PICARD AND BRAUER GROUPS OF $K(h)$-LOCAL SPECTRA VIA PROFINITE GALOIS DESCENT

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Abstract. Using the proétale site, we construct a model for the continuous action of the Morava stabiliser group on the Picard spectrum of Morava E-theory. We show that its descent spectral sequence recovers the computation due to Hopkins, Mahowald and Sadofsky of the group Pic_1. We also show this gives a bound on the Brauer group of $K(h)$-local spectra.

1. Introduction

In [HMS94], Hopkins, Mahowald and Sadofsky study the *Picard group* of a symmetric monoidal category: by definition, this is the group of isomorphism classes of invertible objects with respect to the monoidal product. This is a notion that goes back much further, and gives a useful invariant of a ring or scheme. Its particular relevance to homotopy theory comes from the observation that if the category $\mathcal{C}$ is a Brown category (for example, $\mathcal{C}$ might be the homotopy category of a compactly-generated stable ∞-category), then the representability theorem applies and shows that the Picard group $\mathrm{Pic}(\mathcal{C})$ classifies homological automorphisms of $\mathcal{C}$, each of these being of the form $T \otimes (-)$ for some invertible object. The objective of *op. cit.* is to develop techniques for studying Picard groups in some examples coming from chromatic homotopy: the main theorem is the computation of the Picard group of $K(1)$-local spectra at all primes, where $K(1)$ is Morava K-theory at height one.

The aim of this project is to give a new proof of these computations using *Galois descent*, inspired by the formalism developed in [MS16]. We will write Pic_h for the Picard group of $K(h)$-local spectra. There are still many open questions regarding these groups: for example, it is unknown if they are finitely generated as modules over $\mathbb{Z}_p$. The question of computing Pic_2 has been studied by many authors (for example [KS04, Kar10, GHMR15]); using recent work at the prime 2 [BBG+22a], our results give a new potential approach to the computation of Pic_2 in [BBG+22b]. Following [GL21], we also extend these techniques to the *Brauer group* of $\mathcal{S}p_{K(h)}$, giving a cohomological approach to these, which allows us to bound the size of the group of $K(1)$-local Azumaya algebras trivialised over $E_1 = KU_p$, at all primes.

The notion of Galois descent in algebra is very classical, and says that if $A \to B$ is a Galois extension of rings, then Mod_A can be recovered as the category of descent data in Mod_B: in particular, invertible A-modules can be recovered as invertible B-modules M equipped with isomorphisms $\psi_g : M \cong g^*M$ for each $g \in \mathrm{Gal}(B/A)$, subject to a cocycle condition. This gives an effective way to compute Picard groups and other invariants. One can try to play the same game in higher algebra, and use descent techniques to get a handle on the groups Pic_h. Fundamental to this approach is the notion of a Galois extension of commutative ring spectra, as set down in [Rog08]: this is a direct generalisation of the classical axioms in [AG60]. Given a finite G-Galois extension $A \to B$ which is *faithful*, the analogous descent statement (due to [Mei12, GL21]) is that the canonical functor

$$\mathrm{Mod}_A \to (\mathrm{Mod}_B)^{hG} := \lim\Big(\mathrm{Mod}_B \rightrightarrows \textstyle\prod_G \mathrm{Mod}_B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \Big) \tag{1}$$

is an equivalence of symmetric monoidal ∞-categories. Taking the Picard *spectrum* of a symmetric monoidal ∞-category preserves (homotopy) limits, and therefore any such Galois extension gives rise to an equivalence $\mathfrak{pic}(\mathrm{Mod}_A) \simeq \tau_{\geq 0}(\mathfrak{pic}(\mathrm{Mod}_B)^{hG})$. In particular one gets a homotopy fixed points spectral sequence (hereafter HFPSS), whose 0-stem converges to $\mathrm{Pic}(\mathrm{Mod}_A)$. This technique has proved very fruitful in Picard group

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computations: for example, the Picard groups of KO and Tmf are computed in [MS16], and the Picard group of the higher real K-theories $EO(n)$ in [HMS17]. In each case the starting point is a theorem of Baker and Richter [BR05], which says that the Picard group of an even-periodic ring spectrum E with $\pi_0 E$ regular Noetherian is a $\mathbb{Z}/2$-extension of the Picard group of the ring $\pi_0 E$. One can for example study the action of the Morava stabiliser group $\mathbb{G}_h$ on Morava E-theory:

Theorem 1.1 ([DH04, Rog08])**.** *Write E_h for height h Morava E-theory at the (implicit) prime p. The unit*

$$L_{K(h)}\mathbb{S} \to E_h$$

is a $K(h)$-local profinite Galois extension for the Goerss-Hopkins-Miller action of $\mathbb{G}_h$. That is, there are $K(h)$-local spectra E_h^{hU} for every open subgroup U of $\mathbb{G}_h$ such that the following hold:

1. *each E_h^{hU} is an $\mathbb{E}_\infty$-ring spectrum over which E_h is a commutative algebra,*
2. *choosing a cofinal sequence of open subgroups U yields $E_h \simeq L_{K(h)} \varinjlim_U E_h^{hU}$,*
3. *for any normal inclusion $V \triangleleft U$ of open subgroups, the map $E_h^{hU} \to E_h^{hV}$ is a faithful U/V-Galois extension of $K(h)$-local spectra.*

In order to leverage this to compute the groups Pic_h it is necessary to understand not just the finite Galois descent technology mentioned above, but how this assembles over the entire system of extensions. Our main theorem is the following descent result for Picard groups:

Theorem A. *The unit $L_{K(h)}\mathbb{S} \to E_h$ induces an equivalence of spectra*

$$\mathfrak{pic}(\mathcal{S}p_{K(h)}) \simeq \tau_{\geq 0}\mathfrak{pic}(E_h)^{h\mathbb{G}_h}, \tag{2}$$

where the right-hand side denotes continuous *homotopy fixed points. For any closed subgroup $G \subset \mathbb{G}$, the resulting spectral sequence takes the form*

$$E_2^{s,t} = H^s_{\mathrm{cont}}(G, \pi_t \mathfrak{pic}(E_h)) \implies \pi_{t-s}\mathfrak{pic}(E_h^{hG}). \tag{3}$$

In an explicit range, it agrees with the $K(h)$-local E_h-Adams spectral sequence for the Devinatz-Hopkins fixed points E_h^{hG}, including differentials.

Here $\mathfrak{pic}(E_h)$ denotes the Picard spectrum of $K(h)$-*local* E_h-modules. One of the major tasks is to properly interpret the right-hand side of (2), in order to take into account the profinite topology on the Morava stabiliser group; to do so, we make use of the *proétale* (or *condensed/pyknotic*) formalism of [BS14, Sch19, BH19]. We elaborate on our approach later in the introduction, but will first mention some consequences.

Picard group computations. As a first corollary, we show how to recover the computation of Pic_1. Recall there is a map

$$(E_h)_*^\vee : \mathrm{Pic}_h \to \mathrm{Pic}_h^{\mathrm{alg}},$$

whose domain is the group of invertible *Morava modules*; this is discussed in detail at the end of the introduction.

Theorem B ([HMS94]; Propositions 4.15 and 4.19)**.** *The Picard group of the $K(1)$-local category is as follows:*

1. *At odd primes,* $\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{\mathrm{alg}}$,
2. *When $p = 2$, there is an exact sequence*

$$0 \to \mathbb{Z}/2 \to \mathrm{Pic}_1 \to \mathrm{Pic}_1^{\mathrm{alg}} \to 0.$$

This is obtained by computing the spectral sequence (3) from knowledge of the $K(1)$-local E_1-Adams spectral sequence. In fact, we focus on computing the exotic part of Pic_1; in Theorem 4.4 we show that this is precisely the part detected in (3) in filtration at least two.

We also consider examples at height bigger than two. When combining the results of [HMS94, KS04, Kar10, GHMR15, BBG+22b, Pst22], the following gives an algebraic expression for the first undetermined Picard group:

Theorem C ([CZ23]; Corollary 4.12 and Example 4.13)**.** *If $p = 5$, there is an isomorphism*

$$\kappa_3 \simeq H_0(\mathbb{S}_3, \pi_8 E_3)^{\mathrm{Gal}(\mathbb{F}_{125}/\mathbb{F}_5)}.$$

At height two, spectral sequence (3) implies the (known) result that exotic elements exist only at the primes 2 and 3; the groups κ_2 at these primes were computed in [BBG+22b] and [GHMR15] respectively. Both cases used the filtation on Pic_2 defined in [HMS94, Prop. 7.6], and explicit constructions of Picard elements. On the other hand, Theorem A gives another possible approach to compute κ_2, by computing (3) via comparison with the $K(2)$-local E_2-Adams spectral sequence; the latter is studied for example in [HKM13, BBG+22a]. We hope to return to this in future.

Brauer group computations. In Section 5, we turn our attention to Galois descent computations of the Brauer group of $\mathcal{S}p_{K(h)}$. In [GL21], Gepner and Lawson use Galois descent to compute the relative Brauer group $\mathrm{Br}(KO \mid KU)$ from knowledge of the HFPSS for $\mathfrak{pic}(KO) \simeq \tau_{\geq 0}\mathfrak{pic}(KU)^{hC_2}$, and we show that their method extends to our context. Note that while the Picard group is amenable to computation by a variety of methods, the Brauer group is much more difficult to study in an *ad hoc* manner, and each of [AG14, GL21, AMS22] uses some such form of descent to obtain an upper bound; this follows the classical picture, in which descent computations were pioneered by Grothendieck in [Gro68]. The results of Section 5.1 strengthen existing descent technology, making the Brauer groups of the $K(h)$-local categories computationally tractable.

In analogy with the Picard case, we will write $\mathrm{Br}_h^0 := \mathrm{Br}(\mathcal{S}p_{K(h)} \mid E_h)$ for the group of Brauer classes of $\mathcal{S}p_{K(h)}$ that become trivial (up to $K(h)$-local Morita equivalence) over Morava E-theory. In Section 5.2, we consider the bound on this group imposed by spectral sequence (3).

Theorem D (Corollary 5.16 and Corollary 5.18)**.** *(1) If p is odd, then Br_1^0 is isomorphic to a subgroup of $\mu_{p-1} \subset \mathbb{Z}_p^\times$.*
(2) At the prime 2, we have $|\mathrm{Br}_1^0| \leq 32$.

We also give candidates for populating these groups; in [Mor23], we complete this computation. Combining this with the main theorem of [HL17] would give a description of the full Brauer group Br_h, but we do not pursue this here.

Methods for profinite descent. We now summarise our approach to continuity of the $\mathbb{G}_h$-action on $\mathfrak{pic}(E_h)$. In the case of Morava E-theory itself, this was explored in the work of Davis [Dav03, Dav06] and Quick [Qui11], both of whom (using different methods) gave model-theoretic interpretations for continuous actions of profinite groups. In particular, both approaches recover the $K(h)$-local E_h-based Adams spectral sequence as a type of HFPSS for the action on E_h.

We will make crucial use of the *proétale classifying site* of the profinite group $\mathbb{G}_h$. This is closer in flavour to the approach of Davis, who uses the *étale* classifying site; his strategy is briefly recounted in Section 2, where we make the connection explicit. Namely, Davis makes use of the fact that viewed as an object of the $K(h)$-local category, Morava E-theory is a *discrete* $\mathbb{G}_h$-object, meaning that it is the filtered colimit of the objects E_h^{hU}. Such $\mathbb{G}_h$-actions can be effectively modelled using the site of *finite* $\mathbb{G}_h$-sets, and the requisite model category of simplicial sheaves was developed by Jardine [Jar97] as a way of formalising Thomason's work on descent for $K(1)$-localised algebraic K-theory. Of course, the action on E_h is *not* discrete when we view it as a plain spectrum, as can be seen on homotopy groups. Likewise, the induced action on its Picard spectrum is not discrete. As a model for more general continuous actions of a profinite group G, we therefore use the ∞-category of sheaves on the *proétale* site $BG_{\text{proét}}$, whose objects are the profinite G-sets. This was studied in [BS14]; as shown there, in many cases it gives a site-theoretic interpretation of continuous group cohomology. Even when the comparison fails, proétale sheaf cohomology exhibits many desirable properties absent in other definitions.

The equivalence in Theorem A is therefore interpreted as the existence of a sheaf of connective spectra $\mathfrak{pic}(\underline{\mathrm{E}})$ on the proétale site, having

$$\Gamma(\mathbb{G}_h/*, \mathfrak{pic}(\underline{\mathrm{E}})) \simeq \mathfrak{pic}(\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})) \qquad \text{and} \qquad \Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathfrak{pic}(\underline{\mathrm{E}})) \simeq \mathfrak{pic}(\mathcal{S}p_{K(h)}).$$

To this end, we begin by proving a descent result for Morava E-theory itself.

Theorem A.I (Proposition 2.39, Lemma 2.44 and Proposition 2.48)**.** *There is a hypercomplete sheaf of spectra $\underline{\mathrm{E}}$ on $B(\mathbb{G}_h)_{\text{proét}}$ with*

$$\begin{aligned} \Gamma(\mathbb{G}_h/*, \underline{\mathrm{E}}) &\simeq E_h, \\ \Gamma(\mathbb{G}_h/\mathbb{G}_h, \underline{\mathrm{E}}) &\simeq L_{K(h)}\mathbb{S}. \end{aligned} \tag{4}$$

Its homotopy sheaves are given by $\pi_t\underline{\mathrm{E}} = \underline{\pi_t E_h}$*, the proétale abelian group represented by the profinite topology on* $\pi_t E_h$*. For any closed subgroup* $G \subset \mathbb{G}_h$*, the descent spectral sequence agrees with the* $K(h)$*-local* E_h*-Adams spectral sequence (including differentials).*

This may be of independent interest as it gives a novel construction of the $K(h)$-local E_h-Adams spectral sequence, which may be extended to an arbitrary spectrum X; its E_2-page (given *a priori* in terms of sheaf cohomology on the proétale site) is continuous group cohomology for suitable X (Remark 2.46). Note moreover that proétale cohomology enjoys excellent functoriality properties, and that the category of L-complete abelian sheaves on $B\mathbb{G}_{\text{proét}}$ is abelian, as opposed to L-complete $(E_h)^\vee_* E_h$-comodules.

Next, we deduce a descent result for ∞-categories of $K(h)$-local modules, which is really an extension to the condensed world of the following significant theorem:

Theorem 1.2 ([Mat16])**.** *The diagram of symmetric monoidal* ∞*-categories*

$$\mathcal{S}p_{K(h)} \longrightarrow \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)}) \rightrightarrows \mathrm{Mod}_{L_{K(h)}E_h \wedge E_h}(\mathcal{S}p_{K(h)}) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \cdots$$

is a limit cone.

Namely, in Section 3 we prove the following profinite Galois descent result, which can be seen as the identification $\mathcal{S}p_{K(h)} \simeq \left(\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})\right)^{h\mathbb{G}_h}$ analogous to (1).

Theorem A.II (Theorem 3.1)**.** *There is a hypercomplete sheaf* $\mathrm{Mod}_{\underline{\mathrm{E}}}(\mathcal{S}p_{K(h)})$ *of symmetric monoidal* ∞*-categories on* $B(\mathbb{G}_h)_{\text{proét}}$ *with*

$$\begin{aligned} \Gamma(\mathbb{G}_h/*, \mathrm{Mod}_{\underline{\mathrm{E}}}(\mathcal{S}p_{K(h)})) &\simeq \mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)}), \\ \Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathrm{Mod}_{\underline{\mathrm{E}}}(\mathcal{S}p_{K(h)})) &\simeq \mathcal{S}p_{K(h)}. \end{aligned} \tag{5}$$

One recovers the first part of Theorem A by taking Picard spectra pointwise. For the second part of that theorem, we must identify the E_2-page of the descent spectral sequence, which *a priori* begins which sheaf cohomology on the proétale site. The results of [BS14] allow us to deduce this, as a consquence of the following identification:

Theorem A.III (Theorem 3.11)**.** *There is a hypercomplete sheaf of connective spectra* $\mathfrak{pic}(\underline{\mathrm{E}})$ *on* $B(\mathbb{G}_h)_{\text{proét}}$ *with*

$$\begin{aligned} \Gamma(\mathbb{G}_h/*, \mathfrak{pic}(\underline{\mathrm{E}})) &\simeq \mathfrak{pic}(E_h) \\ \Gamma(\mathbb{G}_h/\mathbb{G}_h, \mathfrak{pic}(\underline{\mathrm{E}})) &\simeq \mathfrak{pic}(\mathcal{S}p_{K(h)}). \end{aligned} \tag{6}$$

The proétale homotopy groups of $\mathfrak{pic}(\underline{\mathrm{E}})$ *are*

$$\pi_t \mathfrak{pic}(\underline{\mathrm{E}}) = \underline{\pi_t \mathfrak{pic}(E_h)},$$

i.e. represented by the homotopy groups of $\mathfrak{pic}(E_h)$ *(with their natural profinite topology). For any closed subgroup* $G \subset \mathbb{G}_h$*, the descent spectral sequence agrees with the Bousfield-Kan spectral sequence for the cosimplicial spectrum* $\mathfrak{pic}(E_h^{hG} \otimes E_h^{\otimes \bullet+1})$*.*

The identification of homotopy groups is immediate for $t \geq 1$ (by comparing with $\pi_t\underline{\mathrm{E}}$), but there is some work to do for $t = 0$; this is the same issue that accounts for the uncertainty in degree zero in the descent spectral sequence of [Hea22].

In fact we go a bit further, relating the spectral sequence of Theorem A to the $K(h)$-local E_h-Adams spectral sequence. In degrees $t \geq 2$, the homotopy groups of the Picard spectrum of an $\mathbb{E}_\infty$-ring are related by a shift to those of the ring itself. It is a result of [MS16] that this identification lifts to one between truncations of the two spectra, in a range that grows with t: that is, for every $t \geq 2$ there is an equivalence

$$\tau_{[t,2t-2]}\mathfrak{pic}(A) \simeq \tau_{[t,2t-2]}\Sigma A,$$

functorial in the ring spectrum A. Using the proétale model, it is quite straightforward to deduce the following comparison result, as proven in *op. cit.* for finite Galois extensions.

Theorem A.IV (Proposition 3.22 and Corollary 3.25; c.f. [MS16])**.** *Let* $G \subset \mathbb{G}_h$ *be a closed subgroup.*

(1) Suppose $2 \leq r \leq t-1$. Under the identification

$$E_2^{s,t} = H^s(G, \pi_t \mathfrak{pic}(E_h)) \cong H^s(G, \pi_{t-1} E_h) = E_2^{s,t-1}(ASS),$$

the d_r-differential on the group $E_r^{s,t}$ in (3) agrees with the differential on the group $E_r^{s,t-1}(ASS)$ on classes that survive to E_r in both.

(2) If $x \in H^t(G, \pi_t \mathfrak{pic}(E_h)) \cong H^t(G, \pi_{t-1} E_h)$ (and x survives to the t-th page in both spectral sequences), then the differential $d_t(x)$ is given by the following formula in the $K(h)$-local E_h-Adams spectral sequence:

$$d_t(x) = d_t^{ASS}(x) + x^2.$$

Comparison with Morava modules. Finally, we will says some words on how to derive Theorems C and D from the main result. Recall that a useful technique for computing Picard groups, originating already in [HMS94], is to use completed E-theory to compare the category of $K(h)$-local spectra to the category $\mathrm{Mod}_{\pi_* E_h}^{\mathbb{G}_h}$ of *Morava modules*, i.e. L-complete $\pi_* E_h$-modules equipped with a continuous action of the Morava stabiliser group $\mathbb{G}_h$:

$$(E_h)_*^\vee(-) \coloneqq \pi_* L_{K(h)}(E_h \wedge (-)) : \mathcal{S}p_{K(h)} \to \mathrm{Mod}_{\pi_* E_h}^{\mathbb{G}_h}. \tag{7}$$

This carries invertible $K(h)$-local spectra to invertible Morava modules, and hence induces a homomorphism on Picard groups. The category on the right-hand side is completely algebraic in nature, and its Picard group $\mathrm{Pic}_h^{\mathrm{alg}}$ can (at least in theory) be computed as a $\mathbb{Z}/2$-extension of $\mathrm{Pic}_h^{\mathrm{alg},0} = H^1(\mathbb{G}_h, (\pi_0 E_h)^\times)$; the strategy is therefore to understand the comparison map and how much of Pic_h it can see. A remarkable theorem of Pstrągowski says that the map $(E_h)_*^\vee : \mathrm{Pic}_h \to \mathrm{Pic}_h^{\mathrm{alg}}$ is an isomorphism if $p \gg n$ (more precisely, if $2p-2 > n^2+n$). This reflects the more general phenomenon that chromatic homotopy theory at large primes is well-approximated by algebra, as is made precise in [BSS20, BSS21].

As noted in [Pst22], the existence of a spectral sequence of the form (3) immediately yields an alternative proof, by sparseness of the $K(h)$-local E_h-Adams spectral sequence at large primes (in fact, this improves slightly the bound on p). Heard gave such a spectral sequence in [Hea22], and our results can be seen as a conceptual interpretation of that spectral sequence, analogous to the relation of [Dav06, Qui11] to [DH04]. Beyond the conceptual attractiveness, our derivation of the spectral sequence also clarifies certain phenomena: for example, we give a proof of the claim made in [Hea22] that the exotic part of the Picard group is given precisely by those elements in filtration at least 2:

Theorem E (Theorem 4.4)**.** *For any pair (h,p), the algebraic Picard group is computed by the truncation to filtration ≤ 1 of (3), and the exotic Picard group κ_h agrees with the subgroup of* Pic_h *in filtration at least 2 for (3).*

For example, when $h^2 = 2p-1$ this leads to the description of the exotic Picard group given in Theorem C.

Note also that the group in bidegree $(s,t) = (1,0)$ of Heard's Picard spectral sequence is undetermined, which is an obstruction to computing Brauer groups; as discussed in Section 3.2, the relevant group in (3) really is $H^1(\mathbb{G}_h, \mathrm{Pic}(E_h))$. As expected, the computation simplifies at sufficiently large primes, and this should give rise to an algebraicity statement for the group Br_h^0. We intend to explore the algebraic analogue $\mathrm{Br}_h^{\mathrm{alg},0}$ in future work.

1.1. Outline.

In Section 2, we collect the results we need on the Devinatz-Hopkins action and the proétale site, showing how to define the sheaf of spectra $\underline{\mathrm{E}}$. One can in fact deduce it is a sheaf from Theorem 3.1. Nevertheless, we wanted to give a self-contained proof of the spectrum-level hyperdescent; we also explain how this compares with Davis' construction of the continuous action on E_h. In the second half of Section 2 we compute the homotopy sheaves of $\underline{\mathrm{E}}$, and explain how this leads to the identification with the $K(h)$-local E_h-Adams spectral sequence; the requisite décalage results are collected in Appendix A.

In Section 3 we categorify, obtaining descent results for the presheaf of $K(h)$-local module ∞-categories over $\underline{\mathrm{E}}$. We discuss how the Picard spectrum functor yields a sheaf of connective spectra exhibiting the identification of fixed points $\mathfrak{pic}(\mathcal{S}p_{K(h)}) \simeq \tau_{\geq 0} \mathfrak{pic}(E_h)^{h\mathbb{G}_h}$, and investigate the resulting descent spectral sequence.

In Section 4 we use the previous results to compute Picard groups. We first identify the algebraic and exotic Picard groups in the descent spectral sequence. Combining this with the well-known form of the $K(1)$-local E_1-Adams spectral sequence allows us to reprove the results of [HMS94]: we are particularly interested in computing the exotic Picard group at the prime 2. We also consider Picard groups in the boundary case $h^2 = 2p-1$. In Appendix B we give a method to compute the height one Adams spectral sequence at $p=2$ using the Postnikov tower for the sheaf $\underline{\mathrm{E}}$.

Finally, in Section 5 we show how to use our results in Brauer group computations. We show that the (-1)-stem in the Picard spectral sequence gives an upper bound for the relative Brauer group, and compute this bound at height one.

1.2. **Relation to other work.** As already mentioned, Heard has obtained a spectral sequence similar to that of Theorem A, and one of our objectives in this work was to understand how to view that spectral sequence as a HFPSS for the Goerss-Hopkins-Miller action. Recently, a result close to Theorem A was proven by Guchuan Li and Ningchuan Zhang [LZ23]. Their approach differs somewhat from ours, using Burklund's result on multiplicative towers of generalised Moore spectra to produce pro-object presentations of $\mathrm{Mod}_{E_h}(\mathcal{S}p_{K(h)})$ and $\mathfrak{pic}_n$; a detailed comparison between the two would certainly be of interest.

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1.4. **Notation and conventions.**

- Throughout, we will work at a fixed prime p and height h, mostly kept implicit. For brevity, we will therefore write E, K and $\mathbb{G}$ for Morava E-theory, Morava K-theory and the extended Morava stabiliser group, respectively. These will be our principal objects of study.
- We will freely use the language of ∞-categories (modeled as quasi-categories) as pioneered by Joyal and Lurie [Lur09, Lur17, Lur18b]. In particular, all (co)limits are ∞-categorical. We will mostly be working internally to the $K(h)$-local category, and as such we stress that the symbol $\otimes$ will denote the $K(h)$**-local** smash product throughout; where we have a need for it, we will use the notation $\wedge$ for the smash product of spectra. On the other hand, to avoid confusion we will distinguish $K(h)$-local colimits by writing for example $L_{\mathrm{K}} \operatorname{colim} X$.
- We use the notation $\varinjlim$ to denote a *filtered* colimit, and similarly for cofiltered limits. In particular, if T is a profinite set, we will use the expression '$T = \varprojlim T_i$' to refer to a presentation of T as a pro-object, leaving implicit that each T_i is finite. We will also assume throughout we have made a fixed choice of decreasing open subgroups $U_i \subset \mathbb{G}$ with trivial intersection; the symbols $\varprojlim_i$ and $\varinjlim_i$ will always refer to the (co)limit over such a family. Likewise, we will assume we have chosen a sequence of ideals $I \subset \pi_0 E_h$ generating the $\mathfrak{m}$-adic topology; without loss of generality, the ideals I will be chosen so that there exists a tower M_I of generalised Moore spectra, with $\pi_* M_I = (\pi_* E_h)/I$.
- If G is a profinite group and M a topological G-module, we write $\underline{M}$ for the sheaf $\mathrm{Cont}_G(-, M)$ on the proétale site (Section 2).
- We only consider spectra with group actions, i.e. functors $BG \to \mathcal{S}p$. When G is a profinite group, we will write $H^*(G, M)$ for *continuous* group cohomology with pro-p (or more generally profinite) coefficients, as defined for example in [SW00] (resp. [Jan88]).
- A few words about spectral sequences. When talking about the 'Adams spectral sequence', we always have in mind the $K(h)$-local E_h-based Adams spectral sequence. We will freely use abbreviations such as 'ASS', 'HFPSS', 'BKSS'. We will also use the name 'descent spectral sequence' for either the t-structure or Čech complex definition, since in the cases of interest we show they agree up to reindexing; when we need to be more explicit, we refer to the latter as the 'Bousfield-Kan' or 'Čech' spectral sequence. The name 'Picard spectral sequence' will refer to the descent spectral sequence for the sheaf $\mathfrak{pic}(\underline{\mathrm{E}})$.
- In Sections 2 and 3 we form spectral sequences using the usual t-structure on spectral sheaves; this is useful for interpreting differentials and filtrations, for example in Theorem 4.4. To obtain familiar charts, we will declare that the spectral sequence associated to a filtered object starts at the E_2-page;

in other words, this is the page given by homotopy groups of the associated graded object. Thus our spectral sequences run

$$E_2^{s,t} = H^s(G, \pi_t E) \implies \pi_{t-s} E^{hG},$$

with differentials d_r of $(s, t-s)$-bidegree $(r, -1)$, and this is what we display in all figures. However, we also make use of the Bousfield-Kan definition of the descent spectral sequence using the Čech complex of a covering, and we relate the two formulations by décalage (see Appendix A): there is an isomorphism between the two spectral sequences that reads

$$E_2^{s,t} \cong \check{E}_3^{2s-t,s}$$

if we use the same grading conventions for each of the underlying towers of spectra. We will always use s for filtration, t for internal degree, and $t-s$ for stem.

- Finally, we largely ignore issues of set-theory, since these are discussed at length in [Sch19] and [BH19] and for the most part do not affect the arguments here. For concreteness, one could fix a hierarchy of strongly inaccessible cardinals $\kappa < \delta_0 < \delta_1$ such that $|\mathbb{G}_h| < \kappa$ and the unit in $\mathcal{S}p_{K(h)}$ is κ-compact, and work throughout over the 'δ_1-topos' of sheaves of δ_1-small spaces on profinite $\mathbb{G}_h$-sets of cardinality less than δ_0.

2. The continuous action on Morava E-theory

2.1. **Discrete Morava E-theory.** Fix an algebraic extension k of $\mathbb{F}_{p^h}$, and a formal group law Γ of height h over k. Let $\mathrm{E} \coloneqq E(k, \Gamma)$ be Morava E-theory based on the pair (k, Γ); this choice will henceforth be fixed, and kept implicit to ease notation. Let $\mathrm{K} \coloneqq K(k, \Gamma)$ be Morava K-theory, its residue field. Recall that E is the K-local Landweber exact spectrum whose formal group is the universal deformation of Γ to the Lubin-Tate ring $\pi_0 \mathrm{E} = \mathbb{W}(k)[[u_1, \dots, u_{h-1}]]$. Functoriality yields an action of the extended Morava stabiliser group $\mathbb{G} \coloneqq \mathrm{Aut}(k, \Gamma)$ on the homotopy ring spectrum E, and celebrated work of Goerss, Hopkins, Miller and Lurie [GH04, Lur18a] promotes E to an $\mathbb{E}_\infty$-ring and the action to one by $\mathbb{E}_\infty$ maps. This action controls much of the structure of the K-local category, and is the central object of study in this document. In this section, we formulate the action of $\mathbb{G}$ on E in a sufficiently robust way for our applications; to do so, we will present E as a sheaf of spectra on the *proétale classifying site* of $\mathbb{G}$. Descriptions of the K-local E-Adams spectral sequence have been previously given, notably in work of Davis and of Quick [Dav03, Dav06, Qui11, BD10, DQ16], who described a number of formulations of this action as the *continuous* action of the *profinite* group $\mathbb{G}$.

Recall that continuous actions and continuous cohomology of a topological group G are generally much more straightforward when we assume our modules to have *discrete* topology. There are notable categorical benefits in this case: for example, it is classical that the category of discrete G-modules is abelian with enough injectives, which is not true of the full category of topological modules. Further, in the discrete context we can understand actions completely by looking at the induced actions of all finite quotients of G. This was pioneered by Thomason in his study of $K(1)$-local descent for algebraic K-theory, and formalised in a model-theoretic sense by Jardine.

