On the rationalization of the $K(n)$-local sphere

Notes by the participants for a reading seminar at MPIM

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0 The isomorphism between the Lubin-Tate tower and the Drinfeld tower (Peter Scholze, 27 May)

We begin with something classical: the *modular curve*

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm},$$

where $\mathbb{H}^{\pm}$ denotes both the upper and lower half planes: that is, $\mathbb{H}^{\pm} := \mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1(\mathbb{R})$. In fact, these are the $\mathbb{C}$-points of an algebraic curve $\mathcal{M}$ over $\mathbb{Q}$. To see this, we identify it as the moduli of elliptic curves:

Theorem 0.1 (Uniformisation Theorem; Riemann)**.** *There is a bijection between:*

1. *Elliptic curves over $\mathbb{C}$,*
2. *Pairs (Λ, W), where Λ is a free $\mathbb{Z}$-module of rank 2 and $W \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ is sub-$\mathbb{C}$-vector space of rank 1 such that $W \cap \overline{W} = 0$.*

Concretely, the equivalence is given as follows: given any elliptic curve E, there is an exponential

$$\exp : \operatorname{Lie} E \to E$$

which exhibits E as the quotient of $\operatorname{Lie} E \cong \mathbb{C}$ by a lattice $\Lambda \subset \mathbb{C}$. Recording this quotient is equivalent to remembering the kernel W of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \twoheadrightarrow \mathbb{C}$; conversely, any pair (Λ, W) defines an elliptic curve $E \coloneqq \mathbb{C}/\Lambda$, provided that W satisfies the condition $W \cap \overline{W} = 0$ that ensures we quotient by a nondegenerate lattice. Fixing the $\mathbb{Z}$-module Λ, this recovers $\mathcal{M}(\mathbb{C})$ as the quotient $\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm}$: the choice of W gives a point in $\mathbb{P}^1_{\mathbb{C}}$, with the condition $W \cap \overline{W}$ cutting out the complement of $\mathbb{P}^1(\mathbb{R})$. In particular, this exhibits $\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^{\pm}$ as an algebraic curve over $\mathbb{Q}$ (in fact, $\mathbb{Z}$). More generally, there is a similar classification for complex tori. Moreover, there is a tower of coverings of $\mathcal{M}(\mathbb{C})$ where we instead take quotients of $\mathbb{H}$ by congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.

Question. Is there a similar 'p-adic uniformization'

$$(\mathcal{M} \otimes \mathbb{C}_p)^{\mathrm{ad}} \cong {}_{\Gamma}\backslash^{\mathbb{H}_p},$$

as the quotient of a 'p-adic symmetric space' $\mathbb{H}_p$ by the action of some arithmetic group Γ?

Answer. In this generality, no. However, there are many related results:

- one variant will give rise to the Lubin-Tate tower for GL_2 (or more generally, GL_h),
- another variant will give rise for the Drinfeld tower.

0.1 The Lubin-Tate tower

Claim. There exists an open subset of $(\mathcal{M} \otimes \mathbb{C}_p)^{\mathrm{ad}}$ that does admit such a p-adic uniformisation.

To see this, let's look first at $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{F}_p$. This has two strata: an open stratum $\mathcal{M}^{\mathrm{ord}}$ of ordinary elliptic curves, and a closed stratum $\mathcal{M}^{\mathrm{ss}}$ of supersingular elliptic curves. Here an elliptic curve E is called *supersingular* if its endomorphism algebra (over $\overline{\mathbb{F}}_p$) is of rank 4; equivalently, the torsion subgroup $E[p^\infty]$ is infinitesimal, or again equivalently, $E[p]$ has no nonzero points over a field.

Remark 0.2. The term 'supersingular' is something of a misnomer, as these are still nonsingular curves. It originates from considering 'singular' values of the j-invariant.

When E is supersingular, its endomorphism algebra is given by $\mathcal{O}_D$, the maximal order in a division algebra $D/\mathbb{Q}$ with

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_v = \begin{cases} M_2(\mathbb{Q}_v), & v \neq p, \infty; \\ \text{nonsplit}, & v = p, \infty. \end{cases}$$

We will give a uniformisation of the rigid-analytic open subset $\mathcal{U} \subset (\mathcal{M} \otimes \mathbb{C}_p)^{\mathrm{ad}}$ of all points having supersingular reduction from $\mathbb{C}_p$ to $\overline{\mathbb{F}}_p$.

Remark 0.3. Being supersingular can be phrased as a vanishing condition over $\overline{\mathbb{F}}_p$; before passing to the residue field this boils down to a certain function on $\mathbb{C}_p$ taking value < 1; in particular an open condition, so $\mathcal{U}$ really is open.

Theorem 0.4. *As rigid-analytic spaces, we have*

$$\mathcal{U} \cong \coprod \mathcal{O}_D^\times \backslash \mathring{\mathbb{D}}_{\mathbb{C}_p},$$

where $\mathring{\mathbb{D}}_{\mathbb{C}_p}$ *denotes the open unit disc.*

Remark 0.5. The coproduct accounts for the (finite) choice of supersingular elliptic curve over $\overline{\mathbb{F}}_p$.

The open unit disc that shows up is the generic fibre of the Lubin-Tate space, and the quotient by $\mathcal{O}_D^\times$ accounts for the automorphisms of a supersingular elliptic curve. As in the complex setting, this moduli space is in fact the zero-th level in a tower of spaces encoding level structures.

To prove the theorem, we will discuss a more canonical definition of Lubin-Tate space. For this, we start with a commutative, one-dimensional formal group $G/\overline{\mathbb{F}}_p$ of height h; in fact, we will pick G so that it is already defined over $\mathbb{F}_p$.

Example 0.6. At height 2, we can take $G = E[p^\infty] = \widehat{E}$, where E is a supersingular elliptic curve.

Warning 0.7. At height $h > 2$, it is no longer the case that we can form G as the completion of an abelian variety.

Serre-Tate theory shows that deforming E (over some p-adic ring) is equivalent to deforming $E[p^\infty]$; the latter is in general much more manageable.

Definition 0.8. The *Lubin-Tate space* $\mathcal{M}_{\mathrm{LT},0}$ is the deformation space of G. In other words, $\mathcal{M}_{\mathrm{LT},0}$ assigns to any Artin local ring R with residue field $\overline{\mathbb{F}}_p$ the set of $\star$-isomorphism classes of deformations of G to R.

Since $\mathcal{M}_{\mathrm{LT},0}$ is smooth and its tangent space is understandable, we obtain the following identification:

Theorem 0.9 (Grothendieck, Illusie)**.** $\mathcal{M}_{\mathrm{LT},0} \cong \operatorname{Spf} W(\overline{\mathbb{F}}_p)[\![u_1, \dots, u_{n-1}]\!]$.

Remark 0.10. The coordinates $u_1, \dots, u_{h-1}$ are not canonical, but u_i is well-defined up to $(p, u_1, \dots, u_{i-1})$.

In particular, we obtain a universal deformation G^{univ} over $W(\overline{\mathbb{F}}_p)[\![u_1, \dots, u_{n-1}]\!]$, satisfying

$$[p]_{G^{\mathrm{univ}}}(X) = X^{p^h} + u_{h-1}X^{p^{h-1}} + \cdots + u_1 X^p + p.$$

The choice of coordinates presents the (geometric) rigid analytic generic fibre $\mathcal{M}_{\mathrm{LT},0,\mathbb{C}_p} \cong \mathring{\mathbb{D}}_{\mathbb{C}_p}$ as the $(h-1)$-dimensional open unit disc. Analogous to the congruence tower in the complex case, we have a tower

$$\mathcal{M}_{\mathrm{LT},m,\mathbb{C}_p} \to \mathcal{M}_{\mathrm{LT},0,\mathbb{C}_p},$$

where $\mathcal{M}_{\mathrm{LT},m,\mathbb{C}_p}$ parameterises deformations (G', ε) together with a trivialisation $\tau\colon G'[p^m] \xrightarrow{\sim} (\mathbb{Z}/p^m\mathbb{Z})^h$ of the p^m-torsion.

Remark 0.11. Number theorists care about this tower because its cohomology is supposed to realise the local Langlands correspondence. A priori it's not so clear how to make sense of $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p} = $ "lim" $\mathcal{M}_{\mathrm{LT},m,\mathbb{C}_p}$, since arbitrary limits don't exist in adic spaces: the issue is that one would like to take a completed colimit of the algebras appearing at finite level, but this completion does not exist canonically. One can nevertheless make sense of its cohomology, as the colimit of the cohomologies at finite level. Weinstein was computing with this gadget and realised that $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ *does* exist as a perfectoid space. More specfically, write $\mathcal{M}_{\mathrm{LT},m} = \mathrm{Spf}\, A_m$, and $\mathfrak{m} \subset A_0$ for the maximal ideal. Weinstein showed that the formal spectrum of $A := (\varinjlim A_m)^{\wedge}_{\mathfrak{m}}$ has the right universal property for maps out of perfectoid spaces [Wei16], [SW13, Prop. 2.4.5]. As a result, these days people usually think of $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ as a *diamond* (more on that in later lectures!).

Returning to local Langlands: note that the group $\mathrm{GL}_h(\mathbb{Z}_p)$ acts on $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$, by the limit of the actions of $\mathrm{GL}_h(\mathbb{Z}/p^m\mathbb{Z}) = \mathrm{Aut}((\mathbb{Z}/p^m\mathbb{Z})^h)$. In fact, one can extend this to an action of the larger group $\mathrm{GL}_n(\mathbb{Q}_p) \times_{\det,\mathbb{Q}_p^\times} \mathbb{Z}_p^\times$. There is also an action of $\mathrm{Aut}(G) = \mathcal{O}_D^\times$, where now $D = D_{1/h}$ is the $\mathbb{Q}_p$-division algebra with $\mathrm{inv}(D) = 1/h$; for example, at height two D is the nonsplit quaternion algebra. Finally there is an action of the Weil group $W_{\mathbb{Q}_p} \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, coming from the fact that G was already defined over $\mathbb{F}_p$. These actions commute, and one has the following:

Theorem (Carayol's Conjecture; Deligne ($h = 2$), Harris-Taylor ($h > 2$))**.** *For $\ell \neq p$, one has an isomorphism*

$$H^*_{\text{ét}}(\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\pi} \pi \otimes \mathrm{JL}(\pi) \otimes \mathrm{LLC}(\pi)$$

as $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$-representations.

(IM) I'm not sure if there is a restriction on which π are allowed in the theorem, or if I have the right adjectives below.

Let us clarify some of the terms in the theorem:

1. JL denotes the *Jacquet-Langlands correspondence*

$$\mathrm{JL} : \left\{ \begin{array}{c} \text{Irreducible discrete series} \\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p) \end{array} \right\} \xrightarrow{\sim} \{\text{Irreducible representations of } D^\times\}.$$

2. LLC denotes the *local Langlands correspondence*

$$\mathrm{LLC} : \left\{ \begin{array}{c} \text{Irreducible admissible} \\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Irreducible } h\text{-dimensional Frobenius} \\ \text{semi-simple representations of } W_{\mathbb{Q}_p} \end{array} \right\}.$$

As such, the space $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ exhibits simultaneously both the Jacquet-Langlands and the local Langlands corresepondence. The case $\ell = p$ is the subject of active research.

0.2 A second variant of p-adic uniformisation

An alternative is to replace the modular curve $\mathcal{M} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm$ by a more general 'Shimura curve'

$$\mathcal{M}' := {}_{\mathcal{O}_D^\times}\backslash^{\mathbb{H}^\pm},$$

where now $D/\mathbb{Q}$ is a quaternion algebra which is *split* at ∞ and nonsplit at some finite place, say p. In particular, $\mathcal{O}_D^\times \hookrightarrow M_2(\mathbb{R})$ gives an action of $\mathcal{O}_D^\times$ on $\mathbb{H}^\pm$.

Theorem (Čerednik)**.** *There is an isomorphism of rigid-analytic spaces*

$$(\mathcal{M}' \otimes_{\mathbb{Q}} \mathbb{C}_p)^{\mathrm{ad}} \cong {}_{\Gamma}\backslash^{(\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p))}$$

for a certain arithmetic group Γ described below.

The space $\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$ appearing in the theorem is *Drinfeld's p-adic upper half-plane.* To describe the group Γ, we denote by D' the $\mathbb{Q}$-division algebra with

$$D' \otimes_{\mathbb{Q}} \mathbb{Q}_v = \begin{cases} M_2(\mathbb{Q}_p), & v = p; \\ \text{nonsplit}, & v = \infty; \\ D \otimes_{\mathbb{Q}} \mathbb{Q}_v, & v \neq p, \infty. \end{cases}$$

Then $\Gamma := \mathcal{O}_{D'}[1/p]^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_p)$, and in particular acts on $\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$. Drinfeld's insight was that $\mathcal{M}'$ also arises from a moduli problem, this time classifying Serre's 'fake elliptic curves': these consist of an abelian surface A, together with an action of $\mathcal{O}_D$ on A.

Remark 0.12. The explanation for the existence of p-adic uniformisation in this case is that in $\mathcal{M}'$, all points have supersingular reduction at p. The moral is that in a general Shimura variety one can only uniformise over the 'basic locus', i.e. supersingular points. Here deformation theory of the abelian variety corresponds to deformation of its p-divisible group, again by Serre-Tate.

The theorem above was first proved by Čerednik using group-theoretic methods. The story is that Manin told Drinfeld to give a talk on this, and he subsequently discovered the following moduli-theoretic interpretation. One fixes $G_0/\overline{\mathbb{F}}_p$ a one-dimensional formal group of height h, so that $\mathrm{End}(G_0) = \mathcal{O}_{D_{1/h}}$. Then $\mathcal{O}_{D_{1/h}}$ also acts on $G := G_0^h$, and the *Drinfeld space at level zero* is defined to be the functor sending a p-power torsion $W(\overline{\mathbb{F}}_p)$-algebra to the set $\mathcal{M}_{\mathrm{Dr},0}(R)$ of triples (H, ρ, ε), where

- H is a formal group of dimension h,
- $\rho\colon \mathcal{O}_{D_{1/h}} \to \mathrm{End}(H)$,
- $\varepsilon\colon H \times_R R/p \sim G \times_{\overline{\mathbb{F}}_p} R/p$ is a quasi-isogeny of height zero.

Remark 0.13. We would like to take for $\mathcal{M}_{\mathrm{Dr},0}$ the deformation space of G (i.e., take ε to be an isomorphism), but the resulting space is not big enough.

Theorem (Drinfeld)**.** *The moduli problem $\mathcal{M}_{\mathrm{Dr},0}$ is representable by a p-adic formal scheme.*

Warning 0.14. The formal scheme that appears in the theorem is quite complicated: for example at height 2, each irreducible component of $\mathcal{M}_{\mathrm{Dr},0,\overline{\mathbb{F}}_p}$ is a $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$, arranged according to the Bruhat-Tits building of $\mathrm{GL}_2(\mathbb{Q}_p)$. A given component $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$ intersects the other components along $\mathbb{P}^1(\mathbb{F}_p)$.

In spite of this, the generic fibre is simple: for general h one has

$$\mathcal{M}_{\mathrm{Dr},0,\mathbb{C}_p} \cong \mathbb{P}^{n-1}_{\mathbb{C}_p} \setminus \bigcup_H \mathbb{P}(H),$$

where the union is over all $\mathbb{Q}_p$-rational hyperplanes. In particular, for $h = 2$ this gives

$$\mathcal{M}_{\mathrm{Dr},0,\mathbb{C}_p} \cong \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p).$$

Once again, this is the zero-th level in a tower of $(\mathcal{O}_{D_{1/h}}/p^m)^\times$-torsors

$$\mathcal{M}_{\mathrm{Dr},m,\mathbb{C}_p} \to \mathcal{M}_{\mathrm{Dr},0,\mathbb{C}_p},$$

where $\mathcal{M}_{\mathrm{Dr},m,\mathbb{C}_p}$ parameterises deformations (H, ρ, ε) together with an isomorphism of $\mathcal{O}_{D_{1/h}}/p^m$-modules

$$H[p^m] \cong \mathcal{O}_{D_{1/h}}/p^m.$$

As in the Lubin-Tate case, we get commuting actions of $\mathcal{O}^\times_{D_{1/h}}$, $(\mathrm{GL}_n(\mathbb{Q}_p) \times_{\det,\mathbb{Q}_p^\times} \mathbb{Z}_p^\times)$ and $W_{\mathbb{Q}_p}$ on the cohomology $H^*(\mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}, \overline{\mathbb{Q}}_p)$ of the limit. These are defined as follows:

- the $\mathcal{O}_{D_{1/n}}^\times$-action is the limit of the actions of $(\mathcal{O}_{D_{1/h}}/p^m)^\times$ on $\mathcal{M}_{\mathrm{Dr},m,\mathbb{C}_p}$,
- the $\mathrm{GL}_h(\mathbb{Q}_p) \times_{\mathbb{Q}_p^\times} \mathbb{Z}_p^\times$-action by coordinate changes on $G \cong G_0^h$,
- the $W_{\mathbb{Q}_p}$-action from the fact that G_0 was already defined over $\mathbb{Q}_p$.

This suggests the question of comparing the cohomologies of the two towers. Indeed we have:

Theorem (Faltings). *There is an isomorphism*

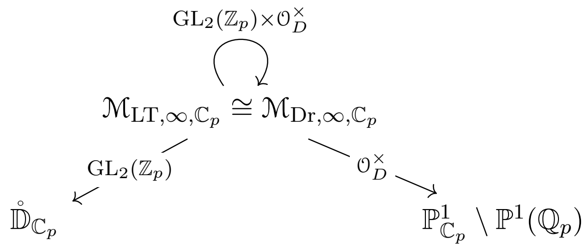
$$\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p} \cong \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p},$$

equivariant for everything in sight.

Remark 0.15.

1. $\mathcal{M}_{\mathrm{LT}}$ and $\mathcal{M}_{\mathrm{Dr}}$ arise from very different deformation problems, so such an isomorphism should be seen as surprising.
2. When Faltings gave his proof of the theorem, the language did not yet exist to talk about $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ and $\mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}$ as geometric objects, so even making good sense of the statement of the theorem was hard work.

Example 0.16. At height 2, we get a diagram



The bottom arrows are proétale torsors for the displayed group, and equivariant for the remaining action on each side.

In the rest of the lecture, we'll try to explain the proof of the isomorphism.

0.3 Comparing the two towers

In both cases we have constructed towers whose $\mathbb{C}_p$-points classify data of the following form:

- a p-divisible group $H/\mathcal{O}_{\mathbb{C}_p}$, possibly with an action;
- an isomorphism $H(\mathbb{C}_p) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h$, possibly equivariant for some action;
- an isogeny $H_{\mathcal{O}_{\mathbb{C}_p}/p} \sim G$, again maybe equivariant.

