projective bundle $P\xi$ which lift to sections of ξ; this is a non-empty connected space. Then $\pi_1(N\xi)$ is a central extension*

$$0 \to \mathbb{Z}/c_n(\xi)[X] \to \pi_1(N\xi) \to H^1(X;\mathbb{Z}) \to 0.$$

In our situation of $X = S^6$ and $\xi = \mathbb{S}^+$, notice that $N\xi$ coincides with our space of sections Γ, as $H^2(S^6;\mathbb{Z}) = 0$ and so every section of $P\mathbb{S}^+$ lifts to a section of $\mathbb{S}^+$. Therefore, to determine $\pi_1(\Gamma)$ we need only determine $c_3(P\mathbb{S}^+)$. To this end, we will use that the total space of our $\mathbb{CP}^3 = SO(6)/U(3)$ bundle over S^6 is homotopy equivalent to $SO(8)/U(4)$ (the space of orientation–compatible orthogonal complex structures on the vector space $\mathbb{R}^8$). The cohomology ring $H^*(SO(8)/U(4);\mathbb{Z})$ is thus generated, as an algebra over the pullback of $H^*(S^6;\mathbb{Z})$ by the projection $SO(8)/U(4) \xrightarrow{p} S^6$, by a single element α in degree 2, subject only to the relation

$$\alpha^4 + \alpha^3 p^*c_1(\mathbb{S}^+) + \alpha^2 p^*c_2(\mathbb{S}^+) + \alpha p^*c_3(\mathbb{S}^+) + p^*c_4(\mathbb{S}^+) = 0,$$

i.e.

$$\alpha^4 + \alpha p^*c_3(\mathbb{S}^+) = 0.$$

By the calculation of the cohomology ring in [Mas61, p.563], this now implies that $p^*c_3(\mathbb{S}^+)$ is (up to sign) twice the degree six generator of $H^*(SO(8)/U(4);\mathbb{Z})$. Since $SO(8)/U(4) \xrightarrow{p} S^6$ admits a section, this further implies $c_3(\xi)$ is (up to sign) twice the fundamental cohomology class of S^6. Applying the theorem of Crabb and Sutherland, we conclude that $\pi_1(\Gamma) = \mathbb{Z}/2$. □

Now one might be led to believe that the space of almost complex structures Γ has the homotopy type of $SO(7)/G_2 \cong RP^7$, given that the rational homotopy types and fundamental groups agree. However, a more careful analysis shows that this is not the case.

Proposition 2.3. *The inclusion $SO(7)/G_2 \hookrightarrow \Gamma$ is not a weak homotopy equivalence.*

Proof. We will consider the Postnikov–like system for the space of sections Γ induced by the Moore–Postnikov system for $SO(8)/U(4) \to S^6$, as in [Haef82, p. 611] (see [Span89, Section 8.3] for an account of Moore–Postnikov systems). First, consider the Moore–Postnikov system for $SO(8)/U(4) \to S^6$, where the Eilenberg–Maclane spaces fibered in at each stage correspond to the homotopy groups of the fiber $\mathbb{CP}^3$:

$$
\begin{array}{ccc}
 & & SO(8)/U(4) \\
 & & \downarrow p \\
 & & \vdots \\
 & & \downarrow \\
K(\mathbb{Z}/2,8) & \longrightarrow & E_3 \\
 & & \downarrow p_3 \\
K(\mathbb{Z},7) & \longrightarrow & E_2 \\
 & & \downarrow p_2 \\
K(\mathbb{Z},2) & \longrightarrow & E_1 \\
 & & \downarrow p_1 \\
 & & S^6
\end{array}
$$

We obtain an induced decomposition of the space Γ, where the fibers are now the spaces of unbased maps (with the compact–open topology) from S^6 to the corresponding fiber of the above Moore–Postnikov system. Choosing a section $s \in \Gamma$, we denote by Γ_i the space of sections of $E_i \xrightarrow{p_i} X$ in the component of $q_i s$, where q_i is the map $SO(8)/U(4) \to E_i$:

$$
\begin{array}{ccc}
 & & \Gamma \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
K(\mathbb{Z}/2,8)^{S^6} & \longrightarrow & \Gamma_3 \\
 & & \downarrow \\
K(\mathbb{Z},7)^{S^6} & \longrightarrow & \Gamma_2 \\
 & & \downarrow \\
 & & \Gamma_1
\end{array}
$$

By a result of Thom (see [Haef82]), for a finite-type space X, there is a homotopy equivalence

$$K(G,m)^X \simeq \prod_{i=0}^{m} K(H^{m-i}(X;G),i).$$

In the case of $X = S^6$, we have

$$K(G,m)^{S^6} \simeq K(G,m) \times K(G,m-6),$$

where we take $K(G,i) \simeq \{*\}$ if i is negative.