Any profinite group G has an *étale classifying site*, denoted $BG_{\text{ét}}$, whose objects are the (discrete) finite G-sets and whose coverings are surjections; as shown in [Jar97, §6], the category of sheaves of abelian groups on $BG_{\text{ét}}$ gives a category equivalent to the category Ab_G^δ of discrete G-modules in the sense of [Ser97]. As noted below, sheaf cohomology on $BG_{\text{ét}}$ corresponds to continuous group cohomology with discrete coefficients, again in the sense of Serre. Motivated by this, Jardine defines a model structure on presheaves of *spectra* on $BG_{\text{ét}}$, which models the category of 'discrete' continuous G-spectra, i.e. those that can be obtained as the filtered colimit of their fixed points at open subgroups. We will more generally refer to objects of $\widehat{\mathcal{S}h}(BG_{\text{ét}}, \mathcal{C})$ as *discrete G-objects* of an arbitrary (cocomplete) ∞-category $\mathcal{C}$. Davis uses this as his starting point, and we observe below that in this formalism it is easy to pass to the ∞-categorical setting.

Theorem 2.1. *There is a sheaf of* K*-local* $\mathbb{E}_\infty$*-rings* $\underline{\mathrm{E}}^\delta$ *on* $B\mathbb{G}_{\text{ét}}$ *with the following properties:*

(1) Any decreasing sequence (U_i) of open subgroups of $\mathbb{G}$ with zero intersection induces

$$L_{\mathrm{K}} \varinjlim_i \underline{\mathrm{E}}^\delta(G/U_i) \simeq \mathrm{E}.$$

(2) On global sections, $\Gamma\underline{\mathrm{E}}^\delta \coloneqq \Gamma(\mathbb{G}/\mathbb{G}, \underline{\mathrm{E}}^\delta) \simeq \mathbf{1}_{\mathrm{K}}$ is the K*-local sphere spectrum.*

(3) For any normal inclusion of open subgroups $V \subset U \subset \mathbb{G}$, the map $\underline{\mathrm{E}}^\delta(\mathbb{G}/U) \to \underline{\mathrm{E}}^\delta(\mathbb{G}/V)$ is a faithful U/V-Galois extension.

Remark 2.2. We quickly justify the choice of notation. In Section 2.3 we construct a sheaf $\underline{\mathrm{E}}$ on the bigger *proétale* site $B\mathbb{G}_{\text{proét}}$, and $\underline{\mathrm{E}}^{\delta}$ is the restriction of E along $B\mathbb{G}_{\text{ét}} \to B\mathbb{G}_{\text{proét}}$ by the results of Section 2.2.3. The choice of notation for $\underline{\mathrm{E}}$ is motivated by Lemma 2.44, which asserts that $\pi_*\underline{\mathrm{E}} = \underline{\pi_*\mathrm{E}}$. In fact, one can view E as an I_h-adic spectrum in the sense of [Lur18b, §7.3], and $\underline{\mathrm{E}}$ should be thought of as the proétale spectrum associated to this pro-spectrum.

We give two proofs. The first, related to the work mentioned above, collects the necessary results from classical literature. Namely, we deduce the theorem by combining work of Devinatz-Hopkins and Rognes, which works in the case $k = \mathbb{F}_{p^h}$ (if only because this is the generality of the results of [DH04]).

Proof 1. The presheaf of spectra $\underline{\mathrm{E}}^{\delta}$ is constructed in [DH04, §4], with (2) being part of Theorem 1 therein and the identification (1) being the trivial case of Theorem 3; see also [BBGS22, §2] for a nice summary. Devinatz and Hopkins construct $\underline{\mathrm{E}}^{\delta}$ by hand (by taking the limit of the *a priori* form of its Amitsur resolution in $K(h)$-local E_h-modules); they denote $\Gamma(\mathbb{G}/U, \underline{\mathrm{E}}^{\delta})$ by E_h^{hU}, but we copy [BD10] and write E_h^{dhU}. By [DH04, Theorem 4], $\underline{\mathrm{E}}^{\delta}$ is a sheaf on $B\mathbb{G}_{\text{ét}}$. Finally item (3) is [Rog08, Theorem 5.4.4]. □

The second proof uses instead the Galois correspondence of [Mat16], and works for general k.

Proof 2. We prove the theorem when $k = \overline{\mathbb{F}}_p$, which implies the result for arbitrary k by restriction. In this case, one has Galois group

$$\pi_1(\mathcal{S}p_{\mathrm{K}}) \simeq \mathbb{G} \in \mathrm{Pro}(\mathrm{Grp}_{\mathrm{fin}})^{\mathrm{op}} \simeq \mathrm{Ind}(\mathrm{Grp}_{\mathrm{fin}}^{\mathrm{op}})$$

by [Mat16, Theorem 10.9]. Unravelling the construction of the Galois correspondence [Mat16, Theorem 5.36], this amounts to the existence of an equivalence of Galois categories

$$\underline{\mathrm{E}}^{\delta} : \mathrm{Fun}(B\mathbb{G}, \mathrm{Fin}) \coloneqq \varinjlim \mathrm{Fun}(B(\mathbb{G}/U), \mathrm{Fin}) \simeq \mathrm{CAlg}^{\mathrm{Gal}}(\mathcal{S}p_{\mathrm{K}})^{\mathrm{op}},$$

where $\mathbb{G} = \varprojlim \mathbb{G}/U$. Note that filtered colimits in GalCat are computed at the level of underlying categories [Mat16, Proposition 5.34], so that $\mathrm{Fun}(B\mathbb{G}, \mathrm{Fin})$ is the category $B\mathbb{G}_{\text{ét}}$ (which is the filtered colimit of the categories $B(\mathbb{G}/U)_{\text{ét}}$). One therefore obtains a functor

$$\underline{\mathrm{E}}^{\delta} : B\mathbb{G}_{\text{ét}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{S}p_{\mathrm{K}}).$$

This sends the terminal object of $B\mathbb{G}_{\text{ét}}$ to the initial object in $\mathrm{CAlg}^{\mathrm{Gal}}(\mathcal{S}p_{\mathrm{K}})$, so that $\Gamma\underline{\mathrm{E}}^{\delta} = \mathbf{1}_{\mathrm{K}}$. Property (4) follows from [Mat16, Corollary 5.18 and Proposition 6.13], and descent for a covering $\mathbb{G}/V \twoheadrightarrow \mathbb{G}/U$ amounts to the Galois condition $\underline{\mathrm{E}}^{\delta}(\mathbb{G}/U) \simeq \underline{\mathrm{E}}^{\delta}(\mathbb{G}/V)^{hU/V}$, at least when V is normal in U. Any transitive finite $\mathbb{G}$-set is covered by one of the form $\mathbb{G}/U$ with U normal, so if need be we may implicitly sheafify the left Kan extension along the inclusion of finite $\mathbb{G}$-sets with normal stabilisers into $B\mathbb{G}_{\text{ét}}$.

It remains to prove claim (1). For brevity we write $A \coloneqq L_{\mathrm{K}} \varinjlim \underline{\mathrm{E}}^{\delta}(\mathbb{G}/U)$, and begin by constructing a particular map $A \to \mathrm{E}$. Note that for U normal in $\mathbb{G}$ and any $X \in \mathrm{CAlg}(\mathcal{S}p_{\mathrm{K}})$ we have

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}p_{\mathrm{K}})}(\underline{\mathrm{E}}^{\delta}(\mathbb{G}/U), X) \simeq \lim \left(\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{E}^{\otimes \bullet+1},\mathrm{K}}}(\underline{\mathrm{E}}^{\delta}(\mathbb{G}/U) \otimes \mathrm{E}^{\otimes \bullet+1}, X \otimes \mathrm{E}^{\otimes \bullet+1})\right)$$

by descendability of E. Every $\underline{\mathrm{E}}^{\delta}(\mathbb{G}/U)$ splits over E, since its Galois theory is algebraic [Mat16, Proposition 6.29] and hence trivial [GR71, Exposé I, Corollaire I.8.4]. As in the proof of [Mat16, Proposition 5.28], the spaces appearing in the limit diagram are *sets* of decompositions

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{E}^{\otimes n},\mathrm{K}}}(\underline{\mathrm{E}}^{\delta}(\mathbb{G}/U) \otimes \mathrm{E}^{n}, X \otimes \mathrm{E}^{\otimes n}) \simeq \left\{ \varphi : \prod_{gU \in G/U} X_{gU,n} \simeq X \otimes \mathrm{E}^{\otimes n} \right\},$$

where each $X_{gU} \in \mathrm{CAlg}_{\mathrm{E}^{\otimes n},\mathrm{K}}$. In particular, for $X = \mathrm{E}$ we can choose the idempotents

$$e_{gU} \in \pi_0 \mathrm{E} \otimes \mathrm{E} \cong \mathrm{Cont}(\mathbb{G}, \pi_0 \mathrm{E})$$

taking 1 on the $gU \subset \mathbb{G}$ and 0 otherwise; since $\sum_{\mathbb{G}/U} e_{gU} = 1$ we have a canonical equivalence

$$\varphi_U : \prod_{\mathbb{G}/U} (\mathrm{E} \otimes \mathrm{E})[e_{gU}^{-1}] \simeq \mathrm{E} \otimes \mathrm{E},$$

and hence a map $\underline{\mathrm{E}}^{\delta}(\mathbb{G}/U) \otimes \mathrm{E} \to \mathrm{E} \otimes \mathrm{E}$ given explicitly by the composite

$$\underline{\mathrm{E}}^{\delta}(\mathbb{G}/U) \otimes \mathrm{E} \simeq \prod_{\mathbb{G}/U} \mathrm{E} \to \prod_{\mathbb{G}/U} (\mathrm{E} \otimes \mathrm{E})[e_{gU}^{-1}] \underset{\varphi_U}{\simeq} \mathrm{E} \otimes \mathrm{E}.$$

This decomposition is fixed by the $\mathbb{G}$-action (since each e_{gU} is), and hence defines a map $\alpha_U\colon \underline{\mathrm{E}}^\delta(\mathbb{G}/U) \to \mathrm{E}$; since the mapping spaces in the diagram are all sets, no higher coherence is required. Note that α_U is defined by the property that

$$\mathrm{E}^\vee_*\alpha_U = p_U^* : \mathrm{Cont}(\mathbb{G}/U, \pi_*\mathrm{E}) = \mathrm{E}^\vee_*\underline{\mathrm{E}}^\delta(\mathbb{G}/U) \to \mathrm{Cont}(\mathbb{G}, \mathrm{E}) = \mathrm{E}^\vee_*\mathrm{E}, \tag{8}$$

where $p\colon \mathbb{G} \to \mathbb{G}/U$ is the projection and $\mathrm{E}^\vee_*(-) \coloneqq \pi_*(\mathrm{E} \otimes (-))$. In particular this choice is compatible in U, and hence defines a map $\alpha\colon A \to \mathrm{E}$.

To verify that α is an equivalence it is enough to show that $\mathrm{E}^\vee_*\alpha$ is an isomorphism, since $\mathbf{1}_{\mathrm{K}} \to \mathrm{E}$ is faithful. We have

$$A \otimes \mathrm{E} \simeq \varinjlim_{U} \prod\nolimits_{\mathbb{G}/U} \mathrm{E} \simeq \lim_I \left(\varinjlim_{U} \prod\nolimits_{\mathbb{G}/U} \mathrm{E} \otimes M_I \right)$$

for $\{M_I\}$ a tower of generalised Moore spectra, and hence $\mathrm{E}^\vee_* A$ may be computed using a Milnor sequence as in Lemma 2.44; this yields

$$\mathrm{E}^\vee_* A \cong \mathrm{Cont}(\mathbb{G}, \pi_t\mathrm{E}).$$

Passing to the limit in (8), we see that $\mathrm{E}^\vee_*\alpha$ is the identity on $\mathrm{Cont}(\mathbb{G}, \pi_t\mathrm{E})$. □

Remark 2.3 (Hypercompleteness)**.** We can further deduce that $\underline{\mathrm{E}}^\delta$ is hypercomplete from the work of Davis and Dugger-Hollander-Isaksen; alternatively, this will be a consequence of hypercompleteness for $\underline{\mathrm{E}}$, which we prove in Section 2.2.3. To reassure the reader that the arguments are not circular, we note that the proof of Lemma 2.27 uses only usual (i.e., not hyper-) descent and nilpotence of the extension $\mathbf{1}_{\mathrm{K}} \to \mathrm{E}$.

More specifically, Davis utilises the Jardine model structure on the category of presheaves of spectra on $B\mathbb{G}_{\text{ét}}$, denoted $\mathrm{Spt}_{\mathbb{G}}$[1]; this is defined in such a way that there is a Quillen adjunction

$$\mathrm{Spt} \underset{(-)^{h\mathbb{G}}}{\overset{\mathrm{const}}{\rightleftarrows}} \mathrm{Spt}_{\mathbb{G}}\,.$$

Recall that the main result of [DHI04] says that the fibrant objects of $\mathrm{Spt}_{\mathbb{G}}$ are precisely those projectively fibrant presheaves that satisfy (i) the (1-categorical) sheaf condition for coverings in $B\mathbb{G}_{\text{ét}}$, and (ii) descent for all hypercovers, and so the ∞-category associated to $\mathrm{Spt}_{\mathbb{G}}$ is a full subcategory of $\widehat{\mathcal{S}h}(B\mathbb{G}_{\text{ét}}, \mathcal{S}p)$.

In this setting, Davis shows that the spectrum $F_h \coloneqq \varinjlim E_h^{dhU}$ (the colimit taken in plain spectra) defines a fibrant object of $\mathrm{Spt}_{\mathbb{G}}$, $\mathcal{F}\colon \mathbb{G}/U \mapsto F_h^{hU}$ [Dav06, Corollary 3.14]; Behrens and Davis show that $E_h^{hU} \simeq L_{\mathrm{K}}F_h^{hU} \simeq E_h^{dhU}$ for any open subgroup $U \subset \mathbb{G}$ [BD10, Discussion above Prop. 8.1.2 and Lemma 6.3.6 respectively]. In fact, [BD10, Theorem 8.2.1] proves the same equivalence for any *closed* subgroup H, but restricting our attention to open subgroups cuts out some of the complexity and makes clear the equivalences happen naturally in U (we remark that they also appeared in Chapter 7 of Davis' thesis [Dav03]). This provides a projective equivalence between $\underline{\mathrm{E}}^\delta$ and $L_{\mathrm{K}}\mathcal{F}$, and hence an equivalence of *presheaves* between $\underline{\mathrm{E}}^\delta$ and a hypercomplete sheaf of spectra.

If we pick a cofinal sequence of open normal subgroups (U_i) we can identify the starting page of the descent spectral sequence for $\underline{\mathrm{E}}^\delta$:

Lemma 2.4. *Let G be a profinite group and $\mathcal{F} \in \widehat{\mathcal{S}h}(BG_{\text{ét}}, \mathcal{C})$, where $\mathcal{C} = \mathcal{S}p$ or $\mathcal{S}p_{\geq 0}$. There is a spectral sequence with starting page*

$$E_2^{s,t} = H^s(G, \pi_t \varinjlim \mathcal{F}(G/U_i)),$$

and converging conditionally to $\pi_{t-s}\Gamma \lim_t \tau_{\leq t}\mathcal{F}$.

Proof. This is the spectral sequence for the Postnikov tower of $\mathcal{F}$, formed as in [Lur17, §1.2.2]. Its starting page is given by sheaf cohomology of the graded abelian sheaf $\pi_*\mathcal{F}$ on $BG_{\text{ét}}$. Explicitly, form the Postnikov tower in sheaves of spectra

$$\begin{tikzcd}
\mathcal{F} \arrow[r] & \cdots \arrow[r] & \tau_{\leq 1}\mathcal{F} \arrow[r] \arrow[d] & \tau_{\leq 0}\mathcal{F} \arrow[r] \arrow[d] & \cdots \\
 & & \Sigma\pi_1\mathcal{F} \arrow[ul] & \pi_0\mathcal{F} \arrow[ul] &
\end{tikzcd}$$

---

[1] We have taken slight notational liberties: in [Dav03], $\mathrm{Spt}_{\mathbb{G}}$ denotes the category of spectra based on discrete $\mathbb{G}$-sets, which is equipped with a model structure lifted from the Jardine model structure on the (equivalent) category **ShvSpt** of sheaves of spectra.

Applying global sections and taking homotopy groups gives an exact couple, and we obtain a spectral sequence with
$$E_2^{s,t} = \pi_{t-s}\Gamma\Sigma^t\pi_t\mathcal{F} = R^s\Gamma\pi_t\mathcal{F} = H^s(B\mathbb{G}_{\text{ét}}, \pi_t\mathcal{F}),$$
with abutment $\pi_{t-s}\Gamma\lim\tau_{\leq t}\mathcal{F}$. To identify this with continuous group cohomology, we make use of the equivalence
$$\begin{aligned}\mathrm{Ab}_G^\delta &\to \mathcal{S}h(BG_{\text{ét}}, \mathrm{Ab})\\ M &\mapsto \left(\coprod G/U \mapsto \prod M^U\right)\end{aligned}$$
of [Jar97], whose inverse sends $\mathcal{F} \mapsto \varinjlim_i \mathcal{F}(G/U_i)$. Under this equivalence the fixed points functor on Ab_G^δ corresponds to global sections, and so taking derived functors identifies sheaf cohomology on the right-hand side with derived fixed points on the left; as in [Ser97, §2.2], when we take discrete coefficients this agrees with the definition in terms of continuous cochains. □

Writing $\mathrm{Mod}^\wedge_{(-)} \coloneqq \mathrm{Mod}_{L_\mathrm{K}(-)}(\mathcal{S}p_\mathrm{K}) = L_\mathrm{K}\,\mathrm{Mod}_{(-)}$, our strategy is is to apply the functor $\mathfrak{pic}\circ\mathrm{Mod}^\wedge_{(-)}$: $\mathrm{CAlg} \to \mathcal{S}p_{\geq 0}$ pointwise to the sheaf $\underline{\mathrm{E}}^\delta$, in order to try to obtain a sheaf $\mathfrak{pic}_\mathrm{K}(\underline{\mathrm{E}}^\delta) \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\text{ét}}, \mathcal{S}p_{\geq 0})$; we'd then like to apply the above lemma to deduce the existence of the descent spectral sequence. This does not quite work for the same reason that the lemma applied to $\underline{\mathrm{E}}^\delta$ does not recover the K-local E-Adams spectral sequence: while E is K-locally discrete, it is certainly not discrete as a $\mathbb{G}$-spectrum (for example, the action of $\mathbb{G}_1 = \mathbb{Z}_p^\times$ on $\pi_2\mathrm{E} = \mathbb{Z}_p(1)$ at height one is free, and in particular not discrete). Nevertheless, it is worth remarking that the first step of this approach does work: since E is a discrete $\mathbb{G}$-object of K-*local spectra*, $\mathrm{Mod}^\wedge_\mathrm{E}$ is discrete as a presentable ∞-category with $\mathbb{G}$-action. This is a consequence of the following two results.

Lemma 2.5. *The composition* $\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta} : {B\mathbb{G}_{\text{ét}}}^{op} \to \mathrm{CAlg}(\mathcal{S}p_\mathrm{K}) \to \mathrm{Pr}^{L,\mathrm{smon}}$ *is a sheaf.*

Proof. To check the sheaf condition for $\mathcal{F} : B\mathbb{G}_{\text{ét}}^{op} \to \mathcal{C}$ we need to show that finite coproducts are sent to coproducts, and that for any inclusion $U \subset U'$ of open subgroups the canonical map $\mathcal{F}(\mathbb{G}/U') \to \operatorname{Tot}\mathcal{F}(\mathbb{G}/U^{\times_{\mathbb{G}/U'}\bullet+1})$ is an equivalence. For the presheaf $\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta}$, the first is obvious (using the usual idempotent splitting), while the second is *finite* Galois descent [Mei12, Proposition 6.2.6] or [GL21, Theorem 6.10], at least after refining U to a normal open subgroup of U'. □

Fixing again a cofinal sequence (U_i), we write $F_{ij} \dashv R_{ij} : \mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta(\mathbb{G}/U_i)} \rightleftarrows \mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta(\mathbb{G}/U_j)}$ and $F_j \dashv R_j$ for the composite adjunction $\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta(\mathbb{G}/U_i)} \rightleftarrows \mathrm{Mod}^\wedge_\mathrm{E}$, and
$$\varinjlim \mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta(\mathbb{G}/U_i)} \underset{R}{\overset{F}{\rightleftarrows}} \mathrm{Mod}^\wedge_\mathrm{E}$$
for the colimit (along the functors F_{ij}) in $\mathrm{Pr}^{L,\mathrm{smon}}$.

Proposition 2.6. *The functors F and R define an adjoint equivalence* $\varinjlim \mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta(\mathbb{G}/U_i)} \simeq \mathrm{Mod}^\wedge_\mathrm{E}$.

More generally:

Proposition 2.7. *Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category. Suppose $A_{(-)} : I \to \mathrm{CAlg}(\mathcal{C})$ is a filtered diagram, and write A for a colimit (formed equivalently in $\mathcal{C}$ or $\mathrm{CAlg}(\mathcal{C})$). Then the induced adjunction*
$$\varinjlim \mathrm{Mod}_{A_i}(\mathcal{C}) \underset{R}{\overset{F}{\rightleftarrows}} \mathrm{Mod}_A(\mathcal{C})$$
is an equivalence of presentable *symmetric-monoidal ∞-categories.*

Proof. This is implied by [Lur17, Corollary 4.8.5.13], since filtered categories are weakly contractible and filtered colimits in $\mathrm{CAlg}_\mathcal{C}(\mathrm{Pr}^L)$ computed in Pr^L. Alternatively, one can give an explicit description of the unit and counit and verify that both are natural equivalences. □

Warning 2.8. The result above *fails* if we consider the same diagram in $\mathrm{Cat}_\infty^{\mathrm{smon}}$ (after restricting to κ-compact objects for κ a regular cardinal chosen such that $\mathbf{1}_\mathrm{K} \in \mathcal{S}p_\mathrm{K}$ is κ-compact, say). Indeed, one can see that the homotopy type of the mapping spectra $\mathrm{map}(\mathbf{1}, X)$ out of the unit in this colimit would be different from that in $\mathrm{Mod}^\wedge_\mathrm{E}$, an artefact of the failure of K-localisation to be smashing.

Applying $\mathfrak{pic} : \mathrm{Pr}^{L,\mathrm{smon}} \to \mathcal{S}p_{\geq 0}$ to the coefficients, one obtains a sheaf of connective spectra on $B\mathbb{G}_{\text{ét}}$ given by

$$G/U_i \mapsto \mathfrak{pic}(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}^{\delta}(G/U_i)}).$$

In particular, $\Gamma\mathfrak{pic}(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}^{\delta}}) = \mathfrak{pic}_h := \mathfrak{pic}(\mathcal{S}p_{\mathrm{K}})$. Unfortunately, this sheaf is unsuitable for the spectral sequence we would like to construct: the E_1-page will be group cohomology with coefficients in the homotopy of $\varinjlim \mathfrak{pic}(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}^{\delta}(G/U_i)})$, which by [MS16, Proposition 2.2.3] is the Picard spectrum of the colimit of module categories computed in $\mathrm{Cat}_{\infty}^{\mathrm{smon}}$; as noted above, this need *not* agree with $\mathfrak{pic}(\mathrm{Mod}^{\wedge}_{\mathrm{E}})$.

2.2. **Proétale homotopy theory.** Considering Warning 2.8, we are led to the following solution: rather than work with the site of *finite* $\mathbb{G}$-sets, we think of Morava E-theory as a sheaf $\underline{\mathrm{E}}$ on *profinite* $\mathbb{G}$-sets.

Definition 2.9. Let G be a δ_0-small profinite group. We denote by $BG_{\text{proét}}$ the *proétale site*, whose underlying category consists of all δ_0-small continuous profinite G-sets and continuous equivariant maps. $BG_{\text{proét}}$ is equipped with the topology whose coverings are collections $\{S_\alpha \to S\}$ for which there is a *finite* subset A with $\coprod_{\alpha\in A} S_\alpha \twoheadrightarrow A$.

The proétale site and resulting 1-topos were extensively studied by Bhatt and Scholze in [BS14]; when G is trivial, one recovers the *condensed/pyknotic* formalism of [Sch19, BH19], up to a choice of set-theoretic foundations. One key feature for our purposes is that sheaf cohomology on $BG_{\text{proét}}$ recovers continuous group cohomology for a wide range of coefficient modules; this will allow us to recover the desired homotopy sheaves, which we saw we could not do using $B\mathbb{G}_{\text{ét}}$—in particular, we will recover the K-local E-Adams spectral sequence as a descent spectral sequence. Moreover, $B\mathbb{G}_{\text{proét}}$ gives a site-theoretic definition of continuous group cohomology, which makes the passage to actions on module categories (and hence Picard spectra) transparent.

To begin, we will discuss some generalities of proétale homotopy theory. Most important for the rest of the document will be Section 2.2.3, in which we give criteria for a sheaf on $BG_{\text{ét}}$ to extend without sheafification to one on $BG_{\text{proét}}$; the other sections will be used only cursorily, but are included for completeness.

2.2.1. *Free G-sets.* A significant difference between the étale and proétale sites is that the latter contains G-torsors. In fact, for any profinite set T we have an object $G \times T \in BG_{\text{proét}}$. It will often be useful to restrict to such free objects.

Lemma 2.10. *If S is a continous G-set, the projection $S \to S/G$ is split: that is, there is a continuous section*

$$S/G \to S.$$

In particular, if the G-action on S is free then there is a shearing isomorphism $S \cong S/G \times G$.

Proof. This is implied by [Ser97, Proposition 1]. □

Definition 2.11. The subsite $\mathrm{Free}_G \subset BG_{\text{proét}}$ is given by the full subcategory of G-sets with *free* G-action; equivalently, these are G-sets isomorphic to ones of the form $T\times G$ for T a profinite set with trivial G-action.

Lemma 2.12. *The subsite* Free_G *generates* $BG_{\text{proét}}$*: any $T \in BG_{\text{proét}}$ admits a covering by a free G-set. Consequently, restriction induces an equivalence*

$$\mathcal{S}h(BG_{\text{proét}}) \xrightarrow{\sim} \mathcal{S}h(\mathrm{Free}_G).$$

Proof. The action map $S \times G \to S$ is surjective, and the G-action on the domain is clearly free. □

Remark 2.13. In fact, one can restrict further to the category $\mathrm{Proj}_G = BG^{\mathrm{wc}}_{\text{proét}}$ of *weakly contractible* G-sets: these are the free G-sets of the form $G \times T$, where T is extremally disconnected. While this is not a site (it does not have pullbacks), Proj_G does generate the *hypercomplete* proétale topos, in the sense that restriction induces an equivalence

$$\widehat{\mathcal{S}h}(BG_{\text{proét}}) \xrightarrow{\sim} \mathcal{P}_{\Sigma}(\mathrm{Proj}_G),$$

where the codomain is the full subcategory of $\mathcal{P}(\mathrm{Proj}_G)$ spanned by multiplicative presheaves—i.e., those that send binary coproducts to products.

2.2.2. *Descent spectral sequence.* Suppose that G is a profinite group. Taking $\mathcal{X} = \mathcal{S}h(BG_{\text{proét}})$ or $\widehat{\mathcal{S}h}(BG_{\text{proét}})$ in [Lur18b, Proposition 1.3.2.7] endows $\mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)$ and $\widehat{\mathcal{S}h}(BG_{\text{proét}}, \mathcal{S}p)$ with t-structures having the following properties:

(1) $\mathcal{F}$ is coconnective if and only if it is pointwise coconnective,
(2) $\mathcal{F}$ is connective if and only if its homotopy sheaves $\pi_t\mathcal{F}$ vanish for $t < 0$,
(3) both t-structures are compatible with colimits and right complete,
(4) in either case, π_0 is an equivalence from the heart of the t-structure to $\mathrm{Ab}(BG_{\text{proét}})$.

In particular, any $\mathcal{F} \in \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)$ has a Postnikov tower

$$\mathcal{F} \to \cdots \to \mathcal{F}_{\leq n} \to \cdots .$$

If $\mathcal{F}$ is hypercomplete, the Postnikov tower converges by [BS14, Proposition 3.2.3]. If H is any subgroup, we may evaluate the Postnikov tower at the G-set G/H, obtaining a tower of spectra and hence a spectral sequence

$$E_2^{s,t} = \pi_{-s}\Gamma(G/H, \pi_t\mathcal{F}) \implies \pi_{t-s}\Gamma(G/H, \mathcal{F}).$$

Since $\mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)^{\heartsuit} \simeq \mathrm{Ab}(BG_{\text{proét}})$, the universal property of the derived category allows us to identify the E_2-page may be identified with sheaf cohomology:

Corollary 2.14. *If $\mathcal{F} \in \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)$, there is a spectral sequence*

$$E_2^{s,t} = H^s(BG_{\text{proét}}, \pi_t\mathcal{F}) \implies \pi_{t-s}\Gamma\mathcal{F}.$$

It is conditionally convergent when $\mathcal{F}$ is hypercomplete.

Proof. Since the t-structures are right-complete and $\mathrm{Ab}(BG_{\text{proét}})$ has enough injectives, the square

$$\begin{tikzcd} \mathrm{Ab}(BG_{\text{proét}}) \arrow[r, "\sim"] \arrow[d, "H^0"'] & \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)^{\heartsuit} \arrow[d, "\Gamma^{\heartsuit}"] \\ \mathrm{Ab} \arrow[r, "\sim"'] & \mathcal{S}p^{\heartsuit} \end{tikzcd}$$

gives rise to a commutative square of ∞-categories

$$\begin{tikzcd} \mathcal{D}^+(BG_{\text{proét}}) \arrow[r] \arrow[d, "H^*"'] & \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p) \arrow[d, "\Gamma"] \\ \mathcal{D}^+(\mathrm{Ab}) \arrow[r] & \mathcal{S}p \end{tikzcd}$$

by [Lur17, Proposition 1.3.3.2] (applied to opposites). The bottom horizontal functor is t-exact, and so gives rise to isomorphisms

$$\pi_{-s}\Gamma\mathcal{F} \cong H^s(BG_{\text{proét}}, \pi_0\mathcal{F})$$

for any $\mathcal{F} \in \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)^{\heartsuit}$. $\square$

Remark 2.15. The fully faithful left adjoint $\mathcal{S}p_{\geq 0} \hookrightarrow \mathcal{S}p$ gives rise to a left adjoint $(-)^+ \colon \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p_{\geq 0}) \to \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)$ with right adjoint $(-)_{\geq 0}$. Given $\mathcal{F} \in \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p_{\geq 0})$, we will consider the spectral sequence above for its image $\mathcal{F}^+ \in \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)$, noting that

$$\Gamma(\mathcal{F}^+)_{\geq 0} \simeq \Gamma((\mathcal{F}^+)_{\geq 0}) \simeq \Gamma\mathcal{F}.$$

The first equivalence follows from the natural equivalence $(-)_{\geq 0} \circ \Gamma \simeq \Gamma \circ (-)_{\geq 0}$, which in turn follows from the fact that both functors are right adjoint to the constant sheaf functor $\mathcal{S}p_{\geq 0} \to \mathcal{S}h(BG_{\text{proét}}, \mathcal{S}p)$.