The key problem is therefore to understand the classification of p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}$ and over $\overline{\mathbb{F}}_p$. The crucial and beautiful fact about p-divisible groups is that they can always be understood in terms of linear algebra, like the Riemann classification over $\mathbb{C}$. This might lead us to some fancy linear algebra, but at the end of the day it's just linear algebra. In close analogy to Riemann's classification, we have the following theorem in the p-adic setting:

Theorem ([SW13], Theorem B). *There is an equivalence between the following categories:*

1. *p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}$,*
2. *Pairs (T, W), where T is a finite free $\mathbb{Z}_p$-module and $W \subset T \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ is a sub-$\mathbb{C}_p$-vector space.*

The correspondence sends a p-divisible group G to the Hodge-Tate filtration *on T_pG.*

On the other hand, the classification over the residue field is classical:

Theorem (Dieudonné)**.** *There is an equivalence between the following categories:*

1. *p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}/p$,*
2. *Triples (M, F, V), where M is a finite free $W(\overline{\mathbb{F}}_p)$-module, $F\colon M \to M$ is* Frob*-linear and $V\colon M \to M$ is* Frob^{-1}*-linear.*

This begs the following question:

Question. Given a pair (T, W) corresponding to a p-divisible group $H_{T,W}/\mathcal{O}_{\mathbb{C}_p}$, what is the Dieudonné module of $H_{T,W} \times_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{\mathbb{C}_p}/p$? In other words, we want a concrete description of the arrow marked '?' below:

$$\begin{array}{ccc} \{p\text{-divisible groups}/\mathcal{O}_{\mathbb{C}_p}\} & \xrightarrow{\sim} & \{(T,W)\} \\ {\scriptstyle /p}\downarrow & & \downarrow{\scriptstyle ?} \\ \{p\text{-divisible groups}/\overline{\mathbb{F}}_p\} & \xrightarrow[\sim]{} & \{(M,F,V)\} \end{array}$$

This requires a suitable formulation of p-adic Hodge theory: T is related to étale cohomology of the generic fibre of H, and M to crystalline cohomology of the special fibre. Note however that we need a form of p-adic Hodge theory over $\mathbb{C}_p$: for example, the infinite level space $\mathcal{M}_{\mathrm{LT},\infty}$ has no points over discretely valued fields (since it's perfectoid).

Constructing such a theory was one of the original motivations for the *Fargues-Fontaine curve*. This is a particular scheme $X_{\mathbb{C}_p}$, which is locally the spectrum of a PID. Its construction and properties will occupy a large part of the first half of the seminar. There is a point

$$\infty = \mathrm{Spec}(\mathbb{C}_p) \hookrightarrow X_{\mathbb{C}_p},$$

and indeed the residue fields at *all* closed points are complete algebraically closed fields—i.e., big. Indeed in one incarnation, $X_{\mathbb{C}_p}$ classifies untilts of $\mathbb{C}_p^\flat$: that is, pairs $(C, C^\flat \cong \mathbb{C}_p^\flat)$. By construction, vector bundles on $X_{\mathbb{C}_p}$ are closely related to isocrystals, i.e. rational Dieudonné modules. The following gives us a way to attack the question above:

Theorem ([SW13], Theorems A and C)**.** *The Dieudonné module functor*

$$\{p\text{-}divisible\ groups\ over\ \mathcal{O}_{\mathbb{C}_p}/p\ up\ to\ isogeny\} \to \mathrm{VB}(X_{\mathbb{C}_p}),$$

is fully faithful, and its essential image consists of those vector bundles with slopes in $[0, 1]$.

Given this, we arrive at a description of '?'. Suppose given a pair (T, W), and form the following cartesian diagram:

$$\begin{array}{ccc} E(T,W) & \longrightarrow & (i_\infty)_* W \\ \big\downarrow & \lrcorner & \big\downarrow \\ T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathbb{C}_p}} & \longrightarrow & (i_\infty)_*(T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathbb{C}_p}}) \end{array} \tag{0.17}$$

Note that $E(T, W)$ is a submodule of $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathbb{C}_p}}$, and so torsion-free, and hence defines a vector bundle on $X_{\mathbb{C}_p}$.

Theorem (Scholze-Weinstein)**.** *If $G/\mathcal{O}_{\mathbb{C}_p}$ corresponds to the pair (T, W), then $G \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{\mathbb{C}_p}/p$ corresponds to $E(T, W)$.*

In fact, one can further describe isogenies between p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}/p$ in terms of vector bundles on $X_{\mathbb{C}_p}$.

Corollary 0.18. *1. $\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}$ classifies the following equivalent data:*

a) Tuples (T, W, α) , where $\alpha\colon T \cong \mathbb{Z}_p^h$ and $E(T, W) \cong \mathcal{O}_{X_{\mathbb{C}_p}}(-1/h)$,

b) Inclusions $\mathcal{O}_{X_{\mathbb{C}_p}}(-1/h) \hookrightarrow \mathcal{O}^h_{X_{\mathbb{C}_p}}$ with cokernel supported at ∞.

2. $\mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}$ classifies D-linear inclusions $\mathcal{O}_{X_{\mathbb{C}_p}}(-1/h)^h \hookrightarrow D \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathbb{C}_p}}$ with cokernel supported at ∞

Combining the above gives the following interpretation of Falting's theorem:

Theorem 0.19 ([SW13], Theorem E)**.** *There is a natural equivariant equivalence of adic spaces*

$$\mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p} \simeq \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}.$$

1 Power Operations on Lubin-Tate theory. (Nikolay Konovalov, 3 June)

1.1 The Morava Stabilizer group

Let us fix a prime p (once and for all) and a one-dimensional formal group Γ_h of height h over $\overline{\mathbb{F}}_p$.

Remark 1.1. Note that we are working over an algebrically closed field so that the choice of such a formal group of height h is unique up to isomorphism.

Definition 1.2. Define $\mathbb{G}_h$ to be the group $\mathrm{Aut}(\Gamma_h, \overline{\mathbb{F}}_p)$ consisting of pairs (f, g) where $f : \Gamma_h \xrightarrow{\sim} \Gamma_h$ is an automorphism of Γ_h that covers an automorphism $g\colon \overline{\mathbb{F}}_p \xrightarrow{\sim} \overline{\mathbb{F}}_p$. This profinite group is called the Morava stabilizer group.

Remark 1.3. Per construction, we have a short exact sequence

$$1 \to \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h) \to \mathbb{G}_h \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1 \tag{1.4}$$

where the latter arrow sends a pair (f, g) to the automorphism g. It is easy to see that this map admits a section $g \mapsto (\mathrm{id}, g)$ so that this short exact sequence splits.

In fact, one can identify the outside terms in this split short exact sequence as follows.

- The Galois group is given by $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$, the profinite completion of the integers,
- The automorphism group is given by $\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h) \cong \mathcal{O}_D^\times$, the the units in the ring of integers of a certain division algebra over $\mathbb{Q}_p$.

Note that central simple algebras over $\mathbb{Q}_p$ are classified by the Brauer group $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$, and the division algebra D appearing above corresponds to the class $1/h$ in this Brauer group, and is of dimension h^2 over $\mathbb{Q}_p$. Taking the semidirect product of the factors above, we conclude that

$$\mathbb{G}_h \cong \widehat{D^\times},$$

i.e. the Morava stabiliser group is the profinite completion of the group of units in D.

Lemma 1.5 (cf. Remark 2.2.5 in [Mor85])**.** *If we let $\mathbb{G}_h$ act on the Witt vectors $W(\overline{\mathbb{F}}_p)$ by the Galois action, there is an identification of the rationalised continuous group cohomology as*

$$H^*_{\text{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^*_{\text{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, \dots, x_h).$$

The right hand side is an exterior algebra on h generators where x_i is in degree $2i-1$.

Proof. The first identification is immediate, as $\mathbb{G}_h$ is compact. Indeed, its semidirect factors are compact; this is immediate for $\widehat{\mathbb{Z}}$, and for $\mathcal{O}_D^\times$ we note that it morally should be like $\mathrm{GL}_h(\mathbb{Z}_p)$ as the group of units in the ring of integers in a division algebra of dimension h^2. Therefore, we can bring the filtered colimit computing rationalisation inside (continuous) group cohomology.

The second identification follows from the Lyndon–Hochschild–Serre spectral sequence associated to the short exact sequence (1.4). Since we are working with rational coefficients, this collapses and takes the form of an isomorphism

$$H^*_{\text{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong H^*_{\text{cont}}(\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h); \mathbb{Q}_p).$$

Now note that the automorphism group on the right hand side is isomorphic to $\mathcal{O}_D^\times$ hence is a p-adic analytic Lie group. A theorem of Lazard then sates that its rational continuous group cohomology can be computed as Lie algebra cohomology as follows.

$$H^*_{\text{cont}}(\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_h); \mathbb{Q}_p) \cong \mathrm{H}^*_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{O}_D^\times); \mathbb{Q}_p).$$

Now the Lie algebra of the units in the ring of integers of D is just D itself with the commutator bracket, we can further identify this as

$$\mathrm{H}^*_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{O}_D^\times); \mathbb{Q}_p) \cong \mathrm{H}^*_{\mathrm{Lie}}(D; \mathbb{Q}_p).$$

Finally, for K sufficiently large over $\mathbb{Q}_p$, Morita theory tells us that we can identify the Lie algebra $D \otimes_{\mathbb{Q}_p} K$ with $\mathfrak{gl}_h(K)$. The Lie algebra cohomology of the latter is precisely the exterior algebra in the statement of the Lemma. □

Remark 1.6. From this computation, we see that $\mathbb{G}_h$ has finite virtual cohomological dimmension, in fact

$$\mathrm{vcd}_{\mathbb{Q}_p}(\mathbb{G}_h) = h^2 + 1.$$

This is also the strict cohomological dimension if $p-1$ does not divide h.

1.2 The Lubin–Tate ring

Definition 1.7. A deformation of a formal group Γ/k is the datum of

- a complete local ring with residue field k, i.e. $(R, \mathfrak{m}_R, R/\mathfrak{m}_R \cong k)$,
- a formal group Γ_R over R, and
- an isomorphism $\Gamma_R \otimes_R k \cong \Gamma$.

Theorem 1.8 (Lubin–Tate)**.** *Consider the functor*

$$\mathrm{Def}_\Gamma \colon \mathrm{CRing}^{\mathrm{cpl,cts}} \to \mathrm{Set}$$

that sends a complete local ring with residue field k to the set of deformations of Γ to R. Then this functor is corepresented by a complete local ring $A(\Gamma, k)$ called the Lubin–Tate ring. In fact, there is a (non-canonical) presentation

$$A(\Gamma, k) \cong W(k)[\![u_1, \dots, u_{h-1}]\!].$$

In the case where $(\Gamma, k) = (\Gamma_h, \overline{\mathbb{F}}_p)$ as above, we will simply write A for the associated Lubin–Tate ring.

Remark 1.9. Several observations can be made immediately from the universal property.

- The group $\mathbb{G}_h$ acts continuously on A.
- This action is horrendous. In particular, the augmentation map $A \to W(\overline{\mathbb{F}}_p)$ is not $\mathbb{G}_h$-equivariant for $h \geq 2$.
- However, the inclusion of constant terms $W(\overline{\mathbb{F}}_p) \to A$ is $\mathbb{G}_h$-equivariant.

We can now state the main theorem of the seminar.

Theorem 1.10. *Let $s \geq 0$. Then the inclusion map $W(\overline{\mathbb{F}}_p) \to A$ as above induces a split injection*

$$\phi\colon H^s_{\text{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \to H^s_{\text{cont}}(\mathbb{G}_h; A).$$

Furthermore, the complement of this split injection is killed by some power of p depending on h and s. In particular, the induced map

$$H^s_{\text{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^s_{\text{cont}}(\mathbb{G}_h; A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

Conjecture 1.11. It is conjectured that the final isomorphism above already holds before rationalisation.

Remark 1.12. The first part of the theorem, namely the fact that ϕ is a split injection, requires nontrivial topological methods including higher ambidexterity to provide transfers along surjective group homomorphisms.

1.3 Morava K-theories

This section largely follows Section 2 of [BB20b].

Definition 1.13. A unital associative ring spectrum A is said to be a division algebra if every A-module M splits as a direct sum of copies of A:

$$M \simeq \bigoplus_i \Sigma^{n_i} A.$$

Definition 1.14. Two division algebras A, B are of the same chromatic characteristic if and only if $A \otimes_{\mathbb{S}} B \neq 0$.

Remark 1.15. We can make the same definition in $\mathrm{Mod}_{\mathbb{Z}}$; in this case we see that two division algebras A, B in $\mathrm{Mod}_{\mathbb{Z}}$ are of the same (chromatic) characteristic if for any prime p we have that $p \cdot 1_A = 0$ if and only if $p \cdot 1_B = 0$. This does not mean that we have classified *all* division algebras in $\mathrm{Mod}_{\mathbb{Z}}$, but it is easy to find minimal representatives, i.e. the finite field $\mathbb{F}_p$ for any prime p and the rational numbers $\mathbb{Q}$.

Proposition 1.16 (Morava)**.** *Let k be a perfect field of characteristic p. Let Γ be a formal group of dimension one and height h (possibly infinite) over k. Then there exists a multiplicative cohomology theory $K(\Gamma, k)^*$ such that*

1. *The coefficients are given by*

$$K(\Gamma,k)^*(\mathrm{pt}) \cong \begin{cases} \mathbb{Q}, & h=0 \\ k[v_h^{\pm}], & 0<h<\infty \\ k, & h=\infty \end{cases}$$

where v_h is in degree $2p^h-2$.

2. *$K(\Gamma,k)^*$ is complex orientable, and we can identify*

$$\mathrm{Spf}(K(\Gamma,k)^0(\mathbb{CP}^\infty)) \cong \Gamma.$$

Note that the ring on the left hand side is given by $k[\![x]\!]$ with the Hopf algebra structure arising from the Pontryagin comultiplication.

3. *$K(\Gamma,k)^*$ satisfies a Künneth formula.*

Remark 1.17. By Brown representability, we see that $K(\Gamma,k)^*$ is represented by a unital associative ring spectrum $K(\Gamma,k)$.

Some basic examples are as follows:

- $K(\widehat{\mathbb{G}}_a,\mathbb{Q})$ and $K(\widehat{\mathbb{G}}_a,\mathbb{F}_p)$ are $\mathrm{H}\mathbb{Q}$ and $\mathrm{H}\mathbb{F}_p$ respectively.
- $K(\widehat{\mathbb{G}}_m;\mathbb{F}_p)$ is a summand of mod p complex K-theory.
- The standard Morava K-theories arise as $K(h,p) \coloneqq K(\Gamma_h,\mathbb{F}_p)$.

Theorem 1.18 (Devinatz–Hopkins–Smith)**.** *The Morava K-theories form a complete and pairwise distinct set of representatives for the chromatic characteristics of division algebras in spectra. Moreover, any division algebra is a module over some $K(h,p)$.*

Remark 1.19. For $0<h<\infty$, the Morava K-theories $K(h,p)$ do not admit the structure of $\mathbb{E}_2$-algebras in spectra.

Remark 1.20. The additional height variable in the minimal division algebras encodes specialisation: If X is a finite spectrum , then $K(h,p)^*X=0$ implies $K(h+1,p)^*X=0$. Therefore $K(h+1,p)$ can be thought of as a specialisation of $K(h,p)$.

1.4 Chromatic fracture

Let $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$ denote the full subcategory of p-local spectra. Since we now fix a prime p, we write $K(h) \coloneqq K(h,p)$. Recall that X is said to be $K(h)$-local if for all spectra Y such that $K(h)^*Y=0$, we have $[Y,X]=0$. The $K(h)$-local spectra span a full subcategory $\mathrm{Sp}_{K(h)} \subset \mathrm{Sp}_{(p)}$.

Theorem 1.21 (Bousfield)**.** *The inclusion $\mathrm{Sp}_{K(h)} \subset \mathrm{Sp}_{(p)}$ admits a left adjoint denoted $L_{K(h)}$, given by the Bousfield localisation with respect to $K(h)$.*

Similarly, define a functor

$$L_h\colon \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$$

as the Bousfield localisation functor with respect to the spectrum $K(0)\oplus\cdots\oplus K(h)$.

Theorem 1.22 (Chromatic convergence, Hopkins–Ravenel)**.** *If $X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ is a p-local finite spectrum, then the map*

$$X \to \varprojlim(\cdots \to L_h X \to L_{h-1} X \to \cdots \to L_0 X)$$

is an equivalence

Theorem 1.23 (Smash product theorem, Hopkins–Ravenel)**.** *The functor $L_h \colon \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$ commutes with colimits, hence is given by $L_h = L_h \mathbb{S} \otimes -$ (everything p-local).*

Remark 1.24. This is to be contrasted with the fact that $L_{K(h)}$ does not commute with colimits.

The smash product theorem gives us a way to reconstruct $L_h X$ from $L_{h-1} X$ and $L_{K(h)} X$.

Corollary 1.25. *For every p-local spectrum X, there exists a pullback square of the form*

$$\begin{array}{ccc} L_h X & \longrightarrow & L_{K(h)} X \\ \downarrow & \lrcorner & \downarrow \\ L_{h-1} X & \longrightarrow & L_{h-1} L_{K(h)} X. \end{array}$$

Remark 1.26. This is to be compared with the fracture squares in quasicoherent sheaves obtained from an open-closed decomposition: L_{h-1} is restriction to the open part, while $L_{K(h)}$ is formal completion along the closed part, and $L_{h-1} L_{K(h)}$ is the glueing datum.

Conjecture 1.27 (Weak chromatic splitting conjecture, Hopkins, Hovey)**.** For $X = \mathbb{S}$, or more generally any finite spectrum, the bottom horizontal map

$$L_{h-1} X \to L_{h-1} L_{K(h)} X$$

splits.

Remark 1.28. There is a stronger splitting conjecture, which will be mentioned later. This gives an explicition prediction of what $L_{h-1} L_{K(h)} \mathbb{S}$ should look like, hence allows us to obtain a better description of L_h.

1.5 Morava E-theories

Definition 1.29. Let FG_h be the (1-)category of pairs (Γ_h, k) of a perfect field k of characteristic $p \neq 0$ and a formal group Γ_h over k of height h. The morphisms are given by pairs

$$(\alpha, \beta) \colon (\Gamma_h, k) \to (\Gamma'_h, k')$$

of a ring map $\beta \colon k \to k'$ and an isomorphism $\alpha \colon \Gamma'_h \xrightarrow{\sim} \Gamma_h \otimes_k k'$.

Theorem 1.30 (Goerss–Hopkins–Miller, Lurie)**.** *There exists a functor*

$$E \colon \mathrm{FG}_h \to \mathrm{CAlg}(\mathrm{Sp}_{K(h)})$$

such that the following hold.

1. *The homotopy groups are given by $\pi_* E(\Gamma, k) \cong A(\Gamma, k)[\beta^{\pm}]$, a Laurent series over the Lubin–Tate ring of (Γ, k) in a variable β of degree two.*

2. *Since this ring spectrum is even, it carries a formal group*

$$\mathrm{Spf}(E(\Gamma,k)^0(\mathbb{CP}^\infty))$$

over $E(\Gamma,k)^0 \cong A(\Gamma,k)$*, which can be identified with the universal deformation of* Γ.