Since $H^3(S^6;\mathbb{Z})$ is trivial, E_1 is homotopy equivalent to the product $S^6 \times K(\mathbb{Z},2)$, and so $\Gamma_1 \simeq K(\mathbb{Z},2)$. As we saw earlier, $\pi_1(\Gamma) = \mathbb{Z}/2$, which informs us that the nontrivial homotopy groups of Γ_2 are $\pi_1(\Gamma_2) = \mathbb{Z}/2$ and $\pi_7(\Gamma_2) = \mathbb{Z}$. Then we see that $\pi_2(\Gamma_3) = \mathbb{Z}_2$, and since all subsequent fibers in the tower are at least 2–connected, we have $\pi_2(\Gamma) = \mathbb{Z}/2$. We can similarly unambiguously determine some further homotopy groups.

□

3 A trichotomy of consequences of holomorphic charts

The action of $SO(7)$ on the orthogonal almost complex structures is the restriction of the much larger group $\mathrm{Diff}^+(S^6)$ of orientation-preserving diffeomorphisms acting on all almost complex structures. For any almost complex structure J and orientation-preserving diffeomorphism J we have

$$(\phi J)_p(v) = \phi_*^{-1} J_{\phi(p)}(\phi_* v).$$

Denote the subspace of Γ consisting of integrable complex structures by C; note that the action of $\mathrm{Diff}^+(S^6)$ restricts to C. Restricting to the connected component of the identity $\mathrm{Diff}_{\mathrm{id}}(S^6)$, we have the folllowing conditional statement. It is an immediate consequence of the properties of the Borel construction; each of the three possibilities can hopefully be analyzed further geometrically. Recall that for a group G acting on a space X, the Borel construction $X/\!\!/G$ is the quotient of $X \times EG$ by the diagonal action of G (we refer the reader to [Hsiang, III.1] for properties of the Borel construction).

Proposition 3.1. *Suppose S^6 admits holomorphic charts. Then at least one of the following statements is true:*

(1) The quotient $C/\mathrm{Diff}_{\mathrm{id}}(S^6)$ is connected, has degree-wise finite dimensional rational cohomology, and has an infinite number of nonzero Betti numbers.

(2) Some complex structure on S^6 has an infinite group of holomorphic automorphisms.

(3) The Borel quotients $\Gamma/\!\!/\mathrm{Diff}_{\mathrm{id}}(S^6)$ and $C/\!\!/\mathrm{Diff}_{\mathrm{id}}(S^6)$ do not have the same Betti numbers.

Proof. First, we observe the following:

Lemma 3.2. *The Borel quotient $\Gamma/\!\!/\mathrm{Diff}_{\mathrm{id}}(S^6)$ has degree-wise finite dimensional rational cohomology, and has an infinite number of non-zero Betti numbers.*

Proof. Consider the fibration $\Gamma \to \Gamma/\!\!/\mathrm{Diff}_{\mathrm{id}}(S^6) \to B\,\mathrm{Diff}_{\mathrm{id}}(S^6)$ associated to the Borel construction, and note that the base space is simply connected. As a consequence of a result of Kupers [Kup19, Corollary 5.4(iii)], $B\,\mathrm{Diff}_{\mathrm{id}}(S^6)$ has degree-wise finite dimensional rational cohomology, and since Γ is rationally S^7, is follows from the Serre spectral sequence that $\Gamma/\!\!/\mathrm{Diff}_{\mathrm{id}}(S^6)$ has degree-wise finite dimensional rational cohomology.

The map $BSO(7) \to B\,\mathrm{Diff}_{\mathrm{id}}(S^6)$ induced by the inclusion is an injection on rational homology, since $BSO \to BSTOP$ is a rational equivalence (see [MadMil79, Chapter 10]); indeed, this implies $BSO(7) \to BSTOP$ is an injection on rational homology (since the rational cohomology of $BSO(7)$ is generated by the stable classes p_1, p_2, p_3), and this map factors through $B\,\mathrm{Diff}_{\mathrm{id}}(S^6)$. Again, from the Serre spectral sequence for the fibration, since $H^*(B\,\mathrm{Diff}_{\mathrm{id}}(S^6); \mathbb{Q})$ contains at least two free polynomial generators (namely p_1^2 and p_2), we see that $H^*(\Gamma/\!\!/\mathrm{Diff}_{\mathrm{id}}(S^6); \mathbb{Q})$ contains a free polynomial algebra on at least one generator in degree 8. □

Now suppose statements (2) and (3) are false. Since Γ is connected, it follows that $\Gamma/\!\!/\mathrm{Diff}_{\mathrm{id}}(S^6)$ is connected and so by the assumption on (3), $C/\mathrm{Diff}_{\mathrm{id}}(S^6)$ is connected. let us show that the naive quotient $C/\mathrm{Diff}_{\mathrm{id}}(S^6)$ has an infinite number of non-zero Betti numbers.