When $\mathcal{F} = \underline{M} := \mathrm{Cont}_G(-, M) \in \mathcal{S}h(BG_{\text{proét}}, \mathrm{Ab})$ is represented by some topological G-module, there is a comparison map

$$\Phi : H^s_{\text{cont}}(G, M) \to H^s(BG_{\text{proét}}, \underline{M})$$

from continuous group cohomology (defined in terms of continuous cochains), which arises as the edge map in a Čech-to-derived functor spectral sequence. In this context, [BS14, Lemma 4.3.9] gives conditions for Φ to be an isomorphism.

2.2.3. *Extending discrete objects.* For a profinite group G, the proétale site $BG_{\text{proét}}$ is related to the site $BG_{\text{ét}}$ of finite G-sets by a map of sites $\nu^{-1} : BG_{\text{ét}} \hookrightarrow BG_{\text{proét}}$, which induces a geometric morphism at the level of topoi. More generally, if $\mathcal{C}$ is any complete and cocomplete ∞-category we write

$$\nu^* : \mathcal{S}h(BG_{\text{ét}}, \mathcal{C}) \rightleftarrows \mathcal{S}h(BG_{\text{proét}}, \mathcal{C}) : \nu_*$$

for the resulting adjunction; the left adjoint is the sheafification of the presheaf extension, given in turn by left Kan extending along ν^{-1}: informally,

$$\nu^p \mathcal{F} : S = \varprojlim_i S_i \mapsto \varinjlim_i \mathcal{F}(S_i) \tag{9}$$

for any presentation $S = \varprojlim S_i$ with S_i finite. When $\mathcal{C} = \text{Set}$, it is a basic result of [BS14] that $\nu^* = \nu^p$: essentially, this is because

(1) the sheaf condition is a finite limit,
(2) it therefore commutes with the filtered colimit in (9).

This fails for sheaves valued in an arbitrary ∞-category $\mathcal{C}$ with limits and filtered colimits, where both properties might fail; inspired by the machinery of [Mat16], we will circumvent this by introducing further finiteness assumptions.

Definition 2.16. (1) Let $Y : \mathbb{Z}_{\geq 0}^{\text{op}} \to \mathcal{C}$ be a pretower in a stable ∞-category $\mathcal{C}$. If $\mathcal{C}$ has sequential limits, we can form the map

$$f : \{\lim Y\} \to \{Y_n\}$$

in the ∞-category $\text{Fun}(\mathbb{Z}_{\geq 0}^{\text{op}}, \mathcal{C})$ (in which the source is a constant tower). Recall that Y is *d-rapidly convergent to its limit* if each d-fold composite in $\text{fib}(f) \in \text{Fun}(\mathbb{Z}_{\geq 0}^{\text{op}}, \mathcal{C})$ is null (c.f. [CM21, Definition 4.8]; see also [HPS99]). More generally, if $Y : (\mathbb{Z}_{\geq 0} \cup \{\infty\})^{\text{op}} \to \mathcal{C}$ is a tower, we say Y is *d-rapidly convergent* if the same condition holds for the map

$$f : \{Y_\infty\} \to \{Y_n\}.$$

In particular, this implies that $Y_\infty \simeq \lim Y$; in fact, $F(Y_\infty) \simeq \lim F(Y)$ for any exact functor $F : \mathcal{C} \to \mathcal{D}$.

(2) Now suppose $\mathcal{C}$ also has finite limits, and $X^\bullet : \Delta_+ \to \mathcal{C}$ is an augmented cosimplicial object. Recall that X is *d-rapidly convergent* if the Tot-tower $\{\text{Tot}_n X^\bullet\}$ is d-rapidly convergent. In particular $X^{-1} \simeq \text{Tot}\, X^\bullet$, and the same is true after applying any exact functor F.

Example 2.17. Given an $\mathbb{E}_1$-algebra A in any presentably symmetric monoidal stable ∞-category $\mathcal{C}$, the Adams tower $T(A, M)$ for $M \in \mathcal{C}$ over A is defined by the property that every map in $A \otimes T(A, M)$ is null. It is well-known to agree with the dual tower to the Tot-tower for the Amitsur complex for M over A, in the sense that

$$T_n(A, M) \simeq \text{fib}(M \to \text{Tot}_n(A^{\otimes \bullet+1} \otimes M)).$$

For example, this is worked out in detail in [MNN17, §2.1]. In particular the cosimplicial object $A^{\otimes \bullet+2} \otimes M$, given by smashing the Amitsur complex with a further copy of A, is always 1-rapidly convergent.

Remark 2.18. Let G be a profinite group and $\mathcal{F}^\delta$ a presheaf on $BG_{\text{ét}}$, valued in a stable ∞-category $\mathcal{C}$. Inspired by [CM21], the following will be the key finiteness assumption we invoke:

($\star$) There exists $d \geq 0$ such that for any normal inclusion $V \subset U$ of open subgroups of G, the Čech complex

$$\mathcal{F}^\delta(G/U) \longrightarrow \mathcal{F}^\delta(G/V) \rightrightarrows \mathcal{F}^\delta(G/V \times_{G/U} G/V) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

is d-rapidly convergent.

When $\mathcal{C} = \mathcal{S}p$ or $\mathcal{S}p_{\text{K}}$, we will show that assumption ($\star$) implies that the left extension of $\mathcal{F}^\delta$ is a hypercomplete sheaf on $BG_{\text{proét}}$ (Proposition 2.20). Recall that a map $f\colon X \to Y$ in $\mathcal{S}p_{\text{K}}$ is *phantom* if any composite $C \to X \xrightarrow{f} Y$ with C compact is null. I'm grateful to Neil Strickland for suggesting the following argument:

Lemma 2.19. *Let I be a filtered category, and suppose that $\mathcal{C}$ is a stable ∞-category with $h\mathcal{C}$ a Brown category in the sense of* [HPS97]*, or $\mathcal{C} = \mathcal{S}p_{\text{K}}$. Let $p\colon I \to \text{Fun}(\Delta, \mathcal{C})$ be a diagram of d-rapidly convergent cosimplicial objects. Then $\varinjlim p\colon \Delta \to \mathcal{S}p$ is $2d$-rapidly convergent.*

Proof. Each $p(i)^\bullet$ is d-rapidly convergent, and in particular for each i, each d-fold composite in the fibre of

$$\{\operatorname{Tot} p(i)^\bullet\} \to \{\operatorname{Tot}_n p(i)^\bullet\}$$

is phantom. The colimit of a filtered diagram of phantom maps is phantom (since its precomposition with any map from a compact object factors through some finite stage), and hence any d-fold composite in the fibre of

$$\{\varinjlim_i \operatorname{Tot} p(i)^\bullet\} \to \{\varinjlim_i \operatorname{Tot}_n p(i)^\bullet\} \simeq \{\operatorname{Tot}_n \varinjlim_i p(i)^\bullet\}$$

is phantom too. By [HPS97, Theorem 4.18] or [HS99, Theorem 9.5] respectively, the composite of any two phantoms in $\mathcal{C}$ is null. Thus $\varinjlim p(i)^\bullet$ is $2d$-rapidly convergent. □

Proposition 2.20. *Let G be a profinite group, and $\mathcal{A}^\delta \in \mathcal{S}h(BG_{\text{ét}}, \mathcal{S}p_{\mathrm{K}})$. If $\mathcal{A}^\delta$ satisfies $(\star)$ then $\nu^p\mathcal{A}^\delta \in \mathcal{P}(BG_{\text{proét}}, \mathcal{S}p_{\mathrm{K}})$ is a sheaf.*

Proof. As above, $\nu^p\mathcal{A}^\delta$ is given by the formula

$$S = \varprojlim S_i \mapsto L_{\mathrm{K}} \varinjlim \mathcal{A}^\delta(S_i).$$

Suppose we are given a surjection $S' = \lim_{j'\in J'} S'_{j'} \xrightarrow{\alpha} S = \lim_{j\in J} S_j$ of profinite G-sets. By [Lur09, Proposition 5.3.5.15] (applied to $A = \Delta^1$ and $\mathcal{C} = \mathrm{Fin}^{\mathrm{op}}$), one can present this as the limit of finite covers $S'_i \to S_i$. We therefore want to show that the following is a limit diagram:

$$L_{\mathrm{K}} \varinjlim \mathcal{A}^\delta(S_i) \longrightarrow L_{\mathrm{K}} \varinjlim \mathcal{A}^\delta(S'_i) \rightrightarrows L_{\mathrm{K}} \varinjlim \mathcal{A}^\delta(S'_i \times_{S_i} S'_i) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots . \tag{10}$$

By 2.19 it follows that it will suffice to show that there is some $d < \infty$ such that

$$\mathcal{A}(S) \longrightarrow \mathcal{A}(S') \rightrightarrows \mathcal{A}(S' \times_S S') \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

is d-rapidly convergent for all *finite* coverings $S' \to S$. Indeed, in that case the Čech complex for an arbitrary covering in $BG_{\text{proét}}$ is $2d$-rapidly convergent, and in particular converges.

In fact, one can further reduce to the case of a covering of orbits $G/V \to G/U$, in which case the Čech complex is d-rapidly convergent by assumption $(\star)$. Indeed, suppose that $S' \to S$ is a finite covering. Decomposing S into transitive G-sets splits (10) into a product of the Čech complexes for $S'_H = S' \times_S G/H \to G/H$, and further writing $S'_H := \coprod_i G/K_i$ we will reduce to the case $G/K \to G/H$, where $K \subset H$ are open subgroups. For this last point, note that if $S'_1, S'_2 \to S$ are coverings then $S'_1 \sqcup S'_2 \to S$ factors as $S'_1 \sqcup S'_2 \to S \sqcup S'_2 \to S$. For the first map we have $(S'_1 \sqcup S'_2)^{\times_{S \sqcup S'_2}\bullet} \simeq S_1'^{\times_S \bullet} \sqcup S'_2$, so d-rapid convergence for $S \to S'_1$ implies the same for $S'_1 \sqcup S'_2 \to S \sqcup S'_2$. On the other hand, the second is split, and so the Čech complex for $S'_1 \sqcup S'_2 \to S$ is a retract of that for $S'_1 \sqcup S'_2 \to S \sqcup S'_2$. In particular, d-rapid convergence for the covering $S'_1 \to S$ implies the same for $S'_1 \sqcup S'_2 \to S$. This allows us to reduce to covers by a single finite G-orbit. This completes the proof that $\nu^p\mathcal{A}^\delta$ is a sheaf. □

In our key example of interest, the starting point will be a *descendable* Galois extension in the sense of [Mat16]. We will therefore show that if $\mathbf{1} \to A$ is a descendable Galois extension in $\mathcal{S}p_{\mathrm{K}}$, then the presheaf $\mathcal{A}^\delta$ satisfies $(\star)$. Applying Proposition 2.20, this implies that its left Kan extension to $BG_{\text{proét}}$ is a sheaf. This relies on some preliminary lemmas.

Definition 2.21. (1) Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is symmetric monoidal, and $Z \in \mathcal{C}$. Then $\mathrm{Thick}^\otimes(Z)$ is the smallest full subcategory of $\mathcal{C}$ containing Z and closed under extensions, retracts, and $(-) \otimes X$ for every $X \in \mathcal{C}$. Note that $\mathrm{Thick}^\otimes(Z)$ is the union of full subcategories $\mathrm{Thick}^\otimes_r(Z)$, for $r \geq 1$, spanned by retracts of those objects that can be obtained by at most r-many extensions of objects $Z \otimes X$; each of these is a $\otimes$-ideal closed under retracts, but not thick.

(2) Suppose now that $A \in \mathrm{Alg}(\mathcal{C})$. Recall that $M \in \mathcal{C}$ is *A-nilpotent* ([Rav00, Definition 7.1.6]) if $M \in \mathrm{Thick}^\otimes(A)$, and that A is *descendable* ([Mat16, Definition 3.18]) if $\mathrm{Thick}^\otimes(A) = \mathcal{C}$. Thus A is descendable if and only if $\mathbf{1}$ is A-nilpotent.

In [Mat16, Proposition 3.20] it is shown that $A \in \mathrm{Alg}(\mathcal{C})$ is descendable if and only if the Tot-tower of the Amitsur complex

$$\mathbf{1} \longrightarrow A \rightrightarrows A \otimes A \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

defines a constant pro-object converging to $\mathbf{1}$. It will be useful to have the following quantitative refinement of this result, which explicates some of the relations between various results in *op. cit.* with [CM21, MNN17, Mat15]:

Lemma 2.22. *Let $\mathcal{C}$ be stable and symmetric monoidal. Consider the following conditions:*

$(1)_d$ *The Amitsur complex for A is d-rapidly convergent to $\mathbf{1}$.*
$(2)_d$ *The canonical map $\mathbf{1} \to \mathrm{Tot}_d\, A^{\otimes \bullet+1}$ admits a retraction.*
$(3)_d$ *The canonical map $\mathrm{Tot}^d\, A^{\otimes\bullet+1} := \mathrm{fib}(\mathbf{1} \to \mathrm{Tot}_d\, A^{\otimes\bullet+1}) \to \mathbf{1}$ is null. In the notation of [MNN17, §4], this says that $\exp_A(\mathbf{1}) = d$.*
$(4)_d$ *For any $X \in \mathcal{C}$, the spectral sequence for the tower of mapping spectra*

$$\cdots \to \mathrm{map}_{\mathcal{C}}(X, \mathrm{Tot}_n\, A^{\otimes\bullet}) \to \cdots \to \mathrm{map}_{\mathcal{C}}(X, \mathrm{Tot}_0\, A^{\otimes\bullet}) \tag{11}$$

collapses at a finite page, with a horizontal vanishing line at height d.

Then we have implications $(1)_d \Leftrightarrow (4)_d$, $(2)_d \Leftrightarrow (3)_d$, and $(1)_d \Rightarrow (2)_d \Rightarrow (1)_{d+1}$.

Proof. We begin with the first three conditions. The implication $(1)_d \Rightarrow (2)_d$ is immediate from the diagram

$$\begin{tikzcd}
\mathrm{Tot}^d\, A^{\otimes\bullet+1} \arrow[r] \arrow[d, "0"'] & \mathbf{1} \arrow[r] \arrow[d, equal] & \mathrm{Tot}_d\, A^{\otimes\bullet+1} \arrow[d] \arrow[dl, dashed, "\exists"] \\
\mathrm{Tot}^0\, A^{\otimes\bullet+1} \arrow[r] & \mathbf{1} \arrow[r] & \mathrm{Tot}_0\, A^{\otimes\bullet+1}
\end{tikzcd}$$

in which the rows are cofibre sequences. The equivalence $(2)_d \Leftrightarrow (3)_d$ is clear, so we now prove $(2)_d \Rightarrow (1)_{d+1}$.

Recall from the proof of [Mat16, Proposition 3.20] that the full subcategory

$$\{X : X \otimes A^{\otimes\bullet+1} \text{ is a constant pro-object}\} \subset \mathcal{C}$$

is a thick $\otimes$-ideal containing A (since $A \otimes A^{\otimes\bullet+1}$ is split), and therefore contains $\mathbf{1}$. We can in fact define full subcategories

$$\mathcal{C}_r := \{X : X \otimes A^{\otimes\bullet+1} \text{ is } r\text{-rapidly convergent}\} \subset \mathcal{C},$$

which are closed under retracts and $(-) \otimes X$ for any $X \in \mathcal{C}$. Each of these is *not* thick, but if $X \to Y \to Z$ is a cofibre sequence with $X \in \mathcal{C}_r$ and $Z \in \mathcal{C}_{r'}$, then $Y \in \mathcal{C}_{r+r'}$ (this follows by contemplating the diagram

$$\begin{tikzcd}
X \otimes \mathrm{Tot}_n\, A^{\otimes\bullet+1} \arrow[r] \arrow[d] & Y \otimes \mathrm{Tot}_n\, A^{\otimes\bullet+1} \arrow[r] \arrow[d] & Z \otimes \mathrm{Tot}_n\, A^{\otimes\bullet+1} \arrow[d, "0"] \\
X \otimes \mathrm{Tot}_{n+r'}\, A^{\otimes\bullet+1} \arrow[r] \arrow[d, "0"'] & Y \otimes \mathrm{Tot}_{n+r'}\, A^{\otimes\bullet+1} \arrow[r] \arrow[d] & Z \otimes \mathrm{Tot}_{n+r'}\, A^{\otimes\bullet+1} \arrow[d] \\
X \otimes \mathrm{Tot}_{n+r+r'}\, A^{\otimes\bullet+1} \arrow[r] & Y \otimes \mathrm{Tot}_{n+r+r'}\, A^{\otimes\bullet+1} \arrow[r] & Z \otimes \mathrm{Tot}_{n+r+r'}\, A^{\otimes\bullet+1}
\end{tikzcd}$$

as in [HPS99, Lemma 2.1] or [Mat15, Proposition 3.5]). In particular, $\mathrm{Tot}_r\, A^{\otimes\bullet+1} \in \mathcal{C}_{r+1}$ since this can be constructed iteratively by taking $r+1$-many extensions by free A-modules (while $A \in \mathcal{C}_1$ by Example 2.17). Now assumption $(2)_d$ implies that $\mathbf{1} \in \mathcal{C}_{d+1}$, that is, $(1)_{d+1}$ holds.

The implication $(1)_d \Leftrightarrow (4)_d$ is proven in [Mat15, Proposition 3.12] (in the case $\mathfrak{U} = \mathcal{C}$), by keeping track of the index d (called N therein). ☐

Remark 2.23. The implications in this lemma are special to the Amitsur/cobar complex. For an arbitrary cosimplicial object (even in spectra), condition $(2)_d$ does not in general imply $(1)_{d'}$ for any d' (although $(1)_d \Rightarrow (2)_d$ still holds).

Lemma 2.24. *Let G be a profinite group, $\mathcal{C}$ a stable homotopy theory, and $\mathbf{1} \to A$ a faithful G-Galois extension in $\mathcal{C}$ corresponding to $\mathcal{A}^\delta \in \widehat{\mathcal{S}h}(BG_{\text{ét}}, \mathrm{CAlg}(\mathcal{C}))$. Then $\mathcal{A}^\delta$ sends pullbacks of finite G-sets to pushouts in $\mathrm{CAlg}(\mathcal{C})$.*

Proof. Let $S_1 \to S_0 \leftarrow S_2$ be a cospan in $BG_{\text{ét}}$, and assume first that each S_i is of the form G/U_i where U_i is an open subgroup.

We can write $S_1 \times_{S_0} S_2 = \coprod_{g \in U_1 \backslash U_0 / U_2} G/U_1 \cap g^{-1}U_2 g$, so that

$$\nu^p \mathcal{A}^\delta(S_1 \times_{S_0} S_2) = \bigoplus_{U_1 \backslash U_0 / U_2} \mathcal{A}^\delta(U_1 \cap g^{-1}U_2 g).$$

This admits an algebra map from the pushout $\mathcal{A}^\delta(S_1) \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_2)$, and we can check this map is an equivalence after base-change to A, a faithful algebra over $\mathcal{A}^\delta(S_0)$. But

$$\begin{aligned} A \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_1) \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_2) &\simeq \bigoplus_{U_0/U_1} A \otimes_{\mathcal{A}^\delta(S_0)} \mathcal{A}^\delta(S_2) \\ &\simeq \bigoplus_{U_0/U_1} \bigoplus_{U_0/U_2} A \\ &\simeq \bigoplus_{U_1 \backslash U_0/U_2} \bigoplus_{U_0/U_1 \cap g^{-1}U_2 g} A \\ &\simeq A \otimes_{\mathcal{A}^\delta(S_0)} \bigoplus_{U_1 \backslash U_0 / U_2} \mathcal{A}^\delta(U_0/U_1 \cap g^{-1}U_2 g), \end{aligned}$$

using the isomorphism of U_0-sets

$$U_0/U_1 \times U_0/U_2 \simeq \coprod_{g \in U_1 \backslash U_0/U_2} U_0/U_1 \cap g^{-1}U_2 g.$$

The result for finite G-sets follows from the above by taking coproducts. □

Corollary 2.25. *Let G be a profinite group, $\mathcal{C}$ a stable homotopy theory, and $\mathbf{1} \to A$ a faithful G-Galois extension in $\mathcal{C}$ corresponding to $\mathcal{A}^\delta \in \widehat{\mathcal{S}h}(BG_{\text{ét}}, \mathrm{CAlg}(\mathcal{C}))$. The presheaf $\nu^p \mathcal{A}^\delta$ sends pullbacks in $BG_{\text{proét}}$ to pushouts in $\mathrm{CAlg}(\mathcal{C})$.*

Proof. This follows from the previous lemma by passing to limits. □

Finally, we prove the desired descent result.

Proposition 2.26. *Let G be a profinite group, and $\mathbf{1} \to A$ a G-Galois extension in $\mathcal{S}p_{\mathrm{K}}$ corresponding to $\mathcal{A}^\delta \in \widehat{\mathcal{S}h}(BG_{\text{ét}}, \mathrm{CAlg}(\mathcal{S}p_{\mathrm{K}}))$. Suppose moreover that $\mathbf{1} \to A$ is descendable. Then $\nu^p \mathcal{A}^\delta \in \mathcal{P}(BG_{\text{proét}}, \mathcal{S}p_{\mathrm{K}})$ is a hypersheaf.*

Proof. By Proposition 2.20 it is enough to prove that $\mathcal{A}^\delta$ satisfies $(\star)$. By Lemma 2.24, the Čech complex in question is the cobar complex

$$A^{hU} \longrightarrow A^{hV} \rightrightarrows A^{hV} \otimes_{A^{hU}} A^{hV} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots . \tag{12}$$

For every $r \geq 1$, we shall consider the following full subcategory of $\mathcal{D} \coloneqq \mathrm{Mod}_{A^{hU}}(\mathcal{C})$:

$$\mathcal{D}_r = \mathcal{D}_r(U, V) \coloneqq \{X : X \otimes_{A^{hU}} (A^{hV})^{\otimes_{A^{hU}} \bullet + 1} \text{ is } r\text{-rapidly convergent}\}.$$

Our aim is to show that $A^{hU} \in \mathcal{D}_d$. To this end, we observe the following properties of $\mathcal{D}_r$:

(1) $\mathcal{D}_r$ is closed under retracts.
(2) $\mathcal{D}_r$ is a $\otimes$-ideal: if $X \in \mathcal{D}_r$ then $X \otimes Y \in \mathcal{D}_r$ for any Y.
(3) If $X \in \mathcal{D}_r$ is an $\mathbb{E}_1$-algebra and Y is an X-algebra, then $Y \in \mathcal{D}_r$ (since $Y \otimes_{A^{hU}} (-) \simeq Y \otimes_X X \otimes_{A^{hU}} (-)$).
(4) $\mathcal{D}_r$ is not thick. However, if $X \to Y \to Z$ is a cofibre sequence such that $X \in \mathcal{D}_r$ and $Z \in \mathcal{D}_{r'}$, then $Y \in \mathcal{D}_{r+r'}$.

But $A^{hV} \in \mathcal{D}_1$ by Example 2.17, and hence $A \in \mathcal{D}_1$. Since A^{hU} splits over A (so that $A \otimes A^{hU}$ contains A as a retract), we have

$$\exp_A(A^{hU}) = \exp_{A \otimes A^{hU}}(A^{hU}) \leq \exp_A(\mathbf{1}) = d,$$

with the inequality following from [MNN17, Corollary 4.13]. Thus $A^{hU} \in \mathcal{D}_d$ as desired. This proves that $\mathcal{A}^\delta$ satisfies $(\star)$, and hence that $\nu^p \mathcal{A}^\delta$ is a sheaf.

For hyperdescent, we consider the unlocalised version

$$\mathcal{A}^{\mathrm{un}} \colon S = \varprojlim S_i \mapsto \varinjlim \mathcal{A}^\delta(S_i) \in \mathrm{CAlg},$$

which is the left Kan extension along $BG_{\text{ét}} \hookrightarrow BG_{\text{proét}}$ formed in *spectra* (equivalently CAlg, since filtered colimits in CAlg are computed on the underlying spectrum). Since $\mathcal{A}^{\text{un}}$ is pointwise E-local, we have $\mathcal{A} = \text{map}(M\mathbb{S}, \mathcal{A}^{\text{un}})$ for $M\mathbb{S}$ the monochromatic sphere at height h [HS99, Theorem 7.10(e)]; thus hyperdescent for $\mathcal{A}^{\text{un}}$ will imply it for $\mathcal{A}$. This is the content of Lemma 2.27, proven immediately below. $\square$

Lemma 2.27. *Let G be a profinite group, and $\mathcal{F}^{\delta} \in \mathcal{S}h(BG_{\text{ét}}, \mathcal{S}p)$ a sheaf satisfying condition $(\star)$. Then its left Kan extension $\mathcal{F} \in \mathcal{P}(BG_{\text{proét}}, \mathcal{S}p)$ is Postnikov complete.*

Proof. The same proof as Proposition 2.20 shows that $\mathcal{F}$ is a sheaf of spectra on $BG_{\text{proét}}$, since $h\mathcal{S}p$ is Brown and hence Lemma 2.19 still applies. To prove Postnikov completeness we can restrict to the subsite $\text{Free}_G \subset BG_{\text{proét}}$, since this generates the proétale ∞-topos; to prevent the notation from becoming too cluttered, we leave the restriction implicit below.

Since Postnikov towers in presheaves always converge, it suffices to prove that the truncations $\tau_{\leq t}\mathcal{F} \in \mathcal{S}h(\text{Free}_G, \mathcal{S}p)$ are given by pointwise truncation. Given a covering of free G-sets $S' \to S$, we claim that the Čech complex $\tau_{\leq t}\mathcal{F}(S) \to \tau_{\leq t}\mathcal{F}(S'^{\times_S \bullet+1})$ is d-rapidly convergent for some $d < \infty$, and in particular converges. Indeed, since

$$\tau_{\leq t}\mathcal{F}(T \times G) \simeq \tau_{\leq t}\left(\varinjlim_i \prod_{T_i} \mathcal{F}(G)\right) \simeq \left(\varinjlim_i \prod_{T_i} \mathbb{S}\right) \otimes \tau_{\leq t}\mathcal{F}(G),$$

it is enough to verify that the cosimplicial object

$$\varinjlim_i \textstyle\prod_{T_i} \mathbb{S} \longrightarrow \varinjlim_i \prod_{T_i'} \mathbb{S} \rightrightarrows \varinjlim_i \prod_{T_i' \times_{T_i} T_i'} \mathbb{S} \Rrightarrow \cdots \tag{13}$$

is rapidly convergent, where $S \cong T \times G$ and $S' \cong T' \times G$. Again, it is enough to find $d < \infty$ such that each d-fold composite in the fibre tower is phantom; by [Mat15, Proposition 3.12], this is equivalent to the existence of a horizontal vanishing line at a finite page in the descent spectral sequence for $\text{Map}(F, \varinjlim_i \prod_{T_i} \mathbb{S})$ for any finite spectrum, at height independent of F. But

$$\pi_t\text{Map}(F, \varinjlim_i \prod_{T_i} \mathbb{S}) \simeq \varinjlim_i \prod_{T_i} \pi_t\text{Map}(F, \mathbb{S}) \simeq \text{Cont}(T, \pi_t\mathbb{D}F),$$

since each group $\pi_t\mathbb{D}F$ is discrete (this is the case for $F = \mathbb{S}$, and follows in general by taking extensions). The E_2-page of the corresponding spectral sequence is thus the cohomology of the Moore complex of

$$\text{Cont}(T', \pi_t\mathbb{D}F) \rightrightarrows \text{Cont}(T' \times_T T', \pi_t\mathbb{D}F) \Rrightarrow \cdots \ \cdots.$$

When T and T' are both finite this complex is exact (by choosing a splitting of $T' \twoheadrightarrow T$), and has

$$\begin{aligned} H^0 &= \text{Eq}\left(\text{Cont}(T', \pi_t\mathbb{D}F) \rightrightarrows \text{Cont}(T' \times_T T', \pi_t\mathbb{D}F)\right) \\ &\cong \text{Cont}(T, \pi_t\mathbb{D}F). \end{aligned}$$

Thus both properties hold for a general covering by passing to limits, since $\pi_t\mathbb{D}F$ is discrete. $\square$

2.2.4. *Change of group.* Let $f\colon H \to G$ be a continuous homomorphism between profinite groups. In this section, we will consider the functoriality of the construction

$$G \mapsto \widehat{\mathcal{S}h}(BG_{\text{proét}}, \mathcal{C}).$$

For example, taking $p\colon G \to *$ we obtain (homotopy) invariants and coinvariants functors; functoriality will imply the iterated fixed points formula

$$(X^{hH})^{hG/H} \simeq X^{hG}$$

for $H \subset G$ a normal subgroup.

Remark 2.28. For any presentable $\mathcal{C}$ we have $\mathcal{S}h(BG_{\text{proét}}, \mathcal{C}) \simeq \mathcal{S}h(BG_{\text{proét}}) \otimes \mathcal{C}$ by [Lur18b, Remark 1.3.1.6], and so we restrict ourselves to sheaves of spaces.

Proposition 2.29. *(1) Any map of profinite groups $f\colon H \to G$ gives rise to a geometric morphism*

$$f^* : \mathcal{S}h(BG_{\text{proét}}) \rightleftarrows \mathcal{S}h(BH_{\text{proét}}) : f_*$$

This defines a functor $\text{Grp}(\text{Profin})^{\text{op}} \to \mathcal{L}\mathcal{T}\text{op}_\infty$ *to the ∞-category of ∞-topoi and left adjoints.*

(2) The left adjoint $f^*\colon \mathcal{S}h(BG_{\text{proét}}) \to \mathcal{S}h(BH_{\text{proét}})$ admits a further left adjoint $f_!$.

Proof. Given $f\colon H \to G$, restriction determines a morphism of sites

$$\mathrm{res}_f : BG_{\text{proét}} \to BH_{\text{proét}},$$

hence a geometric morphism

$$f^* : \mathcal{S}h(BG_{\text{proét}}) \rightleftarrows \mathcal{S}h(BH_{\text{proét}}) : f_*$$

whose right adjoint is given by precomposition with res_f.