Remark 1.31. This should be thought of as a spectral version of the Lubin–Tate ring.

1. By functoriality, $\mathbb{G}_h$ acts on $E_h \coloneqq E(\Gamma_h, \overline{\mathbb{F}}_p)$ by $\mathbb{E}_\infty$ ring maps.
2. Let $I = (p, u_1, \dots, u_{h-1}) \subset A = A(\Gamma_h, \overline{\mathbb{F}}_p)$ be the augmentation ideal over $W(\overline{\mathbb{F}}_p)$. Then there is a decomposition
$$E_h/I \simeq \bigoplus_{0 \le i \le p^h - 2} \Sigma^{2i} K(h,p).$$
3. The Bousfield localisation functors L_{E_h} and $L_h = L_{K(0)\oplus\cdots\oplus K(h)}$ are equivalent.
4. This construction has further universal properties which we will not discuss.

Theorem 1.32 (Devinatz–Hopkins)**.** *Let us work in* $\mathrm{Sp}_{K(h)}$ *and let* $\widehat{\otimes}$ *denote the localised tensor product on* $\mathrm{Sp}_{K(h)}$*, i.e.* $X \widehat{\otimes} Y = L_{K(h)}(X \otimes Y)$*.*

1. *We can identify* $E_h \widehat{\otimes} E_h \simeq C_{\mathrm{cts}}(\mathbb{G}_h, E_h)$*, where the right hand side denotes continuous cochains on* $\mathbb{G}_h$*, i.e.*
$$C_{\mathrm{cts}}(\mathbb{G}_h, E_h) = \varinjlim_{\alpha} C(\mathbb{G}_h/H_\alpha, E_h),$$
where H_α *ranges over all finite index subgroups of* $\mathbb{G}_h$*.*
2. *Let* $E_h^{\widehat{\otimes}\bullet+1}$ *denote the Amitsur complex of* E_h*, i.e. the cosimplicial object with coface and codegeneracy maps given by multiplications and units. Then the map*
$$L_{K(h)}\mathbb{S} \to \mathrm{Tot}(E_h^{\widehat{\otimes}\bullet+1})$$
is an equivalence.

Corollary 1.33. *The Bousfield–Kan spectral sequence of the Amitsur complex above therefore takes the form*

$$E_2^{s,t} \cong H^s_{\mathrm{cont}}(\mathbb{G}_h; \pi_t E_h) \implies \pi_{t-s} L_{K(h)}\mathbb{S}.$$

When $h = 1$, this spectral sequence recovers Adams' computations of the image of the J-homomorphism. In general for any (finite) h, there is a horizontal vanishing line on some page. In particular, let us note that $A = \pi_0 E_h$ so that the cohomology groups from the main theorem of the talk appear as $E_2^{*,0}$. Consider the span

$$\begin{array}{ccccc}
H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) & \xrightarrow{\quad\phi\quad} & & & H^*_{\mathrm{cont}}(\mathbb{G}_h; \pi_0 E_h) = E_2^{*,0}, \\
\downarrow & & \widetilde{x}_i & \longmapsto & \phi(\widetilde{x}_i) \\
 & & \uparrow & & \\
H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & & x_i & &
\end{array}$$

where x_i denotes the i-th generator in $H^*_{\mathrm{cont}}(\mathbb{G}_h; W(\overline{\mathbb{F}}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \Lambda_{\mathbb{Q}_p}(x_1, \dots, x_h)$, and $\widetilde{x}_i$ is any lift of x_i.

Conjecture 1.34 (Strong splitting conjecture)**.** Let p be odd and fix an integer $1 \le i \le h$.

1. $\phi(\widetilde{x}_i) \in E_2^{*,0}$ persists to a nontrivial class

$$e_i \in \pi_{1-2i} L_{K(h)}\mathbb{S}.$$

2. The composite

$$\mathbb{S}^{1-2i} \xrightarrow{e_i} L_{K(h)}\mathbb{S} \to L_{h-1}L_{K(h)}\mathbb{S}$$

factors through $\mathbb{S}^{1-2i} \to L_{h-i}\mathbb{S}^{1-2i}$. The resulting map is denoted

$$\overline{e}_i \colon L_{h-i}\mathbb{S}^{1-2i} \to L_{h-1}L_{K(h)}\mathbb{S}.$$

3. The maps $\overline{e}_i$ induce an equivalence

$$\bigwedge_{i=1}^{n}(L_{h-i}\mathbb{S}^{1-2i}) \xrightarrow{\sim} L_{h-1}L_{K(h)}\mathbb{S}.$$

The left hand side in the final statement is an exterior algebra indexed on the $\mathbb{Z}_p$-module generators of $\Lambda_{\mathbb{Z}_p}(\overline{e}_1, \dots, \overline{e}_h)$, defined as

$$\bigwedge_{i=1}^{n}(L_{h-i}\mathbb{S}^{1-2i}) = \bigoplus_{0\le j\le h}\ \bigoplus_{1\le i_1<\cdots<i_j\le h}\ \bigotimes_{k=1}^{j} L_{h-i_k}\mathbb{S}^{1-2i_k}.$$

Remark 1.35. The strong chromatic splitting conjecture is known to be true for $h \le 2$ and $p \ge 3$.

- If $p = 2$, the statement given above is not true, but slight modifications are to be made.
- If the strong conjecture is true, the map

$$L_{h-1}\mathbb{S} \to L_{h-1}L_{K(h)}\mathbb{S}$$

from the chromatic fracture square is the unit of this exterior algebra hence splits (which implies the weak chromatic splitting conjecture).
- Upon rationalisation, the strong chromatic splitting conjecture implies that

$$(\pi_* L_{K(h)}\mathbb{S}) \otimes \mathbb{Q} \cong \Lambda_{\mathbb{Q}_p}(\overline{e}_1, \dots, \overline{e}_h),$$

where we recall that $\overline{e}_i$ is of degree $1 - 2i$.

1.6 Power operations and splitting

Recall that we want to obtain a splitting in continuous cohomology of the map induced by the inclusion $W(\overline{\mathbb{F}}_p) \to A = \pi_0 E_h$. We denote the latter by E^0 for brevity, the height being implicit.

Definition 1.36. For $m \ge 0$, we define a multiplicative (but not additive!) map

$$P^m \colon E \to E^0(\mathrm{B}\Sigma_m)$$

by

$$\begin{aligned} E^0 =& [\mathbb{S}, E], \\ &\to [(\mathbb{S})^{\otimes m}_{h\Sigma_m}, E^{\otimes m}_{h\Sigma_m}], \\ &\to [\mathbb{S}_{h\Sigma_m}, E] \\ &\cong E^0(\mathrm{B}\Sigma_m), \end{aligned}$$

where we used the multiplication map on E.

Remark 1.37. One easily checks that P^0 is constant at 1, and that P^1 is the identity.

1. P^m is $\mathbb{G}_h$-equivariant for all $m \geq 0$,
2. $E^0(\mathrm{B}\Sigma_m)$ is a free E^0-module of finite rank.
3. $E^0(\mathrm{B}\Sigma_m)$ can be equipped with a unique linear topology compatible with the structure of an E^0-module.

Lemma 1.38. *For $m \geq 0$, the map*

$$P^m \colon E^0 \to E^0(\mathrm{B}\Sigma_m)$$

is continuous with respect to the topology generated by the ideal $I = (p, u_1, \ldots, u_{h-1})$.

Proof. This is a surprisingly difficult lemma. Let $I_{\mathrm{tr}} \subset E^0(\mathrm{B}\Sigma_m)$ denote the transfer ideal, i.e. the ideal generated by the images of the transfers along inclusions of the form $\Sigma_j \times \Sigma_i \subset \Sigma_m$. Then the composite map

$$\overline{P}^m \colon E^0 \to E^0(\mathrm{B}\Sigma_m) \to E^0(\mathrm{B}\Sigma_m)/I_{\mathrm{tr}}$$

is a ring map (i.e. it is furthermore additive) and continuous, with target a finitely generated free E^0-module. We then use Hopkins–Kuhn–Ravenel character theory to embed

$$E^0(\mathrm{B}\Sigma_{p^k}) \to \prod_{i=0}^{k} E^0(\mathrm{B}\Sigma_{p^i})/I_{\mathrm{tr}}. \qquad \square$$

Remark 1.39. As we will see below, the paper uses a $K(h)$-local transfer along the surjective group homomorphism $\Sigma_m \to e$ to obtain a map

$$\mathrm{Tr}^e_{\Sigma_m} \colon E^0(\mathrm{B}\Sigma_m) \to E^0.$$

They don't particularly elaborate on this, so Nikolay gave us a sketch of how to construct this: Let $f' \colon G \to H$ be any morphism of finite groups, inducing a map of anima

$$f \colon \mathrm{B}G \to \mathrm{B}H$$

such that the fibre of f can be identified with a coproduct of anima $\mathrm{B}K_i$ for finite groups K_i. Furthermore, f induces an adjunction

$$\mathrm{LocSys}(\mathrm{B}H; \mathrm{Sp}_{K(h)}) \underset{\overset{\longleftarrow f_* \longrightarrow}{}}{\overset{\overset{\longleftarrow f_! \longrightarrow}{\longrightarrow f^* \longrightarrow}}{}} \mathrm{LocSys}(\mathrm{B}G; \mathrm{Sp}_{K(h)}).$$

Due to the higher semiadditivity of $\mathrm{Sp}_{K(h)}$ we see that there is an equivalence $f_! \simeq f_*$, which allows us to produce a transfer map

$$\mathrm{Tr}_f \colon L_{K(h)}\Sigma^\infty_+ \mathrm{B}H \to L_{K(h)}\Sigma^\infty_+ \mathrm{B}G.$$

If we hadn't taken values in $K(h)$-local spectra, this transfer would only exist for injective group homomorphisms.

Proposition 1.40. *There exists a $\mathbb{G}_h$-equivariant continuous map*

$$E^0 \to W(\overline{\mathbb{F}}_p)$$

splitting the inclusion $W(\overline{\mathbb{F}}_p) \to E^0$.

Proof. Set

$$\beta_m\colon E^0 \xrightarrow{P^m} E^0(\mathrm{B}\Sigma_m) \xrightarrow{\mathrm{Tr}^e_{\Sigma_m}} E^0.$$

For every m, these assemble into a map

$$\beta\colon E^0 \to E^0[\![x]\!], \beta(a) = \sum_m \beta_m(a)x^m.$$

Now remark that by the lemma above, β is continuous and $\mathbb{G}_h$-equivariant. Further, since P^0 is constant at the unit (and the transfer map is multiplicative), we see that

$$\beta(a) \in 1 + xE^0[\![x]\!].$$

Since the power operations define an additive map modulo the transfer ideal, we see that $\beta(a+b) = \beta(a)\beta(b)$. This tells us that β defines an additive map

$$\gamma\colon E^0 \xrightarrow{\beta} 1 + xE^0[\![x]\!] \to 1 + x\overline{\mathbb{F}}_p[\![x]\!] \cong W_{\mathrm{big}}(\overline{\mathbb{F}}_p) \to W(\overline{\mathbb{F}}_p).$$

The second map is the quotient by the maximal ideal I in E^0. Note that the last map sits in a further composite

$$1 + x\overline{\mathbb{F}}_p[\![x]\!] \cong W_{\mathrm{big}}(\overline{\mathbb{F}}_p) \to W(\overline{\mathbb{F}}_p) \to \overline{\mathbb{F}}_p$$

with the reduction mod p map from the Witt vectors. This map simply sends a power series to its coefficient at x. Since P^1 acts by the identity, we see that the composite

$$\overline{\mathbb{F}}_p \hookrightarrow E^0 \xrightarrow{\gamma} W(\overline{\mathbb{F}}_p) \to \overline{\mathbb{F}}_p$$

is just the identity. Let f denote the composite $W(\overline{\mathbb{F}}_p) \hookrightarrow E^0 \xrightarrow{\beta} W(\overline{\mathbb{F}}_p)$. Then our observation that $\gamma(x)$ reduces to x modulo p tells us that f is a homomorphism of p-adically complete abelian groups which reduces to the identity modulo p hence must be an isomorphism. Then define $\alpha := f^{-1} \circ \gamma$ as a map $E^0 \to W(\overline{\mathbb{F}}_p)$. Per construction this is a $\mathbb{G}_h$-equivariant, additive, continuous section of the inclusion. □

2 Adic Spaces (Ningchuan Zhang, 17 June)

The main references for this talk are section 1 of J. Weinstein's lectures at the 2017 AWS ([Wei17]) and sections 2 and 3 of Scholze–Weinstein ([SW13]).

2.1 Motivation

Last time, we proved that we have a split injection of the form

$$H^*_{\mathrm{cont}}(\mathbb{G}_h; W) \longleftrightarrow H^*_{\mathrm{cont}}(\mathbb{G}_h; A),$$

where $W = W(\overline{\mathbb{F}}_p)$ and $A = \pi_0 E(\Gamma_h, \overline{\mathbb{F}}_p)$ is the Lubin–Tate ring. The goal of introducing adic spaces is to be able to describe why this is a rational isomorphism.

Remark 2.1. Recall that a stronger result is conjectured. Namely the Vanishing conjecture (Conjecture 1.11) states that the split injection above is an isomorphism before rationalisation.

Note that the right hand side of this split injection can be described conceptually as the sheaf cohomology of the formal stack

$$\mathrm{Spf}(A) /\!\!/ \mathbb{G}_h.$$

However, the rational continuous cohomology of $\mathbb{G}_h$ does not admit such a description in terms of sheaf cohomology over (a quotient of) a formal scheme. Indeed formal schemes are set up such that the global sections of their structure sheaf is always complete for the topology on the underlying ring, but it is clear that $\mathbb{Q}_p$ is not p-complete. Adic spaces therefore give us a way of simultaneously incorporating the generic fibre as geometric information. To describe the right hand side, we will explicitly need to understand the geometry of the adic space $\mathrm{LT}^{\mathrm{ad}}_{W\otimes_{\mathbb{Z}_p}\mathbb{Q}_p}$ obtained as the generic fibre of the Lubin–Tate space.

Remark 2.2. Adic spaces form a general framework that should encompass classical schemes, formal schemes, and rigid analytic spaces. In particular, it should respect the inclusion of schemes into formal schemes an the construction of a rigid analytic space from a formal scheme.

Let us begin with a philosophical question: what is a space? The answer we follow in the construction of adic spaces is that it is a topological space (i.e. set of points with a topology) with a structure sheaf of rings. In some cases, we will not actually have a topological space, but rather a site.

Example 2.3.

- Smooth manifolds are such an example, which are locally isomorphic to an open subset of $U \subset \mathbb{R}^n$, with structure sheaf given by $U \mapsto C^\infty(U)$.
- Schemes are an example as well, being locally isomorphic to $\mathrm{Spec}(A)$ for a ring A and with structure sheaf $\mathcal{O}_A$ as usual.
- An adic space will be a topologically ringed space with valuations on the stalks, which is locally of the form $\mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+).

2.2 Formal schemes

Remark 2.4. All ideals of definition in a topological ring will be assumed to be finitely generated throughout.

Definition 2.5. A (linearly) topological ring A is called *adic* if there exists an open ideal $I \subset A$, called an *ideal of definition*, such that the open subsets $\{I^n\}_{n\geq 0}$ form a basis of open neighbourhoods of the point $0 \in A$. Further, A is assumed to be complete and separated with respect to this topology.

Example 2.6.

- We can let A be a discrete ring and $I = (0)$.
- Let $A = \mathbb{Z}_p$ be the p-adic integers with the usual p-adic topology, then we can choose $I = (p^k)$ for any $k \geq 1$.

Remark 2.7. Two ideals of definition I, J generate the same topology if and only if their radicals agree, i.e. $\sqrt{I} = \sqrt{J}$.

Example 2.8. Let $A = \mathbb{Z}_p[\![T]\!]$. Then A can be viewed as an adic ring in several distinct ways, namely

$$I = (p), \qquad I = (T), \qquad I = (p, T).$$

In either case, A will be complete for the I-adic topology, but it is clear that their radicals are different.

Definition 2.9. Let A be an adic ring. Then define $\mathrm{Spf}(A)$ as the set of open prime ideals of A with topology generated by the basic opens

$$D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$$

as f ranges over elements of A and structure sheaf defined by

$$\Gamma(D(f); \mathcal{O}_{\mathrm{Spf}(A)}) = (A[f^{-1}])^{\wedge}_{I}.$$

Remark 2.10. Note that in particular the global sections of a formal scheme are always I-complete, which precludes obtaining a non p-complete $\mathbb{Z}_p$-module like $\mathbb{Q}_p$ as global sections of some p-adic formal scheme as mentioned in the motivation section.

Remark 2.11. The open condition on the prime ideals changes the underlying topological space. In the case of $\mathbb{Z}_p$ we have

$$\mathrm{Spec}(\mathbb{Z}_p) = \{(0), (p)\}, \qquad \mathrm{Spf}(\mathbb{Z}_p) = \{(p)\}.$$

In fact, $\mathrm{Spf}(A)$ is homeomorphic to $\mathrm{Spec}(A/I)$ but the structure sheaf accounts for the additional "fuzz".

Definition 2.12. A formal scheme is a locally ringed space which is locally isomorphic to $\mathrm{Spf}(A)$ for an adic ring A.

2.3 Rigid spaces

Let K be a non-archimedean field, so that it is in particular complete with respect to some valuation denoted $|\cdot|$. Examples include $\mathbb{Q}_p$, where $|p| = p^{-1}$.

Definition 2.13. The *Tate algebra* $K\langle T_1, \dots, T_n\rangle$ is the completion of the polynomial ring $K[T_1, \dots, T_n]$ with respect to the *Gauß norm*. The latter is defined by

$$\left\| \sum_I a_I T^I \right\| = \sup_I |a_I|.$$

Elements of the Tate algebra are to be thought of as formal power series $f = \sum_I a_I T^I$ such that the sequence $|a_I|$ converges to zero.

Definition 2.14. A K-*affinoid algebra* is a quotient of the Tate algebra over K by some closed ideal[1]. For a K-affinoid algebra A, we define the associated *affinoid space* by the following.

- As a set, it is given by $\mathrm{mSpec}(A)$, the set of maximal ideals in A.
- The topology is generated by the rational opens. For $x \in \mathrm{mSpec}(A)$ a maximal idea, A/x will be a finite extension of K, so that the valuation on K extends uniquely to a valuation on this residue field. For an element $f \in A$, write $|f(x)|$ for the norm of f in A/x. For $f_1, \dots, f_n, g \in A$ we then define the associated rational open by

$$D\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{mSpec}(A) \mid \forall i = 1, \dots, n, |f_i(x)| \le |g(x)|\}.$$

[1] Actually, all ideals of the Tate algebra are closed, ([Wei17, p. 3]).

- The structure sheaf is defined on rational opens by

$$\Gamma\left(D\left(\frac{f_1,\ldots,f_n}{g}\right);\mathcal{O}\right)=A\langle T_1,\ldots,T_n\rangle/(f_i-gT_i\mid i=1,\ldots n).$$

Remark 2.15. This need not always form a topological space, but in general the rational opens form a basis for a site.