Consider the Leray spectral sequence associated to the natural projection $C /\!/ \operatorname{Diff}_{\mathrm{id}}(S^6) \to C/\operatorname{Diff}_{\mathrm{id}}(S^6)$; the stalk over the orbit $[J]$ of a point $J \in C$ in the coefficient sheaf is isomorphic to the rational cohomology of the classifying space of the stabilizer of J in $\operatorname{Diff}_{\mathrm{id}}(S^6)$. By the assumption on (2), each of these is trivial. By our assumption on (3) and the above lemma, $C /\!/ \operatorname{Diff}_{\mathrm{id}}(S^6)$ has infinite dimensional rational cohomology that is degree-wise finite dimensional, and so the spectral sequence tells us that so does $C/\operatorname{Diff}_{\mathrm{id}}(S^6)$ (see e.g. [Bri98, Remark 1.2]).

□

Remark 3.3. Note that the argument shows that we can upgrade (2) in the statement above to say that for some complex structure, the classifying space of the group of holomorphic automorphisms has some non-vanishing rational cohomology in positive degree. For any putative complex structure on S^6, the group of holomorphic automorphisms is a complex Lie group of dimension at most two, and this group cannot have an open orbit on S^6 [HKP00]; if the connected component of the identity of this group is not abelian, then this connected component must be the universal cover of the complex affine group $Aff(\mathbb{C})$ [Bru99]. Note that the validity of (3) would mean there is no $\operatorname{Diff}_{\mathrm{id}}(S^6)$–equivariant deformation retract of Γ onto C. The existence of an equivariant deformation retract and the rational acyclicity of the classifying spaces of all holomorphic automorphism groups would imply that the space $C/\operatorname{Diff}_{\mathrm{id}}(S^6)$, though locally finite–dimensional by Kodaira–Spencer and Kuranishi theory [Kur62], would be globally infinite–dimensional (see [LeB17] for examples of such behavior).

Remark 3.4. Due to Cerf we know that $\operatorname{Diff}^+(S^6)$ has 14 connected components (enumerated by half of the group of homotopy 7–spheres). We can show that the Borel quotient $\Gamma /\!/ \operatorname{Diff}(S^6)$ has infinite-dimensional rational cohomology as well, as follows. Consider the fibration $\Gamma \to \Gamma /\!/ \operatorname{Diff}^+(S^6) \to B\operatorname{Diff}^+(S^6)$. from the Borel construction. Consider the induced map on universal covers (which are finite covers) of the latter two spaces. (Note, our spaces might not admit universal covers a priori. Since all of our desired conclusions will be about homotopy or cohomology, we can replace a considered space by a weakly homotopy equivalent CW-complex when necessary.) Since Γ has finite fundamental group, so does the homotopy fiber F of this map on universal covers, by the four lemma. Note that the universal cover of $B\operatorname{Diff}^+(S^6)$ is $B\operatorname{Diff}_{\mathrm{id}}(S^6)$.

$$\begin{array}{ccccc} F & \longrightarrow & \widetilde{\Gamma /\!/ \operatorname{Diff}^+(S^6)} & \longrightarrow & B\operatorname{Diff}_{\mathrm{id}}(S^6) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \Gamma /\!/ \operatorname{Diff}^+(S^6) & \longrightarrow & B\operatorname{Diff}^+(S^6) \end{array}$$

Since F has the same rational higher homotopy groups as Γ, the rational cohomology of F is that of a seven–sphere (since its universal cover, a finite cover, is rationally a seven–sphere). Now from the spectral sequence for the upper fibration we see that $\widetilde{\Gamma /\!/ \operatorname{Diff}^+(S^6)}$ has infinite-dimensional cohomology; let us argue that there is an infinite dimensional subspace of invariants under the deck transformations, and so $\Gamma/\operatorname{Diff}(S^6)$ will have infinite dimensional cohomology.

First of all, the fundamental group of $\Gamma /\!/ \operatorname{Diff}^+(S^6)$ is either cyclic or a direct sum of two cyclic groups (recall that $\pi_1(\Gamma) = \mathbb{Z}/2$). We look at the degree 12 vector space of the rational cohomology of the universal cover: we can split it into a sum of two vector spaces – the first one consisting of infinite order elements, and the second consisting of finite order elements. Now, by [Kup19, Corollary C], $B\,\mathrm{Diff}(S^6)$ is of finite type, so these two vector spaces are finite dimensional. Taking sufficiently high powers of the first vector space we may thus assume that the action of the fundamental group leaves a polynomial algebra on some number of generators (all of the same even degree) invariant as a set. There is at least one such generator, namely the appropriate power of p_3. Now if the fundamental group is cyclic, upon passing to complex coefficients, this polynomial algebra splits into a sum of invariant polynomial algebras each generated by one element; If the fundamental group is the direct sum of two cyclic groups, then since the actions of each summand commute, each summand induces an action on the invariants of the other, and so again we are in the previous situation. Now we see that there is an infinite-dimensional invariant subspace in each of these singly generated polynomial algebras, since a finite-order automorphism of $\mathbb{C}$ is a root of unity of bounded order, so a sufficiently high power of any element will be invariant.

Notice that, due to [Kup19, Corollary C], we have that the rational cohomology of both $\Gamma /\!\!/ \mathrm{Diff}(S^6)$ and $\Gamma /\!\!/ \mathrm{Diff}_{\mathrm{id}}(S^6)$ are degree–wise finite dimensional.

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