For (ii), we apply Lemma 2.30 to the adjunction

$$(-) \times_H G : BH_{\text{proét}} \rightleftarrows BG_{\text{proét}} : \mathrm{res}_f,$$

noting that if $S \to S'$ is a surjection of H-sets then $S \times_H G \to S' \times_H G$ is also surjective. □

Lemma 2.30. *Given a morphism of sites $f^{-1}\colon \mathcal{C} \to \mathcal{C}'$, if f^{-1} admits a left adjoint g^{-1} then f^* is given by the sheafification of g_*,*

$$f^*\mathcal{F} = (g_*\mathcal{F})^+.$$

If g^{-1} is itself a morphism of sites, then $f^* = g_*$ admits a further left exact left adjoint $f_! = g^*$.

Remark 2.31. We have used the notation f^{-1} so that the right adjoint on sheaves, given by precomposing with f^{-1}, composes correctly: that is,

$$f_*g_*\mathcal{F} = \mathcal{F} \circ g^{-1} \circ f^{-1} = \mathcal{F} \circ (fg)^{-1} = (fg)_*\mathcal{F}.$$

Proof. For the first claim it's enough to work at the level of presheaves, and show that the functors

$$g_* : \mathcal{P}(\mathcal{C}) \rightleftarrows \mathcal{P}(\mathcal{C}') : f_*$$

are adjoint. The unit and counit of the $g^{-1} \dashv f^{-1}$ adjunction give us maps

$$id_{\mathcal{P}(\mathcal{C})} \xrightarrow{\eta_*} (fg)_* \simeq f_*g_* \quad \text{and} \quad g_*f_* \simeq (gf)_* \xrightarrow{\varepsilon_*} id_{\mathcal{P}(\mathcal{C})}.$$

The triangle identities follow from those for η and ε.

Finally, if moreover g preserves coverings then $g_*\mathcal{F}$ is already a sheaf, and so $f^* \simeq g_*$ admits g^* as a left adjoint. □

Example 2.32. If $i\colon H \subset G$ is a closed subgroup, then $\mathrm{res}_i\colon BG_{\text{proét}} \to BH_{\text{proét}}$ admits a left adjoint ind^i which preserves coverings; thus $i^* \simeq \mathrm{ind}^i_*$. In particular, for any $M \in \mathrm{Ab}(B\mathbb{G}_{\text{proét}})$ we have

$$i^*\underline{M} \cong \mathrm{Cont}_G(\mathrm{ind}^i(-), M) \cong \underline{\mathrm{res}_i M}.$$

Lemma 2.33 ($(X^{hH})^{hG/H} \simeq X^{hG}$)**.** *Suppose that $i\colon H \subset G$ is a normal subgroup with quotient $p\colon G \to G/H$, and $X \in \mathcal{S}h(BG_{\text{proét}}, \mathcal{C})$. Then*

$$\Gamma((G/H)/(G/H), p_*X) \simeq \Gamma(G/G, X).$$

Proof. Write $\pi^G\colon G \to *$ and likewise for G/H. We in fact have an equivalence of condensed spectra

$$\pi^G_* X \simeq \pi^{G/H}_* p_* X$$

by functoriality, and the claim follows by evaluating on $* \in B*_{\text{proét}}$. □

2.2.5. *Comparison with pyknotic G-objects.* Given a profinite group G and a base ∞-category $\mathcal{C}$, we have described the ∞-category $\widehat{\mathcal{S}h}(BG_{\text{proét}})$ as a good model for the ∞-topos of continuous G-spaces. Using the theory of pyknotic objects, there are two other natural candidates:

(1) G has image $\underline{G} \in \mathrm{Pyk}(\mathrm{Set})$ under the embedding of compactly generated spaces in pyknotic sets, and $\underline{G}$ is a pyknotic group. Since $\mathcal{S}$ is tensored over sets (and hence $\mathrm{Pyk}(\mathcal{S})$ over $\mathrm{Pyk}(\mathrm{Set})$), we can form a category of $\underline{G}$-modules in $\mathrm{Pyk}(\mathcal{S})$.
(2) G has a *pyknotic classifying space* $BG \in \mathrm{Pyk}(\mathcal{S})$, given by the limit in $\mathrm{Pyk}(\mathcal{S})$ of the classifying spaces $BG_i \in \mathcal{S} \hookrightarrow \mathrm{Pyk}(\mathcal{S})$, where $G = \lim G_i$ is a presentation of G as a profinite group. We can therefore consider the ∞-topos $\mathrm{Pyk}(\mathcal{S})_{/BG}$.

In this section we show that all these notions agree. See [Lur18b, Theorem A.5.6.1] for a related result in the context of *pro-π-finite spaces.*

Remark 2.34. We have chosen to use the ∞-topos $\mathrm{Pyk}(\mathcal{S})$ in this section, as some of our results depend on [Wol22]. Note however that a translation of these results to $\mathrm{Cond}(\mathcal{S})$ is given in [Wol22, Remark 4.19].

Proposition 2.35. *For any profinite group G, there are natural equivalences*

$$\widehat{\mathcal{S}h}(BG_{\text{proét}}) \simeq \operatorname{Fun}^{\text{cts}}(BG, \operatorname{Pyk}(\mathcal{S})) \simeq \operatorname{Pyk}(\mathcal{S})_{/BG} \simeq \operatorname{Mod}_{\underline{G}}(\operatorname{Pyk}(\mathcal{S})).$$

Proof. This is a very mild modification of [Wol22, Corollary 1.2]. In the trivially stratified case, the main theorem of *op. cit.* provides a natural *exodromy* equivalence

$$\mathcal{X}^{\text{Pyk}} := \widehat{\mathcal{S}h}_{\text{eff}}(\operatorname{Pro}(\mathcal{X}^{\text{coh}}_{<\infty})) \xrightarrow[\text{ex}]{\sim} \operatorname{Fun}^{\text{cts}}(\widehat{\Pi}_\infty \mathcal{X}, \operatorname{Pyk}(\mathcal{S})) \tag{14}$$

between the *pyknotification* of an ∞-topos $\mathcal{X}$ and the ∞-category of *pyknotic presheaves* on its profinite shape, viewed as a pyknotic space[2]. Any $X \in \operatorname{Pro}(\mathcal{S}h(BG_{\text{ét}})^{\text{coh}}_{<\infty})$ may be covered by a profinite G-set: this follows just as in the case $G = *$, which is [BGH20, Proposition 13.4.9]. Thus the subcategory $j\colon BG_{\text{proét}} = \operatorname{Pro}(BG_{\text{ét}}) \hookrightarrow \operatorname{Pro}(\mathcal{S}h(BG_{\text{ét}})^{\text{coh}}_{<\infty})$ generates the same topos, and we obtain natural equivalences

$$\widehat{\mathcal{S}h}(BG_{\text{proét}}) \xleftarrow[j_*]{\sim} \widehat{\mathcal{S}h}(\operatorname{Pro}(\mathcal{S}h(BG_{\text{ét}})^{\text{coh}}_{<\infty})) = \mathcal{S}h(BG_{\text{ét}})^{\text{Pyk}} \xrightarrow[\text{ex}]{\sim} \operatorname{Fun}^{\text{cts}}(\widehat{\Pi}_\infty \mathcal{S}h(BG_{\text{ét}}), \operatorname{Pyk}(\mathcal{S})),$$

where j_* is restriction and ex denotes the pyknotic exodromy equivalence (14). We now identify $\widehat{\Pi}_\infty \mathcal{S}h(BG_{\text{ét}}) \in \mathcal{S}^\wedge_\pi = \operatorname{Pro}(\mathcal{S}_\pi)$: if we write $G = \lim G_i$, then by [CM21, Construction 4.5] there is an equivalence in $\mathcal{LT}\text{op}_\infty$,

$$\mathcal{S}h(BG_{\text{ét}}) \simeq \varprojlim \mathcal{S}_{/BG_i}.$$

By [Lur18b, Theorem E.2.4.1], the pro-extension $\Psi\colon \mathcal{S}^\wedge_\pi \to \mathcal{LT}\text{op}_\infty$ of $X \mapsto \mathcal{S}_{/X}$ is a fully faithful right adjoint to $\widehat{\Pi}_\infty$, so

$$\widehat{\Pi}_\infty \mathcal{S}h(BG_{\text{ét}}) \simeq \widehat{\Pi}_\infty \varprojlim \Psi(BG_i) \simeq \widehat{\Pi}_\infty \Psi(\{BG_i\}) \simeq \{BG_i\} \in \mathcal{S}^\wedge_\pi.$$

By definition, BG is the image of $\{BG_i\}$ in $\operatorname{Pyk}(\mathcal{S})$, so continuous unstraightening [Wol22, Corollary 3.20] yields the desired equivalence

$$\widehat{\mathcal{S}h}(BG_{\text{proét}}) \simeq \operatorname{Fun}^{\text{cts}}(BG, \operatorname{Pyk}(\mathcal{S})) \simeq \operatorname{Pyk}(\mathcal{S})_{/BG}. \qquad \square$$

Corollary 2.36. *For any presentable $\mathcal{C}$, the assignment $G \mapsto \widehat{\mathcal{S}h}(BG_{\text{proét}}, \mathcal{C})$ extends to a Wirthmüller context on profinite groups and continuous homomorphisms.*

Remark 2.37. Recall [FHM03] that this says that there is a 6-functor formalism, with $f^! \simeq f^*$ for any f and f^* strong symmetric monoidal. It does not assert the existence of a Wirthmüller isomorphism, but produces a candidate map in the presence of a dualising object.

Proof. By [Man22, Prop. A.5.10], it remains to check:

(1) For every map $f\colon H \to G$ of profinite groups, the functor f^* admits a left adjoint $f_!\colon \widehat{\mathcal{S}h}(BH_{\text{proét}}) \to \widehat{\mathcal{S}h}(BG_{\text{proét}})$.

(2) If moreover $g\colon G' \to G$ and $H' := H \times_G G'$, then the following square is left adjointable:

$$\begin{tikzcd} \widehat{\mathcal{S}h}(BG_{\text{proét}}) \arrow[r, "f^*"] \arrow[d, "g^*"'] & \widehat{\mathcal{S}h}(BH_{\text{proét}}) \arrow[d, "g'^*"] \\ \widehat{\mathcal{S}h}(BG'_{\text{proét}}) \arrow[r, "f'^*"'] & \widehat{\mathcal{S}h}(BH'_{\text{proét}}) \end{tikzcd}$$

That is, the push-pull transformation $f'_! g'^* \to g^* f_!$ is an equivalence.

(3) If $M \in \widehat{\mathcal{S}h}(BG_{\text{proét}})$ and $N \in \widehat{\mathcal{S}h}(BH_{\text{proét}})$, the natural map

$$f_!(N \times f^*M) \to f_!N \times M$$

is an equivalence.

In fact, these properties will all follow formally from Proposition 2.35. The functor $G \mapsto \widehat{\mathcal{S}h}(BG_{\text{proét}})$ is the restriction to classifying spaces of profinite groups of the functor $X \mapsto \operatorname{Pyk}(\mathcal{S})_{/X}$, which in the language of [Mar22] is the *universe* $\Omega_{\operatorname{Pyk}(\mathcal{S})}$. But the universe Ω of any ∞-topos $\mathcal{B}$ satisfies properties (i) to (iii). Explicitly:

[2]Recall [BGH20, Remark 13.4.4] that the inclusion $y\colon \mathcal{S}^\wedge_\pi \hookrightarrow \widehat{\mathcal{S}h}_{\text{eff}}(\mathcal{S}^\wedge_\pi) \simeq \operatorname{Pyk}(\mathcal{S})$ preserves all small limits. Combining this with [BGH20, 13.4.11] we see that $y\{X_i\} = \lim yX_i = \lim \Gamma^* X_i$ is the limit in $\operatorname{Pyk}(\mathcal{S})$ of the π-finite spaces X_i, viewed as constant pyknotic spaces.

(1) For any $f\colon A \to B$ in $\mathcal{B}$, the étale geometric morphism $f^* : \mathcal{B}_{/B} \to \mathcal{B}_{/A}$ is given by basechange along f, and admits a left adjoint $f_! = f \circ -$.
(2) For a pullback square

$$\begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

in $\mathcal{B}$, the evaluation of the push-pull transformation $f'_! g'^* \to g^* f_!$ on $X \to B$ may be identified with the left-hand vertical map in the extended diagram

$$\begin{tikzcd} X \times_B A \arrow[r] \arrow[d] & A \arrow[r, "g'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ X \times_D C \arrow[r] & C \arrow[r, "g"'] & D \end{tikzcd}$$

and is therefore an equivalence.
(3) For $f\colon A \to B$ in $\mathcal{B}$, the projection formula is given in [Lur09, Remark 6.3.5.12].

□

Corollary 2.38. *For any profinite group G there is an equivalence*

$$\widehat{\mathcal{S}h}(BG_{\text{proét}}) \simeq \mathrm{Mod}_{\underline{G}}(\mathrm{Pyk}(\mathcal{S})).$$

Proof. The results of [Lur18b, Appendix E.5] imply that the profinite group G, as a 0-truncated object of $\mathrm{Grp}(\mathcal{S}^{\wedge}_{\pi})$, has a profinite classifying space; that is, a delooping in $\mathcal{S}^{\wedge}_{\pi}$. In fact, in the proof of [Lur18b, Theorem E.5.0.4] ([Lur18b, E.5.6]) Lurie observes that this is $\{BG_i\}$; passing to $\mathrm{Pyk}(\mathcal{S})$ we see that BG is a delooping of $\underline{G}$. The map $e\colon * \to BG$ gives rise to an adjunction

$$e_! : \mathrm{Pyk}(\mathcal{S}) \rightleftarrows \mathrm{Pyk}(\mathcal{S})_{/BG} : e^*$$

which is monadic since e^* is conservative and admits a right adjoint. Given $X \in \mathrm{Pyk}(\mathcal{S})$ one has the following diagram, in which all squares are Cartesian:

$$\begin{tikzcd} e_! e^* X \arrow[r] \arrow[d] & \underline{G} \arrow[r] \arrow[d] & * \arrow[d, "e"] \\ X \arrow[r] & * \arrow[r, "e"'] & BG \end{tikzcd}$$

In particular, $e_! e^* X \simeq \underline{G} \times X$. □

2.3. **Morava E-theory as a proétale spectrum.** The aim of this section is to show that the proétale site allows us to capture the continuous action on E as a sheaf of spectra on $B\mathbb{G}_{\text{proét}}$. Our first task is to exhibit Morava E-theory itself as a proétale sheaf of spectra, so as to recover the K-local E-Adams spectral sequence as a descent spectral sequence. This will allow us to compare it to the descent spectral sequence for the Picard spectrum.

Proposition 2.39. *The presheaf of* K*-local spectra*

$$\underline{\mathrm{E}} \coloneqq \nu^p \underline{\mathrm{E}}^{\delta} : S = \varprojlim_i S_i \mapsto L_{\mathrm{K}} \varinjlim_i \underline{\mathrm{E}}^{\delta}(S_i)$$

is a hypercomplete sheaf on $B\mathbb{G}_{\text{proét}}$.

Proof. This is immediate from Proposition 2.26: by [Mat16, Proposition 10.10], Morava E-theory $\mathrm{E} \in \mathrm{CAlg}(\mathcal{S}p_{\mathrm{K}})$ is descendable. This is a consequence of descendability of $L_h\mathbb{S} \to \mathrm{E}$, which is proven in [Rav00]. □

Remark 2.40. The same formula defines a sheaf of spectra (or even of $\mathbb{E}_\infty$-rings) before K-localisation. We stress however that ν^p will always refer to the K-**local** version (i.e., the Kan extension into $\mathcal{S}p_{\mathrm{K}}$).

As an aside, note that the proof of Proposition 2.26 allows us likewise to consider E-homology of any spectrum X:

Corollary 2.41. *If X is any* K*-local spectrum, then the presheaf $\underline{\mathrm{E}} \otimes X$ is a hypercomplete sheaf.*

Proof. Given a covering $S' \to S$ in $B\mathbb{G}_{\text{proét}}$, the proof of Proposition 2.26 showed that the Amitsur complex for $S' \to S$ is rapidly convergent. Since the functor $(-) \otimes X : \mathcal{S}p_{\mathrm{K}} \to \mathcal{S}p_{\mathrm{K}}$ preserves finite limits, the same is true for the augmented cosimplicial object given by tensoring everywhere by X. □

We can now define a descent spectral sequence for the sheaf $\underline{\mathrm{E}}$.

Remark 2.42. Proposition 2.39 gives us a sheaf $\underline{\mathrm{E}} \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\text{proét}}, \mathcal{S}p_{\mathrm{K}})$. On applying the forgetful functor $\mathcal{S}p_{\mathrm{K}} \hookrightarrow \mathcal{S}p$, we obtain $\underline{\mathrm{E}} \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\text{proét}}, \mathcal{S}p)$; note that this sheaf is *not* left Kan extended from $B\mathbb{G}_{\text{ét}}$. Nevertheless, applying Corollary 2.14, we obtain the following result.

Corollary 2.43. *For any closed subgroup $i\colon G \subset \mathbb{G}$, there is a conditionally convergent spectral sequence of the form*

$$E_{2,+}^{s,t} = H^s(BG_{\text{proét}}, i^*\pi_t\underline{\mathrm{E}}) \implies \pi_{t-s}\mathrm{E}^{hG}, \tag{15}$$

where E^{hG} is the Devinatz-Hopkins fixed points construction.

Proof. The spectral sequence is the descent spectral sequence for the sheaf $i^*\underline{\mathrm{E}}$; for conditional convergence, we need to show that $i^*\underline{\mathrm{E}}$ is a hypercomplete sheaf on $BG_{\text{proét}}$. This follows from Lemma 2.30, which implies that $i^*\underline{\mathrm{E}} = \mathrm{ind}_*^i \underline{\mathrm{E}}$, and the fact the ind_*^i preserves hypersheaves.

The abutment is given by

$$\Gamma i^*\underline{\mathrm{E}} = \Gamma(\mathbb{G}/G, \underline{\mathrm{E}}).$$

To identify this with E^{hG} it is enough to so when $G \subset \mathbb{G}$ is open, since both sheaves are Kan extended from $B\mathbb{G}_{\text{ét}}$; for E^{hG} this is clear from [DH04, Definition 1.5]. In this case, we compute $\underline{\mathrm{E}}(\mathbb{G}/G)$ using descent for the covering $\mathbb{G}/G \times \mathbb{G} \to \mathbb{G}/G$: this is the complex

$$\mathbf{C}(\mathbb{G}/G, \bullet) = \left(\prod\nolimits_{\mathbb{G}/G} \mathrm{E} \rightrightarrows \prod\nolimits_{\mathbb{G}/G} \mathrm{E} \otimes \mathrm{E} \,\substack{\rightarrow\\[-3pt]\rightarrow\\[-3pt]\rightarrow}\, \cdots \right)$$

constructed in [DH04, Construction 4.11], whose totalisation is E^{hG} by definition. □

To identify the E_2-page, we need to identify the proétale homotopy groups of $\underline{\mathrm{E}}$.

Lemma 2.44. *The proétale homotopy groups of $\underline{\mathrm{E}}$ are given by*

$$\pi_t\underline{\mathrm{E}} = \underline{\pi_t\mathrm{E}}, \tag{16}$$

i.e. represented by the homotopy groups of E *with their profinite topology.*

Proof. To prove the lemma, it is enough to prove that the homotopy *presheaves* $\pi_t^p\underline{\mathrm{E}}$ take the form $\underline{\pi_t\mathrm{E}}$, since the topology is subcanonical. Moreover, since $\mathrm{Free}_{\mathbb{G}}$ generates the proétale topos, it is enough to show that $(\pi_t^p\underline{\mathrm{E}})|_{\mathrm{Free}_{\mathbb{G}}} : S \mapsto \mathrm{Cont}_{\mathbb{G}}(S, \pi_t\mathrm{E}) = \mathrm{Cont}(S/\mathbb{G}, \pi_t\mathrm{E})$.

Using Lemma 2.24, we have for any free $\mathbb{G}$-set of the form $S = T \times \mathbb{G}$ (with trivial action on T),

$$\underline{\mathrm{E}}(S) \simeq \underline{\mathrm{E}}(T) \otimes \underline{\mathrm{E}}(\mathbb{G}) \simeq L_{\mathrm{K}} \varinjlim_j \prod_{T_j} \mathrm{E}. \tag{17}$$

Since E-localisation is smashing, the spectrum $\varinjlim_j \prod_{T_j} \mathrm{E}$ is E-local, and so its K-localisation can be computed by smashing with a tower of generalised Moore spectra M_I; see for example [HS99]. Thus

$$\pi_t\left(\underline{\mathrm{E}}(S)\right) = \pi_t \lim_I (\varinjlim_j \prod_{T_j} \mathrm{E} \otimes M_I),$$

and we obtain a Milnor sequence

$$0 \to {\lim_I}^1 \pi_t \varinjlim_j \prod_{T_j} \mathrm{E} \otimes M_I \to \pi_t\left(\underline{\mathrm{E}}(S)\right) \to \lim_I \pi_t \varinjlim_j \prod_{T_j} \mathrm{E} \otimes M_I \to 0.$$

Now observe that

$$\begin{aligned}\pi_t \varinjlim_j \prod_{T_j} \mathrm{E} \otimes M_I &= \varinjlim_j \prod_{T_j} \pi_t \mathrm{E} \otimes M_I \\ &= \varinjlim_j \prod_{T_j} (\pi_t \mathrm{E})/I \\ &= \varinjlim_j \mathrm{Cont}(T_j, (\pi_t\mathrm{E})/I) \\ &= \mathrm{Cont}(T, (\pi_t\mathrm{E})/I),\end{aligned}$$

using for the last equality that the target is finite. In particular, each of the maps $\mathrm{Cont}(T,(\pi_t\mathrm{E})/I') \to \mathrm{Cont}(T,(\pi_t\mathrm{E})/I)$ is surjective, since the inclusion $I' \subset I$ induces a surjection of finite sets $(\pi_t\mathrm{E})/I' \twoheadrightarrow (\pi_t\mathrm{E})/I$, and so admits a (set-theoretic) splitting. Thus $\lim^1$ vanishes and

$$\pi_t\left(\underline{\mathrm{E}}(S)\right) \cong \lim_I \mathrm{Cont}(T,(\pi_t\mathrm{E})/I) \cong \mathrm{Cont}(T,\pi_t\mathrm{E}). \tag{18}$$

We write $\theta\colon \pi_t\underline{\mathrm{E}}(T\times\mathbb{G}) \to \mathrm{Cont}(T,\pi_t\mathrm{E})$ for the map above and conclude by proving that θ is natural in the $\mathbb{G}$-set $T\times\mathbb{G}$. Given a $\mathbb{G}$-map $T\times\mathbb{G}\to T'\times\mathbb{G}$, we need to show that

$$\begin{array}{ccc} \pi_t\mathcal{E}(T'\times\mathbb{G}) & \xrightarrow{\theta} & \mathrm{Cont}_{\mathbb{G}}(T'\times\mathbb{G},\pi_t\mathrm{E}) \\ {\scriptstyle \pi_t\mathcal{E}(j)}\big\downarrow & & \big\downarrow{\scriptstyle j^*} \\ \pi_t\mathcal{E}(T\times\mathbb{G}) & \xrightarrow[\theta]{} & \mathrm{Cont}_{\mathbb{G}}(T\times\mathbb{G},\pi_t\mathrm{E}) \end{array}$$

commutes. Since $\mathrm{Cont}_{\mathbb{G}}(T\times\mathbb{G},\pi_t\mathrm{E})$ embeds in $\prod_{T^\delta}\pi_t\mathrm{E}$, we may reduce to the case $T=*$, in which case j is the composite of a $\mathbb{G}$-map $\mathbb{G}\to\mathbb{G}$ with the inclusion of the fibre $\mathbb{G}\to T'\times\mathbb{G}$ at $x\in T'$. Naturality for the latter is clear from the construction of θ: indeed, θ^{-1} is the colimit comparison map

$$L_0 \varinjlim_i \prod_{T_i} \pi_t\mathrm{E} \to \pi_t L_{\mathrm{K}}\pi_t \varinjlim_i \prod_{T_i} \mathrm{E}$$

where L_0 denotes derived I_h-completion (recalled in Section 4), so natural in maps induced from $T'\to T$. It thus remains to show that θ is $\mathbb{G}$-equivariant in the case $T=*$. Unwrapping definitions, we are asking whether the identity on $\pi_t\mathrm{E}$ is $\mathbb{G}$-equivariant for

1. the action on the source given on homotopy groups by the $\mathbb{G}$-action on $\underline{\mathrm{E}}(\mathbb{G}) \coloneqq L_{\mathrm{K}}\varinjlim_U \underline{\mathrm{E}}(\mathbb{G}/U)$ and the Goerss-Hopkins-Miller action on $\underline{\mathrm{E}}(\mathbb{G}/U)$,
2. the standard action on the target (i.e., that coming from automorphisms of (k,Γ)).

In this case, it is enough to observe that the Goerss-Hopkins-Miller action on E induces the standard action on homotopy groups, and that each map $\underline{\mathrm{E}}(\mathbb{G}/U)\to\mathrm{E}$ is $\mathbb{G}$-equivariant. □

Corollary 2.45. *For any closed subgroup $G\subset\mathbb{G}$, the starting page of the spectral sequence (15) is given by continuous group cohomology:*

$$E_{2,+}^{s,t} = H^s(G,\pi_t\mathrm{E}). \tag{19}$$

Proof. First note that if $i\colon G\to\mathbb{G}$, then

$$\pi_t i^*\underline{\mathrm{E}} \cong \underline{\mathrm{res}_i\,\pi_t\mathrm{E}}$$

by Example 2.32. Now the claim follows from [BS14, Lemma 4.3.9(4)], since each of the rings $\mathbb{W}_i(k)$ of truncated Witt vectors is discrete and the maps in the diagram $\mathbb{W}(k)=\lim\mathbb{W}_i(k)$ are all split. Thus each $\pi_t\mathrm{E}$ is in the subcategory $\mathcal{C}$ of *loc. cit.*, and in particular the map

$$\Phi : H^*(G,\pi_t E) \to H^*(BG_{\text{proét}},\underline{\pi_t\mathrm{E}})$$

is an isomorphism. □

Remark 2.46 (c.f. [BH16]). By virtue of Corollary 2.41, one obtains for any K-local spectrum X a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(B\mathbb{G}_{\text{proét}},\underline{\mathrm{E}}_t^\vee X) \implies \pi_{t-s}X,$$

where $\underline{\mathrm{E}}_t^\vee X$ denotes the sheaf $\pi_t(\underline{\mathrm{E}}\otimes X)$. We briefly comment on the E_2-page:

(1) To compare with group cohomology, one would first like to assert that

$$H^s(B\mathbb{G}_{\text{proét}}, \mathrm{E}_t^\vee X) \cong H^s\left(B\mathbb{G}_{\text{proét}}, \underline{\mathrm{E}_t^\vee X}\right), \tag{20}$$

i.e. that $\pi_t\underline{\mathrm{E}} \otimes X$ is represented by the topological $\mathbb{G}$-module $\mathrm{E}_t^\vee X$. As in Lemma 2.44, there is an exact sequence

$$0 \to \lim_I{}^1 \operatorname{Cont}_{\mathbb{G}}(S, (\mathrm{E}/I)_t X) \to \pi_t\Gamma(S, \underline{\mathrm{E}} \otimes X) \to \operatorname{Cont}_{\mathbb{G}}(S, \lim_I(\mathrm{E}/I)_t X) \to 0$$

for any free $\mathbb{G}$-set S, and this implies for example that (20) holds if $\mathrm{E}_*(X/I) \simeq (\mathrm{E}_*X)/I$ for each of the ideals I (i.e., E_*X is Landweber exact). We expect that a more general comparison can be made.

(2) If moreover each of the $\mathbb{G}$-modules $\mathrm{E}_t^\vee X$ satisfies one of the assumptions of [BS14, Lemma 4.3.9], then the E_2-page is given by the continuous group cohomology $H^s(\mathbb{G}, \mathrm{E}_t^\vee X)$. For example, this happens if $\mathrm{E}_*^\vee X$ is degreewise profinite; it also holds for $\mathrm{E}_*^\vee X[p^{-1}]$ whenever it holds for X.

We have thus identified the E_2-page of the descent spectral sequence for $\underline{\mathrm{E}}$ with the E_2-page of the K-local E-Adams spectral sequence. The next step is to show this extends to an identification of the spectral sequences. We do this by using the décalage technique originally due to Deligne [Del71]; the following theorem is standard, but for the sake of completeness (and to fix indexing conventions, one of the great difficulties in the subject) we include the argument in Appendix A.

Proposition 2.47 (Lemma A.3)**.** *Let $\mathcal{F}$ be a sheaf of spectra on a site $\mathcal{C}$, and let $X \twoheadrightarrow *$ be a covering of the terminal object. Suppose that for every t and $q > 0$ we have $\Gamma(X^q, \Sigma^t\pi_t\mathcal{F}) = \Sigma^t\pi_t\Gamma(X^q, \mathcal{F})$. Then there is an isomorphism between the descent and Bousfield-Kan spectral sequences, up to reindexing: for all r,*

$$E_r^{s,t} \cong \check{E}_{r+1}^{2s-t,s}.$$

Proposition 2.48. *Let $G \subset \mathbb{G}$ be a closed subgroup. Then décalage of the Postnikov filtration induces an isomorphism between the following spectral sequences:*

$$E_{2,+}^{s,t} = \pi_s\Gamma\Sigma^t\pi_t\underline{\mathrm{E}} = H^s(G, \pi_t\mathrm{E}) \implies \pi_{t-s}\Gamma(\mathbb{G}/G, \underline{\mathrm{E}}),$$
$$\check{E}_{3,+}^{2s-t,s} = \pi^s(\pi_t\mathrm{E}^{\otimes_{\mathrm{E}(\mathbb{G}/G)}\bullet+1}) = H^s(G, \pi_t\mathrm{E}) \implies \pi_{t-s}\mathrm{E}^{hG}.$$

The first is the descent spectral sequence for the sheaf $\underline{\mathrm{E}}$, and the second is the K*-local* E*-Adams spectral sequence converging to the Devinatz-Hopkins fixed points.*

Proof. By Lemma 2.24, the cosimplicial spectrum $\Gamma(\mathbb{G}/G \times \mathbb{G}^\bullet, \underline{\mathrm{E}})$ is obtained by smashing E^{hU} with the K-local Amitsur complex for E; its Tot-tower is the K-local E-Adams tower for E^{hU} by definition.