Definition 2.16. A *rigid analytic space* over K is a locally topologically ringed space that is locally isomorphic to the affinoid space associated to a K-affinoid algebra.

Example 2.17.

- The affinoid space associated to the $\mathbb{Q}_p$-affinoid algebra $\mathbb{Q}_p\langle T\rangle$ is called the rigid closed disc. Indeed, if $K/\mathbb{Q}_p$ is a finite extension, we see that its functor of points satisfies

$$\mathrm{mSpec}(\mathbb{Q}_p\langle T\rangle)(K)=\{x\in K\mid\{x^n\}_n\text{ is bounded}\}.$$

  The latter is equivalently the set $\{x\in K\mid |x|\le 1\}$ whence the name.

- Given a finite type formal scheme, we can construct an analytic space by taking its generic fibre.

Let us now mention an application of the theory of rigid analytic spaces in chromatic homotopy theory, given by the Gross–Hopkins period map.

Theorem 2.18 (Gross–Hopkins, [HG94])**.** *There is a $\mathbb{G}_h$-equivariant map*

$$\pi_{GH}\colon \mathrm{LT}_K^{\mathrm{rig}}\longrightarrow\mathbb{P}_K^{h-1}$$

of rigid analytic spaces which is étale surjective and such that the pullback $\pi_{GH}^*\mathcal{O}(1)$ is the Lie algebra of the universal deformation Γ_h over $\mathrm{LT}_K^{\mathrm{rig}}$.

This is used to study Brown–Comenetz duality in $K(n)$-local spectra by Strickland in [Str00].

2.4 Huber rings

Definition 2.19. A *Huber ring* is a topological ring A such that one can find an open subring $A_0\subset A$ carrying the I-adic topology for some finitely generated ideal $I\subset A_0$. A Huber ring A is said to be *Tate* if A contains a topologically nilpotent unit, called the *pseudo-uniformiser*.

Example 2.20.

1. We can let $A=A_0$ be a discrete ring and $I=(0)$.
2. We can let $A=A_0$ be an I-adically complete ring for $I\subset A$ a finitely generated ideal of definition.
3. Let $A=K$ be a non-archimedean field, then we can set $A_0=\mathcal{O}_K=\{x\in K\mid |x|\le 1\}$. This admits a pseudo-uniformiser ϖ given by any element of $\mathcal{O}_K$ such that $0<|\varpi|<1$.
4. Let $A=K\langle T_1,\ldots,T_n\rangle$ be the Tate algebra over K, and set $A_0=\mathcal{O}_K\langle T_1,\ldots,T_n\rangle$. This has a pseudo-uniformiser given by the same ϖ as above.
5. Let K be a non-archimedean perfect field of characteristic p, and set $A=A_0=W(\mathcal{O}_K)$ where now $I=(p,[\varpi])$ for $[\varpi]$ the multiplicative lift of $\varpi\in\mathcal{O}_K$.
6. For a non-example, note that $A=\mathbb{Q}_p[\![T]\!]$ is not a Huber ring. One would like to set $A_0=\mathbb{Z}_p[\![T]\!]$, but this subring is not open. Indeed, the sequence $\{p^{-1}T^n\}$ is not contained in A_0 but it converge to $0\in A_0$.

Remark 2.21. In the examples above, the first two are not Tate, while the next three are.

2.5 The underlying topological space

For a Huber ring A, we want to set $\mathrm{Spa}(A)$ to be a subset of the set of continuous valuations on A. We will see below that the actual definition is more involved, but let us first note why this is a sensible definition. Recall that if K is a field with a subring A, we defined the Zariski–Riemann space $\mathrm{Zar}(K, A)$ of the pair (K, A) to be the space of continuous valuations on K such that for all $a \in A$, $|a| \leq 1$. This defined a quasicompact ringed space, and it is actually a scheme in nice cases. For example, if K/k is a field extension of transcendence degree one, then

$$\mathrm{Zar}(K, k) \cong \text{smooth projective curve with function field } K \text{ over } k.$$

The easiest example is $K = k(x)$, which corresponds to $\mathbb{P}^1_k$.

Remark 2.22. Note that we do not restrict the target of our valuations, they can land in any ordered abelian group. Further, note that if Γ is an ordered abelian group, then $\Gamma \cup \{0\}$ is an ordered monoid with minimum 0. Examples of such ordered abelian groups include $\mathbb{R}^{\times n}_{>0}$ for $n \geq 1$ with multiplication and lexicographical order.

Definition 2.23. A *continuous valuation* on a topological ring A is a continuous map

$$|\cdot|\colon A \longrightarrow \Gamma \cup \{0\}$$

such that

- $|ab| = |a||b|$,
- $|a + b| \leq \max(|a|, |b|)$,
- $|1| = 1$,
- $|0| = 0$, and
- for all $\gamma \in \Gamma$, the set $\{a \in A \mid |a| < \gamma\} \subset A$ is open.

As seems to be standard, we will usually denote valuations by $x\colon A \to \Gamma \cup \{0\}$, and write $|f(x)| \coloneqq x(f)$ for $f \in A$.

Definition 2.24. Given a topological ring A, we define the set $\mathrm{Cont}(A)$ of equivalence classes of continuous valuations on A: the *value group* of a valuation $x\colon A \to \Gamma \cup \{0\}$ is the subgroup of Γ generated by the image $x(A)$, and often we implicitly replace $\Gamma \cup \{0\}$ with the value group (though even in this case $|-|$ need not be surjective unless A is a field). The equivalence relation between valuations identifies x and x' if there is a commutative diagram

$$\begin{array}{ccc} & \overset{x}{\nearrow} & \Gamma \cup \{0\} \\ A & & \big\downarrow \cong \\ & \underset{x'}{\searrow} & \Gamma' \cup \{0\} \end{array}$$

where Γ, Γ' are the value groups and the vertical map is an order preserving isomorphism.

Let us now topologise $\mathrm{Cont}(A)$ with a basis given by the rational opens

$$D\left(\frac{f_1, \ldots, f_n}{g}\right) = \{x \in \mathrm{Cont}(A) \mid |f_i(x)| \leq |g(x)| \neq 0 \text{ for } i = 1, \ldots, n\}.$$

Remark 2.25. Note that in the definition above, setting $f_i = 1$ recovers the Zariski opens, while allowing $g = 0$ recovers the basis of opens in the affinoid rigid analytic space associated to A.

Definition 2.26. A *Huber pair* is a pair of a Huber ring A and a subring A^+ of integral elements such that every element of A^+ is power bounded, and $A^+ \subset A$ is open and integrally closed. Define the space

$$\mathrm{Spa}(A, A^+)$$

to be the subspace of $\mathrm{Cont}(A)$ on valuations x such that for all $f \in A^+$, $|f(x)| \leq 1$.

Remark 2.27. The subring of power-bounded elements of a Huber ring is denoted A°. In fact, we see that A° is the union of all possible choices of $A_0 \in A$.

Example 2.28. If C is a non-archimedean algebraically closed field, such as $\mathbb{C}_p$, then $X = \mathrm{Spa}(C\langle T\rangle, C^\circ\langle T\rangle)$ has five types of points.

1. For any $\alpha \in C$, such that $|\alpha| \leq 1$, so that in particular $\alpha \in C^\circ$, we obtain a point

$$f \longmapsto |f(\alpha)|.$$

2. Let α be as above, and let $D = D(\alpha, r)$ be the disc centred at α with radius $0 < r < 1$ in the image of the valuation map on C. Then we obtain a point by

$$f \longmapsto \sup\{|f(\beta)| \mid \beta \in D\}.$$

3. The construction above also works when $0 < r < 1$ is not in the image of the valuation map on C.
4. If C is not spherically complete (e.g. when $C = \mathbb{C}_p$), i.e. there exists a decreasing sequence of discs with radius < 1 of the form

$$D_1 \supset D_1 \supset D_3 \supset \cdots$$

such that $\bigcap_i D_i \neq \varnothing$, then we obtain a point by

$$f \longmapsto \inf_{i \geq 1} \sup_{\beta_i \in D_i} |f(\beta_i)|.$$

5. Let α be as above, and now $0 < r \leq 1$. Pick a sign $\pm$ (excluding $+$ if $r = 1$) and let $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, be the ordered abelian group generated by $\mathbb{R}_{>0}$ and an element γ which is infinitesimally less than or greater than r (depending on the sign we chose). Then we obtain a point by

$$f = \sum_{n=0}^{\infty} a_n (T - \alpha)^n \longmapsto \sup_n |a_n| \gamma^n.$$

Remark 2.29. Points of type 2 or 3 are called Gaußpoints, while the points of type 5 are said to be of rank two.

2.6 The structure sheaf

Let $U \subset X \coloneqq \mathrm{Spa}(A, A^+)$ be a rational open subset, we want to specify the value $\Gamma(U; \mathcal{O}_X)$. This is done perhaps rather indirectly using the following theorem of Huber.

Theorem 2.30 (Huber). *Let U and (A, A^+) be as above, then there is a complete Huber pair with a map*

$$(A, A^+) \longrightarrow (\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$$

such that the induced map on adic spectra factors through $U \subset X$ and the induces map

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \longrightarrow U$$

is a homeomorphism, and terminal among all such factorisations. In particular, U is quasi-compact.

Remark 2.31. The hard part is not defining $\mathcal{O}_X(U)$, since this is defined in the same way as before, i.e. for $U = D(\frac{f}{g})$, we see that $\mathcal{O}_X(U) = A\langle T\rangle/(fT - g)$. However, $\mathcal{O}_X^+(U)$ is the integral closure of $A^+[\frac{f}{g}]$ in this ring, which is less evident. We will see below that $\mathcal{O}_X^+(U)$ can be recovered from $\mathcal{O}_X(U)$.

Definition 2.32. Let $X = \mathrm{Spa}(A, A^+)$ be the adic spectrum of a Huber pair as above. Define a presheaf on X by sending an open $W \subset X$ to

$$\mathcal{O}_X = \varprojlim_{U \subset W} \mathcal{O}_X(U),$$

where the limit ranges over rational opens U contained in W. We say that a Huber pair $(A, A+)$ is *sheafy* if this presheaf is actually a sheaf on X.

Proposition 2.33. *We can identify*

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid \forall x \in U, |f(x)| \leq 1\}.$$

In particular, this is a sheaf if $\mathcal{O}_X$ is. Furthermore, if (A, A^+) is a complete Huber pair, then on global sections we obtain

$$\mathcal{O}_X(X) = A, \qquad \mathcal{O}_X^+(X) = A^+.$$

Definition 2.34. An *adic space* is a triple $(X, \mathcal{O}_X, |\cdots|_{x\in X})$ of a topologically ringed space equipped with valuations on its stalks, such that it is locally of the form $\mathrm{Spa}(A, A^+)$ for a sheafy Huber pair.

Remark 2.35. We see that a Huber pair (A, A^+) is sheafy if

- $A = A^+$ is discrete,
- $A = A^+$ is finitely generated over a Noetherian ring, or
- A is Tate and such that the Tate algebras $A\langle T_1, \ldots, T_n\rangle$ are Noetherian for all $n \geq 0$.

We conclude that an affine scheme $\mathrm{Spec}(A)$ can be viewed as an adic space $\mathrm{Spa}(A, A)$, a formal scheme $\mathrm{Spf}(A)$ can be viewed as an adic space $\mathrm{Spec}(A, A)$, and a K-affinoid rigid analytic space $\mathrm{mSpec}(A)$ can be viewed as an adic space $\mathrm{Spa}(A, A^\circ)$.

3 Examples of adic spaces (Itamar Mor, 24 June)

3.1 Complements

Recall that a Huber ring A is a topological ring admitting an open I-adic subring A_0 for some $I \trianglelefteq A_0$. A subset $T \subseteq A$ is call *bounded* if for any open subset $U \subseteq A$, there is an open subset $V \subseteq A$ such that $TV \subseteq U$. This definition makes sense for any topological ring, and in the Huber case happens if and only if for any ideal I of definition of A and $n \geq 0$, $I^m T \subseteq I^n$ for $m \gg 0$.

Definition 3.1. We define the subsets of *power bounded* and *topologically nilpotent* elements of a topological ring A as:

$$A^\circ := \{f \in A \mid f^{\mathbb{N}} \text{ is bounded}\},$$
$$A^{\circ\circ} := \{f \in A \mid f^n \to 0\}.$$

Also recall a Huber pair (A, A^+) is a Huber ring A together with an integrally closed and open subring A^+.

Observation 3.2. Here are some facts about Huber rings and pairs:

1. Any ideal of definition I is contained in $A^{\circ\circ}$ since $I^n \to 0$.
2. Any ring of definition A_0 is bounded and therefore contained in A°.
3. The set A° itself does not need to be bounded. For example, take $A = \mathbb{Q}_p[T]/T^2$. Then $A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p\{T\}$ is not bounded.
4. In fact, $A^\circ = \bigcup A_0$ is the union of all rings of definition in A. Essentially, for any power bounded element $f \in A$, there is a ring of definition A_0 containing f (in fact, this is a filtered colimit).
5. For any Huber pair (A, A^+), we have $A^{\circ\circ} \subseteq A^+$. This is because for any topologically nilpotent element f, we have $f^N \in A^+$ when $N \gg 0$ since $f^n \to 0$ and A^+ is open. The element f is then contained in A^+, since A^+ is integrally closed.

Definition 3.3. A Huber ring A is called *Tate* if there is an element $\varpi \in A^{\circ\circ} \cap A^\times$. Such ϖ is called a *pseudo-uniformizer*. If we fix a ring of definition, we may assume without loss of generality that $\varpi \in A_0$ (replace ϖ by some power) and $I = (\varpi)$, in which case and $A = A^+[\varpi^{-1}]$.

3.2 Completeness and sheafiness

Next, we will talk a bit about sheafiness. To this end, we first explain the role of completeness in Huber rings.

We've defined the space $\mathrm{Spa}(A, A^+)$, and want to equip it with a structure sheaf $\mathcal{O}$. In (usual) algebraic geometry, we do so by localising, and one key point for doing so is Nullstellensatz: for a discrete ring A there are one-to-one correspondences

$$\{\text{Radical ideals of } A\} \longleftrightarrow \{\text{Closed subsets of } \operatorname{Spec} A\} \longleftrightarrow \{\text{Open subsets of } \operatorname{Spec} A\}$$

the right-hand given by taking complements and the left by $I \mapsto V(I)$ and $V \mapsto I(V)$. In turn, this depends on the fact that $\operatorname{Spec} A = \emptyset$ if and only if $A = 0$. For example, one consequence is that we can recover the units from $\operatorname{Spec} A$:

Corollary 3.4. *If $f \in A$ has $\varphi(f) \neq 0$ for any map $\varphi\colon A \to k$ to a field, then $f \in A^\times$.*

To define $\mathcal{O}$, we want a similar picture with valuations replacing prime ideal. The starting point is:

Theorem 3.5 ([Hub93], Proposition 3.6)**.** *Given a Huber pair (A, A^+), recall* $\mathrm{Spa}(A, A^+)$ *is the set of equivalence classes of continuous valuations on A such that $|f(x)| \leq 1$ for any $f \in A^+$. We have*

- $\mathrm{Spa}(A, A^+) = \emptyset \iff A/\overline{\{0\}} = 0$. *In particular, $A = 0$ if it is Hausdorff.*

- $A^+ = \{f \in A \mid |f(x)| \leq 1, \ \forall x \in \mathrm{Spa}(A, A^+)\}$.

See also the notes by Morel.

Corollary 3.6 ([SW20], Proposition 2.3.10)**.** *Let (A, A^+) be a Huber pair.*

1. *An element $f \in A$ lives in A^+ if and only if $|f(x)| \leq 1$ for any $x \in \mathrm{Spa}(A, A^+)$*
2. *If (A, A^+) is* complete *and* Hausdorff*, then*

$$|f(x)| \neq 0, \ \forall x \quad \Longrightarrow \quad f \in A^\times.$$

Proof. Suppose $A \neq 0$ and $f \notin A^\times$. Then f must be contained in some (proper) maximal ideal $\mathfrak{m}$. We claim $\mathfrak{m}$ is closed. Then $A/m \neq 0$ is Hausdorff and nonzero. Picking any valuation $x \in \mathrm{Spa}(A/\mathfrak{m}, A^+/\mathfrak{m})$ and extending to A, we have $|f(x)| = 0$.

To see $\mathfrak{m}$ is closed, we notice that $A^{\circ\circ} \subseteq A$ is open since it contains the ideal of definition I. Then $A^\times$ is also open as it contains an open subset $1 + A^{\circ\circ}$ (since A is complete). We then have $\mathfrak{m}$ is contained in the closed subset $A \backslash A^\times$. Therefore its closure $\overline{\mathfrak{m}}$ must be a proper subset of A. One can further check that $\overline{\mathfrak{m}}$ is also an ideal. Hence, $\overline{\mathfrak{m}} = \mathfrak{m}$ since it is a maximal ideal. ☐

Remark 3.7.

1. Henceforth, 'complete' will mean 'Hausdorff and complete'.
2. We can weaken the completeness assumption, for example to allow Henselian A—see the notes by Bhatt. However, we will see later that we have to make *some* restriction.

The following result tells us how to compute the topological completion from the adic completion:

Lemma 3.8 ([Hub93], Lemma 1.6)**.** *Let A_0 be a ring of definition for A. Then $\widehat{A} \cong \widehat{A}_0 \otimes_{A_0} A$.*

Warning 3.9. In general, Huber rings/pairs do not have pushouts. The situation is better for pushouts along *adic* maps, that is, maps of Huber rings $\varphi\colon A \to B$ for which there is an ideal of definition I such that $\varphi(I)$ generates the topology on B. Then Huber proves under finiteness assumptions that the pushout exists [Hub96, Proposition 1.2.2].

Let (A, A^+) be a Huber pair. Recall rational open subsets of $\mathrm{Spa}(A, A^+)$ are defined to be:

$$U = U\left(\frac{f_1, \cdots, f_n}{g}\right) = \{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| \leq |g(x)| \neq 0\}.$$

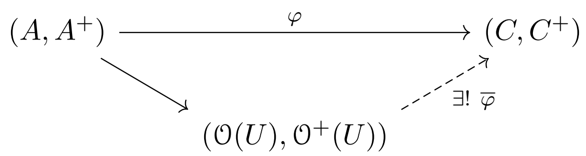
We want to define the sections $\mathcal{O}(U)$ functorially in U. One idea is that we should have $g \in \mathcal{O}(U)^\times$ and $\frac{f_i}{g} \in \mathcal{O}^+(U)$. A natural guess is then to define

$$(B, B^+) = \left(A\left[1/g\right], \overline{A^+[f_1/g, \cdots, f_n/g]}\right), \tag{3.10}$$

where $\overline{A^+[f_1/g, \cdots, f_n/g]}$ is the integral closure of $A^+[f_1/g, \cdots, f_n/g]$ in $A[1/g]$.