According to Lemma 2.47, all that remains to check is that each spectrum

$$\Gamma(\mathbb{G}/G \times \mathbb{G}^q, \Sigma^t\pi_t\underline{\mathrm{E}}), \qquad q > 0$$

is Eilenberg-Mac Lane, which we will deduce from Lemma 2.44. Indeed, we know that $\Gamma(\mathbb{G}/G \times \mathbb{G}^q, \Sigma^t\pi_t\underline{\mathrm{E}})$ is t-truncated, while for $s \le t$ we have

$$\begin{aligned}\pi_s\Gamma(\mathbb{G}/G \times \mathbb{G}^q, \Sigma^t\pi_t\underline{\mathrm{E}}) &= H^{t-s}(B\mathbb{G}_{\text{proét}/(\mathbb{G}/G\times\mathbb{G}^q)}, \pi_t\underline{\mathrm{E}}) \\ &= H^{t-s}(\mathrm{Profin}_{/(\mathbb{G}/G\times\mathbb{G}^{q-1})}, \underline{\pi_t\mathrm{E}}) \\ &= H^{t-s}_{\mathrm{cond}}(\mathbb{G}/G \times \mathbb{G}^{q-1}, \underline{\pi_t\mathrm{E}}),\end{aligned}$$

where H^*_{cond} denotes condensed cohomology. Here we have used the equivalence $B\mathbb{G}_{\text{proét}/(\mathbb{G}/G\times\mathbb{G}^q)} \simeq \mathrm{Profin}_{/(\mathbb{G}/G\times\mathbb{G}^{q-1})}$ sending $S \mapsto S/\mathbb{G}$. We now argue that the higher cohomology groups vanish, essentially as in the first part of [Sch19, Theorem 3.2]. Namely, the sheaves $\underline{\pi_t\mathrm{E}}$ on $B\mathbb{G}_{\text{proét}/(\mathbb{G}/G\times\mathbb{G}^q)}$ satisfy the conditions of [BS14, Lemma 4.3.9(4)], and so the cohomology groups in question can be computed by Čech cohomology: the Čech-to-derived spectral sequence collapses, since the higher direct images of $\underline{\pi_t\mathrm{E}}$ vanish. As a result, to check they vanish it will be enough to check that the Čech complex

$$\operatorname{Cont}_{\mathbb{G}}(\mathbb{G}/G \times \mathbb{G}^q, \pi_t\mathrm{E}) \to \operatorname{Cont}_{\mathbb{G}}(S, \pi_t\mathrm{E}) \to \operatorname{Cont}_{\mathbb{G}}(S \times_{\mathbb{G}/G\times\mathbb{G}^q} S, \pi_t\mathrm{E}) \to \cdots \tag{21}$$

is exact for any surjection $S \twoheadrightarrow \mathbb{G}/G \times \mathbb{G}^q$. Writing this as a limit of surjections of finite $\mathbb{G}$-sets $S_i \twoheadrightarrow S_i'$ (with $\lim S_i' = \mathbb{G}/G \times \mathbb{G}^q$ and $\lim S_i = S$), and writing $A_{i,I}^j \coloneqq \operatorname{Cont}_{\mathbb{G}}(S_i^{\times_{S_i'} j}, \pi_t\mathrm{E}/I)$ for brevity, (21) is the complex

$$\lim_I \operatorname{colim}_i A^0_{i,I} \to \lim_I \operatorname{colim}_i A^1_{i,I} \to \lim_I \operatorname{colim}_i A^2_{i,I} \to \cdots .$$

Its cohomology therefore fits in a Milnor sequence[3]

$$0 \to {\lim_I}^1 H^{j-1}(\operatorname{colim}_i A^*_{i,I}) \to H^j(\lim_I \operatorname{colim}_i A^*_{i,I}) \to \lim_I H^j(\operatorname{colim}_i A^*_{i,I}) \to 0. \tag{22}$$

But for each fixed pair (i, I), the complex $A^*_{i,I}$ is split; the same is therefore also true of the colimit, which thus has zero cohomology. Now (22) shows that (21) is exact, and so

$$\pi_s\Gamma(\mathbb{G}/G \times \mathbb{G}^q, \Sigma^t\pi_t\underline{\mathrm{E}}) = \begin{cases} \pi_t\Gamma(\mathbb{G}/G \times \mathbb{G}^q, \underline{\mathrm{E}}) & s = t \\ 0 & s < t \end{cases}$$

That is, $\Gamma(\mathbb{G}/G \times \mathbb{G}^q, \Sigma^t\pi_t\underline{\mathrm{E}}) = \Sigma^t\pi_t\Gamma(\mathbb{G}/G \times \mathbb{G}^q, \underline{\mathrm{E}})$ as required. □

Remark 2.49. The same proof shows that $H^*_{\mathrm{cond}}(T, \underline{M}) = 0$ in positive degrees, for any profinite set T and profinite abelian group M with a presentation as a directed limit $M = \lim_{\mathbb{N}^{\mathrm{op}}} M_n$ satisfying the Mittag-Leffler condition.

Remark 2.50. There are alternative ways to obtain a proétale or condensed object from E, and it is sometimes useful to make use of these. Since we have proven that $\underline{\mathrm{E}} = \nu^p\underline{\mathrm{E}}^\delta$ is Kan extended from the étale classifying site, it is uniquely determined as an object of $\widehat{\mathcal{S}h}(BG_{\text{proét}}, \mathcal{S}p)$ by the following properties:

1. it is pointwise K-local,
2. it has underlying K-local spectrum $\Gamma(\mathbb{G}/*, \underline{\mathrm{E}}) = \mathrm{E}$,
3. its proétale homotopy groups are $\underline{\pi_t\mathrm{E}}$, where $\pi_t\mathrm{E}$ is viewed as a topological $\mathbb{G}$-module with its I_h-adic topology.

3. Descent for modules and the Picard spectrum

In the previous section, we showed that Morava E-theory defines a hypercomplete sheaf of spectra $\mathcal{E}$ on the proétale classifying site of $\mathbb{G}$. Our next aim is to improve this to a statement about its module ∞-category and therefore its Picard spectrum. The main result of this section is the construction of a hypercomplete sheaf of connective spectra $\mathfrak{pic}(\mathcal{E})$, with global sections $\Gamma\mathfrak{pic}(\mathcal{E}) = \mathfrak{pic}_h := \mathfrak{pic}(\mathcal{S}p_{\mathrm{K}})$. The functoriality of the construction via the proétale site will allow us to compare the resulting descent spectral sequence to the K-local E-Adams spectral sequence, including differentials.

3.1. Descent for $K(n)$-local module categories.

A similar strategy to that of Proposition 2.39 fails for the sheaf of module categories, since it is not clear that filtered colimits in $\mathrm{Pr}^{L,\mathrm{smon}}$ are exact. This section is devoted to the following result, which builds on the descent results of [MS16]:

Theorem 3.1. *The presheaf $\nu^p\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta} : B\mathbb{G}^{op}_{\text{proét}} \to \mathrm{Pr}^{L,\mathrm{smon}}$ satisfies hyperdescent.*

Remark 3.2. By Proposition 2.7, $\nu^p\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}^\delta} = \mathrm{Mod}^\wedge_{\nu^p\underline{\mathrm{E}}^\delta} = \mathrm{Mod}^\wedge_{\underline{\mathrm{E}}}$. Thus taking endomorphisms of the unit gives an alternative proof of Proposition 2.39 as an immediate corollary.

We will deduce Theorem 3.1 from a more general result, true for any descendable Galois extension in a sufficiently nice stable homotopy theory:

Theorem 3.3. *Let $\mathcal{C}$ be a compactly assembled stable homotopy theory, and G a profinite group. Suppose that $\mathbf{1} \to A$ is a descendable G-Galois extension in $\mathcal{C}$, corresponding to $\mathcal{A}^\delta \in \mathcal{S}h(BG_{\text{ét}}, \mathcal{C})$. Then the presheaf*

$$\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}(\mathcal{C}) \in \mathcal{P}(BG_{\text{proét}}, \mathrm{Pr}^{L,\mathrm{smon}})$$

satisfies hyperdescent.

Before giving the proof, let us make a few remarks.

Remark 3.4. (1) By [Mat16, Proposition 6.15], descendable Galois extensions are faithful.

[3] According to [Wei94, Prop. 3.5.8], this is true whenever the system $A^j_{i,*} \to A^j_{i,*}$ is Mittag-Leffler for each fixed j. But we have already noted that any inclusion $J \subset I$ induces a surjection $A^j_{i,J} \twoheadrightarrow A^j_{i,I}$.

(2) Recall [Lur18b, §21.1.2] that a presentable ∞-category is said to be *compactly assembled* if it is a retract in Pr^L of a compactly generated presentable ∞-category; if $\mathcal{C}$ is stable, this is equivalent to being dualisable in the symmetric monoidal structure on $\mathrm{Pr}^L_{\mathrm{st}}$.
(3) For the proof, we do *not* require that $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_\omega)$. In particular, $\mathcal{S}p_{\mathrm{K}}$ is compactly generated (though not by the unit) and so is an example of such an ∞-category, and Theorem 3.1 follows since $\mathbf{1}_{\mathrm{K}} \to \mathrm{E}$ is descendable.

The functor $\mathfrak{pic}$ preserves limits of symmetric monoidal ∞-categories by [MS16, Proposition 2.2.3], and so we obtain the first part of our main result:

Corollary 3.5. *There is a hypercomplete sheaf*

$$\mathfrak{pic}(\underline{\mathrm{E}}) \coloneqq \mathfrak{pic} \circ \mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}} \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\text{proét}}, \mathcal{S}p_{\geq 0})$$

with

$$\Gamma(\mathbb{G}/G, \mathfrak{pic}(\underline{\mathrm{E}})) \simeq \mathfrak{pic}(\mathrm{Mod}^{\wedge}_{\mathrm{E}^{hG}}).$$

In particular, we get a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(B\mathbb{G}_{\text{proét}}, \pi_t\mathfrak{pic}(\underline{\mathrm{E}})) \implies \pi_{t-s}\mathfrak{pic}_h. \tag{23}$$

Remark 3.6. In Section 3.2, we will evaluate the homotopy sheaves $\pi_t\mathfrak{pic}(\underline{\mathrm{E}})$ and hence identify the E_2-page with continuous cohomology $H^s(\mathbb{G}, \pi_t\mathfrak{pic}(\mathrm{E}))$.

We begin the proof of Theorem 3.1 with the following observation, stated for later use in slightly greater generality than is needed for this section.

Lemma 3.7. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category with geometric realisations, and $1 \leq k \leq \infty$. Suppose that $A \in \mathcal{C}$ is an $\mathbb{E}_k$-algebra, and that $B \in \mathrm{CAlg}(\mathcal{C})$ is such that*

$$\mathcal{C} \simeq \lim\Big(\mathrm{Mod}_B(\mathcal{C}) \rightrightarrows \mathrm{Mod}_{B\otimes B}(\mathcal{C}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big). \tag{24}$$

Then

$$\mathrm{RMod}_A(\mathcal{C}) \simeq \lim\Big(\mathrm{RMod}_{A\otimes B}(\mathcal{C}) \rightrightarrows \mathrm{RMod}_{A\otimes B\otimes B}(\mathcal{C}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big).$$

Remark 3.8. We call $B \in \mathrm{CAlg}(\mathcal{C})$ a *descent algebra* if (24) holds.

Proof. Write $B' \coloneqq A \otimes B$; this is itself an $\mathbb{E}_k$-algebra. We will verify the hypotheses of the Barr-Beck-Lurie theorem. Namely,

(1) $(-) \otimes_A B' \simeq (-) \otimes B$ is conservative, by virtue of the equivalence (24).
(2) $\mathrm{RMod}_A(\mathcal{C})$ has limits of B'-split cosimplicial objects: given $M^\bullet : \mathbf{\Delta} \to \mathrm{RMod}_A(\mathcal{C})$ with $M^\bullet \otimes_A B'$ split, we can form the limit M in $\mathcal{C}$, by the descendability assumption. Since $\mathrm{RMod}_A(\mathcal{C}) \subset \mathcal{C}$ is closed under limits, M is also a limit in $\mathrm{RMod}_A(\mathcal{C})$. This limit is preserved by $(-) \otimes_A B'$, again by (24). □

Lemma 3.9. *Let $\mathcal{C}$ be a compactly assembled stable homotopy theory and $A \in \mathrm{Alg}(\mathcal{C})$. Then $\mathrm{Mod}_A(\mathcal{C})$ is compactly assembled.*

Proof. We need to exhibit $\mathrm{Mod}_A(\mathcal{C})$ as a retract in Pr^L of some compactly generated ∞-category. To this end, recall [Lur18b, Theorem 21.1.2.10] that $\mathcal{C}$ is a retract in Pr^L of $\mathrm{Ind}(\mathcal{C}) = \mathrm{Ind}_\omega(\mathcal{C})$. More precisely, write $\varinjlim\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ for the Ind-extension of the identity; this is left adjoint to the Yoneda embedding y, and admits a further left adjoint $\widehat{y}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ under the assumption that $\mathcal{C}$ is compactly assembled. In principle, $\varinjlim$ can be made symmetric monoidal with respect to Day convolution on $\mathrm{Ind}(\mathcal{C})$, which implies that $\mathrm{Mod}_A(\mathcal{C})$ is also compactly assembled. To make this precise we must proceed with some care, as $\mathcal{C}$ is not small.

Let κ be a regular cardinal such that $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_\kappa)$. In particular $\mathbf{1} \in \mathcal{C}^\kappa$, and $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$. At the expense of enlarging κ, we can moreover assume that $\mathcal{C}^\kappa$ is closed under the tensor product on $\mathcal{C}$. Indeed, for this it suffices to choose $\kappa' > \kappa$ large enough that each object $C_1 \otimes C_2$, where $C_1, C_2 \in \mathcal{C}^\kappa$, is κ'-compact: then $C_1' \otimes C_2' \in \mathcal{C}^{\kappa'}$ for each $C_1', C_2' \in \mathcal{C}^{\kappa'}$, since this may be written as a κ'-small colimit of κ'-compact objects. Enlarging κ further, we assume that $A \in \mathcal{C}^\kappa$.

Since $\mathcal{C}$ is compactly assembled, the left adjoint $\varinjlim\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\widehat{y}$. Since $\widehat{y}$ preserves colimits and $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$, this factors through an adjunction

$$\widehat{y} : \mathcal{C} \rightleftarrows \mathrm{Ind}(\mathcal{C}^\kappa) : \varinjlim .$$

Moreover, the right adjoint is the Ind-extension of the inclusion $\mathcal{C}^\kappa \hookrightarrow \mathcal{C}$, and can therefore be canonically made symmetric monoidal for Day convolution on the source by [Lur17, Corollary 4.8.1.14]. Since $\varinjlim yA = A$, we obtain a functor

$$\varinjlim : \mathrm{Mod}_{yA}(\mathrm{Ind}(\mathcal{C}^\kappa)) \to \mathrm{Mod}_A(\mathcal{C}),$$

which preserves limits since these are computed on underlying objects. Both source and target are presentable, and so we obtain a left adjoint $\widehat{y}$ at the level of modules. This gives the desired retraction in Pr^L. □

Equipped with this and Lemma 2.24, we can now prove the main theorem of this section. I am grateful to Dustin Clausen for the strategy of the following result, which forms the key complement to the results of [Mat16].

Proposition 3.10. *Let $\mathcal{C}$ be a compactly assembled stable homotopy theory, and G a profinite group. Suppose that $\mathbf{1} \to A$ is a descendable G-Galois extension in $\mathcal{C}$, corresponding to $\mathcal{A}^\delta \in \mathcal{S}h(BG_{\text{ét}}, \mathcal{C})$. Then the restriction*

$$\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}(\mathcal{C})|_{\mathrm{Free}_G} \in \mathcal{P}(\mathrm{Free}_G, \mathrm{Pr}^L)$$

is a hypercomplete sheaf.

Proof. We do this in a number of steps.

(1) As noted in Section 2.2.1, any G-set is covered by a free one and every free G-set is split. A consequence is that for any free G-set S, the functor $S' \mapsto S'/G$ is an equivalence $(BG_{\text{proét}})_{/S} \simeq \{G\} \times \mathrm{Profin}_{/(S/G)}$, since any G-set over S is itself free. Thus suppose $T = \lim_i T_i$ is a profinite set over S/G, and choose a convergent sequence of neighbourhoods $U_j \subset G$ of the identity; applying Lemmas 2.7 and 2.24 we deduce the equivalences

$$\begin{aligned}\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}(T \times G) &= \varinjlim_{i,j} \mathrm{Mod}_{\mathcal{A}^\delta(T_i \times G/U_j)} \\ &\simeq \varinjlim_{i,j} \mathrm{Mod}_{\mathcal{A}^\delta(T_i) \otimes \mathcal{A}^\delta(G/U_j)} \\ &\simeq \varinjlim_{i} \mathrm{Mod}_{\varinjlim_j \mathcal{A}^\delta(T_i) \otimes \mathcal{A}^\delta(G/U_j)} \\ &\simeq \varinjlim_{i} \mathrm{Mod}_{\mathcal{A}^\delta(T_i) \otimes A} \\ &\simeq \varinjlim_{i} \mathrm{Mod}_{\prod_{T_i} A} \\ &\simeq \varinjlim_{i} \prod_{T_i} \mathrm{Mod}_A.\end{aligned}$$

Under the aforementioned equivalence we think of this as a presheaf on $\mathrm{Profin}_{/(S/G)}$: that is, if T is a profinite set over S/G, then

$$T = \lim_i T_i \mapsto \varinjlim_i \prod_{T_i} \mathrm{Mod}_A.$$

(2) Since T_i is a finite set, $\nu^p\mathrm{Mod}_{\mathcal{A}}(T_i \times G) \simeq \prod_{T_i} \mathrm{Mod}_A \simeq \mathcal{S}h(T_i, \mathrm{Mod}_A)$; we claim that the same formula holds for arbitrary T. Writing $T = \lim_i T_i$, it will suffice to prove that the adjunction

$$q^* : \varinjlim \mathcal{S}h(T_i, \mathrm{Mod}_A) \leftrightarrows \mathcal{S}h(T, \mathrm{Mod}_A) : q_*,$$

induced by the adjunctions $(q_i)^* \dashv (q_i)_*$ for each projection $q_i : T \to T_i$, is an equivalence; then the claim follows by passing to a limit of finite T_i. In fact, since the adjunction is obtained by tensoring the adjunction

$$q^* : \varinjlim \mathcal{S}h(T_i) \leftrightarrows \mathcal{S}h(T) : q_*, \tag{25}$$

with Mod_A (combine [Lur18b, Remark 1.3.1.6 and Prop. 1.3.1.7]), it will suffice to prove that (25) is an equivalence.

Let us assume for notational convenience that the diagram T_i is indexed over a filtered poset J, which we may do without loss of generality. Then the claim is a consequence of the fact that the topology on T is generated by subsets $q_i^{-1}(x_i)$, where $q_i\colon T \to T_i$ is a finite quotient. Indeed, let $\mathcal{O} := \mathrm{Open}(T)$; subsets of the form $q_i^{-1}(x_i)$ form a clopen basis, and we will write $\mathcal{B} \subset \mathcal{O}$ for the full subcategory spanned by such. Note that

$$\mathcal{S}h(T) \simeq \mathcal{P}_\Sigma(\mathcal{B}),$$

where the right-hand side denotes the full subcategory of presheaves that send binary coproducts to products. On the other hand, an object of $\varinjlim \mathcal{S}h(T_i)$ is a Cartesian section of the fibration determined by $i \mapsto \mathcal{S}h(T_i)$; abusively, we will denote such an object by $(\mathcal{F}_i)$, where $\mathcal{F}_i \in \mathcal{S}h(T_i)$, leaving the coherence data implicit. Write also

$$(q_{j,\infty})^* : \mathcal{S}h(T_j) \rightleftarrows \varinjlim \mathcal{S}h(T_i) : (q_{j,\infty})_*$$

for the colimit adjunction. By applying Yoneda, one verifies that the map

$$\varinjlim (q_i)^*\mathcal{F}_i \to q^*((\mathcal{F}_i)),$$

obtained by adjunction from the maps $(q_{j,\infty})_*(\eta)\colon \mathcal{F}_j = (q_{j,\infty})_*((\mathcal{F}_i)) \to (q_{j,\infty})_*q_*q^*((\mathcal{F}_j)) = (q_j)_*q^*((\mathcal{F}_i))$ as j varies, is an equivalence. Likewise, adjunct to $q^*((q_i)_*\mathcal{F}) \simeq \varinjlim (q_i)^*(q_i)_*\mathcal{F} \to \mathcal{F}$ is an equivalence

$$((q_i)_*\mathcal{F}) \xrightarrow{\sim} q_*\mathcal{F}.$$

Restricting to $\mathcal{B}$ (where no sheafification is required for forming the left adjoint), we will show that the unit and counit of the adjunction are equivalences. For the counit $q^*q_*\mathcal{F} \to \mathcal{F}$, this is clear:

$$\begin{aligned}
[q^*q_*\mathcal{F}]\,(q_i^{-1}(x_i)) &\simeq \left[\varinjlim_{j\geq i} (q_j)^*(q_j)_*\mathcal{F}\right](q_i^{-1}(x_i)) \\
&\simeq \varinjlim_{j\geq i}\left[(q_j)^*(q_j)_*\mathcal{F}(q_i^{-1}(x_i))\right] \\
&\simeq \varinjlim_{j\geq i}\left[(q_j)_*\mathcal{F}(q_{ij}^{-1}(x_i))\right] \\
&\simeq \varinjlim_{j\geq i}\left[\mathcal{F}(q_i^{-1}(x_i))\right] \simeq \mathcal{F}(q_i^{-1}(x_i)).
\end{aligned}$$

For the unit $(\mathcal{F}_j) \to q_*q^*(\mathcal{F}_j)$, it will suffice to prove for each i that the canonical map

$$\mathcal{F}_i \to (q_i)_* \varinjlim_{j\geq i} (q_j)^*\mathcal{F}_j \tag{26}$$

is an equivalence. But

$$\begin{aligned}
\left[(q_i)_* \varinjlim_{j\geq i}(q_j)^*\mathcal{F}_j\right](x_i) &\simeq \left[\varinjlim_{j\geq i}(q_j)^*\mathcal{F}_j\right](q_i^{-1}(x_i)) \\
&\simeq \varinjlim_{j\geq i}\left[(q_j)^*\mathcal{F}_j(q_i^{-1}x_i)\right] \\
&\simeq \varinjlim_{j\geq i}\left[\mathcal{F}_j(q_{ij}^{-1}x_i)\right] \\
&\simeq \varinjlim_{j\geq i}\left[(q_{ij})_*\mathcal{F}_j(x_i)\right],
\end{aligned}$$

with respect to which (26) is the structure map for $j = i$. This is an equivalence: since each of the coherence maps

$$\mathcal{F}_i \to (q_{ij})_*\mathcal{F}_j$$

defining the colimit is an equivalence by definition of $\varinjlim \mathcal{S}h(T_i)$, the diagram $j \mapsto (q_{ij})_*\mathcal{F}_j(x_i)$ factors through the groupoid completion $J_{i/}^{\mathrm{gpd}}$. But since $J_{i/}$ is filtered, both inclusions

$$J_{i/} \hookrightarrow J_{i/}^{\mathrm{gpd}} \hookleftarrow \{i\}$$

are cofinal by [Lur09, Corollary 4.1.2.6].

(3) We are left to prove that $T \mapsto \mathcal{S}h(T, \mathrm{Mod}_A) \in \mathrm{Pr}^{L,\mathrm{smon}}$ is a hypercomplete sheaf on $\mathrm{Profin}_{/(S/G)}$. This is precisely the content of [Hai22, Theorem 0.5], noting that

(*i*) limits in $\mathrm{Pr}^{L,\mathrm{smon}}$ are computed in Cat_∞;
(*ii*) $\mathcal{C}$, and so $\mathrm{Mod}_A(\mathcal{C})$, is compactly assembled;
(*iii*) any profinite set T has homotopy dimension zero by [Lur09, Theorem 7.2.3.6 and Remark 7.2.3.3], so that Postnikov towers in $\mathcal{S}h(T)$ converge: $\mathcal{S}h(T) \simeq \lim_n \mathcal{S}h(T, \mathcal{S}_{\leq n})$ by [Lur09, Theorem 7.2.1.10]. □

The proof of Theorem 3.3 follows by combining the previous proposition with [Mat16, Proposition 3.22]:

Proof (Prop. 3.3). Let $T_\bullet \twoheadrightarrow T_{-1} = S$ be a hypercovering in $BG_{\text{proét}}$, and form the diagram

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downdownarrows\!\downarrow & & \downdownarrows\!\downarrow & & \downdownarrows\!\downarrow & & \\
S \times G \times G & \longleftarrow & T_0 \times G \times G & \leftleftarrows & T_1 \times G \times G & \Lleftarrow & \cdots \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \\
S \times G & \longleftarrow & T_0 \times G & \leftleftarrows & T_1 \times G & \Lleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
S & \longleftarrow & T_0 & \leftleftarrows & T_1 & \Lleftarrow & \cdots
\end{array}$$

To prove descent for the hypercover, it is enough to show descent for each column and for each non-negative row. For each such row we are in the context of Proposition 3.10, and so obtain a limit diagram after applying $\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}$. On the other hand, writing $T_j = \lim T_{ij}$ and applying $\nu^p\mathrm{Mod}_{\mathcal{A}^\delta}$ to a column we obtain the complex

$$\varinjlim_i \mathrm{Mod}_{\mathcal{A}^\delta(T_{ij})} \longrightarrow \varinjlim_{i,k} \mathrm{Mod}_{\mathcal{A}^\delta(T_{ij}\times G/U_k)} \rightrightarrows \varinjlim_{i,k} \mathrm{Mod}_{\mathcal{A}^\delta(T_{ij}\times G/U_k \times G/U_k)} \Rrightarrow \cdots$$

Using Lemma 2.7 and the equivalences

$$\varinjlim_{i,k} \mathcal{A}^\delta(T_{ij} \times (G/U_k)^n) \simeq \mathcal{A}(T_j) \otimes A^{\otimes n}$$

one identifies this with the complex

$$\mathrm{Mod}_{\mathcal{A}(T_j)} \longrightarrow \mathrm{Mod}_{\mathcal{A}(T_j)\otimes A} \rightrightarrows \mathrm{Mod}_{\mathcal{A}(T_j)\otimes A \otimes A} \Rrightarrow \cdots . \tag{27}$$

When $T_j = *$, this is the complex

$$\mathcal{C} \longrightarrow \mathrm{Mod}_A \rightrightarrows \mathrm{Mod}_{A\otimes A} \Rrightarrow \cdots$$

which is a limit diagram according to [Mat16]. For general T, (27) is a limit diagram by combining the $T = *$ case with Lemma 3.7. □

3.2. **The Picard spectrum as a proétale spectrum.** Since limits in $\mathrm{Pr}^{L,\mathrm{smon}}$ are computed in $\mathrm{Cat}_\infty^{\mathrm{smon}}$, the functor

$$\mathfrak{pic} : \mathrm{Pr}^{L,\mathrm{smon}} \to \mathcal{S}p_{\geq 0},$$

preserves them [MS16, Prop. 2.2.3], and so the composite $\mathfrak{pic}(\underline{\mathrm{E}}) = \mathfrak{pic} \circ \mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}}$ is immediately seen to be a sheaf of connective spectra. As a result, the 0-stem in its descent spectral sequence (23) converges conditionally to Pic_h. Its E_2-page consists of cohomology of the proétale homotopy groups $\pi_*\mathfrak{pic}(\underline{\mathrm{E}})$, and as in Corollary 2.45 we'd like to identify this with group cohomology with coefficients in the continuous $\mathbb{G}$-modules $\pi_*\mathfrak{pic}(\mathrm{E})$. In order to deduce this once again from [BS14, Lemma 4.3.9], we need to show that

$$\pi_t\mathfrak{pic}(\underline{\mathrm{E}}) = \underline{\pi_t\mathfrak{pic}(\mathrm{E})}. \tag{28}$$

for $t = 0, 1$; for $t \geq 2$ the result follows from Lemma 2.44 and the isomorphism $\pi_t\mathfrak{pic}(A) \simeq \pi_{t-1}A$, natural in the ring spectrum A. The first aim of this section is to prove (28). Having done this, we will evaluate the resulting spectral sequence.

Theorem 3.11. *The proétale homotopy groups of the Picard sheaf are given by*

$$\pi_t\mathfrak{pic}(\underline{\mathrm{E}}) = \underline{\pi_t\mathfrak{pic}(\mathrm{E})} = \begin{cases} \underline{\mathrm{Pic}(\mathrm{E})} & t = 0 \\ \underline{(\pi_0\mathrm{E})^\times} & t = 1 \\ \underline{\pi_{t-1}\mathrm{E}} & t \geq 2 \end{cases} \tag{29}$$

Before proving this, we give the desired identification of the E_2-page in (23):

Corollary 3.12. *For any closed subgroup $G \subset \mathbb{G}$, the starting page of the descent spectral sequence for the Picard spectrum $i^*\mathfrak{pic}(\underline{\mathrm{E}})$ is given by continuous group cohomology:*

$$E_2^{s,t} = H^s(G, \pi_t\mathfrak{pic}(\mathrm{E})). \tag{30}$$

Proof. As in Corollary 2.45, it follows that the homotopy sheaves of $i^*\mathfrak{pic}(\underline{\mathrm{E}})$ are represented by the restricted G-action on $\pi_t\mathfrak{pic}(\mathrm{E})$. As in Lemma 2.44 we appeal to [BS14, Lemma 4.3.9(4)]. We have already noted that the requisite conditions hold for the sheaves $\pi_t\mathfrak{pic}(\underline{\mathrm{E}}) \cong \pi_{t-1}\underline{\mathrm{E}}$ when $t \geq 2$; all that remains to justify is what happens at $t = 0$ and 1. But $\pi_0\mathfrak{pic}(\mathrm{E}) \cong \mathbb{Z}/2$ certainly satisfies the hypotheses of [BS14, Lemma 4.3.9], while $\pi_1\mathfrak{pic}(\mathrm{E}) = (\pi_0\mathrm{E})^\times$ is the limit of finite $\mathbb{G}$-modules $(\pi_0\mathrm{E}/I)^\times$. $\square$

We now focus on the proof Theorem 3.11.