Warning 3.11. This is not obviously independent of f_i and g. In fact, we will see this definition is *not* independent of f_i and g later in Example 3.25.

Given a map of Huber pairs $\varphi\colon (A, A^+) \to (C, C^+)$, denote by $\varphi^\#\colon \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ the induced map on adic spaces. When $\mathrm{Im}(\varphi^\#) \subseteq U$, we want a factorization:



Theorem 3.12. *If (C, C^+) is a complete Huber pair, there exists a unique factorization $\overline{\varphi}$.*

Proof. To get a map $\overline{\varphi}\colon B \to C$ we need to show that $\varphi(g) \in C^\times$; what we know is that for any $x \in \mathrm{Spa}(C, C^+)$,

$$|\varphi(g)(x)| = |g(\varphi^\# x)| \neq 0,$$

since $\varphi^\#$ lands in U. Since C is complete, we deduce that $\varphi(g) \in C^\times$. To check that $\overline{\varphi}$ restricts to $B^+ \to C^+$ we need to check that $|f_i/g(x)| \leq 1$ for each i and $x \in \mathrm{Spa}(C, C^+)$, which again follows from Corollary 3.6 □

Lemma/Definition 3.13 ([Mor19] Lemma III.4.2.3)**.** *Let (B, B^+) be a Huber pair. The completion of B^+ defines a ring of integral elements of $\widehat{B}$, and the pair $(\widehat{B}, \widehat{B}^+)$ is the* completion *of (B, B^+).*

Corollary 3.14. *The pair $(\widehat{B}, \widehat{B}^+)$ has the universal property that it is initial among* complete *Huber pairs (C, C^+) such that $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ factors through $U = U\left(\frac{f_1,\ldots,f_n}{g}\right)$.*

Definition 3.15. Set $(\mathcal{O}(U), \mathcal{O}^+(U)) := (\widehat{B}, \widehat{B}^+)$. This depends only on the rational subset U, and so defines a presheaf on $\mathrm{Spa}(A, A^+)$. Recall that (A, A^+) is called *sheafy* if $\mathcal{O}$ happens to be a sheaf.

As a sanity check, we have:

Theorem 3.16 ([Hub93], Proposition 3.9)**.** *Completion induces a homeomorphism* $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \cong \mathrm{Spa}(A, A^+)$.

Therefore, we may for most purposes replace an arbitrary Huber pair by its completion. The payoff for introducing completion in the definition of $\mathcal{O}$ is that in full generality it ruins any chance of $\mathcal{O}$ remaining a sheaf. The following omnibus theorem (see [SW20, Theorem 3.1.8 and Theorem 5.2.5] or [Mor19, Theorem IV.1.1.5]) nevertheless gives sheafiness in all cases we care about:

Theorem/Definition 3.17 ([Hub94], [BV18], [KL15],...)**.** *Let (A, A^+) be a complete Huber pair. The assignment $U \mapsto \mathcal{O}(U)$ is a sheaf in the following cases:*

1. *A is discrete.*
2. *A is finitely generated over a Noetherian ring of definition A_0.*
3. *A is Tate and* strongly Noetherian*: that is, the Tate algebras $A\langle X_1, \cdots, X_n\rangle$ are Noetherian for all $n \geq 0$* [2].
4. *A is* stably uniform*: that is, A° is bounded (such A is called uniform) and the same is true for $\mathcal{O}(U)^\circ$ for every rational subset U.*

Example 3.18. Theorem 3.17 covers the following cases of interest:

- $\mathbb{Q}_p$ satisfies item (2).
- $\mathbb{C}_p$ satisfies item (3).
- Perfectoids satisfy item (4).

[2] Caution: Unlike polynomial algebras, the Hilbert Basis Theorem fails for Tate algebras $A\langle - \rangle$.

Remark 3.19. If a Huber pair (A, A^+) is sheafy, then for any $x \in \mathrm{Spa}(A, A^+)$, we get a valuation $|\cdot|_x$ on the stalk

$$\mathcal{O}_{X,x} := \operatorname*{colim}_{U \ni x} \mathcal{O}(U)$$

Then for $f \in \mathcal{O}(U)$, we have

$$f \in \mathcal{O}^+(U) \quad \Longleftrightarrow \quad |f(x)|_x \le 1 \text{ at } \mathcal{O}_{X,x} \text{ for any } x.$$

We are now ready to define the category of adic spaces.

Definition 3.20. Consider the category $\mathcal{V}$ with

- objects are triples $(X, \mathcal{O}_X, (|\cdot|_x)_{x\in X})$.
- morphisms are maps of topologically ringed spaces that are compatible with the valuations on stalks.

Exercise 3.21. Check that any such ringed space is locally ringed, and that any such morphism is a map of locally ringed spaces.

Definition 3.22. An object $X \in \mathcal{V}$ is called an *adic space* if X is locally of the form $\mathrm{Spa}(A, A^+)$ for some *sheafy* Huber pair (A, A^+). It is called *pre-adic* if (A, A^+) is not necessarily sheafy. Denote by CAff the category of sheafy complete Huber pairs, where C stands for "complete".

Theorem 3.23 ([Hub94], Proposition 2.1)**.** *The adic space construction gives a fully faithful embedding* $\mathrm{Spa}\colon \mathrm{CAff}^{\mathrm{op}} \hookrightarrow \mathcal{V}$. *This gives rise the functor of points for adic spaces:*

$$\begin{aligned} \mathrm{AdicSp} &\hookrightarrow \mathrm{Sh}(\mathrm{CAff}^{\mathrm{op}}) \\ X &\mapsto \mathrm{Hom}(\mathrm{Spa}(-), X), \end{aligned}$$

where $\mathrm{CAff}^{\mathrm{op}}$ *has a "Zariski" topology.*

This perspective will be very useful in computing with adic spaces. Moreover, any pre-adic space also has a functor of points, though now the assignment is not fully faithful; in some cases, we will nevertheless use pre-adic spaces as auxiliary spaces when identifying adic spaces.

3.3 Examples

Now we switch gears and work out some examples.

3.3.1 The terminal object

The first example of adic spaces is the terminal object $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$. As a set, this adic space contains the following points:

- For each prime p, there is a closed point $x_p\colon \mathbb{Z} \to \mathbb{F}_p \to \{0, 1\}$, where $|n|_{x_p} = 0$ iff $p \mid n$.
- For each prime p, there is another point $\eta_p\colon \mathbb{Z} \to \mathbb{Q}_p \to p^{\mathbb{Z}} \cup \{0\}$, where $|n|_{\eta_p} = p^{-v_p(n)}$. The closure of η_p is $\{\eta_p, x_p\}$.
- A generic point $\eta\colon \mathbb{Z} \to \mathbb{Q} \to \{0, 1\}$, where $|n|_\eta = 0$ iff $n = 0$.

Below is a picture of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$, where each squiggly arrow denotes a specialization and the blue paths encircle closed subsets of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$. Note that as a topological space, $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Spec}\, \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

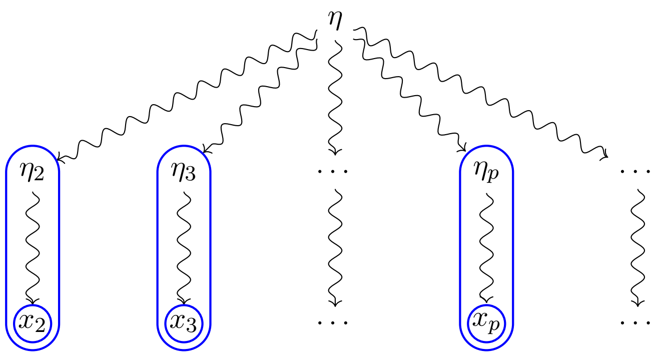


Figure 1: $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$

3.3.2 $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$

Consider a complete local non-archimedean local field K with ring of integers $\mathcal{O}_K$ and a uniformizer ϖ. Then we have:

$$\begin{array}{ccccc} \mathrm{Spa}(K, \mathcal{O}_K) & = & \{\eta_\varpi\} & \cong & \mathrm{Spec}\, K \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K) & = & \{\eta_\varpi \rightsquigarrow x_\varpi\} & \cong & \mathrm{Spec}\, \mathcal{O}_K, \end{array}$$

where $|\varpi(\eta_\varpi)| = \varpi^{-1}$ and $|\varpi(x_\varpi)| = 0$ and the $\cong$'s on the right hand side are isomoprhisms of locally ringed spaces. In this sense, $\mathrm{Spa}(K, \mathcal{O}_K) \hookrightarrow \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$ is the inclusion of the generic point. Restricting along this map (e.g. at the level of functors of points), we get the generic fiber.

Remark 3.24.
- A map of Huber pairs $\varphi\colon (\mathcal{O}_K, \mathcal{O}_K) \to (A, A^+)$ is the condition that $\varphi(\varpi) \in A$ is topologically nilpotent, and this factors through $(K, \mathcal{O}_K)$ precisely when $\varphi(\varpi)$ is also invertible. In this case $\varphi(\varpi) \in A^{\circ\circ} \cap A^\times$, and hence A is Tate.
- The above examples indicate a general phenomenon: for any Huber pair (A, A^+) we have maps

$$\begin{aligned} \mathrm{Spa}(A, A^+) &\longrightarrow \mathrm{Spec}\, A, & x &\longmapsto \ker(x), \\ \mathrm{Spec}\, A &\longrightarrow \mathrm{Spa}(A^\delta, A^\delta), & \mathfrak{p} &\longmapsto (A \to A_\mathfrak{p}/\mathfrak{p} \to \{0, 1\}). \end{aligned}$$

  This exhibits $\mathrm{Spec}\, A$ as a retract of $\mathrm{Spa}(A^\delta, A^\delta)$ (as topological spaces).

3.3.3 Closed unit disc

Consider $\mathbb{D} := \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}[T])$. Define

$$\begin{aligned} \mathbb{D}_K &:= \mathbb{D} \times_{\mathrm{Spa}(\mathbb{Z},\mathbb{Z})} \mathrm{Spa}(K, \mathcal{O}_K) \\ &\cong \mathrm{Spa}(K\langle T \rangle, \mathcal{O}_K\langle T \rangle). \end{aligned}$$

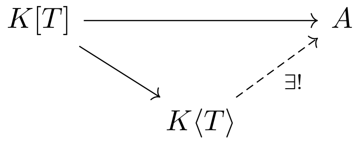
This is the *rigid analytic closed disc* [3], and is an affinoid space if either K is strictly noetherian or $\mathcal{O}_K$ is noetherian. To see the isomorphism claimed above we used the functor of points: for

[3]For example, its coordinate ring is analogous to the ring of *overconvergent* holomorphic functions on the closed unit disc in $\mathbb{C}$.

any complete Huber pair (A, A^+) we have $\mathbb{D}(A, A^+) = A^+$, and if $(K, \mathcal{O}_K) \to (A, A^+)$ then in particular A is Tate and hence

$$\mathbb{D}_K(A, A^+) = A^+ = \hom((K\langle T\rangle, \mathcal{O}_K\langle T\rangle), (A, A^+)).$$

Here we used completeness of A in the factorization:



Example 3.25 (Counterexample)**.** Here is an example where (B, B^+) in (3.10) depends on $f_1, \cdots, f_n, g$. Take $(A, A^+) = (\mathbb{Q}_p[T], \mathbb{Z}_p[T])$ with the p-adic topology on $\mathbb{Z}_p[T]$ and consider $f = 1 + pT \notin A^\times$.

Exercise 3.26. Using the description of the points of $\mathbb{D}_K$ given in the previous lecture, check that $|f(x)| \neq 0$ for any $x \in \mathrm{Spa}(A, A^+) \cong \mathrm{Spa}(\widehat{A}, \widehat{A}^+) = \mathbb{D}_{\mathbb{Q}_p}$. For example, given a Type I point $\alpha \in \mathbb{Z}_p$ in Example 2.28, we have $|f(\alpha)|_p = |1 + p\alpha|_p = 1 \neq 0$.

In particular, this implies

$$X = \mathrm{Spa}(A, A^+) = U(1/f).$$

However, there is no map from $A[1/f]$ to A.

3.3.4 Adic affine line

Consider $\mathbb{A}^1 := \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z})$. If $K, \mathcal{O})$ is a nonarchimedean field, then we have

$$\mathbb{A}^1_K(A, A^+) = A \cong A^+[\pi^{-1}] \cong \mathrm{colim}(A^+ \xrightarrow{\pi} A^+ \xrightarrow{\pi} \cdots)$$

It follows that

$$\begin{aligned}\mathbb{A}^1_K &\cong \operatorname*{colim}_k(\mathbb{D}_K \xrightarrow{\pi^\#} \mathbb{D}_K \xrightarrow{\pi^\#} \cdots) \\ &\cong \operatorname*{colim}_n \mathrm{Spa}(K\langle \pi^n T\rangle, \mathcal{O}_K\langle \pi^n\rangle),\end{aligned}$$

i.e. the affine line is the union of closed discs along inclusions of increasing radius.

3.3.5 Adic circle, punctured line, and projective line

We define:

$$\begin{aligned}\partial\mathbb{D} &:= \mathrm{Spa}(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{\pm 1}]) \\ \mathbb{G}_m &:= \mathrm{Spa}(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}) \\ \mathbb{P}^1 &:= \mathbb{D} \cup_{\partial\mathbb{D}} \mathbb{D}\end{aligned}$$

where the gluing for $\mathbb{P}^1$ is $T \mapsto T^{-1}$ [4]. The functors of points are respectively $\partial\mathbb{D}(A, A^+) = (A^+)^\times$, $\mathbb{G}_m(A, A^+) = A^\times$ and $\mathbb{P}^1(A, A^+) = \{[a_0 : a_1] \mid a_i \in A^+\}$. As an example, let's compute the coordinate ring of $\partial\mathbb{D}_K$: we know that

$$\partial\mathbb{D}_K = U(1/f) = \mathrm{Spa}\left(\widehat{K\langle T\rangle\left[\frac{1}{T}\right]}, \widehat{\mathcal{O}_K\langle T\rangle\left[\frac{1}{T}\right]}\right),$$

[4]The notation $\mathbb{G}_m$ may be non-standard—we could not find a reference.

and so

$$\widehat{K\langle T\rangle}\left[\frac{1}{T}\right] = \left(\mathcal{O}_K\langle T\rangle\left[\frac{1}{T}\right]\right)^{\wedge}_{(\pi)}\left[\frac{1}{\pi}\right]$$
$$= \left\{\sum_{n\in\mathbb{Z}} a_n T^n \,\middle|\, a_n \in \mathcal{O}_K, \lim_{|n|\to\infty} |a_n| = 0\right\}\left[\frac{1}{\pi}\right]$$
$$= \left\{\sum_{n\in\mathbb{Z}} a_n T^n \,\middle|\, a_n \in K, \lim_{|n|\to\infty} |a_n| = 0\right\}.$$

Another name for this would be $K\langle T^{\pm 1}\rangle$.

3.3.6 Open unit disc

Consider

$$\mathbb{D}^\circ := \operatorname{Spa}(\mathbb{Z}[\![T]\!], \mathbb{Z}[\![T]\!]).$$

Then $\mathbb{D}^\circ(A, A^+) = A^{\circ\circ}$. We want to describe the points in $\mathbb{D}^\circ$. Here are some obvious ones:

- $x_{\mathbb{F}_p}\colon \mathbb{Z}_p[\![T]\!] \to \mathbb{F}_p \to \{0,1\}$, $|p| = |T| = 0$.
- $x_{\mathbb{F}_p(\!(T)\!)}\colon \mathbb{Z}_p[\![T]\!] \to \mathbb{F}_p(\!(T)\!) \to T^{\mathbb{Z}} \cup \{0\}$, $|p| = 0, |T| = T^{-1}$.
- $x_{\mathbb{Q}_p}\colon \mathbb{Z}_p[\![T]\!] \to \mathbb{Q}_p \to p^{\mathbb{Z}} \cup \{0\}$, $|p| = p^{-1}, |T| = 0$.

What other points can we find?

Definition 3.27. A point x in $\operatorname{Spa}(A, A^+)$ is called *analytic* if $\ker x$ is not open.

For example, $x_{\mathbb{F}_p} \in \mathbb{D}^\circ_{\mathbb{Z}_p}$ is not analytic, and is the unique non-analytic point. To go further, we use the following lemma:

Lemma 3.28. *Let Γ be a totally ordered abelian group with an element $\gamma \in \Gamma$ such that*

- $0 < \gamma < 1$ *in* $\Gamma \cup \{0\}$*;*
- *for any $\gamma' \in \Gamma$, there is an n such that $\gamma^n < \gamma'$.*

Then any continuous valuation $x\colon (A, A^+) \to \Gamma \cup \{0\}$ has a maximal generalization

$$\begin{array}{ccc} (A, A^+) & \xrightarrow{x} & \Gamma \cup \{0\} \\ & \searrow^{\tilde{x}} & \downarrow \\ & & \mathbb{R}_{\geq 0}. \end{array}$$

in the sense that any other such $\tilde{x}'$ factors uniquely through $\tilde{x}$.

Proof. Pick any $0 < \delta < 1$ in Γ and a real number $r \in (0,1)$ set for any $\gamma \in \Gamma$

$$\varphi_n(\gamma) = r^{m(n)/n}, \qquad \text{where } m(n) = \max\{i \mid \delta^i \geq \gamma^n\}.$$

As the sequence is increasing and bounded, we can define

$$\varphi(\gamma) = \lim_{n\to\infty} \varphi_n(\gamma) \in \mathbb{R}_{\geq 0}.$$

For example, $\varphi_n(\delta^i) = r^i$ for any n and i.

Exercise 3.29. Show that this makes sense, and that $\tilde{x} = \varphi \circ x$ does the job. $\square$

For example, if $x\colon A \twoheadrightarrow \Gamma \cup \{0\}$ is analytic and A is a Huber ring, then there is a $\gamma \in \Gamma$ as in the Lemma. Now suppose $x\colon \mathbb{Z}_p[\![T]\!] \to \Gamma \cup \{0\}$ is an analytic point in $\mathbb{D}^\circ$. Then either $|p(x)| \neq 0$ or $|T(x)| \neq 0$, and moreover both $|p(x)|$ and $|T(x)| < 1$ (since p and T are topologically nilpotent). Set

$$\kappa(x) := \frac{\log|T(\tilde{x})|}{\log|p(\tilde{x})|} \in [0, \infty].$$

For example, we have $\kappa(x_{\mathbb{Q}_p}) = \infty$ and $\kappa(x_{\mathbb{F}_p(\!(T)\!)}) = 0$. Observe that

$$\kappa(x) \leq \frac{s}{t} \iff |p(x)|^s \geq |T(x)|^t.$$

Likewise, for any positive real number r, we have $\kappa(x) \leq r$ iff for any $\frac{s}{t} > r$, $|p(x)|^s \geq |T(x)|^t$. The implication still holds with the directions of all inequalities reversed. This property characterizes κ. In this way, we have constructed a map $\kappa\colon \mathbb{D}^\circ \backslash \{x_{\mathbb{F}_p}\} \to [0, \infty]$. As $\mathbb{D}^\circ \backslash \{x_{\mathbb{F}_p}\} \subseteq \mathbb{D}^\circ$ is spectral, hence compact, its image $\kappa(\mathbb{D}^\circ \backslash \{x_{\mathbb{F}_p}\})$ is also compact (closed) in $[0, \infty]$.

Exercise 3.30. Prove that $\kappa(\mathbb{D}^\circ \backslash \{x_{\mathbb{F}_p}\})$ is dense in $[0, \infty]$ by constructing an x such that $\kappa(x) = \frac{s}{t}$.

Hint: Given a pair of coprime natural numbers $s, t \in \mathbb{N}$, construct:

$$x\colon \mathbb{Z}_p[\![T]\!] \longrightarrow \mathbb{Z}_p[\![T]\!]/(T^t - p^s) \longrightarrow E \longrightarrow \pi^{\mathbb{Z}} \cup \{0\}.$$

Then $|T|^t = |p|^s$ and $\kappa(x) = \frac{s}{t}$.

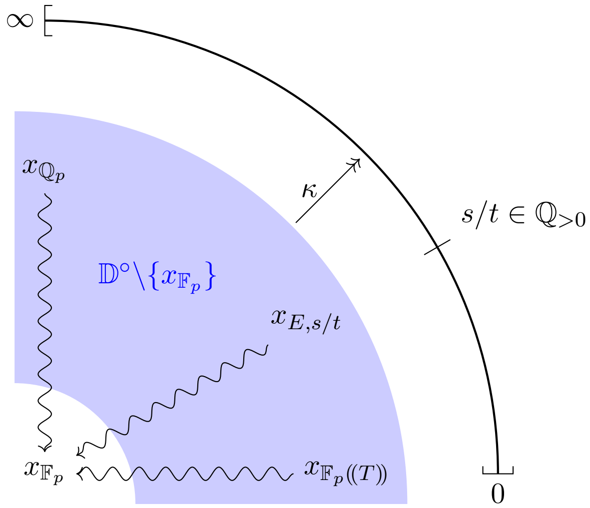


Figure 2: The map $\kappa\colon \mathbb{D}^\circ \backslash \{x_{\mathbb{F}_p}\} \to [0, \infty]$

Proposition 3.31. *$\mathbb{D}^\circ \backslash \{x_{\mathbb{F}_p}\}$ is an adic space.*

Warning 3.32. $\kappa^{-1}((0, \infty])^\circ$ is *not* affinoid: else by the functor of points, we'd have $\kappa^{-1}((0, \infty])^\circ = \mathrm{Spa}(B, B^+)$ with

$$(B, B^+) = (\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!]) \otimes_{(\mathbb{Z}_p, \mathbb{Z}_p)} (\mathbb{Q}_p, \mathbb{Z}_p). \tag{3.33}$$

But in that case we'd have $1/p \in B$ and $T \in B^{\circ\circ}$, so $T^n/p \to 0$. In particular $T^N/p \in A^+$ for $N \gg 0$. On the other hand, by the universal property there should be a map $(B, B^+) \to (\mathbb{Q}\langle T, T^{N+1}/p\rangle, \mathbb{Z}_p\langle T, T^{N+1}/p\rangle)$, and this is a contradiction. In particular, the pushout Eq. (3.33) does not exist in the category of Huber pairs: the problem is that the map $\mathbb{Z}_p \to \mathbb{Z}_p[\![T]\!]$ is not adic.

Proof (Proposition 3.31). We may cover $\mathbb{D}^\circ \setminus \{x_{\mathbb{F}_p}\}$ by rational subsets

$$U(p/T) = \kappa^{-1}([0,1])^\circ$$
$$U(T^n/p) = \kappa^{-1}([1/n,\infty])^\circ$$

for $n \geq 1$: indeed, if $\kappa(x) \neq 0$ then $|p(x)| \neq 0$, so $|T(x)|^N < |p(x)|$ for $N \gg 0$. We'll prove that these are affinoid: that is, we need to show their coordinate rings are sheafy.

1. $(U(p/T))$. We have $U(p/T) = \operatorname{Spa}(B, B^+)$, where
$$B = \mathbb{Z}_p[\![T]\!][p/T]^\wedge_{(p,T)}[T^{-1}] = \mathbb{Z}_p[\![T]\!][p/T]^\wedge_{(T)}[T^{-1}] = \mathbb{Z}_p[\![T]\!]\langle p/T\rangle[T^{-1}].$$
A ring of definition is $\mathbb{Z}[\![T]\!]\langle p/T\rangle$, which is the T-adic completion of $\mathbb{Z}[T, p/T]$ and hence noetherian; thus item (2) of Theorem 3.17 applies.
2. $(U(T^n/p))$. In this case,
$$\mathcal{O}(U(T^n/p)) = \mathbb{Z}_p[\![T]\!][T^n/p]^\wedge_{(p)}[p^{-1}] = \mathbb{Q}_p\langle T, T^n/p\rangle.$$
This is again Tate, and topologically finitely generated over $\mathbb{Z}_p$, which is noetherian. □

Variant 3.34. (A preview of things to come). Suppose that R is a complete Tate ring of characteristic $p > 0$, and moreover R^+ is perfect. In particular, this implies (using the Banach Open Mapping theorem) that R is uniform [SW20, Proposition 6.1.6]. Set $S = \operatorname{Spa}(R, R^+)$.

Definition 3.35. The ring $\mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$ is the ring of p-typical Witt vectors,

$$\mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p} := W(R^+).$$

Note that we have elements $p, [\varpi] \in \mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$, where $[-]$ denotes the multiplicative lift. We equip $\mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$ with the $(p,[\varpi])$-adic topology.

Definition 3.36. $\mathcal{Y}_{S,\mathbb{Q}_p} := \operatorname{Spa} \mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p} \setminus \{[\varpi] = 0\}$.

Remark 3.37. Note that $\operatorname{Spa} \mathbb{A}_{\mathrm{inf},S,\mathbb{Q}_p}$ has a unique non-analytic point

$$x_{\mathrm{na}} : \mathbb{A}_{\mathrm{inf}} \to R^+/\varpi \to \{0,1\}.$$

The subspace $\mathcal{Y}$ looks like a "ϖ-generic fibre" (over S), except that the map $[-]\colon R^+ \to \mathbb{A}_{\mathrm{inf}}$ is not additive and so $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}}$ does not really live over S.

The space $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}}$ looks very similar to $\mathbb{D}^\circ_{\mathbb{Z}_p}$: see Fig. 3. In particular, one can define a continuous surjective function $\kappa\colon \operatorname{Spa} \mathbb{A}_{\mathrm{inf}} \setminus \{x_{\mathrm{na}}\} \to [0,\infty]$ just as we did above. The following, which is for example [SW20, Proposition 11.2.1], is the analogue of Proposition 3.31:

Theorem 3.38. *$\mathcal{Y}$ is an adic space.*

The idea is again to cover $\mathcal{Y}$ by rational subsets $U = U(p/[\varpi^{1/p^n}])$, which cover since $|p(x)| < 1$ for any x. Then

$$\mathcal{O}(U) = W(R^+)\left[\frac{p}{[\varpi^{1/p^n}]}\right]^\wedge_{([\varpi])}\left[\frac{1}{[\varpi]}\right].$$

To show that $\mathcal{Y}$ is adic we need to show these rings are sheafy, which is very non-obvious. The strategy will be to construct split coverings of $\mathcal{O}(U)$ by perfectoids, and use this to deduce sheafiness.

Remark 3.39. Scholze and Weinstein remark that it is probably the case that $\operatorname{Spa} \mathbb{A}_{\mathrm{inf}} \setminus \{x_{\mathrm{na}}\}$ is in fact adic too. In any case, note that $\mathcal{Y}$ is not affinoid, since no finite subcover of the $U(p/[\varpi^{1/p^n}])$ covers.

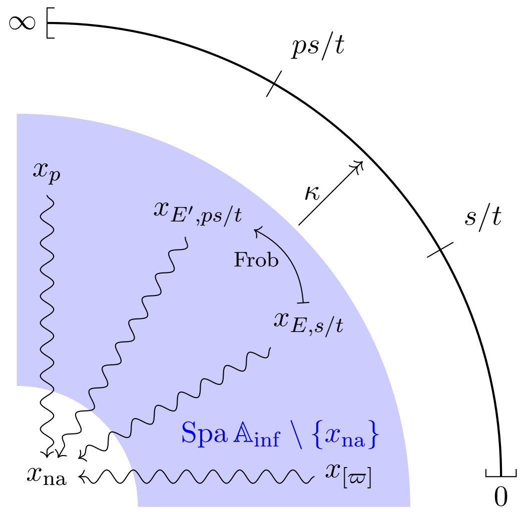


Figure 3: The map $\operatorname{Spa}\mathbb{A}_{\mathrm{inf}} \setminus \{x_{\mathrm{na}}\} \to [0,\infty]$. Here $|p(x_p)| = p^{-1}$ and $|[\varpi](x_p)| = 0$, while $|p(x_{[\varpi]})| = 0$ and $|[\varpi](x_{[\varpi]})| = 0$. In this case the Frobenius on R^+ induces one on $\operatorname{Spa}\mathbb{A}_{\mathrm{inf}}$, and $\kappa(\operatorname{Frob} x) = p\kappa(x)$.

4 The proétale site of an adic space (Emma Brink, 1 July)

4.1 The proétale site

Ultimately our goal in this seminar is to understand the E_2-page of the rationalized $K(h)$-local E_h-Adams spectral sequence,

$$H^*_{\mathrm{cont}}(\mathbb{G}_h, \pi_* E_h),$$

by viewing it as the cohomology of $[\mathrm{LT}_\eta/\mathbb{G}_h]$, the stack quotient of the adic generic fibre of Lubin-Tate space by the Morava stabiliser action. Since $\mathbb{G}_h$ is a profinite group, to make sense of this stack we will view it as a sheaf on the *proétale site* of LT_η, which is what we discuss today. To begin with, we define the relevant sites.

Definition 4.1. A map $X \to Y$ of adic spaces is

1. *analytic* if $|X| \to |Y|$ is an open embedding and $\mathcal{O}_X = \mathcal{O}_Y|_X$.
2. *finite étale* if for any open affinoid $\operatorname{Spa}(A, A^+) \subset Y$, $X \times_Y \operatorname{Spa}(X, X^+) \cong \operatorname{Spa}(B, B^+)$ is affinoid and
   - the map $A \to B$ exhibits B as a finitely generated A-module with induced topology,
   - B^+ is the integral closure of A^+ in B,
   - The A-algebra B has a presentation as $A[X_1, \dots, X_n]/(f_1, \dots, f_n)$ with $\det(\partial f_i/\partial X_j)_{ij} \neq 0$.
3. *étale* if there is an open covering $X = \bigcup_i U_i$ and affinoid opens $f(U_i) \subset V_i \subset_o Y$ such that each $f|_{U_i} \colon U_i \to V_i$ is finite étale.
4. *proétale* if there exists a covering $X = \bigcup_i U_i$ and $f(U_i) \subset V_i \subset_o Y$ such that
   - $U_i = \varprojlim_{J_i} \operatorname{Spa}(A_{ij}, A_{ij}^+)$ is the cofiltered limit of affinoids and
   - $f|_{U_i} \colon U_i \to V_i$ is the limit of étale maps $f_{ij} \colon \operatorname{Spa}(A_{ij}, A_{ij}^+) \to V_i$.

Definition 4.2. A family of morphisms of adic spaces $\{f_i \colon X_i \to Y\}_I$ is an *qc cover* if for all quasicompact opens $U \subset Y$, there exists a quasicompact open $U \subset \coprod_{i\in I} X_i$ such that $U \subset f(C)$.

Definition 4.3. For an adic space X, we define the following sites:

1. $X_{\mathrm{Zar}} = X_{\mathrm{an}}$ is the category of adic spaces equipped with an analytic map to X, with the topology given by analytic qc covers.
2. $X_{\text{ét}}$ is the category of adic spaces equipped with an étale map to X, with the topology given by étale qc covers.
3. $X_{\text{proét}}$ is the category of adic spaces equipped with a proétale map to X, with the topology given by proétale qc covers.

In particular there are obvious inclusions $X_{\mathrm{Zar}} \xrightarrow{\iota} X_{\text{ét}} \xrightarrow{\nu} X_{\text{proét}}$, which give rise to geometric morphisms

$$\mathrm{Sh}(X_{\mathrm{Zar}}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathrm{Sh}(X_{\text{ét}}) \underset{\nu_*}{\overset{\nu^*}{\rightleftarrows}} \mathrm{Sh}(X_{\text{proét}})$$

on the associated 1- or ∞-topoi.

Remark 4.4. If X is affinoid then it is essentially by definition that $X_{\text{ét}}$ is generated under colimits by the subcategory $X_{\text{ét}}^{\mathrm{aff}}$ of *affinoids* étale over X; likewise, $X_{\text{proét}}$ is generated under colimits by the subcategory $X_{\text{proét}}^{\mathrm{aff}}$, and there is an equivalence

$$\varprojlim : \mathrm{Pro}(X_{\text{ét}}^{\mathrm{aff}}) \xrightarrow{\sim} X_{\text{proét}}^{\mathrm{aff}}.$$

As a warning, this is no longer true when we take X to be a general adic space, since we might not be able to glue pro-affinoid presentations.

4.2 The case of a point

We will consider in detail the case that $X = \mathrm{Spa}(K, K^\circ)$ is the adic spectrum of a nonarchimedean field, in which case

$$X \simeq \mathrm{Cont}(K) \simeq \mathrm{Spv}(K^\circ/K^{\circ\circ}),$$

which is a point when $K^\circ/K^{\circ\circ}$ is finite (e.g. when $K = \mathbb{Q}_p$). See [Mor19] for the details of these equivalences. Write $G = G_K := \mathrm{Gal}(K^s/K)$ for the Galois group of the separable closure K^s of K.

1. $X_{\mathrm{Zar}} = \{\emptyset, *\}$, hence a sheaf on X_{Zar} is just an abelian group.
2. If $K \subseteq L$ is a finite separable field extension, denote by $L^+ := \overline{K}^L$ the integral closure of K in L and equip $L \cong K^n$ with the topology induced from K. Then the map $\mathrm{Spa}_K(L) := \mathrm{Spa}(L, L^+) \to \mathrm{Spa}(K, K^+)$ induced by $L \hookrightarrow K$ is étale. Every affinoid adic space étale *over* $\mathrm{Spa}(K, K^+)$-scheme is a coproduct $\sqcup_{i\in I} \mathrm{Spa}_K(L_i, L_i^+)$ for finite separable field extensions $K \subseteq L_i$. Denote by Set_G the category of discrete G-sets with a continuous G-action and G-equivariant maps.

   Sending an orbit G/H for a finite-index subgroup $H \subseteq G$ to $\mathrm{Spa}_K((K^s)^H)$ defines an equivalence
   $$\mathrm{Set}_G \xrightarrow{\sim} X_{\text{ét}}^{\mathrm{aff}}.$$

   Thus,
   $$\begin{aligned}\mathrm{Sh}(X_{\text{ét}}) &\simeq \mathrm{Sh}(X_{\text{ét}}^{\mathrm{aff}}) \\ &\simeq \mathrm{Sh}(\mathrm{Set}_G).\end{aligned}$$

   The topology on Set_G is generated by jointly surjective families of maps. Via restriction, the right hand side is further equivalent to sheaves on the site of finite G-sets with jointly surjective covers, which are automatically refined by a finite subfamily.[5] In particular, this implies that cohomology of étale sheaves agrees with continuous group cohomology of *discrete* G-modules.

3. By the same reasoning,

$$\begin{aligned}\mathrm{Sh}(X_{\text{proét}}) &\simeq \mathrm{Sh}(X^{\text{aff}}_{\text{proét}})\\ &\simeq \mathrm{Sh}(\mathrm{Pro}(\mathrm{Fin}_G))\\ &\simeq \mathrm{Sh}(\mathrm{Profin}_G)\\ &\simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Cond}(\mathrm{Set}))\end{aligned}$$

where Profin_G denotes the category of profinite sets with continuous G-action, and the topology in the second and third lines is given in both cases by families refined by some finite surjective subfamily.

The equivalence $\mathrm{Sh}(\mathrm{Pro}(\mathrm{Fin}_G)) \simeq \mathrm{Sh}(\mathrm{Profin}_G)$ holds since

$$\varprojlim : \mathrm{Pro}(\mathrm{Fin}_G) \to \mathrm{Profin}_G$$

is fully faithful and its image generates Profin_G under colimits.

Remark 4.5. The final equivalence follows from the following general fact: if G is a group object in a site $\mathcal{C}$ (i.e. a group object in the underlying category), then $\mathrm{Sh}(B_{\mathcal{C}}G) \simeq B_{\mathrm{Sh}(\mathcal{C})}G$.

We want to relate sheaf cohomology on $X_{\text{proét}}$ to continuous group cohomology. Suppose that M is a continuous locally profinite G-module, or more generally any $T1$ topological G-module for which the associated condensed abelian group $\underline{M}$ is solid. We will show that

$$H^*(X_{\text{proét}}, M) \cong H^*_{\text{cont}}(G, M).$$

The cover $\{\underline{G} \to *\} \in \mathcal{B}_{\mathrm{Cond}(\mathrm{Set})}\underline{G}$ yields a Čech-to-cohomology spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond}(\mathrm{Set})}\underline{G})}(\mathbb{Z}[\underline{G}^{p+1}], -) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond}(\mathrm{Set})})\underline{G}}(\mathbb{Z}, -) = H^*(X_{\text{proét}}, -).$$

Denote by S^G_* the simplicial complex associated with the Čech nerve of $\{G \to *\}$. If M is a condensed $\mathbb{Z}[\underline{G}]$-module with

$$\mathrm{Ext}^j_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], \underline{M}) = 0 \text{ for } i, j \in \mathbb{N}_1,$$

then the Čech-to-cohomology spectral sequence collapses and yields

$$H^*(\mathrm{Hom}_{\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^*], M)) = \mathrm{Ext}^*_{\mathcal{B}_{\underline{G}}(\mathrm{Cond}(\mathrm{Set}))}(\mathbb{Z}, M) = H^*(X_{\text{proét}}, M).$$

As G^i is compactly generated for all $i \in \mathbb{N}_0$, for a $T1$ continuous G-module M,

$$\mathrm{Hom}_{\mathbb{Z}[\underline{G}]}(S^{\underline{G}}_*, \underline{M}) \cong \mathrm{Cont}(G^{\times *}, M)$$

is the complex of continuous cochains, which by definition computes the continuous group cohomology.

For a condensed $\mathbb{Z}[\underline{G}]$-module M and $i \in \mathbb{N}_1$,

$$\mathrm{Ext}^j_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], M) \cong \mathrm{Ext}^j_{\mathrm{Cond}(\mathrm{Ab}))}(\mathbb{Z}[G^{i-1}], M).$$

[5] Here the following fact is used: if $\mathcal{D}$ is a site and $\mathcal{C}$ a full subcategory closed under pullbacks, equipped with a topology $\tau_{\mathcal{C}} \subset \tau_{\mathcal{D}}$ and generating in the sense that any covering in $\mathcal{D}$ is refined by one in $\mathcal{C}$, then restriction induces an equivalence $\mathrm{Sh}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{C})$.