Remark 3.13. As in Section 2, it will suffice to prove that the homotopy sheaves take this form on $\mathrm{Free}_{\mathbb{G}}$. If T is a finite set, then $\underline{\mathrm{E}}(T \times \mathbb{G}) \simeq \bigoplus_T \mathrm{E}$, and so

$$\mathrm{Cont}(T, \mathrm{Pic}(\mathrm{E})) \cong \bigoplus_T \mathrm{Pic}(\mathrm{E}) \cong \mathrm{Pic}(\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}(T\times\mathbb{G})}).$$

If T is an arbitrary profinite set, the isomorphisms above induce a canonical map

$$\chi : \mathrm{Cont}(T, \mathrm{Pic}(\mathrm{E})) \to \mathrm{Pic}(\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}(T\times\mathbb{G})}),$$

which may be described explicitly as follows: any continuous map $f\colon T \to \mathrm{Pic}(\mathrm{E}) = \mathbb{Z}/2\{\Sigma\mathrm{E}\}$ defines a clopen decomposition $T = T^0 \sqcup T^1$, with $T^i = f^{-1}(\Sigma^i\mathrm{E})$. Projecting to finite quotients gives $T_i = T_i^0 \sqcup T_i^1$ for i sufficiently large, and since $\underline{\mathrm{E}}(T \times \mathbb{G}) = L_{\mathrm{K}} \varinjlim_i \bigoplus_{T_i} \mathrm{E}$ we set

$$\chi(f) = L_{\mathrm{K}} \varinjlim_i \left(\bigoplus_{T_i^0} \mathrm{E} \oplus \bigoplus_{T_i^1} \Sigma\mathrm{E} \right) \in \mathrm{Pic}(\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}(T\times\mathbb{G})}). \tag{31}$$

More formally, χ is the natural transformation

$$\chi \in \mathrm{Hom}(\underline{\mathrm{Pic}(\mathrm{E})}, \mathrm{Pic}(\underline{\mathrm{E}})) \cong \mathrm{Pic}(\underline{\mathrm{E}}(\mathbb{Z}/2)) = \mathrm{Pic}(\mathbf{1}_{\mathrm{K}}) \times \mathrm{Pic}(\mathbf{1}_{\mathrm{K}})$$

corresponding to the invertible $\underline{\mathrm{E}}(\mathbb{Z}/2)$-module $(\mathbf{1}_{\mathrm{K}}, \Sigma\mathbf{1}_{\mathrm{K}})$.

We will deduce Theorem 3.11 by showing that the map χ is an isomorphism. The key point will be the following:

Proposition 3.14. *Let T be a profinite set and $X \in \mathrm{Pic}(\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}(T\times\mathbb{G})})$. Then X is in the image of χ.*

Given this, it is straightforward to prove the main result of the section:

Proof (Theorem 3.11). For $t \geq 1$, this follows from the equivalence

$$\Omega\mathfrak{Pic}\left(\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}}\right) \simeq \mathrm{GL}_1(\underline{\mathrm{E}}).$$

and the identification of the proétale homotopy groups $\pi_t\underline{\mathrm{E}}$ in Lemma 2.44.

For $t = 0$, Proposition 3.14 implies that χ is surjective, while injectivity is clear from (31). This yields isomorphisms of sheaves on $\mathrm{Free}_{\mathbb{G}}$,

$$\underline{\mathrm{Pic}(\mathrm{E})} \simeq \mathrm{Cont}((-)/\mathbb{G}, \mathrm{Pic}(\mathrm{E})) \simeq \pi_0\mathfrak{pic}(\underline{\mathrm{E}}).$$ $\square$

To prove Proposition 3.14 it will be convenient to work in the context of sheaves of E-modules, using the equivalence

$$\mathrm{Mod}^\wedge_{\underline{\mathrm{E}}(T\times\mathbb{G})} \simeq \mathcal{S}h(T, \mathrm{Mod}^\wedge_{\mathrm{E}})$$

from the proof of Theorem 3.3. We begin by recording some basic lemmas.

Lemma 3.15 ([pa23], Tag 0081)**.** *Let T be a topological space, and A a set. The constant sheaf A_T on the T takes the form*

$$U \mapsto \mathrm{LC}\,(U, A)\,, \tag{32}$$

that is, locally constant functions $U \to A$.

Lemma 3.16. *For any $X \in \mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge})$,*

$$[\mathrm{E}_T, X] \simeq \mathrm{Hom}(\pi_*\mathrm{E}_T, \pi_*X)$$

Proof. We have isomorphisms

$$\begin{aligned}[\mathrm{E}_T, X] &\simeq \pi_0\mathrm{Map}(\mathrm{E}_T, X)\\ &\simeq \pi_0\Gamma X\\ &\simeq \Gamma\pi_0 X\\ &\simeq \mathrm{Hom}(\pi_*\mathrm{E}_T, \pi_*X),\end{aligned}$$

because the descent spectral sequence for ΓX collapses immediately to the 0-line. Indeed, profinite sets have homotopy dimension zero, and therefore cohomological dimension zero [Lur09, Corollary 7.2.2.30]. □

Remark 3.17. After sheafification, postcomposition with the functor $\mathrm{K}_*^{\mathrm{E}}(-) := \pi_*(\mathrm{K} \otimes_{\mathrm{E}} -) \colon \mathrm{Mod}_{\mathrm{E}}^{\wedge} \to \mathrm{Mod}_{\pi_*\mathrm{K}}$ defines a functor

$$\mathrm{K}_*^{\mathrm{E}} : \mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge}) \to \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathrm{K}}).$$

Similarly to [HMS94, BR05], our strategy will be to use monoidality of this functor to deduce results about invertible objects in $\mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge})$ by first proving them at the level of $\mathrm{K}_*^{\mathrm{E}}(-)$.

Lemma 3.18. *Let T be a profinite set, and $X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge}))$. Then $\mathrm{K}_*^{\mathrm{E}}X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathrm{K}}))$.*

Proof. The functor $\mathrm{K}_*^{\mathrm{E}}(-)$ on spectra is (strict) monoidal: this is [BR05, Corollary 33]. It follows that the induced functor

$$\mathrm{K}_*^{\mathrm{E}} : \mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge}) \to \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathrm{K}})$$

is also strictly monoidal, and hence preserves invertible objects. □

Lemma 3.19. *Let T be a profinite set and k a field graded by an abelian group A. Then any $X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_k))$ is locally free of rank one.*

Proof. Since $\mathcal{S}h(T, \mathrm{Mod}_k) = \mathrm{Mod}_{k_T}$ for k_T the constant sheaf, the result follows from [pa23, Tag 0B8M] in the ungraded case. In the graded case, [pa23, Tag 0B8K] still shows that X is locally a summand of a finite free module (though not necessarily one in degree zero); since k is a field it follows that X is locally a shift of k_T by some $a \in A$. □

Proof (Proposition 3.14). Note that under the equivalence

$$\mathrm{Mod}_{\underline{\mathrm{E}}(T\times\mathbb{G})}^{\wedge} \simeq \mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge}),$$

the image of χ in $\mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge})$ consists of those invertible sheaves that are locally free of rank one: indeed, T has a basis of finite clopen covers $T = \bigsqcup_{x\in T_i} U_x$, and $\chi(\mathrm{Cont}(T_i, \mathrm{Pic}(\mathrm{E})))$ is the subgroup of invertible sheaves that are constant along $\bigsqcup_{x\in T_i} U_x$.

We will deduce that any $X \in \mathrm{Pic}(\mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge}))$ is locally free of rank one by proving in turn each of the following statements:

(1) $\mathrm{K}_*^{\mathrm{E}}X \in \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathrm{K}})$ is locally free of rank one,
(2) for every $i_0, \dots, i_{n-1} \geq 1$, the sheaf $\pi_*(X/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}})) \in \mathcal{S}h(T, \mathrm{Mod}_{\pi_*\mathrm{E}})$ is locally constant with value $\Sigma^{\varepsilon}\pi_*\mathrm{E}/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}})$ [4],
(3) $X \in \mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge})$ is locally free of rank one.

Essentially, this follows the proof of [BR05, Theorem 3.7].

(1) This follows immediately by combining Lemmas 3.18 and 3.19. Since all our claims are local, we will assume for simplicity that $\mathrm{K}_*^{\mathrm{E}}X \simeq \pi_*\mathrm{K}_T$.
(2) For $i_0, \dots, i_{n-1} \geq 1$, we first show that that $\pi_*(X/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}}))$ admits a surjection by $\pi_*\mathrm{E}_T$. This follows by induction on $\sum_j i_j$, with the base case being (i). For the induction step, note that $\mathcal{S}h(T, \mathrm{Mod}_{\mathrm{E}}^{\wedge})$ is tensored over $\mathrm{Mod}_{\mathrm{E}}^{\wedge}$, and so the cofibre sequences in [BR05, Lemma 34] give rise to cofibre sequences

$$X/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}}) \xrightarrow{u_j} X/(p^{i_0}, \dots, u_{n-1}^{i_{n-1}}) \to X/(p^{i_0}, \dots, u_j, \dots u_{n-1}^{i_{n-1}}) \oplus \Sigma X/(p^{i_0}, \dots, u_j, \dots u_{n-1}^{i_{n-1}}) \tag{33}$$

[4] Here $\varepsilon \in \{0, 1\}$ may vary on T.

and

$$X/(p^{i_0},\ldots,u_j^{i_j-1},\ldots,u_{n-1}^{i_{n-1}}) \xrightarrow{u_j} X/(p^{i_0},\ldots,u_j^{i_j},\ldots,u_{n-1}^{i_{n-1}}) \to X/(p^{i_0},\ldots,u_j,\ldots u_{n-1}^{i_{n-1}}) \tag{34}$$

for $i_j > 1$; in particular, using (34) and the inductive hypothesis, we can assume that $\pi_*(X/(p^{i_0},\ldots,u_{n-1}^{i_{n-1}}))$ is concentrated in even degrees. Thus (33) implies we have an exact sequence

$$\begin{aligned} 0 \to u_j\pi_{2t}(X/(p^{i_0},\ldots,u_j^{i_j},\ldots,u_{n-1}^{i_{n-1}})) &\to \pi_{2t}(X/(p^{i_0},\ldots,u_j^{i_j},\ldots,u_{n-1}^{i_{n-1}})) \\ &\to \pi_{2t}(X/(p^{i_0},\ldots,u_j,\ldots,u_{n-1}^{i_{n-1}})) \to 0 \end{aligned} \tag{35}$$

for any t. Since $\pi_*\mathrm{E}_T$ is free, we can choose a lift α below:

$$\begin{tikzcd} & \pi_*(X/(p^{i_0},\ldots,u_j^{i_j},\ldots,u_{n-1}^{i_{n-1}})) \arrow[d] \\ \pi_*\mathrm{E}_T \arrow[ur, dashed, "\alpha"] \arrow[r, two heads] & \pi_*(X/(p^{i_0},\ldots,u_j,\ldots,u_{n-1}^{i_{n-1}})) \end{tikzcd}$$

Now (35) implies that α is a surjection, since Nakayama's lemma implies that it is so on stalks. As in [BR05, Lemma 36], this implies (ii) by induction on $\sum i_j$.

(3) Since X is invertible (and in particular dualisable),

$$\lim_{i_0,\ldots,i_{n-1}}(\mathrm{E}/(p^{i_0},\ldots,u_{n-1}^{i_{n-1}})\otimes_{\mathrm{E}} X) \simeq \lim_{i_0\ldots,i_{n-1}}(\mathrm{E}/(p^{i_0},\ldots,u_{n-1}^{i_{n-1}}))\otimes_{\mathrm{E}} X \simeq X.$$

Now (ii) implies that all $\lim^1$-terms vanish and that

$$\begin{aligned} \pi_*X &\cong \pi_*\lim_{i_0,\ldots,i_{n-1}}(\mathrm{E}/(p^{i_0},\ldots,u_{n-1}^{i_{n-1}})\otimes_{\mathrm{E}} X) \\ &\cong \lim_{i_0,\ldots,i_{n-1}}\pi_*\mathrm{E}_T/(p^{i_0},\ldots,u_{n-1}^{i_{n-1}}) \cong \pi_*\mathrm{E}_T. \end{aligned}$$

By Lemma 3.16 we can lift the inverse to a map $\mathrm{E}_T \to X$, which is necessarily an equivalence. □

This completes the construction of the Picard sheaf $\mathfrak{pic}(\underline{\mathrm{E}})$, and the proof that its proétale homotopy groups take the desired form; Corollary 3.12 follows. We will now study the resulting descent spectral sequence, as we did for the descent spectral sequence of $\underline{\mathrm{E}}$ in Section 2. The construction of the descent spectral sequence using a Postnikov-style filtration is useful in making the comparison with the Picard spectral sequence (30). Namely, we make the following observation:

Lemma 3.20. *Suppose that $\mathcal{C}$ is a site, and $\mathcal{D}$ a presentable ∞-category. Let $X \in \mathcal{S}h(\mathcal{C},\mathcal{D})$, and $p,q\colon \mathcal{D} \to \mathcal{S}p_{\geq 0}$ two limit-preserving functors related by a natural equivalence $\tau_{[a,b]}p \simeq \tau_{[a,b]}q$. Then the descent spectral sequences for pX and qX' satisfy the following:*

(1) The E_2-pages agree in a range: $E_{2,pX}^{s,t} \simeq E_{2,qX}^{s,t}$ if $t \in [a,b]$.
(2) Under the isomorphism induced by (i), we have $d_{r,pX}^{s,t} = d_{r,qX}^{s,t}$ if $2 \leq r \leq b-t+1$.

Proof. This can be seen directly by comparing the Postnikov towers, since both claims depend only on their $[a,b]$-truncations. □

Remark 3.21. We are not asserting that $x \in E_{2,pX}$ survives to E_r if and only if the corresponding class in $E_{2,qX}$ does; this should really be taken as another assumption on the class x.

In our setting, the equivalence

$$\tau_{[t,2t-2]}\mathfrak{pic}(A) \simeq \tau_{[t,2t-2]}\Sigma A \tag{36}$$

of [HMS17], valid for $t \geq 2$ and *functorial* in the ring spectrum A, implies immediately the following corollary:

Proposition 3.22. *Let $A \in \mathcal{S}h(B\mathbb{G}_{\text{proét}},\mathrm{CAlg})$, and consider the two spectral sequences (15) and (30). Then*

(1) $E_2^{s,t} \simeq E_{2,+}^{s,t-1}$ if $t \geq 2$.
(2) The differentials d_r and $d_{r,+}$ on $E_r^{s,t} \simeq E_{r,+}^{s,t-1}$ agree as long as $r \leq t-1$ (whenever both are defined).

Finally, we want to prove décalage for the sheaf $\mathfrak{pic}(\underline{\mathrm{E}})$, as we did for the sheaf $\underline{\mathrm{E}}$ itself. This will allow us to determine the differential d_t on classes in $E_2^{s,t}$ too.

Proposition 3.23. *For any closed subgroup $G \subset \mathbb{G}$, décalage of the Postnikov filtration induces an isomorphism between the following spectral sequences:*

$$E_2^{s,t} = \pi_{t-s}\Gamma\Sigma^t\pi_t i^*\mathfrak{pic}(\underline{\mathrm{E}}) = H^s(G, \pi_t\mathfrak{pic}(\mathrm{E})) \implies \pi_{t-s}\mathfrak{pic}(\mathrm{E}^{hG}),$$
$$\check{E}_3^{2s-t,s} = \pi^s\pi_t\mathfrak{pic}(\mathrm{E}^{hG} \otimes \mathrm{E}^{\otimes\bullet+1}) \implies \pi_{t-s}\mathfrak{pic}(\mathrm{E}^{hG}).$$

The first is the Picard spectral sequence (30), and the second is the Bousfield-Kan spectral sequence for the cosimplicial spectrum

$$\Gamma(\mathbb{G}/G \times \mathbb{G}^{\bullet+1}, \mathfrak{pic}(\underline{\mathrm{E}})) = \mathfrak{pic}(\mathrm{Mod}^{\wedge}_{\mathrm{E}^{hG}\otimes\mathrm{E}^{\otimes\bullet+1}}).$$

Remark 3.24. The Bousfield-Kan spectral sequence of Proposition 3.23 is by definition Heard's spectral sequence [Hea22, Theorem 6.13].

Proof. As in Proposition 2.48, this follows from Proposition 2.47 once we prove the following equivalences:

$$\Gamma(\mathbb{G}^q, \pi_0\mathfrak{pic}(\underline{\mathrm{E}})) \simeq \pi_0\Gamma(\mathbb{G}^q, \mathfrak{pic}(\underline{\mathrm{E}})),$$
$$\Gamma(\mathbb{G}^q, \Sigma\pi_1\mathfrak{pic}(\underline{\mathrm{E}})) \simeq \Sigma\pi_1\Gamma(\mathbb{G}^q, \mathfrak{pic}(\underline{\mathrm{E}})).$$

Using Theorem 3.11, we compute that

$$H^{t-s}(\mathbb{G}^q, \mathrm{Pic}(\mathrm{E})) \cong H^{t-s}(\mathbb{G}^q, (\pi_0\mathrm{E})^\times) = 0$$

for $t - s > 0$, since condensed cohomology with profinite coefficients vanishes (Remark 2.49). □

Following the method of [MS16], we can now identify the first new differential in the Picard spectral sequence:

Corollary 3.25. *Suppose that $t \geq 2$ and $x \in E_2^{t,t}$. We abuse notation to identify x with its image in the starting page of the descent spectral sequence for $\underline{\mathrm{E}}$, and assume that both classes survive to the respective E_t-pages. The first nonadditive differential on x in the Picard spectral sequence is*

$$d_t x = d_t^{ASS} x + x^2 \tag{37}$$

The ring structure in the right hand side is that of the K*-local* E*-Adams spectral sequence (15).*

Proof. The same proof that appears in [MS16] goes through: namely, this formula holds for the universal cosimplicial $\mathbb{E}_\infty$-ring having a class in $E_2^{t,t}$ of its Bousfield-Kan spectral sequence, and so for the cosimplicial spectrum $\underline{\mathrm{E}}(\mathbb{G}/G \times \mathbb{G}^{\bullet+1}) = \mathrm{E}^{hG} \otimes \mathrm{E}^{\otimes\bullet+1}$ too. □

4. Picard group computations

In the previous parts we constructed proétale models for the continuous action of $\mathbb{G}$ on Morava E-theory, its K-local module ∞-category and its Picard spectrum. Respectively, these are $\underline{\mathrm{E}}$ (constructed in Section 2), $\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}}$ (in Section 3.1), and $\mathfrak{pic}(\underline{\mathrm{E}})$ (in Section 3.2). We also described the resulting spectral sequences. In this section, we compute the Picard spectral sequence in the height one case and use this to give a new proof of the results of [HMS94] at all primes. As is common at height one, this splits into two cases: the case of odd primes, and the case $p = 2$. In both cases, the strategy is first to compute the descent spectral sequence for $\underline{\mathrm{E}}$ (which by Proposition 2.48 is the K-local E-Adams spectral sequence) and then to use this to compute the Picard spectral sequence. However, the spectral sequences look somewhat different in the two cases. In this section we fix $k = \mathbb{F}_{p^h}$, and start with some generalities true uniformly in h and p.

4.1. Morava modules and the algebraic Picard group.

A productive strategy for studying Pic_h is to compare it to a certain algebraic variant $\mathrm{Pic}_h^{\mathrm{alg}}$ first defined in [HMS94, §7]; we recall its definition below. In [Pst22], Pstrągowski shows that the algebraic approximation is precise if $p \gg h$, a consequence of the fact that in this case the vanishing line in the Adams spectral sequence occurs at the starting page. While there is always a vanishing line, when $p - 1 \mid h$ it appears only at a later page, since $\mathbb{G}$ no longer has finite cohomological dimension mod p. In this section we will show that this leaves a possibility for *exotic* Picard elements, and more importantly explain how to identify these in the Picard spectral sequence (Theorem 4.4).

To do so, we recall that the *completed E-homology* of a K-local spectrum X is

$$\mathrm{E}^\vee_* X \coloneqq \pi_*(\mathrm{E} \otimes X) = \pi_* L_{\mathrm{K}}(\mathrm{E} \wedge X).$$

This is naturally a $\pi_*\mathrm{E}$-module: indeed K-localisation is symmetric monoidal, and therefore sends the E-module $\mathrm{E} \wedge X$ to a module over $L_{\mathrm{K}}\mathrm{E} = \mathrm{E}$. As we discuss below, the abelian group $\mathrm{E}^\vee_* X$ has significantly more structure, and is a crucial tool in understanding the K-local category. It was first studied by Hopkins, Mahowald and Sadofsky in [HMS94], where it is denoted $\mathcal{K}_{h,*}(-)$; it is almost a homology theory, but fails to preserve infinite coproducts as a result of the failure of K-localisation to be smashing. It is nevertheless an extremely effective invariant for Picard group computations, by virtue of the following theorem:

Theorem 4.1 ([HMS94], Theorem 1.3)**.** *$M \in \mathcal{S}p_{\mathrm{K}}$ is invertible if and only if $\mathrm{E}^\vee_* M$ is a free $\pi_*\mathrm{E}$-module of rank one.*

In particular, Theorem 4.1 implies that completed E-homology is not a useful invariant of invertible K-local spectra when we only remember its structure as a $\pi_*\mathrm{E}$-module: since $\pi_0\mathrm{E}$ is a Noetherian local ring, its Picard group is trivial. To get a more interesting invariant, we should remember the equivariant structure coming from the Morava action on E. That is, if X is a K-local spectrum then $\mathbb{G}$ acts on $\mathrm{E} \otimes X$ by acting on the first factor, and therefore acts on $\mathrm{E}^\vee_* X$. This action makes $\mathrm{E}^\vee_* X$ into a *twisted* $\mathbb{G}$-$\pi_*\mathrm{E}$-*module*, which means by definition that

$$g(a \cdot x) = ga \cdot gx$$

for $x \in \mathrm{E}^\vee_* X$, $a \in \pi_*\mathrm{E}$ and $g \in \mathbb{G}$. We will write $\mathrm{Mod}^{\mathbb{G}}_{\pi_*\mathrm{E}}$ for the category of twisted $\mathbb{G}$-$\pi_*\mathrm{E}$-modules.

Remark 4.2. For any twisted $\mathbb{G}$-$\pi_*\mathrm{E}$-module M, the $\mathbb{G}$-action is continuous for the I_h-adic topology: if $gx = y \in M$, then Section 4.1 implies that

$$g(x + ax') = y + ga \cdot gx' \in y + I_h^k M$$

for $a \in I_h^k$ and $x' \in M$, since the action of $\mathbb{G}$ on $\pi_*\mathrm{E}$ fixes the I_h-adic filtration; that is, $g^{-1}(y + I_h^k)$ contains the open neighbourhood $x + I_h^k M$ (compare [BH16, Lemma 5.2]).

Definition 4.3. The *algebraic Picard group* of $\mathcal{S}p_{\mathrm{K}}$ is

$$\mathrm{Pic}_h^{\mathrm{alg}} := \mathrm{Pic}\left(\mathrm{Mod}^{\mathbb{G}}_{\pi_*\mathrm{E}}\right).$$

The *exotic Picard group* of $\mathcal{S}p_{\mathrm{K}}$ is defined by the exact sequence of abelian groups

$$0 \to \kappa_h \to \mathrm{Pic}_h \xrightarrow{\mathrm{E}^\vee_*} \mathrm{Pic}_h^{\mathrm{alg}},$$

whose existence follows from Theorem 4.1. Restricting both Picard groups to their subgroups of elements concentrated in even degrees, one can equally obtain κ_h as the kernel of the map

$$\mathrm{Pic}_h^0 \xrightarrow{\mathrm{E}^\vee_*} \mathrm{Pic}_h^{\mathrm{alg},0} \simeq \mathrm{Pic}(\mathrm{Mod}^{\mathbb{G}}_{\pi_0\mathrm{E}}).$$

One of the main theorems of [HMS94] is the computation $\kappa_1 \simeq \mathbb{Z}/2$ at the prime 2 (this is Theorem 3.3 therein). We want to show how the descent spectral sequence for $\mathfrak{pic}(\underline{\mathrm{E}})$ recovers this computation, and we begin by identifying the algebraic Picard group in the spectral sequence. The aim of this subsection is therefore to prove the following result:

Theorem 4.4. *At arbitrary height h, the 1-line of the descent spectral sequence for $\mathfrak{pic}(\underline{\mathrm{E}})$ computes the image of Pic_h^0 in $\mathrm{Pic}_h^{\mathrm{alg},0}$. The exotic Picard group κ_h is computed by the subgroup in filtration ≥ 2.*

Proving Theorem 4.4 will require a short discussion of derived complete modules. Firstly, recall that $\pi_0\mathrm{E}$ is a regular Noetherian local ring, with maximal ideal $I_h = (p, u_1, \ldots, u_{h-1})$. If R is any such (classical) ring and $\mathfrak{m}$ its maximal ideal, the $\mathfrak{m}$-adic completion functor $(-)^\wedge_{\mathfrak{m}}$ has left-derived functors L_i, defined for example in [GM92]; this is in spite of $(-)^\wedge_{\mathfrak{m}}$ not being right-exact. For any R-module M the completion map $M \to M^\wedge_{\mathfrak{m}}$ factors through the zero-th derived functor

$$M \xrightarrow{\eta_M} L_0 M \xrightarrow{\epsilon_M} M^\wedge_{\mathfrak{m}},$$

and one says that M is *L-complete* or *derived* $\mathfrak{m}$-*complete* if η_M is an isomorphism. Hovey and Strickland prove the following facts about L-completion:

(1) For any R, the full subcategory spanned by the L-complete modules is a thick abelian subcategory of $\mathrm{Mod}^{\heartsuit}_R$, with enough projective generators [HS99, Theorem A.6 and Corollary A.12]. This is denoted $\widehat{\mathcal{M}}$ in *op. cit.*, but we will write $\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_R$. The projective objects are precisely those R-modules which are *pro-free*, that is $M = L_0 \bigoplus_S R$ for some (possibly infinite) set S.

(2) The functor L_0 from modules to L-complete modules is a localisation. In particular, colimits in L-complete modules are computed as
$$L_0 \operatorname{colim} M,$$
where $\operatorname{colim} M$ denotes the colimit at the level of modules. Thus $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ is still generated under colimits by the L-completion of the unit, and in particular κ-presentable where κ is chosen so that L_0R is κ-compact (note that we may have to take $\kappa > \omega$).

(3) Any $\mathfrak{m}$-adically complete module is L-complete [HS99, Theorem A.6]. In particular, $\pi_*\mathrm{E}$ is an L-complete module over itself.

(4) The category $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ admits a unique symmetric monoidal product making L_0 a monoidal functor. This is given by the formula
$$M\widehat{\otimes}_{L_0R}N = L_0(M \otimes_R N)$$
for M and N L-complete. For $\mathfrak{m}$-adically complete modules, one can also define the $\mathfrak{m}$-complete module
$$(M \otimes_R N)^\wedge_{\mathfrak{m}} \simeq (M\widehat{\otimes}_{L_0R}N)^\wedge_{\mathfrak{m}},$$
but we will have no use for this.

(5) Derived completion agrees with ordinary completion on finitely generated modules, and on projective modules: in other words, ϵ_M is an isomorphism in either of these cases. Moreover, for any N the composition
$$M \otimes_R L_0N \to L_0M \otimes_R L_0N \to L_0(M \otimes_R N) = L_0M\widehat{\otimes}_{L_0R}L_0N$$
is an isomorphism when M is finitely generated [HS99, Proposition A.4]. In particular, if R is itself L-complete then finitely generated modules are complete, i.e. $M = L_0M = M^\wedge_{\mathfrak{m}}$ and $L_iM = 0$ for $i > 0$.

If A is an L-complete R-algebra (not necessarily Noetherian), we can define a category of A-modules which are L-complete with respect to $\mathfrak{m} \subset R$, $\mathrm{Mod}_A^{\heartsuit\mathrm{cpl}} := \mathrm{Mod}_A(\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}})$.

We now specialise to the case $(R, \mathfrak{m}) = (\pi_0\mathrm{E}, I_h)$, and work towards the proof of Theorem 4.4. It is shown in [HS99, Proposition 8.4] that for any K-local spectrum X, the $\pi_0\mathrm{E}$-module $\mathrm{E}_0^\vee X$ is L-complete, and that $\mathrm{E}_*^\vee X$ is finitely generated over $\pi_*\mathrm{E}$ if and only if X is K-locally dualisable [HS99, Theorem 8.6]. Our first task is to prove that the presheaf of 1-categories
$$S \mapsto \mathrm{Mod}_{\pi_*\underline{\mathrm{E}}(S)}^{\heartsuit\mathrm{cpl}}$$
is a stack on $\mathrm{Free}_{\mathbb{G}}$.

Warning 4.5. We would like to proceed as in Section 3.1, but there is a small subtlety: namely, to deduce descent from the results of [Hai22] (as in part (3) of the proof of Proposition 3.10), we would need to show that $\mathrm{Mod}_{\pi_*\mathrm{E}}^{\heartsuit\mathrm{cpl}}$ is compactly generated; in fact, it would suffice to show it is compactly assembled. This is not clear: for example, Barthel and Frankland observe [BF15, Appendix A] that the unit in $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ (which *is* a generator) cannot be compact. For our purposes it is enough to find *any* (small) set of compact generators: by comparison, the K-local category is compactly generated by the K-localisation of any finite type-h spectrum, even though its unit is not compact.

At this point it will be useful to pass to the ∞-category of complete modules over the discrete rings $\pi_0\underline{\mathrm{E}}(S)$, as defined in [Lur18b, §7.3] or [BS14, §3.4]. This *can* be seen to be compactly generated by virtue of local (=Greenlees-May) duality; we will make use of this observation in the proof of Proposition 4.8. Given a discrete commutative ring R complete with respect to a finitely generated maximal ideal $\mathfrak{m}$, we will view R as a (connective) $\mathbb{E}_\infty$-ring and write $\mathrm{Mod}_R^{\mathrm{cpl}} \subset \mathrm{Mod}_R$ for the sub-∞-category of complete objects [Lur18b, Definition 7.3.1.1], which is a localisation of Mod_R. There is a unique symmetric monoidal product on $\mathrm{Mod}_R^{\mathrm{cpl}}$ for which the localisation is a monoidal functor, and to avoid confusion we denote this by $\widehat{\otimes}^{\mathbb{L}}_R$. Moreover, the abelian category $\mathrm{Mod}_R^{\heartsuit\mathrm{cpl}}$ of L-complete discrete R-modules includes as the heart of $\mathrm{Mod}_R^{\mathrm{cpl}}$ for a t-structure constructed in [Lur18b, Proposition 7.3.4.4], and the ∞-categorical localisation functor $L : \mathrm{Mod}_R \to \mathrm{Mod}_R^{\mathrm{cpl}}$ agrees upon restriction with the (total) left derived functor of L-completion [Lur18b, Corollary 7.3.7.5]; in particular, $L_0 \simeq \pi_0 L$.

Example 4.6. To give an example which makes the difference between 1-categories and ∞-categories apparent, one can consider the colimit along the multiplication-by-p maps

$$\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \cdots$$

over $\mathbb{Z}_p$. The colimit in $\mathrm{Mod}_{\mathbb{Z}_p}^{\heartsuit\mathrm{cpl}}$ is $L_0(\mathbb{Z}/p^\infty) = 0$; on the other hand, in the ∞-categorical setting one has $L \varinjlim \mathbb{Z}/p = \Sigma L_1(\mathbb{Z}/p^\infty) = \Sigma\mathbb{Z}_p$.

In particular, this example shows that the t-structure on $\mathrm{Mod}_R^{\mathrm{cpl}}$ is generally *not* compatible with colimits in the sense of [Lur17, Definition 1.2.2.12].

Lemma 4.7. *For any profinite set T, we have an equivalence*

$$\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(\mathbb{G}\times T)} \simeq \mathcal{S}h(T, \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathrm{E}}), \tag{38}$$

Proof. According to [Hov08, Corollary 2.5], if $i \mapsto X_i \in \mathcal{S}p_{\mathrm{K}}$ is a filtered diagram such that $\mathrm{E}^\vee_* X_i$ is *pro-free* for each $i \in I$, then the natural map

$$L_0 \varinjlim_i \mathrm{E}_0^\vee X_i \to \mathrm{E}_0^\vee \varinjlim_i X_i$$

is an isomorphism. In particular this applies to give the middle equivalence in

$$\pi_0\underline{\mathrm{E}}(\mathbb{G}\times T) \simeq \mathrm{E}_0^\vee\left(\varinjlim_i \bigoplus_{T_i} \mathbf{1}_{\mathrm{K}}\right) \simeq L_0 \varinjlim_i \bigoplus_{T_i} \pi_0\mathrm{E} = L_0 \varinjlim_i \pi_0\underline{\mathrm{E}}(\mathbb{G}\times T_i), \tag{39}$$

since $\mathrm{E}_0^\vee\left(\bigoplus_{T_i}\mathbf{1}_{\mathrm{K}}\right) = \bigoplus_{T_i}\mathrm{E}_0^\vee\mathbf{1}_{\mathrm{K}} = \bigoplus_{T_i}\pi_0\mathrm{E}$ is certainly pro-free, each T_i being finite. As a result, item (2) implies that $\pi_0\underline{\mathrm{E}}(\mathbb{G}\times T)$ is the colimit in $\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\mathrm{E}}$ of the algebras $\pi_0\underline{\mathrm{E}}(\mathbb{G}\times T_i) = \bigoplus_{T_i}\pi_0\mathrm{E}$. In fact this particular example *is* also the limit in $\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathrm{E}}$: indeed, for $s > 0$ we see that $L_s \varinjlim \pi_0\underline{\mathrm{E}}(\mathbb{G}\times T_i) = 0$ by [HS99, Theorem A.2(b)], since $\varinjlim \bigoplus_{T_i}\pi_0\mathrm{E}$ is projective in $\mathrm{Mod}^{\heartsuit}_{\pi_0\mathrm{E}}$.

This yields the first of the following equivalences of (presentably symmetric monoidal) ∞-categories:

$$\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(\mathbb{G}\times T)} \simeq \mathrm{Mod}^{\mathrm{cpl}}_{L\varinjlim \pi_0\underline{\mathrm{E}}(\mathbb{G}\times T_i)} \simeq \varinjlim \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(\mathbb{G}\times T_i)} \simeq \mathcal{S}h(T, \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathrm{E}}).$$

The second equivalence follows from Proposition 2.7, and the third can be proved identically to part (2) of the proof of Proposition 3.10 (replacing $\underline{\mathrm{E}}$ there by $\pi_0\underline{\mathrm{E}}$). ☐

Proposition 4.8. *The presheaf*

$$S \mapsto \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S)}$$

it is a stack of 1-categories; in other words, it is a sheaf of 1-truncated symmetric monoidal ∞-categories on $\mathrm{Free}_{\mathbb{G}}$.

Proof. We will proceed in a few steps: we will begin by showing that $S \mapsto \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S)}$ is a sheaf of ∞-categories, and then deduce the desired result at the level of 1-categories.

(1) Since any covering in $\mathrm{Free}_{\mathbb{G}}$ is a composite of an isomorphism with a map of the form $p\times\mathbb{G}\colon T'\times\mathbb{G} \to T\times\mathbb{G}$ for $p : T' \to T$ a covering of profinite sets, we can restrict attention to the presheaf

$$T \mapsto \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(T\times\mathbb{G})} \tag{40}$$

on $\mathrm{Profin} \simeq B\mathbb{G}_{\mathrm{pro\acute{e}t}/\mathbb{G}}$. The local duality equivalence

$$\mathrm{Mod}^{\mathrm{tors}}_{\pi_0\mathrm{E}} \simeq \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathrm{E}}$$

of [BHV18, Theorem 3.7] or [Lur18b, Proposition 7.3.1.3] implies that $\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathrm{E}}$ is compactly generated, and hence dualisable in Pr^L. Thus the presheaf

$$T \mapsto \mathcal{S}h(T, \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathrm{E}}) \simeq \mathcal{S}h(T) \otimes \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\mathrm{E}}$$

is a sheaf on Profin by the main theorem of [Hai22], and so Lemma 4.7 implies that (40) is a sheaf too.

(2) Next, we deduce descent for the presheaf of 1-categories $S \mapsto \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S)}$. Given a covering $p : T' \to T$ of profinite sets, we form the diagram

$$\begin{tikzcd}
\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S)} \arrow[r, "\theta"] \arrow[d, hook] & \lim \arrow[r] \arrow[d, hook] & \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S')} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S'\times_S 2)} \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[d, hook] & \mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S'\times_S 3)} \arrow[r] \arrow[d, hook] & \cdots \\
\mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S)} \arrow[r, "\sim"] & \lim \arrow[r] & \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S')} \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S'\times_S 2)} \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S'\times_S 3)} \arrow[r] & \cdots
\end{tikzcd}$$

where $S^{(i)} = T^{(i)} \times \mathbb{G}$ as usual. Note that the limit of the top row can be computed as the limit of the truncated diagram $\Delta_{\leq 3} \hookrightarrow \Delta \to \mathrm{Cat}_\infty$, since each term is a 1-category. Moreover, the diagram shows that θ is fully faithful, and so to prove descent it remains to show that θ is essentially surjective. That is, given $M \in \mathrm{Mod}^{\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S)}$ with $M\widehat{\otimes}^{\mathbb{L}}_{\pi_0\underline{\mathrm{E}}(S)}\pi_0\underline{\mathrm{E}}(S')$ discrete, we must show that M was discrete to begin with.

To this end, we claim first that $\pi_0\underline{\mathrm{E}}(S')$ is projective over $\pi_0\underline{\mathrm{E}}(S)$; since the graph of p exhibits $\pi_0\underline{\mathrm{E}}(S') = \mathrm{Cont}(T', \pi_0\mathrm{E})$ as a retract of $\mathrm{Cont}(T' \times T, \pi_0\mathrm{E}) = \mathrm{Cont}(T', \mathrm{Cont}(T, \pi_0\mathrm{E}))$, it will suffice for this part to show that the latter is projective. But $\mathrm{Cont}(T, \pi_0\mathrm{E})$ is pro-discrete, which implies that

$$\begin{aligned}
\mathrm{Cont}(T', \mathrm{Cont}(T, \pi_0\mathrm{E})) &\cong \lim_I \varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathrm{E}/I) \\
&\cong \lim_I \varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathrm{E}) \otimes_{\pi_0\mathrm{E}} \pi_0\mathrm{E}/I \\
&\cong \lim_I \left[\varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathrm{E})\right] \otimes_{\pi_0\mathrm{E}} \pi_0\mathrm{E}/I \\
&\cong L_0 \varinjlim_i \bigoplus_{T'_i} \mathrm{Cont}(T, \pi_0\mathrm{E})
\end{aligned}$$

is pro-free. To obtain the final isomorphism, we've used the fact that each term in the colimit is free over $\mathrm{Cont}(T, \pi_0\mathrm{E})$, so that the (uncompleted) colimit is projective. As a result, for any complete $\pi_0\underline{\mathrm{E}}(S)$-module spectrum M we have

$$\pi_*\left(M\widehat{\otimes}^{\mathbb{L}}_{\pi_0\underline{\mathrm{E}}(S)}\pi_0\underline{\mathrm{E}}(S')\right) = (\pi_*M)\widehat{\otimes}_{\pi_0\underline{\mathrm{E}}(S)}\pi_0\underline{\mathrm{E}}(S').$$

Since $\pi_0\underline{\mathrm{E}}(S')$ is faithful over $\pi_0\underline{\mathrm{E}}(S)$, we deduce that M is discrete whenever its basechange is. ☐

Remark 4.9. For any graded ring $\bigoplus_{a\in A} R_a$ with R_0 a complete noetherian local ring, invertible objects in $\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_R$ are locally free: this follows from Lemma 3.19 and Nakayama's lemma.

It is convenient at this stage to work with Picard spaces, which as usual we denote $\mathfrak{Pic}$.

Corollary 4.10. *The presheaf*

$$S \mapsto \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_*\underline{\mathrm{E}}(S)}\right)$$

is a sheaf of groupoids on $\mathrm{Free}_{\mathbb{G}}$.

Proof. By Proposition 4.8, the assignment

$$S \mapsto \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_0\underline{\mathrm{E}}(S)}\right) \tag{41}$$

is a sheaf. Since invertible objects in $\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_*\underline{\mathrm{E}}(S)}$ are locally free, this extends to the graded case. ☐

We are now equipped to prove the promised result, identifying the algebraic elements in the Picard spectral sequence.

Proof. (Theorem 4.4). In the 0-stem of the descent spectral sequence, the bottom two lines compute the image of the map $\pi_0\Gamma\mathfrak{pic}(\underline{\mathrm{E}}) \to \pi_0\Gamma\tau_{\leq 1}\mathfrak{pic}(\underline{\mathrm{E}})$. We will argue by computing the target, identifying it with the algebraic Picard group.

First recall that by definition, $\mathfrak{pic}(\underline{\mathrm{E}}) = \mathfrak{pic}(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}})$. Observe that $\tau_{\leq 1}\mathfrak{Pic}(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)}) = \mathfrak{Pic}(h\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)})$, and so $\tau_{\leq 1}\mathfrak{Pic}(\underline{\mathrm{E}})$ is the sheafification of

$$S \mapsto \mathfrak{Pic}\left(h\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)}\right).$$

Given $\underline{\mathrm{E}}(S)$-modules M and M' with $M \in \mathrm{Pic}(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)})$, we saw in Proposition 3.14 that M and so π_*M is locally free, and therefore the latter is projective over $\pi_*\underline{\mathrm{E}}(S)$. The universal coefficient spectral sequence [EMKM97, Theorem 4.1] over $\underline{\mathrm{E}}(S)$ therefore collapses. Thus

$$[M, M']_{\underline{\mathrm{E}}(S)} \simeq \mathrm{Hom}_{\pi_*\underline{\mathrm{E}}(S)}(\pi_*M, \pi_*M'),$$

and we see that the functor

$$\pi_* : \mathfrak{Pic}\left(h\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)}\right) \to \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_*\underline{\mathrm{E}}(S)}\right) \tag{42}$$

is fully faithful. It is in fact an equivalence: any invertible L-complete module over $\pi_*\underline{\mathrm{E}}(S)$ is locally free, and so projective, and in particular lifts to $\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)}$ [Wol98, Theorem 3].

By Corollary 4.10, no sheafification is therefore required when we restrict $\tau_{\leq 1}\mathfrak{Pic}\left(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}}\right)$ to $\mathrm{Free}_{\mathbb{G}}$ and so we obtain

$$\begin{aligned}
\Gamma\tau_{\leq 1}\mathfrak{Pic}(\underline{\mathrm{E}}) &\simeq \mathrm{Tot}\left[\mathfrak{Pic}\left(h\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(\mathbb{G}^{\bullet+1})}\right)\right] \\
&\simeq \mathrm{Tot}\left[\mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\mathrm{Cont}(\mathbb{G}^{\bullet},\pi_*E)}\right)\right] \\
&\simeq \mathrm{Tot}\left[\mathfrak{Pic}\left(\mathrm{Mod}_{\mathrm{Cont}(\mathbb{G}^{\bullet},\pi_*E)}\right)\right] \\
&\simeq \mathrm{Tot}_3\left[\mathfrak{Pic}\left(\mathrm{Mod}_{\mathrm{Cont}(\mathbb{G}^{\bullet},\pi_*E)}\right)\right]
\end{aligned}$$

This is the groupoid classifying twisted $\mathbb{G}$-$\pi_*\mathrm{E}$-modules with invertible underlying module; in particular, $\pi_0\Gamma\tau_{\leq 1}\mathfrak{Pic}(\underline{\mathrm{E}}) \simeq \mathrm{Pic}^{\mathrm{alg}}_h$. On free $\mathbb{G}$-sets, the truncation map

$$\mathfrak{Pic}\left(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)}\right) \to \tau_{\leq 1}\mathfrak{Pic}\left(\mathrm{Mod}^{\wedge}_{\underline{\mathrm{E}}(S)}\right) \simeq \mathfrak{Pic}\left(\mathrm{Mod}^{\heartsuit\mathrm{cpl}}_{\pi_*\underline{\mathrm{E}}(S)}\right)$$

is just $M \mapsto \pi_*M$. On global sections, it therefore sends M to the homotopy groups of the associated descent datum for the covering $\mathbb{G} \to *$; this is its Morava module $\mathrm{E}^\vee_* M = \pi_*\mathrm{E} \otimes M$, by definition. □

Remark 4.11. As a consequence, the map $\mathrm{E}^\vee_* : \mathrm{Pic}_h \to \mathrm{Pic}^{\mathrm{alg}}_h$ is:

- (i) injective if and only if the zero stem in the E_∞ term of the Picard spectral sequence is concentrated in filtration ≤ 1;
- (ii) surjective if and only if there are no differentials in the Picard spectral sequence having source $(0, 1)$ (the generator of the $\mathbb{Z}/2$ in bidegree $(0, 0)$ certainly survives, and represents $\Sigma\mathbf{1}_{\mathrm{K}}$).

This refines the algebraicity results in [Pst22], although in practise it is hard to verify either assertion without assuming a horizontal vanishing line at E_2 in the ASS (which is what makes the results of *op. cit.* go through).

Corollary 4.12 (([CZ23], Proposition 1.25))**.** *If $p > 2, p-1 \nmid h$ and $h^2 \leq 4p-4$, there is an isomorphism*

$$\kappa_h \cong H^{2p-1}(\mathbb{G}, \pi_{2p-2}\mathrm{E}).$$

Proof. This follows from sparsity in the Adams spectral sequence. By [Hea14, Prop. 4.2.1], the lowest-filtration contribution to the exotic Picard group comes from $E_\infty^{2p-1,2p-1}$. If $p-1 \nmid h$, the vanishing line in the Adams spectral sequence occurs at the starting page, and if moreover $h^2 \leq 4p-4$ then the group $E_\infty^{2p-1,2p-1}$ is the only possibly nonzero entry in this region, and hence fits in the exact sequence

$$1 \to H^0(\mathbb{G}, (\pi_0\mathrm{E})^\times) \xrightarrow{d_{2p-1}} H^{2p-1}(\mathbb{G}, \pi_{2p-2}\mathrm{E}) \to E_\infty^{2p-1,2p-1} \to 0.$$

In fact, the differential must vanish. Indeed, a weak form of chromatic vanishing proven in [BG18, Lemma 1.33] shows that

$$H^0(\mathbb{G}, \pi_0\mathrm{E}) \simeq H^0(\mathbb{G}, \mathbb{W}(\mathbb{F}_{p^h})) \simeq H^0(\mathrm{Gal}(\mathbb{F}_{p^h}/\mathbb{F}_p), \mathbb{W}(\mathbb{F}_{p^h})) \simeq \mathbb{Z}_p.$$

This isomorphism is inverse to a component of the map on the E_2-pages of Adams spectral sequences associated to the diagram

$$\begin{tikzcd} \mathbf{1}_p \arrow[r] \arrow[d] & \mathbf{1}_{\mathrm{K}} \arrow[d] \\ \mathbb{SW}_h \arrow[r] & \mathrm{E} \end{tikzcd}$$

Here $\mathbf{1}_p$ denotes the p-complete sphere, and $\mathbb{SW}_h$ the spherical Witt vectors of $\mathbb{F}_{p^h}$; the bottom map is defined under the universal property of $\mathbb{SW}_h$ [Lur18a, Definition 5.2.1] by the inclusion $\mathbb{F}_{p^h} \hookrightarrow (\pi_0\mathrm{E})/p$. Since $(\pi_0\mathbb{SW}^{h\mathrm{Gal}})^\times = (\pi_0\mathbf{1}_p)^\times = \mathbb{Z}_p^\times$, the map on Picard spectral sequences induced by the above square implies that the group $E_2^{0,1} = H^0(\mathbb{G}, (\pi_0\mathrm{E})^\times) \cong \mathbb{Z}_p^\times$ in the Picard spectral sequence consists of permanent cycles. □

Example 4.13. In the boundary case $2p-1 = \mathrm{cd}_p(\mathbb{G}) = h^2$, we can use Poincaré duality to simplify the relevant cohomology group: this gives

$$\kappa_h \cong H^{2p-1}(\mathbb{G}, \pi_{2p-2}\mathrm{E}) \cong H_0(\mathbb{S}, \pi_{2p-2}\mathrm{E})^{\mathrm{Gal}}. \tag{43}$$

Examples of such pairs (h,p) are $(3,5)$, $(5,13)$, $(9,41)$ and $(11,61)$; in each case, this is the first prime for which [Pst22, Remark 2.6] leaves open the possibility of exotic Picard elements. The case $(h,p)=(3,5)$ case was considered by Culver and Zhang, using different methods; however, they show as above that Heard's spectral sequence combined with the conjectural vanishing

$$H_0(\mathbb{S}, \pi_{2p-2}\mathrm{E}) = 0,$$

would imply that $\kappa_h = 0$ [CZ23, Corollary 1.27].

To our knowledge, it is not known if there are infinitely primes p for which $2p-1$ is a perfect square; this is closely tied to Landau's (unsolved) fourth problem, which asks if there are infinitely many primes of the form h^2+1.

4.2. **Picard groups at height one.** We now specialise to height one, so $(k,\Gamma) = (\mathbb{F}_p, \widehat{\mathbb{G}}_m)$. It is well-known that Morava E-theory at height one (and a fixed prime p) is p-completed complex K-theory KU_p, and as such its homotopy is given by

$$\pi_*\mathrm{E} = \mathbb{Z}_p[u^{\pm 1}], \tag{44}$$

with $u \in \pi_2\mathrm{E}$ the Bott element. In this case, the Morava stabiliser group is isomorphic to the p-adic units $\mathbb{Z}_p^\times$, acting on KU_p by Adams operations

$$\psi^a : u \mapsto au.$$

The K-local E-Adams spectral sequence therefore reads

$$E_{2,+}^{s,t} = H^s(\mathbb{Z}_p^\times, \mathbb{Z}_p(t/2)) \implies \pi_{t-s}\mathbf{1}_{\mathrm{K}}, \tag{45}$$

where $\mathbb{Z}_p(t/2)$ denotes the representation

$$\mathbb{Z}_p^\times \xrightarrow{t/2} \mathbb{Z}_p^\times \to \mathbb{Z}_p$$

when t is even, and zero when t is odd. Note that these are never discrete $\mathbb{Z}_p^\times$-modules, except at $t=0$. Nevertheless, cohomology of continuous pro-p modules is sensible for profinite groups G of type p-FP_∞ [SW00, §4.2]. The *small* Morava stabiliser groups $\mathbb{S}$ are in general p-adic Lie groups, and so satisfy this assumption; this implies that $\mathbb{G}$ is type p-FP_∞, since $\mathbb{S}$ is a finite index subgroup. In this case cohomology is continuous, in the sense that its value on a pro-p module is determined by its value on finite quotients:

$$H^*(G, \varprojlim M_i) = \varprojlim H^*(G, M_i).$$

Under the same assumption on G, there is also a Lyndon-Hochschild-Serre spectral sequence for any closed normal subgroup $N <_o G$ [SW00, Theorem 4.2.6]: that is,

$$E_2^{i,j} = H^i(G/N, H^j(N,M)) \implies H^{i+j}(G,M). \tag{46}$$

See also [Jan88, Theorem 3.3] for a similar result for profinite coefficients that are not necessarily pro-p: one obtains a Lyndon-Hochschild-Serre spectral sequence by replacing the Galois covering $X' \to X$ therein by a map of sites $BG_{\text{ét}} \to BN_{\text{ét}}$.

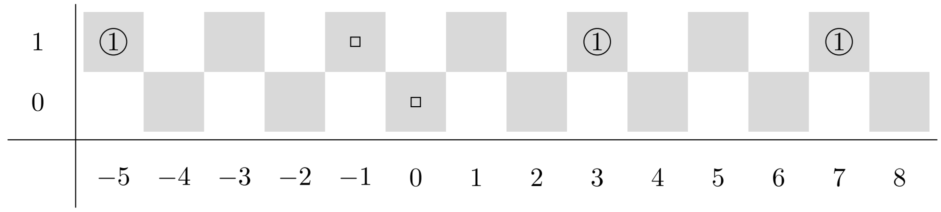


FIGURE 1. The E_2-page of the descent spectral sequence for E at odd primes (implicitly at $p = 3$). Squares denote $\mathbb{Z}_p$-summands, and circles are p-torsion summands (labelled by the degree of the torsion). This recovers the well-known computation of $\pi_*\mathbf{1}_{\mathrm{K}}$ at height 1 and odd primes.

The descent spectral sequence for $\mathfrak{pic}(\underline{\mathrm{E}})$ at height one (and all primes) therefore has starting page

$$E_2^{s,t} = \begin{cases} H^s(\mathbb{Z}_p^\times, \mathbb{Z}/2) & t = 0 \\ H^s(\mathbb{Z}_p^\times, \mathbb{Z}_p(0)^\times) & t = 1 \\ H^s(\mathbb{Z}_p^\times, \mathbb{Z}_p(\frac{t-1}{2})) & t \geq 2 \end{cases} \tag{47}$$

The results of Section 3.2 also tell us how to discern many differentials in the Picard spectral sequence from those in the Adams spectral sequence: in particular, we will make use of Proposition 3.22 and Corollary 3.25. Our input is the well-known computation of the K-local E-Adams spectral sequence at height one. A convenient reference is [BGH22, §4], but for completeness a different argument is presented in Appendix B.

4.2.1. *Odd primes.* When $p > 2$, the Adams spectral sequence collapses immediately:

Lemma 4.14 (Lemma B.1)**.** *The starting page of the descent spectral sequence for* $\underline{\mathrm{E}}$ *is given by*

$$E_{2,+}^{s,t} = H^s(\mathbb{Z}_p^\times, \pi_t\mathrm{E}) = \begin{cases} \mathbb{Z}_p & t = 0 \text{ and } s = 0, 1 \\ \mathbb{Z}/p^{\nu_p(t)+1} & t = 2(p-1)t' \neq 0 \text{ and } s = 1 \end{cases} \tag{48}$$

and zero otherwise. The result is displayed in Fig. 1.

As a result of the vanishing line, the computation of $\mathrm{Pic}(\mathcal{S}p_{\mathrm{K}})$ in this case depends *only* on $H^*(\mathbb{Z}_p^\times, \mathrm{Pic}(\mathrm{E}))$ and $H^*(\mathbb{Z}_p^\times, (\pi_0\mathrm{E})^\times)$. This recovers the computation in [HMS94, Proposition 2.7].

Proposition 4.15. *The height one Picard group is algebraic at odd primes:*

$$\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{\mathrm{alg}}. \tag{49}$$

4.2.2. *The case* $p = 2$. At the even prime, the Morava stabiliser group contains 2-torsion, and therefore its cohomology with 2-complete coefficients is periodic.

Lemma 4.16. *The starting page of the descent spectral sequence for* $\underline{\mathrm{E}}$ *is given by*

$$E_{2,+}^{s,t} = H^s(\mathbb{Z}_2^\times, \pi_t\mathrm{E}) = \begin{cases} \mathbb{Z}_2 & t = 0 \text{ and } s = 0, 1 \\ \mathbb{Z}/2 & t \equiv_4 2 \text{ and } s \geq 1 \\ \mathbb{Z}/2^{\nu_2(t)+1} & 0 \neq t \equiv_4 0 \text{ and } s = 1 \\ \mathbb{Z}/2 & t \equiv_4 0 \text{ and } s > 1 \end{cases} \tag{50}$$

and zero otherwise.

This time we see that the spectral sequence can support many differentials. These can be computed by various methods, as for example in [BGH22]. We give another proof, more closely related to our methods, in Appendix B.

Proposition 4.17. *The descent spectral sequence collapses at* E_4 *with a horizontal vanishing line. The differentials on the third page are displayed in Fig. 2.*

As a result of the previous subsection, we can compute the groups of exotic and algebraic Picard elements at the prime 2. We will need one piece of the multiplicative structure: write η for the generator in bidegree $(s,t) = (1,2)$, and $u^{-2}\eta^2$ for the generator in bidegree $(s,t) = (2,0)$.

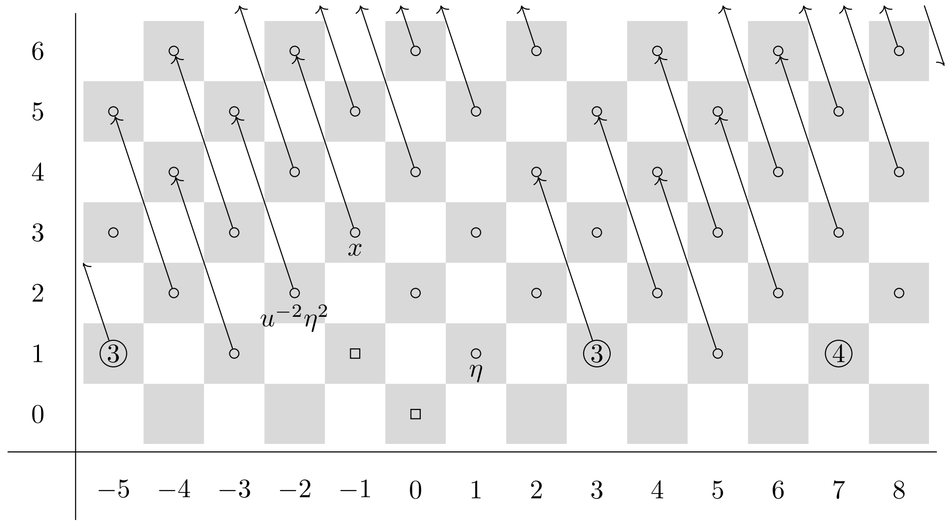


FIGURE 2. The E_3-page of the descent spectral sequence for E at $p = 2$.

Lemma 4.18. *In the descent spectral sequence for $\underline{\mathrm{E}}$, the class*

$$x := u^{-2}\eta^2 \cdot \eta \in E_2^{3,2}$$

is non-nilpotent. In particular, x^j generates the group in bidegree $(s,t) = (3j, 2j)$ of the descent spectral sequence for $\underline{\mathrm{E}}$.

Proof. The classes $u^{-2}\eta^2$ and η are detected by elements of the same name in the HFPSS for the conjugation action on KU_2, under the map of spectral sequences induced by the square of Galois extensions

$$\begin{tikzcd} \mathbf{1}_{\mathrm{K}} \arrow[r] \arrow[d] & KO_2 \arrow[d] \\ KU_2 \arrow[r, equal] & KU_2 \end{tikzcd}$$

To see this, one can trace through the computations of Appendix B: indeed, the proof of Proposition B.3 identifies the map of spectral sequences induced by the map of C_2-Galois extensions

$$\begin{tikzcd} \mathbf{1}_{\mathrm{K}} \arrow[r] \arrow[d] & KO_2 \arrow[d] \\ KU_2^{h(1+4\mathbb{Z}_2)} \arrow[r] & KU_2 \end{tikzcd}$$

and Lemma B.5 identifies the descent spectral sequence for $\underline{\mathrm{E}}$ with the HFPSS for $KU_2^{h(1+4\mathbb{Z}_2)}$, up to a filtration shift. But the starting page of the HFPSS for KU_2 is

$$H^*(C_2, \pi_* KU_2) = \mathbb{Z}_2[\eta, u^{\pm 2}]/2\eta,$$

and here in particular $(u^{-2}\eta^2 \cdot \eta)^j = u^{-2j}\eta^{3j} \neq 0$. ∎

Proposition 4.19. *At the prime 2, the exotic Picard group κ_1 is $\mathbb{Z}/2$.*

Proof. We will deduce this from Proposition 4.17, which implies that the descent spectral sequence for $\mathfrak{pic}(\underline{\mathrm{E}})$ takes the form displayed in Figure 4. According to Theorem 4.4, the only differential remaining for the computation of κ_1 is that on the class in bidegree $(s,t) = (3,0)$, which corresponds to the class $x \in E_{2,+}^{3,2}$ of the Adams spectral sequence. Applying the formula from Lemma 3.25,

$$d_3(x) = d_3^{ASS}(x) + x^2 = 2x^2 = 0.$$

After this there is no space for further differentials on x, so $\kappa_1 = \mathbb{Z}/2$. ∎

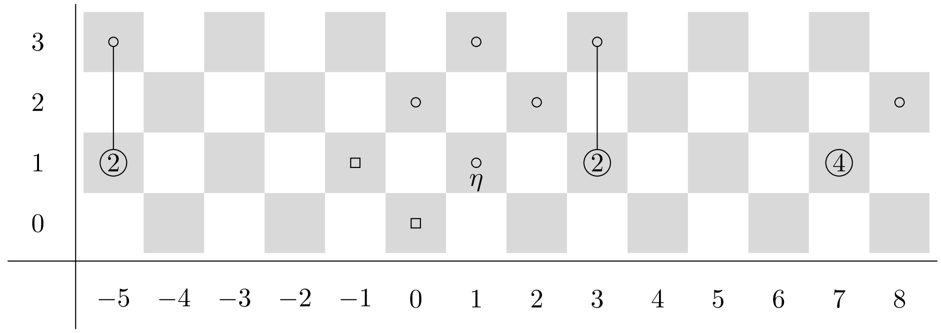


FIGURE 3. The $E_4 = E_\infty$-page of the descent spectral sequence for E at $p = 2$.