If G is not discrete, then $\mathbb{Z}[G^2]$ is not projective, so $\mathrm{Ext}^1_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], -) \neq 0$.

But for solid abelian groups M, $\mathrm{Ext}^j_{\mathrm{Cond(Ab))}}(\mathbb{Z}[G^{i-1}], M) = \mathrm{Ext}^j_{\mathrm{Solid(Ab)}}(\mathbb{Z}[G^{i-1}]^{L\square}, M)$ and we have:

Proposition 4.6 ([Sch19], Theorem 5.8)**.** *For any profinite space X, we have that*

$$\mathbb{Z}[\underline{X}]^{L\square} \simeq \mathbb{Z}[\underline{X}]^{\square}$$

is concentrated in degree zero, and this is a projective object of Solid(Ab)*.*

In particular, for a $T1$ continuous G-module M with $\underline{M}$ solid, (e.g. M locally profinite),

$$\mathrm{Ext}^j_{\mathrm{Cond}(\mathbb{Z}[\underline{G}]}(\mathbb{Z}[\underline{G}^i], \underline{M}) = 0 \text{ for } i, j \in \mathbb{N}_1,$$

whence

$$H^*(X_{\text{proét}}, M) \cong H^*_{cont}(G, M).$$

The idea of proof for the proposition is to use the fact, due to Nöbeling and Specker, that $C(X, \mathbb{Z})$ is a free abelian group for any profinite set X, and that $\mathbb{Z}[\underline{X}]^{\square} = \underline{\mathrm{Hom}}_{\mathrm{Cond(Ab)}}(C(X, \mathbb{Z}), \mathbb{Z})$.

All in all, we obtain for any locally profinite G-module M, maps

$$H^*(X_{\text{Zar}}, M) \to H^*(X_{\text{ét}}, M) \to H^*(X_{\text{proét}}, M),$$

where moreover

$$\begin{aligned} H^*(X_{\text{Zar}}, M) &\cong \begin{cases} M^G & * = 0 \\ 0 & * \neq 0 \end{cases} \\ H^*(X_{\text{ét}}, M) &\cong H^*_{cont}(G, M^\delta) := \varinjlim H^*(G/U, M^U) \\ H^*(X_{\text{proét}}, M) &\cong H^*_{cont}(G, M) \end{aligned}$$

where the colimit in the second line runs over all open normal subgroups of G.

Remark 4.7. In general, for a condensed group $\mathcal{G}$ and a condensed $\mathcal{G}$-module $M \in \mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})$, one has a Čech-to-cohomology spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})}(\mathbb{Z}[\mathcal{G}^{p+1}], -) = \mathrm{Ext}^q_{\mathrm{Cond(Ab)}}(\mathbb{Z}[\mathcal{G}^p], -) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})}(\mathbb{Z}, -)$$

which converges to the condensed group cohomology of $\mathcal{G}$. Condensed group cohomology is therefore in general a finer invariant than continuous group cohomology which ignores all terms $E_2^{p,q}$ for $p, q > 1$ in the above spectral sequence. For example, one can show that for a $T1$ topological abelian group G and a locally profinite abelian group M with trivial $\underline{G}$-action,

$$\mathrm{Ext}^*_{\mathrm{Ab}(\mathcal{B}_{\mathrm{Cond(Set)}}\mathcal{G})}(\mathbb{Z}, \underline{M}) = \mathrm{Ext}^{p+q}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[\underline{BG}], -)$$

is the condensed cohomology of the classifying space BG of G, whereas

$$H^*_{cont}(G, M) = H^*_{cont}(\pi_0 G, M) = \mathrm{Ext}^{p+q}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[\underline{B\pi_0 G}], M)$$

is the condensed cohomology of the classifying space $B\pi_0 G$. However, Proposition 4.6 implies that continuous group cohomology with solid coefficients can (for large classes of groups) be realised as Ext-groups in the condensed world.

Denote by $\mathrm{Solid}(\mathbb{Z}[\underline{G}]) \subseteq \mathrm{Cond}(\mathbb{Z}[\underline{G}])$ the full subcategory on condensed $\mathbb{Z}[\underline{G}]$-modules whose underlying condensed abelian group is solid. This is an abelian subcategory closed under limits.

The inclusion $\mathrm{Solid}(\mathbb{Z}[\underline{G}]) \subseteq \mathrm{Cond}(\mathbb{Z}[\underline{G}])$ has a left adjoint $(-)^{\square G}$ which sends a $\mathbb{Z}[\underline{G}]$-module to its solidification with the induced $\mathbb{Z}[\underline{G}]$-module structure. Solidification is not an exact functor, but one can show that for a $T1$ topological group, the degreewise solidificaion $(S_*^{\underline{G}})^{\square G}$ of the simplicial resolution is a resolution of $\mathbb{Z} \in \mathrm{Solid}(\mathbb{Z}[\underline{G}])$. If this is a projective resolution and G^i is compactly generated for all $i \in \mathbb{N}_0$), we obtain that

$$\mathrm{Ext}^*_{\mathrm{Solid}(\mathbb{Z}[\underline{G}])}(\mathbb{Z}, \underline{-}) \cong H^*_{cont}(G, -)$$

on continuous G-modules whose associated condensed abelian group is solid, like locally profinite continuous G-modules.

Since solid tensor products of projectives in Solid(Ab) are projective, $(S_*^{\underline{G}})^{\square G}$ is a projective resolution of $\mathbb{Z}$ in $\mathrm{Solid}(\mathbb{Z}[\underline{G}])$ if and only if $\mathbb{Z}[\underline{G}]^{\square}$ is a projective solid abelian group. This holds in many cases, for example if G is locally connected or if G is a coproduct of compact spaces or a product of two such groups. It works in slightly larger generality, and we currently do not know an example where it fails. It might fail for groups which are totally disconnected but not locally compact like the rationals with euclidean topology.

4.3 Cohomology of $\hat{\mathcal{O}}_X$

We now return to the case of a general affinoid X. There is a functor

$$\begin{aligned} X_{\text{ét}}^{\text{aff}} &\to \mathrm{Top\,Ring} \\ (A, A^+) &\mapsto A^+ \end{aligned}$$

and this satisfies étale descent. We obtain a sheaf of topological rings $\mathcal{O}^+_{X_{\text{ét}}}$ on $X_{\text{ét}}$, and hence a sheaf of topological rings

$$\mathcal{O}^+ := \nu^* \mathcal{O}^+_{X_{\text{ét}}} \in \mathrm{Sh}(X_{\text{proét}}; \mathrm{Top}(\mathrm{Ring})).$$

Denote by $\hat{\mathcal{O}}^+ := \varprojlim_n \mathcal{O}^+/p^n \in \mathrm{Sh}(X_{\text{proét}}; \mathrm{Top}(\mathrm{Ring}))$ the p-completion of $\mathcal{O}^+$, i.e. the sheafification of $T \mapsto \varprojlim_n \mathcal{O}^+(T)/p^n$.

By pointwise passing to condensed sets, we obtain sheaves of condensed rings $\underline{\mathcal{O}^+}$ and $\underline{\hat{\mathcal{O}}^+}$. Since $\underline{-}\colon \mathrm{Top} \to \mathrm{Cond(Set)}$ is a right adjoint, no further sheafification is necessary, i.e.

$$\underline{\mathcal{O}^+} = \underline{-} \circ \mathcal{O}^+ \text{ and } \underline{\hat{\mathcal{O}}^+} = \underline{-} \circ \hat{\mathcal{O}}^+.$$

Example 4.8. If $Y = \mathrm{Spa}(R, R^+) = [\varprojlim_{i \in I} \mathrm{Spa}(Y_i, Y_i^+)]_p^\wedge$ is perfectoid affinoid,

$$\hat{\mathcal{O}}^+(Y) = R^+ = [\varinjlim_{i \in I} Y_i^+]_p^\wedge = [\varinjlim_{i \in I} \hat{\mathcal{O}}^+(Y_i)]_p^\wedge \tag{4.9}$$

is the p-completion of the topological ring $\varinjlim_{i \in I} Y_i^+$.

In the rest of the talk, we discuss:

Proposition 4.10 ([Bar+24], Lemma 3.7.1)**.** $\mathbf{R}\Gamma(X_{\text{proét}}, \underline{\hat{\mathcal{O}}^+}) \simeq \left[\underline{\mathbf{R}\Gamma(X_{\text{proét}}, \mathcal{O}_\delta^+)}\right]_p^\wedge$ *for X affinoid perfectoid, where $\mathcal{O}_\delta^+$) denotes the sheaf of discrete rings underlying $\mathcal{O}^+$.*

Remark 4.11. The functor

$$\begin{aligned} j : *_{\text{proét}} &\to X_{\text{proét}} \\ \varprojlim S_i &\mapsto \varprojlim (S_i \times X) \end{aligned}$$

induces an adjunction

$$j^* : \mathrm{Cond(Set)} \rightleftarrows \mathrm{Sh}(X_{\text{proét}}) : j_*.$$

For every sheaf of abelian groups $F \in \mathrm{Sh}(X_{\text{proét}}, \mathrm{Ab})$, $\mathbf{R}j_*F \in \mathcal{D}(\mathrm{Cond(Ab)})$ has $\mathbf{R}\Gamma(X_{\text{proét}}, F)$ as underlying complex of abelian groups.

But as we will see below, this yields no ambiguity in the condensed structure:

Lemma 4.12. *For any affinoid X there is a natural equivalence*

$$\mathbf{R}j_*\hat{\mathcal{O}}_\delta^{\,+} \simeq \mathbf{R}\Gamma(X_{\text{proét}}, \underline{\hat{\mathcal{O}}}^{+}).$$

The proof of this lemma requires some preparation, see below.

Definition 4.13. An adic space X is *strictly totally disconnected* (abbreviated to *std*) if for any étale cover $\{X_i \to X\}$ there is a finite subset $F \subset I$ such that

$$\coprod_{i\in F} X_i \to X$$

admtits a section. In particular, this implies that $\Gamma\colon \mathrm{Sh}(X_{\text{ét}}; \mathcal{A}) \to \mathcal{A}$ is exact for any abelian $\mathcal{A}$.

Lemma 4.14. *For any std perfectoid affinoid Y, $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+) = \hat{\mathcal{O}}^+(Y)$ and $\mathbf{R}j_*\hat{\mathcal{O}}^+ = j_*\hat{\mathcal{O}}^+$ are concentrated in degree zero.*

Proof. Let $\Gamma_{\text{cond}}\colon \mathrm{Cond(Ab)} \to \mathrm{Ab}$ denote the global section/evaluation at $*$. This is an exact functor and in particular induces a functor $\mathbf{R}\Gamma_{\text{cond}}\colon \mathcal{D}(\mathrm{Cond(Ab)}) \to \mathcal{D}(\mathrm{Ab})$, given by *componentwise application*. It suffices to show that $\mathbf{R}\Gamma_{\text{cond}}\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+)$ and $\mathbf{R}\Gamma_{\text{cond}}(\mathbf{R}j_*\hat{\mathcal{O}}^+)$ are concentrated in degree 0.

Now

$$\mathbf{R}\Gamma_{\text{cond}} \circ \mathbf{R}j_* = \mathbf{R}(\Gamma_{\text{cond}} \circ j_*) = \mathbf{R}\Gamma(Y_{\text{proét}}, -)$$

and

$$\mathbf{R}\Gamma_{\text{cond}} \circ \mathbf{R}\Gamma(Y_{\text{proét}}, -) = \mathbf{R}(\Gamma_{\text{cond}} \circ \Gamma(Y_{\text{proét}}, -)) = \mathbf{R}\Gamma(Y_{\text{proét}}, -)$$

as functors $\mathrm{Sh}(Y_{\text{proét}}; \mathrm{Ab}) \to \mathcal{D}(\mathrm{Ab})$. So it suffices to show that $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+)$ is concentrated in degree 0.

By [BS15, Corollary 5.16], the sheafification

$$\nu^*\colon \mathrm{Sh}(Y_{\text{ét}}; \mathrm{Ab}) \to \mathrm{Sh}(Y_{\text{proét}}; \mathrm{Ab})$$

is fully faithful and for any étale sheaf F,

$$\mathbf{R}\Gamma(Y_{\text{proét}}, \nu^*F) = \mathbf{R}\Gamma(Y_{\text{ét}}, F) = F(Y)$$

is concentrated in degree 0 as Y is strictly totally disconnected.

For all $m \in \mathbb{N}_0$ we have an exact sequence

$$0 \to \lim{}^1 H^{m-1}(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) \to H^m(Y_{\text{proét}}, R\lim \hat{\mathcal{O}}^+/p^n) \to \varprojlim_n H^m(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) \to 0.$$

By definition of perfectoid rings, for all affinoid perfectoid Y, $\hat{\mathcal{O}}^+(Y) = R\lim \hat{\mathcal{O}}^+/p^n(Y)$ is derived p-complete. As we will see next week, perfectoid affinoids generate the topology on

$X_{\text{proét}}$, hence $R\lim \hat{\mathcal{O}}^+/p^n = \hat{\mathcal{O}}^+$ as sheaves on $X_{\text{proét}}$. Since X is affinoid perfectoid, for all $n \in \mathbb{N}_1$, $\hat{\mathcal{O}}^+/p^n = \nu^* O^+/p^n$. Hence, for $m \in \mathbb{N}_1$:

$$\varprojlim_n H^m(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) = \varprojlim_n H^m(Y_{\text{ét}}, O^+/p^n) = 0$$

and for $m \in \mathbb{N}_2$,

$$\lim{}^1 H^{m-1}(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) = \lim{}^1 H^{m-1}(Y_{\text{ét}}, O^+/p^n) = 0.$$

This implies that $H^m(Y_{\text{proét}}, \hat{\mathcal{O}}^+) = H^{m-1}(Y_{\text{proét}}, R\lim \hat{\mathcal{O}}^+/p^n) = 0$ for $m \geq 2$ and

$$H^1(Y_{\text{proét}}, \hat{\mathcal{O}}^+) = \lim{}^1 H^0(Y_{\text{proét}}, \hat{\mathcal{O}}^+/p^n) = \lim{}^1 R^+(Y)/p^n = 0$$

since $\hat{\mathcal{O}}^+(Y) = R^+(Y)$ is derived p-complete. This shows that $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+)$ is concentrated in degree 0. □

We will see next week that any X admits a proétale covering by strongly totally disconnected perfectoid affinoids. We may therefore compute proétale cohomology of perfectoid affinoid spaces using any hypercovers by std perfectoid affinoids. This now implies Lemma 4.12:

Proof (Lemma 4.12). Since every rigid analytic space X admits a proétale covering by strongly totally disconnected perfectoid affinoids, we can compute $\mathbf{R}\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+)$ and $\mathbf{R}j_*\hat{\mathcal{O}}^+$ using a hypercover by strongly totally disconnected perfectoid affinoids. Whence it suffices to show that for totally disconnected affinoid perfectoid Y, $\mathbf{R}\Gamma(Y_{\text{proét}}, \hat{\mathcal{O}}^+) \cong \mathbf{R}j_*\hat{\mathcal{O}}^+$ naturally with respect to pro-étale morphisms.

By Lemma 4.14, for Y totally disconnected affinoid perfectoid, $\mathbf{R}\Gamma(Y_{\text{proét}}, \underline{\hat{\mathcal{O}}^+}) \cong \underline{\hat{\mathcal{O}}^+(Y)}$ and $\mathbf{R}j_*\hat{\mathcal{O}}^+ = j_*\hat{\mathcal{O}}^+$ are concentrated in degree 0, so we are reduced to showing that for Y strongly totally disconnected affinoid perfectoid, $j_*\hat{\mathcal{O}}^+(Y) \cong \Gamma(Y_{\text{proét}}, \underline{\hat{\mathcal{O}}^+})$, naturally with respect to pro-étale morphisms. We will show that this holds for all affinoid perfectoid spaces Y.

For $S = \lim_{i\in I} S_i \in \text{Pro(Fin)}$ and $X = \text{Spa}(R, R^+)$ perfectoid affinoid,

$$X \times S := [\varprojlim_{i\in I} S_i \times Y]^\wedge_p = [\varprojlim_{i\in I} \text{Spa}(R^{S_i}, (R^+)^{S_i})]^\wedge_p$$

is perfectoid affinoid and $X \times S \in X_{\text{proét}}$.

Hence, by definition of j_*,

$$\begin{aligned} j_*\hat{\mathcal{O}}^+(\varprojlim_{i\in I} S_i) &= \text{Hom}_{X_{\text{proét}}}(\varprojlim_{i\in I} X \times S_i, \hat{\mathcal{O}}^+) \\ &= \hat{\mathcal{O}}^+(\varprojlim_{i\in I} X \times S_i) \\ &\overset{4.8}{=} [\varinjlim_{i\in I} \hat{\mathcal{O}}^+(X \times S_i)]^\wedge_p \\ &= [\varinjlim_{i\in I} (\hat{\mathcal{O}}^+(X))^{S_i})]^\wedge_p \\ &= [\varinjlim_{i\in I} \mathcal{C}(S_i, \hat{\mathcal{O}}^+(X))]^\wedge_p \\ &= [\varinjlim_{i\in I} \mathcal{C}(S_i, R^+(X))]^\wedge_p. \end{aligned}$$

Since $\hat{\mathcal{O}}^+(X) = R^+ = \varprojlim_n R^+/p^n$ is profinite and R^+/p^n is finite discrete for all $n \in \mathbb{N}_1$ by definition of perfectoid rings,

$$\underline{R^+}(\varprojlim_{i \in I} S_i) = \mathcal{C}(\varprojlim_{i \in I} S_i, R^+) = \mathcal{C}(\varprojlim_{i \in I} S_i, \varprojlim_n R^+/p^n) = \varprojlim_n \varinjlim_i \mathcal{C}(S_i, R^+/p^n).$$

As for all finite sets F and $n \in \mathbb{N}$, $C(F, R^+/p^n) = \mathcal{C}(F, R^+)/p^n$,

$$\varprojlim_n \varinjlim_i \mathcal{C}(S_i, R^+/p^n) = \varprojlim_n \varinjlim_i (\mathcal{C}(S_i, R^+)/p^n) = \varprojlim_n (\varinjlim_i \mathcal{C}(S_i, R))/p^n = [\varinjlim_{i \in I} (\mathcal{C}(S_i, R^+)]_p^\wedge.$$

This shows that

$$\underline{R^+}(\varprojlim_{i \in I} S_i) \cong [\varinjlim_{i \in I} (\mathcal{C}(S_i, R^+)]_p^\wedge \cong j_* \hat{\mathcal{O}}^+(\varprojlim_{i \in I} S_i)$$

for all $S = \varprojlim_{i \in I} S_i \in \mathrm{Pro}(\mathrm{Fin})$. These identifications obviously define isomorphism of condensed sets/abelian groups/rings and are natural in the perfectoid affinoid space X with respect to pro-étale morphisms. □

In fact, we do not need complicated hypercovers but can even work with Čech nerves of covers to compute $\mathbf{R}\Gamma(Y, \underline{\hat{\mathcal{O}}}^+)$:

Lemma 4.15 ([Bar+24], Lemmas 3.7.3-4)**.** *Let Y be perfectoid affinoid. There exists a proétale cover $X \to Y$ with X std perfectoid affinoid. All terms $X^{(i+1)} := X^{\times_Y i}$ in the Čech nerve are also std perfectoid affinoid.*

Proof (Proposition 4.10). Given this, we obtain the desired identification

$$\begin{aligned}
\mathbf{R}\Gamma(Y, \underline{\hat{\mathcal{O}}}^+) &\simeq \mathrm{Tot}\, \underline{\hat{\mathcal{O}}}^+(X^{(\bullet)}) \\
&\simeq \mathrm{Tot} \varprojlim_n (\underline{\mathcal{O}}^+(X^{(\bullet)})/p^n) \\
&\simeq \mathrm{Tot} \varprojlim_n (\underline{\mathcal{O}_\delta^+(X^{(\bullet)})}/p^n) \\
&\simeq \varprojlim_n \mathrm{Tot}(\underline{\mathcal{O}_\delta^+(X^{(\bullet)})})/p^n \\
&\simeq \left[(\mathrm{Tot}\, \underline{\mathcal{O}_\delta^+}(X^{(\bullet)}))\right]_p^\wedge \\
&\simeq \left[\mathbf{R}\Gamma(Y_{\text{proét}}, \underline{\mathcal{O}_\delta^+})\right]_p^\wedge \\
&\simeq \left[\underline{\mathbf{R}\Gamma(Y_{\text{proét}}, \mathcal{O}_\delta^+)}\right]_p^\wedge .
\end{aligned}$$

where $\mathcal{O}_\delta^+$ denotes the sheaf of discrete rings underlying $\hat{\mathcal{O}}^+$.

Here we used that for X perfectoid affinoid, $\hat{\mathcal{O}}^+(X) = [\mathcal{O}^+(X)]_p^\wedge = \lim_n \mathcal{O}^+(X)/p^n$ and $\mathcal{O}^+(X)/p^n$ is discrete for all $n \in \mathbb{N}_0$, whence

$$\underline{\hat{\mathcal{O}}^+(X)} = \lim_n \underline{\mathcal{O}_\delta^+(X)}/p^n.$$ □

5 Perfectoid spaces (Christian Kremer, 8 July)

5.1 History

Let us recall a classical theorem that admits a generalisation in the language of perfectoid spaces.

Theorem 5.1 (Fontaine–Wintenberger)**.** *There is an isomorphism of absolute Galois groups*

$$\mathrm{Gal}(\mathbb{Q}_p[p^{1/p^\infty}]) \cong \mathrm{Gal}(\mathbb{F}_p(\!(t)\!)[t^{1/p^\infty}]).$$

While these two fields are radically different, we can write any element of $\mathbb{Q}_p$ as a Laurent series in the variable p with coefficients in $\{0, \ldots, p-1\}$. The same thing is true for $\mathbb{F}_p(\!(t)\!)$ per construction, replacing p by the variable t, and we will see that the adjunction of sufficiently many p-th roots of this variable allows us to obtain an isomorphism on absolute Galois groups.

5.2 Perfectoid rings

Let us fix a prime p throughout the rest of this talk, and Let R be a Tate ring, i.e. a Huber ring such that one can choose a topologically nilpotent unit, called the pseudo-uniformiser ϖ. As usual, let $R^\circ \subset R$ denote the subring of power-bounded elements.

Definition 5.2. A Tate ring A is perfectoid if it is complete, uniform (i.e. $A^\circ \subset A$ is a bounded subset), and one can choose a pseudouniformiser ϖ such that

1. $\varpi^p \mid p$
2. The Frobenius map $\mathrm{Frob}\colon A^\circ/\varpi \to A^\circ/\varpi^p$ is an isomorphism.

Remark 5.3. The first condition $\varpi^p \mid p$ implies that A°/ϖ is of characteristic p so the Frobenius map is a ring map.

Remark 5.4. If A is of characteristic p, the conditions $\varpi^p \mid p$ does not make sense prima facie, but we simply set it to be true.

Lemma 5.5. *Let A be a Tate ring with a pseudo-uniformiser ϖ such that $\varpi^p \mid p$. Then*

1. *The Frobenius map* $\mathrm{Frob}\colon A^\circ/\varpi \to A^\circ/\varpi^p$ *is injective.*
2. *If A is complete and uniform, then*

$$\mathrm{Frob}\colon A^\circ \to A^\circ/\varpi^p$$

is surjective if and only if

$$\mathrm{Frob}\colon A^\circ \to A^\circ/p$$

is surjective.

An upshot of the second part of this lemma is that if a Tate ring is complete and uniform and we already have a pseudo-uniformiser ϖ such that $\varpi^p \mid p$, then the final condition in asking for a certain Frobenius map to be an isomorphism does not depend on the choice of ϖ. Furthermore, the map that we want to be an isomorphism is always injective so it suffices to check it is surjective.

Proof.

1. Suppose $a \in A^\circ$ is (such that its reduction modulo ϖ is) in the kernel of the Frobenius map $A^\circ/\varpi \to A^\circ/\varpi^p$, i.e. there exists a y in A° such that $a^p = y\varpi^p$. Then we have

$$\frac{a^p}{\varpi^p} = y \in A^\circ.$$

By the definition of power-bounded elements, we see that a/ϖ is also contained in A°, whence $a = 0$ in A°/ϖ.

2. Since $\varpi^p \mid p$ by assumption, there is a commutative diagram

$$\begin{array}{ccc} A^\circ & \xrightarrow{\phi_1} & A^\circ/p \\ & \searrow^{\phi_2} & \downarrow \\ & & A^\circ/\varpi^p, \end{array}$$

where the vertical map is reduction, hence clearly surjective. It is therefore clear that ϕ_1 being surjective implies that ϕ_2 is surjective, which proves one direction of the statement. For the converse, assume that ϕ_2 is surjective. Let $a \in A^\circ$ be an element of which we want to find a lift along the Frobenius. Since ϕ_2 is surjective, we can write

$$a = a_0^p + b_0\varpi^p.$$

Let us iterate this process for b_0 obtain a power series expansion for a in the variable ϖ^p. Since A is assumed to be Tate, complete, and uniform, A° will be ϖ-adically complete so that the power series expansion above converges, and we write

$$a = \sum_{n\geq 0} a_n^p \varpi^{np}.$$

If we now define

$$c = \sum_{n\geq 0} a_n \varpi^n \in A^\circ,$$

then it is clear by the binomial formula and the fact that $\varpi^p \mid p$ that

$$c^p = a + p(\cdots)$$

whence c is a lift of the class of a along ϕ_1. □

Let us now relate the notion of perfectoid rings in characteristic p to the notion of perfect $\mathbb{F}_p$-algebras (we will see that they are quite related, but still quite different).

Proposition 5.6. *Let A be Tate of characteristic p, then the following are equivalent.*

1. *A is perfectoid.*
2. *A is complete and perfect.*

Remark 5.7. Note that this proposition applies to Tate algebras, in particular while $\mathbb{F}_p$ is a perfect $\mathbb{F}_p$-algebra, it is not perfectoid at all since it is not Tate.

Proof. To see that 2 implies 1, note that A is a complete Tate ring by assumption, and that it is uniform[6]. The condition $\varpi^p \mid p$ is set to be vacuously true in characteristic p, so it suffices to check that

$$\mathrm{Frob}\colon A^\circ \to A^\circ/\varpi^p$$

[6]More generally, and Tate ring in characteristic p which is complete is automatically uniform

is an isomorphism. By Lemma 5.5 we see that this is equivalent to requiring that Frobenius

$$\text{Frob}\colon A^\circ \to A^\circ/p$$

to be an isomorphism, which is true by the assumption that A was perfect. to prove the converse, just apply Lemma 5.5 again. □

A special class of perfectoid rings are those that happen to be fields, called perfectoid fields[7]. These admit a more concrete description, as in the theorem below.

Theorem 5.8 (Kedlaya)**.** *Let K be a complete topological field, then the followings are equivalent.*

1. *K is a perfectoid Tate ring.*
2. *The topology on K is induced by a rank one valuation*

$$|\cdot|\colon K \to \mathbb{R}_{\geq 0}$$

such that

- *The image of the valuation is non-discrete, i.e. $\text{im}(|\cdot|) \cap \mathbb{R}_{>0}$ is not discrete*[8]*.*
- *$|p| < 1$.*
- *If $K^{\leq 1}$ denotes the subring of elements with valuation ≤ 1, then $K^{\leq 1}/p$ is semiperfect, i.e. has surjective Frobenius.*

Example 5.9.

- $\mathbb{Q}_p$ is not perfectoid, indeed its topology is induced by the usual p-adic valuation which has image $\{0\} \cup p^{\mathbb{Z}}$ (with the convention that $|p| = p^{-1}$). This is discrete as a subset of $\mathbb{R}_{>0}$. Note that $\mathbb{Q}_p^{\leq 1} \cong \mathbb{Z}_p$ is however semiperfect after reduction modulo p. Alternatively, note that one can not choose a pseudo-uniformiser ϖ of $\mathbb{Q}_p$ such that $\varpi^p \mid p$, since there are not enough p-th roots of unity.
- Define the Tate ring

$$\mathbb{Q}_p[p^{1/p^\infty}] = (\bigcup_n \mathbb{Q}_p[p^{1/p^n}])^\wedge,$$

where each of the rings in the union is equipped with the valuation inherited from $\mathbb{Q}_p$, and the completion on the outside is taken with respect to this valuation. Note that $|p| = p^{-1}$ in any of these valuations. However, we see that the image of the valuation is not discrete, since the sequence

$$|p^{1/p^k}| = p^{-p^{-k}} \xrightarrow{k\to\infty} 1$$

converges to the accumulation point 1 in the image. It suffices to check that $\mathbb{Q}_p[p^{1/p^\infty}]^{\leq 1}/p \cong \mathbb{F}_p$, which is clearly (semi-)perfect.

- Define the Tate ring

$$\mathbb{Q}_p^{\text{cycl}} = (\bigcup_n \mathbb{Q}_p[\mu_{p^n}])^\wedge$$

with the valuation once again inherited from $\mathbb{Q}_p$ under finite extensions and filtered colimits. This is once again perfectoid.

[7] This terminology is not entirely universal

[8] We want to exclude zero, since this will often be a trivial accumulation point of the image in cases that are not perfectoid.

5.3 Tilting and Witt vectors

To distnguish between the notion of tilting for p-adically complete rings and perfectoid rings, we will introduce some nonstandard terminology.

Definition 5.10. Let R be a p-adically complete ring, then define the (integral) tilt of R to be the ring

$$R^\flat = \varprojlim_{\phi} R/p,$$

with the limit being taken along Frobenius maps.

It is clear that $R^\flat$, also known as the inverse limit perfection, is a perfect $\mathbb{F}_p$-algebra. Indeed, per construction the map

$$R^\flat \to R^\flat, (a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots)$$

is inverse to the Frobenius map on $R^\flat$.

Remark 5.11. The integral tilting construction above in fact induces an adjunction

$$\{\text{perfect } \mathbb{F}_p\text{-algebras}\} \underset{(-)^\flat}{\overset{W(-)}{\rightleftarrows}} \{p\text{-complete } \mathbb{Z}_p\text{-algebras}\},$$

where $W(-)$ denotes the p-typical Witt vectors. Furthermore, this adjunction is such that the unit

$$S \to W(S)^\flat$$

is always an equivalence. Given a p-complete $\mathbb{Z}_p$-algebra R, the counit map

$$\theta \colon W(R^\flat) \to R$$

is Fontaine's map.

Example 5.12. We have the canonical example

$$W(\mathbb{F}_p) = \mathbb{Z}_p,$$

and

Can't find a reference for this statement about A_{inf}

On the other hand, we can also define a notion of tilting for perfectoid rings.

Definition 5.13. Let A be a perfectoid ring, then define the (perfectoid) tilt of A to be the ring whose multiplicative monoid is given by

$$A^\flat = \varprojlim_{\phi} A,$$

the limit along the Frobenius, and addition defined by

$$(a^{(n)})_n + (b^{(n)})_n = (\lim_{k\to\infty} (a^{(n+k)} + b^{(n+k)})^{p^k})_n.$$

Lemma 5.14. *If A is perfectoid, then $A^\flat$ is perfectoid of characteristic p. If A itself was already of characteristic p, then $A \cong A^\flat$ is isomorphic to its tilt.*

Example 5.15. There is a chain of isomorphisms

$$(\mathbb{Q}_p^{\mathrm{cycl}})^\flat \cong (\mathbb{F}_p(\!(t)\!)[t^{1/p^\infty}])^\wedge \cong (\mathbb{Q}_p[p^{1/p^\infty}])^\flat,$$

even though the two outside rings are not isomorphic before tilting.

Let us now extend the tilting construction to affinoid objects in perfectoid geometry.

Definition 5.16. A perfectoid Huber pair is a Huber pair (A, A^+) such that

1. A is perfectoid (and in particular Tate),
2. $A^+ \subset A^\circ$ is integral and open.

Define the tilt of a perfectoid Huber pair above as the perfectoid Huber pair $(A^\flat, A^{\flat+} = \varprojlim_\phi A^+)$.

Remark 5.17. In fact, we see that $A^{\flat+}$ is isomorphic to the integral tilt of A^+/ϖ.

Remark 5.18. If (A, A^+) is a perfectoid Huber pair, Fontaine's map

$$\theta\colon W(A^{\flat+}) \to A^+$$

is surjective, with kernel primitive of degree one, i.e. generated by an element of the form

$$p + [\varpi]\alpha$$

for $[\varpi]$ the multiplicative lift of ϖ to the Witt vectors, and α some element in $W(A^{\flat+})$. In fact, this assembles to an equivalence of categories

$$\{\text{perfectoid Huber pairs}\} \cong \{\text{perfect prisms over } \mathbb{F}_p\}$$
$$(A, A^+) \mapsto (A^\flat, A^{\flat+}, \ker(\theta)),$$

where the right hand side is the category of triples consisting of a perfectoid Huber pair (R, R^+) in characteristic p and a primitive ideal $I \subset W(R^+)$ of degree one.

Let us remark that if (R, R^+, I) is a perfect prism over $\mathbb{F}_p$ and (S, S^+) is a perfectoid Huber pair with a map $f\colon (R, R^+) \to (S, S^+)$, then the image $f(I)$ is still primitive of degree one in $W(S^{\flat+})$. Let us now extract the main immediate consequence of this equivalence between perfectoid Huber pairs and perfect prisms.

Corollary 5.19 (Tilting equivalence)**.** *Let (A, A^+) be a perfectoid Huber pairs, then there are equivalences of categories (pHp = perfectoid Huber pair)*

$$\textit{pHp over } (A, A^+) \simeq \textit{perfect prisms over } (A^\flat, A^{\flat+}, \ker(\theta)) \simeq \textit{pHp over } (A^\flat, A^{\flat+}).$$

Proof. Tilting a perfectoid Huber pair that is already in characteristic p gives back the same result, so we just apply the tilting equivalence between perfectoid Huber pairs and perfect prisms twice. □

5.4 Perfectoid spaces

Now that we have discussed perfectoid Huber pairs, our affinoid perfectoids, let us globalise the theory to perfectoid spaces. First, note that if (A, A^+) is a perfectoid Huber pair, then it is automatically sheafy so that $\mathrm{Spa}(A, A^+)$ forms an affinoid adic space.

Definition 5.20. A perfectoid space is an adic space locally of the form $\mathrm{Spa}(A, A^+)$ for (A, A^+) a perfectoid Huber pair.

By applying the tilting equivalence on every perfectoid Huber pair and gluing this back together, we see that the tilting equivalence globalises to an equivalence

$$\mathrm{Pfd}_{/X} \simeq \mathrm{Pfd}_{/X^\flat}$$

of perfectoid spaces over a base perfectoid space X and its tilt $X^\flat$. In fact, we can strengthen this to the almost purity result of Faltings.

Theorem 5.21 (Almost purity)**.** *Let X be a perfectoid space, then there is an equivalence of sites*

$$X_{\text{ét}} \simeq X^\flat_{\text{ét}}.$$

Remark 5.22. Taking fundamental groups then recovers the Fontaine–Wintenberger theorem 5.1.

5.5 Pro-étale cohomology

Theorem 5.23. *Let (R, R^+) be a perfectoid Huber pair, and denote by $X = \mathrm{Spa}(R, R^+)$ the associated affinoid perfectoid adic space. Then*

- $\mathrm{H}^i(X; \mathcal{O}_X) = 0$ *for* $i > 0$,
- $\mathrm{H}^i(X; \mathcal{O}_X^+)$ *is almost zero for* $i > 0$, *and*
- $\mathrm{H}^i_{\text{proét}}(X; \widehat{\mathcal{O}}_X^+)$ *is almost zero for* $i > 0$.

Recall that an R^+-module M is almost zero, if for *any* pseudo-uniformiser[9] $\varpi \in R^+$, $\varpi M = 0$. We can use this to prove a surprising theorem about the interplay between the cohomology of adic spaces and affinoid perfectoids.

Theorem 5.24. *Let X be a locally Noetherian adic space over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. *Then the class of $U \in X_{\text{proét}}$ such that U is affinoid perfectoid forms a basis for the proétale topology on $X_{\text{proét}}$.*

Remark 5.25. A slightly stronger version of this, combined with Lemma 4.14 gives rise to the comparison between condensed and proétale cohomology in the previous talk.

5.6 How many untilts?

As we saw before, in the case of the Fontaine–Wintenberger isomorphism, nonisomorphic fields can have isomorphic tilts. It is therefore reasonable to ask for a moduli problem of untilts of a given perfectoid field of characteristic p.

Theorem 5.26 (Fargues–Fontaine)**.** *Let L be a perfectoid field of characteristic p. Then there exists a curve over $\mathbb{Q}_p$ (i.e. a regular Noetherian scheme of Krull dimension one) whose closed points are in bijection with (Frobenius orbits of isomorphism classes of) untilts of L, i.e. pair (K, ι) of a perfectoid field K of characteristic zero and a finite extension $\iota\colon L \to K^\flat$. The degree of a closed points is the degree of this extension.*

[9]Note the word *any*, this helps us e.g. recover the more classical notions of being almost zero over a local ring in terms of all powers of the maximal ideal.

Note that we are taking Frobenius orbits, indeed there is a natural $\mathbb{Z}$-action on an untilt (K, ι), by sending ι to $\iota \circ \mathrm{Frob}^n$.

Proposition 5.27. *Given a perfectoid field L of characteristic p as above, there are infinitely many points of degree one on the Fargues–Fontaine curve over L.*

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