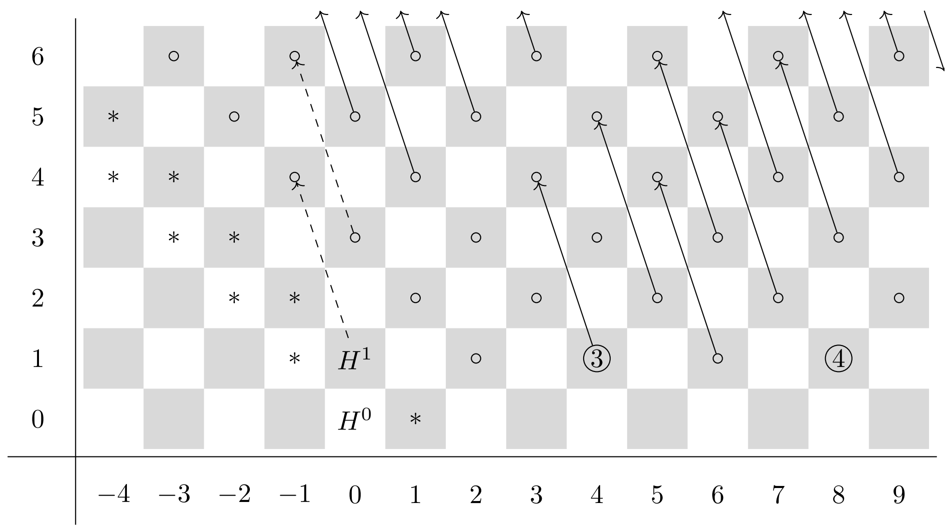


FIGURE 4. The E_3-page of the Picard spectral sequence at $p = 2$. H^0 denotes $H^0(\mathbb{Z}_2^\times, \mathrm{Pic}(\mathrm{E})) = \mathbb{Z}/2$, and $H^1 = H^1(\mathbb{Z}_2^\times, (\pi_0\mathrm{E})^\times) = \mathrm{Pic}_1^{\mathrm{alg},0}$. The possible dashed differentials affects the Picard group; we have omitted some possible differentials with source $t - s \leq -1$.

5. Brauer groups

In the previous sections, we considered Galois descent for the Picard spectrum of the K-local category. Recall that the Picard spectrum deloops to the *Brauer spectrum*, which classifies derived Azumaya algebras. In this section we consider Galois descent for the K-local Brauer spectrum; see [BRS12, AG14, GL21, AMS22] for related work. Unlike most of these sources, the unit in our context is *not* compact, and this makes the corresponding descent statements slightly more delicate. We begin with the basic definitions:

Definition 5.1 ([HL17], Definition 2.2.1)**.** Suppose $\mathcal{C}$ is a stable homotopy theory, and $R \in \mathrm{CAlg}(\mathcal{C})$.

(1) Two R-algebras $A, B \in \mathrm{Alg}_R(\mathcal{C})$ are *Morita equivalent* if there is an R-linear equivalence $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{LMod}_B(\mathcal{C})$. We will write $A \sim B$.
(2) An algebra $A \in \mathrm{Alg}_R(\mathcal{C})$ is an *Azumaya R-algebra* if there exists $B \in \mathrm{Alg}_R(\mathcal{C})$ such that $A \otimes_R B \sim R$.

We will mostly be interested in the case $\mathcal{C} = \mathcal{S}p_{\mathrm{K}}$ and $R = \mathbf{1}_{\mathrm{K}}$.

Remark 5.2. Hopkins and Lurie show [HL17, Corollary 2.2.3] that this definition is equivalent to a more familiar definition in terms of intrinsic properties of A. Using [HL17, Prop. 2.9.6] together with faithfulness of the map $\mathbf{1}_{\mathrm{K}} \to \mathrm{E}$, one sees that $A \in \mathcal{S}p_{\mathrm{K}}$ is Azumaya if and only if

(1) A is nonzero;
(2) A is dualisable in $\mathrm{Mod}_R^\wedge$;

(3) The left/right-multiplication action $A \otimes_R A^{op} \to \mathrm{End}_R(A)$ is an equivalence.

We will define the space $\mathfrak{Az}(R) \subset \iota\mathrm{Alg}_{R,\mathrm{K}}$ to be the subgroupoid spanned by K-local Azumaya algebras over R, and $\mathrm{Az}(R) \coloneqq \pi_0\mathfrak{Az}(R)$.

Definition 5.3. The K-*local Brauer group* of R is the set

$$\mathrm{Br}(R) \coloneqq \mathrm{Az}(R)/\sim,$$

equipped with the abelian group structure $[A] + [B] = [A \otimes_R B]$.

Remark 5.4. This notation should not be confused with the group of Azumaya R-algebras in spectra, which is potentially different.

In analogy with Picard groups we will write $\mathrm{Br}_h \coloneqq \mathrm{Br}(\mathbf{1}_\mathrm{K})$. Our objective is to show how the Picard spectral sequence (23) can be used to compute this. First recall the main theorem of *op. cit.*:

Theorem 5.5 ([HL17], Theorem 1.0.11)**.** *There is an isomorphism*

$$\mathrm{Br}(\mathrm{E}) \simeq \mathrm{BW}(\kappa) \times \mathrm{Br}'(\mathrm{E}),$$

where $\mathrm{BW}(\kappa)$ *is the* Brauer-Wall *group classifying* $\mathbb{Z}/2$*-graded Azumaya algebras over* $\kappa \coloneqq \pi_0\mathrm{E}/I_h$*, and* $\mathrm{Br}'(\mathrm{E})$ *admits a filtration with associated graded* $\bigoplus_{k\geq 2} I_h^k/I_h^{k+1}$.

We will not discuss this result, and instead focus on the orthogonal problem of computing the group of K-local Brauer algebras over the sphere which become Morita trivial over Morava E-theory; in the terminology of [GL21] this is the *relative* Brauer group, and will be denoted by Br_h^0. The full Brauer group can then (at least in theory) be obtained from the exact sequence

$$1 \to \mathrm{Br}_h^0 \to \mathrm{Br}_h \to \mathrm{Br}(\mathrm{E})^{\mathbb{G}}.$$

While interesting, the problem of understanding the action of $\mathbb{G}$ on $\mathrm{Br}(\mathrm{E})$ is somewhat separate, and again we do not attempt to tackle this.

5.1. **Brauer groups and descent.** In this subsection we show that the Picard spectral sequence gives an upper bound on the size of the relative Brauer group, as proven in [GL21, Theorem 6.32] for finite Galois extensions of unlocalised ring spectra. To this end, we define a 'cohomological' Brauer group; this might also be called a 'Brauer-Grothendieck group' of the K-local category, as opposed to the 'Brauer-Azumaya' group discussed above. The ideas described in this section goes back to work of Toën on Brauer groups in derived algebraic geometry [Toë12].

Given $R \in \mathrm{CAlg}(\mathcal{S}p_\mathrm{K})$, the ∞-category $\mathrm{Mod}_R^\wedge$ is symmetric monoidal, and therefore defines an object of $\mathrm{CAlg}_{\mathcal{S}p_\mathrm{K}}(\mathrm{Pr}^L)$. We will consider the symmetric monoidal ∞-category

$$\mathrm{Cat}_R^\wedge \coloneqq \mathrm{Mod}_{\mathrm{Mod}_R^\wedge}(\mathrm{Pr}_\kappa^L),$$

where κ is chosen to be large enough that $\mathrm{Mod}_R^\wedge \in \mathrm{CAlg}(\mathrm{Pr}_\kappa^L)$.

Definition 5.6. The *cohomological* K-*local Brauer space* of R is the Picard space

$$\mathfrak{Br}^{\mathrm{coh}}(R) \coloneqq \mathfrak{Pic}(\mathrm{Cat}_R^\wedge).$$

Since Pr_κ^L is presentable [Lur17, Lemma 5.3.2.9(2)], this is once again a small space. In analogy with the Picard case, we also write $\mathfrak{Br}_h^{0,\mathrm{coh}}$ for the full subspace of $\mathfrak{Br}_h^{\mathrm{coh}} \coloneqq \mathfrak{Br}^{\mathrm{coh}}(\mathbf{1}_\mathrm{K})$ spanned by invertible $\mathcal{S}p_\mathrm{K}$-modules $\mathcal{C}$ for which there is an E-linear equivalence $\mathcal{C} \otimes_{\mathcal{S}p_\mathrm{K}} \mathrm{Mod}_\mathrm{E}^\wedge \simeq \mathrm{Mod}_\mathrm{E}^\wedge$.

Definition 5.7. Passing to ∞-categories of left modules defines a functor $\mathrm{Alg}(\mathrm{Mod}_R^\wedge) \to \mathrm{Cat}_R^\wedge$, with algebra maps acting by extension of scalars. This restricts by the Azumaya condition to

$$\mathfrak{Az}(R) \to \mathfrak{Br}^{\mathrm{coh}}(R) \tag{51}$$

We define $\mathfrak{Br}(R) \subset \mathfrak{Br}^{\mathrm{coh}}(R)$ to be the full subgroupoid spanned by the essential image of $\mathfrak{Az}(R)$. We moreover define $\mathfrak{Br}_h \coloneqq \mathfrak{Br}(\mathbf{1}_\mathrm{K})$ and $\mathfrak{Br}_h^0 \coloneqq \mathfrak{Br}_h \cap \mathfrak{Br}_h^{0,coh}$.

Warning 5.8. When working with plain $\mathbb{E}_\infty$-rings, one can take $\kappa = \omega$ in the definition of the cohomological Brauer group, and so the two groups agree by Schwede-Shipley theory. Indeed, a cohomological Brauer class is then an invertible compactly generated R-linear ∞-category $\mathcal{C}$, and in particular admits a *finite set* $\{C_1, \dots, C_n\}$ of compact generators [AG14, Lemma 3.9]. Thus $\mathcal{C} \simeq \mathrm{Mod}_A$ for $A \coloneqq \mathrm{End}_{\mathcal{C}}(\bigoplus_i C_i)$ an Azumaya algebra. This argument fails in Pr^L_κ for $\kappa > \omega$; on the other hand, $\mathcal{S}p_{\mathrm{K}} \notin \mathrm{CAlg}(\mathrm{Pr}^L_\omega)$ since the unit is not compact. For *relative* Brauer classes, we refer to [Mor23, §3] for an alternative solution.

We now provide a descent formalism suitable for our context, based on the approach of [GL21]. If $\mathcal{C}$ is a presentably symmetric-monoidal ∞-category with κ-compact unit and $R \in \mathrm{CAlg}(\mathcal{C})$, we will write $\mathrm{Cat}_R(\mathcal{C}) \coloneqq \mathrm{Mod}_{\mathrm{Mod}_R(\mathcal{C})}\left(\mathrm{Pr}^L_\kappa\right)$. We will be interested in descent properties of the functor $\mathrm{Cat}_{(-)}(\mathcal{C})\colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{Pr}^{L,\mathrm{smon}}$: that is, if $R \to R'$ is a map of commutative algebras, we would like to know how close the functor θ below is to an equivalence:

$$\mathrm{Cat}_R(\mathcal{C}) \xrightarrow{\theta} \lim \Big[\, \mathrm{Cat}_{R'}(\mathcal{C}) \rightrightarrows \mathrm{Cat}_{R'\otimes_R R'}(\mathcal{C}) \; \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \; \cdots \,\Big]. \tag{52}$$

Lemma 5.9. *If R' is a descent R-algebra, then θ is fully faithful when restricted to the full subcategory spanned by ∞-categories of left modules.*

Proof. Let $A, A' \in \mathrm{Alg}_R(\mathcal{C})$. Writing $\mathrm{LMod}_A = \mathrm{LMod}_A(\mathcal{C})$, we have equivalences of ∞-categories

$$\begin{aligned} \mathrm{Fun}_R(\mathrm{LMod}_A, \mathrm{LMod}_{A'}) &\simeq \mathrm{RMod}_{A \otimes_R A'^{\mathrm{op}}} \\ &\simeq \lim \mathrm{RMod}_{A\otimes_R A'^{\mathrm{op}} \otimes_R R'^\bullet} \\ &\simeq \lim \mathrm{RMod}_{(A \otimes_R R'^\bullet) \otimes_{R'^\bullet} (A' \otimes_R R'^\bullet)^{\mathrm{op}}} \\ &\simeq \lim \mathrm{Fun}_{R'^\bullet}(\mathrm{LMod}_{A\otimes_R R'^\bullet}, \mathrm{LMod}_{A'\otimes_R R'^\bullet}). \end{aligned} \tag{53}$$

Here we have twice appealed to [Lur17, Theorem 4.8.4.1] (applied to A^{op} and A'^{op}), and to Lemma 3.7 for the second equivalence. Passing to cores, we obtain

$$\begin{aligned} \mathrm{Map}_{\mathrm{Cat}_R(\mathcal{C})}(\mathrm{LMod}_A, \mathrm{LMod}_{A'}) &\simeq \lim \mathrm{Map}_{\mathrm{Cat}_{R'^\bullet}(\mathcal{C})}(\mathrm{LMod}_{A\otimes_R R'^\bullet}, \mathrm{LMod}_{A'\otimes_R R'^\bullet}) \\ &\simeq \mathrm{Map}_{\lim \mathrm{Cat}_{R'^\bullet}(\mathcal{C})}(\mathrm{LMod}_{A\otimes_R R'^\bullet}, \mathrm{LMod}_{A'\otimes_R R'^\bullet}). \end{aligned}$$ ☐

Corollary 5.10. *For any covering $S' \to S$ in $B\mathbb{G}_{\text{proét}}$, the functor*

$$\theta : \mathrm{Cat}^\wedge_{\underline{\mathrm{E}}(S)} \to \lim \mathrm{Cat}^\wedge_{\underline{\mathrm{E}}(S' \times_S \bullet)}$$

is fully faithful when restricted to left module ∞-categories. ☐

Using this, we can give a bound on the size of Br^0_h.

Proposition 5.11. *The group Br^0_h is a subgroup of $\pi_0 \lim B\mathfrak{Pic}(\mathrm{E}^{\bullet+1})$.*

Proof. If $S' \to S$ is a covering in $B\mathbb{G}_{\text{proét}}$, we will write $R \to R'$ for the extension $\underline{\mathrm{E}}(S) \to \underline{\mathrm{E}}(S')$. Writing $\mathfrak{Br}(R \mid R')$ for the full subgroupoid of $\mathfrak{Br}(R)$ spanned by objects $\mathrm{LMod}^\wedge_A$ such that $\mathrm{LMod}^\wedge_A \otimes_{\mathrm{Mod}^\wedge_R} \mathrm{Mod}^\wedge_{R'} \simeq \mathrm{Mod}^\wedge_{R'}$, we will exhibit $\mathfrak{Br}(R \mid R')$ as a full subspace of $\lim B\mathfrak{Pic}(R'^\bullet)$. Taking the covering $\mathbb{G} \to *$ gives the desired result on π_0.

By definition, $\mathfrak{Br}(R)$ is the full subcategory of $\mathfrak{Br}^{\mathrm{coh}}(R)$ spanned by module categories. Using the inclusion of $B\mathfrak{Pic}(R')$ as the component of the unit in $\mathfrak{Br}(R')$ we can form the diagram

$$\begin{tikzcd}
\mathfrak{Br}(R \mid R') \arrow[dd, hook] \arrow[dr, dashed] \arrow[drr, bend left] \arrow[drrr, bend left] & & & & \\
& \lim B\mathfrak{Pic}(R'^\bullet) \arrow[r] \arrow[d, hook] & B\mathfrak{Pic}(R') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & B\mathfrak{Pic}(R' \otimes_R R') \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, hook] & \cdots \\
\mathfrak{Br}(R) \arrow[r, hook, "(\star)"] \arrow[d, hook] & \lim \mathfrak{Br}(R'^\bullet) \arrow[r] \arrow[d, hook] & \mathfrak{Br}(R') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \mathfrak{Br}(R' \otimes_R R') \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, hook] & \cdots \\
\iota\,\mathrm{Cat}^\wedge_R \arrow[r, "\theta"] & \lim \iota\,\mathrm{Cat}^\wedge_{R'^\bullet} \arrow[r] & \iota\,\mathrm{Cat}^\wedge_{R'} \arrow[r, shift left] \arrow[r, shift right] & \iota\,\mathrm{Cat}^\wedge_{R'\otimes_R R'} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd} \tag{54}$$

Here hooked arrows denote fully faithful functors: this is essentially by definition in most cases, with the starred functor being fully faithful by virtue of Corollary 5.10 (and the fact that passing to the core preserves limits). The dashed arrow, which clearly exists, is fully faithful by 2-out-of-3. ☐

In particular, the (-1)-stem in the descent spectral sequence for the Picard sheaf $\mathfrak{pic}(\underline{\mathrm{E}})$ gives an upper bound on the size of the relative Brauer group. We will draw consequences from this in the next subsection.

As we now discuss, the cohomological Brauer space also admits a description in terms of the proétale site. We will not use this, but include a proof for completeness.

Lemma 5.12. *The restriction of the presheaf $B\mathfrak{Pic}(\underline{\mathrm{E}})$ to $\mathrm{Free}_{\mathbb{G}}$ is a sheaf.*

Proof. We will prove descent of the Čech nerve for a covering $T' \times \mathbb{G} \to T \times \mathbb{G}$. That is, we would like to show that the following is a limit diagram:

$$B\mathfrak{Pic}(\underline{\mathrm{E}}(T)) \longrightarrow B\mathfrak{Pic}(\underline{\mathrm{E}}(T')) \rightrightarrows B\mathfrak{Pic}(\underline{\mathrm{E}}(T' \times_T T')) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

We will consider the Bousfield-Kan spectral sequence for the limit of the Čech complex, which reads

$$E_2^{s,t} = \pi^s \pi_t B\mathfrak{Pic}(\underline{\mathrm{E}}(T'^{\times_T \bullet+1})) \implies \pi_{t-s} \lim B\mathfrak{Pic}(\underline{\mathrm{E}}(T'^{\times_T \bullet+1})) \tag{55}$$

By Theorem 3.11, the homotopy presheaves of $B\mathfrak{Pic}(\underline{\mathrm{E}})$ are

$$\pi_t B\mathfrak{Pic}(\underline{\mathrm{E}}) = \underline{\pi_t B\mathfrak{Pic}(\mathrm{E})}.$$

We claim that $E_2^{s,s} = 0$ for $s > 0$, so that the map

$$B\mathfrak{Pic}(\underline{\mathrm{E}}(T)) \to \lim B\mathfrak{Pic}(\underline{\mathrm{E}}(T'^{\times_T \bullet+1}))$$

is an equivalence; note that the only possible difference is on π_0, since $\tau_{\geq 1} B\mathfrak{Pic}(\underline{\mathrm{E}})$ is a sheaf of connected spaces. We are therefore interested in the cosimplicial abelian groups

$$\mathrm{Cont}(T', \pi_s B\mathfrak{Pic}(\mathrm{E})) \rightrightarrows \mathrm{Cont}(T' \times_T T', \pi_s B\mathfrak{Pic}(\mathrm{E})) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots . \tag{56}$$

This is the Čech complex computed in Proposition 2.48, and was proven there to be exact. ☐

Definition 5.13. Write $\mathfrak{Br}(\underline{\mathrm{E}} \mid \mathrm{E}) \in \mathcal{S}h(B\mathbb{G}_{\text{proét}}, \mathcal{S})$ for the sheafification of $S \mapsto B\,\mathrm{Pic}(\underline{\mathrm{E}}(S))$ on $B\mathbb{G}_{\text{proét}}$. By Lemma 5.12, no sheafification is required on the subsite $\mathrm{Free}_{\mathbb{G}}$.

Corollary 5.14. *For any closed subgroup $U \subset \mathbb{G}$, we have*

$$\mathfrak{Br}^{\mathrm{coh}}(\mathrm{E}^{hU} \mid \mathrm{E}) \simeq \Gamma(\mathbb{G}/U, \mathfrak{Br}(\underline{\mathrm{E}} \mid \mathrm{E})).$$

In particular, $\mathfrak{Br}^{\mathrm{coh}}(\mathbf{1} \mid \mathrm{E}) \simeq \Gamma\mathfrak{Br}(\underline{\mathrm{E}} \mid \mathrm{E})$. ☐

5.2. **Brauer groups at height one.** Using the descent results of the Section 5.1, we now give bounds on the size of the relative Brauer groups Br_1^0. As usual, the story differs depending on the parity of the prime. In [Mor23] we show that these bounds are in fact tight, by producing explicit generators.

5.2.1. *Odd primes.* We first consider the case $p > 2$. The starting page of the Picard spectral sequence is recorded in Lemma 4.16 (and computed in Lemma B.1). Using this, we obtain the following:

Lemma 5.15. *At odd primes, the starting page of the Picard spectral sequence is given by*

$$E_2^{s,t} = H^s(\mathbb{Z}_p^\times, \pi_t \mathfrak{pic}(\mathrm{E})) = \begin{cases} \mathbb{Z}/2 & t = 0 \text{ and } s \geq 0 \\ \mathbb{Z}_p^\times & t = 1 \text{ and } s = 0, 1 \\ \mu_{p-1} & t = 1 \text{ and } s \geq 2 \\ \mathbb{Z}/p^{\nu_p(t)+1} & t = 2(p-1)t' + 1 \neq 1 \text{ and } s = 1 \end{cases} \tag{57}$$

This is displayed in Fig. 5. In particular, the spectral sequence collapses for degree reasons at the E_3-page.

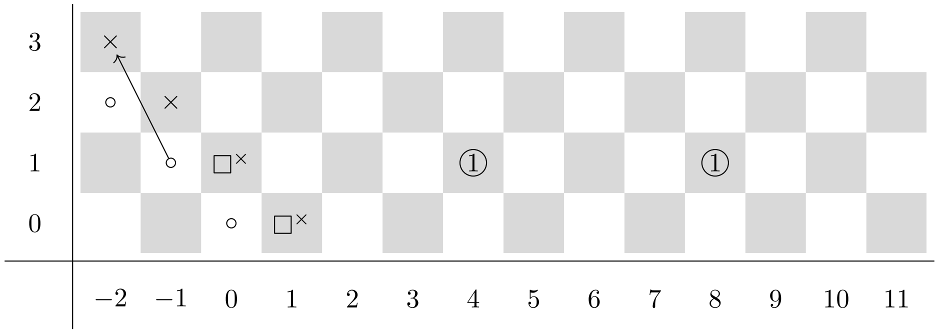


FIGURE 5. The height one Picard spectral sequence for odd primes (implicitly at $p = 3$). Classes are labelled as follows: $\circ = \mathbb{Z}/2$, $\square^\times = \mathbb{Z}_p^\times$, $\times = \mu_{p-1}$, and circles denote p-power torsion (labelled by the torsion degree). Since $\mathrm{Pic}_1 \cong \mathrm{Pic}_1^{\mathrm{alg}} \cong \mathbb{Z}_p^\times$, no differentials can hit the (-1)-stem. Differentials with source in stem $t - s \leq -2$ have been omitted.

Proof. All that remains to compute is internal degrees $t = 0, 1$. We invoke the Lyndon-Hochschild-Serre spectral sequence, which collapses since the extension is split. In particular,

$$\begin{aligned} H^*(\mathbb{Z}_p^\times, \pi_0\mathfrak{pic}(E)) &\simeq H^*(\mu_{p-1}, H^*(\mathbb{Z}_p, \mathbb{Z}/2)) \\ &\simeq H^*(\mu_{p-1}, \mathbb{Z}/2), \\ H^*(\mathbb{Z}_p^\times, \pi_1\mathfrak{pic}(E)) &\simeq H^*(\mu_{p-1}, H^*(\mathbb{Z}_p, \mathbb{Z}_p^\times)) \\ &\simeq H^*(\mu_{p-1}, \mathbb{Z}_p^\times \oplus \Sigma\mathbb{Z}_p). \end{aligned}$$ ☐

Using the results of Section 5.1, we obtain an upper bound on the relative Brauer group:

Corollary 5.16. *At odd primes,* Br_1^0 *is isomorphic to a subgroup of* μ_{p-1}.

Proof. The only possible differentials are d_2-differentials on classes in the (-1)-stem; note that there are no differentials *into* the (-1)-stem, since every E_2-class in the 0-stem is a permanent cycle. The generator in $E_2^{1,0}$ supports a d_2, since this is the case for the class in $E_2^{1,0}$ of the descent spectral sequence for the C_2-action on KU [GL21, Prop. 7.15] (this is displayed in Fig. 7), and the span of Galois extensions

$$\begin{tikzcd} \mathbf{1}_{\mathrm{K}} \arrow[r] \arrow[d] & KO_p \arrow[d] & KO \arrow[l] \arrow[d] \\ KU_p \arrow[r] & KU_p & KU \arrow[l] \end{tikzcd}$$

allows us to transport this differential. Note that the induced span on E_2-pages is

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, equal] \arrow[d, "d_2"] & \mathbb{Z}/2 \arrow[r, equal] \arrow[d, "d_2"] & \mathbb{Z}/2 \arrow[d, "d_2", "="'] \\ \mu_{p-1} \arrow[r, equal] & \mu_{p-1} & \mathbb{Z}/2 \arrow[l, hook'] \end{tikzcd}$$

in bidegrees $(s, t) = (1, 0)$ and $(3, 1)$. Thus

$$\pi_0 \lim B\mathfrak{Pic}(\mathrm{Mod}^{\wedge}_{\mathrm{E}^{\bullet+1}}) \cong \mu_{p-1}.$$ ☐

5.2.2. *The case* $p = 2$. We now proceed with the computation of the (-1)-stem at the prime two.

Lemma 5.17. *We have*

$$H^s(\mathbb{Z}_2^\times, \mathrm{Pic}(\mathrm{E})) = \begin{cases} \mathbb{Z}/2 & s = 0 \\ (\mathbb{Z}/2)^2 & s \geq 1 \end{cases}$$

$$H^s(\mathbb{Z}_2^\times, (\pi_0\mathrm{E})^\times) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2 & s = 0 \\ \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^2 & s = 1 \\ (\mathbb{Z}/2)^3 & s \geq 2 \end{cases}$$

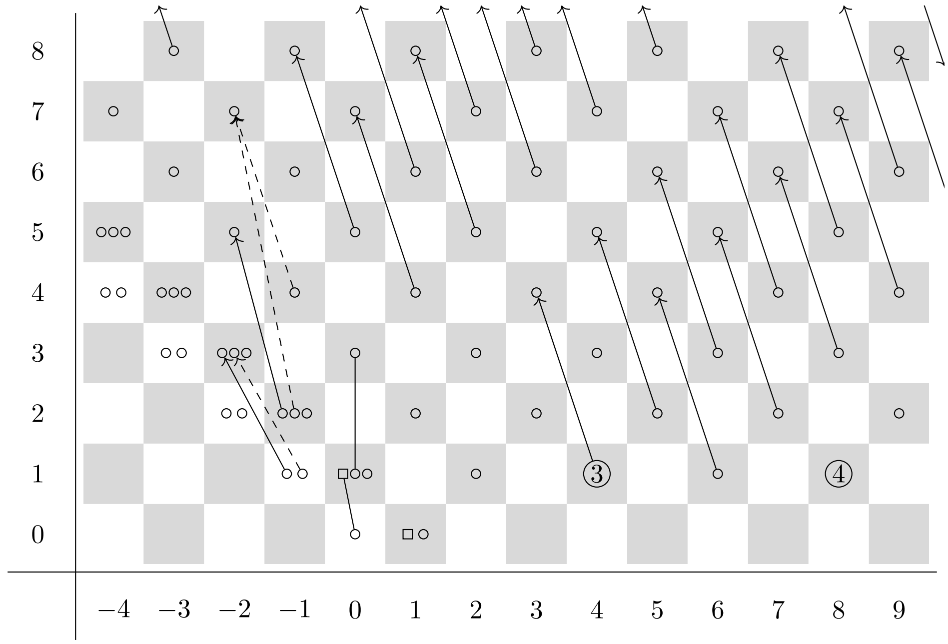


FIGURE 6. The Picard spectral sequence at $p = 2$. Possible differentials out of stems $t - s \leq -2$ are omitted. We know that all remaining classes in the 0-stem survive, by comparing to the algebraic Picard group. Thus the only differentials that remain to compute are those *out* of the (-1)-stem; some of these can be transported from the descent spectral sequence for $\mathfrak{Pic}(KO)^{hC_2}$—see Figs. 7 and 8.

The resulting spectral sequence is displayed in Figure 6.

Proof. We need to compute $H^*(\mathbb{Z}_2^\times, \mathbb{Z}/2)$ and $H^*(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times)$. Again we will use the LHSSS; for the first this reads

$$E_2^{i,j} = H^i(C_2, H^j(\mathbb{Z}_2, \mathbb{Z}/2)) \cong \left[\mathbb{Z}/2[x,y]/x^2\right]^{i,j} \implies H^{i+j}(\mathbb{Z}_2^\times, \mathbb{Z}/2),$$

The generators have (i,j)-bidegrees $|x| = (0,1)$ and $|y| = (1,0)$ respectively, and the spectral sequence is therefore determined by the differential $d_2(x) = \lambda y^2$, where $\lambda \in \mathbb{Z}/2$; since $H^1(\mathbb{Z}_2^\times, \mathbb{Z}/2) = \mathrm{Hom}(\mathbb{Z}_2^\times, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, we deduce that $\lambda = 0$. Finally, the potential extension $2x = y$ is ruled out by the fact that $\mathbb{Z}/2[y] = H^*(C_2, \mathbb{Z}/2)$ is a split summand.

For the second group, the splitting $\mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times \mathbb{Z}/2$ gives a summand isomorphic to $H^*(\mathbb{Z}_2^\times, \mathbb{Z}/2)$. The complement is computed by the LHSSS

$$E_2^{i,j} = H^i(C_2, H^j(\mathbb{Z}_2, \mathbb{Z}_2)) = \left[\mathbb{Z}_2[w,z]/(w^2, 2z)\right]^{i,j} \implies H^{i+j}(\mathbb{Z}_2^\times, \mathbb{Z}_2),$$

where now $|w| = (0,1)$ and $|z| = (2,0)$. In this case, the spectral sequence collapses immediately (again by computing H^1), with no space for extensions. □

Corollary 5.18. *At the prime two,* $|\mathrm{Br}_1^0| = 2^j$ *for* $j \leq 5$.

Proof. Once again, what remains is to determine the possible differentials on classes in the (-1)-stem; these are displayed in Fig. 6. Of these, two can be obtained by comparison to the Picard spectral sequence for the Galois extension $KO_2 \to KU_2$. There are three remaining undetermined differentials, which appear as the dashed arrows in Fig. 6. Invoking Proposition 5.11 now gives the claimed upper bound. □

Remark 5.19. In [Mor23], we show that the dashed differentials in Fig. 6 do not occur, and use Galois descent from KO_2 to compute $\mathrm{Br}_1^0 \cong \mathbb{Z}/8 \times \mathbb{Z}/4$.

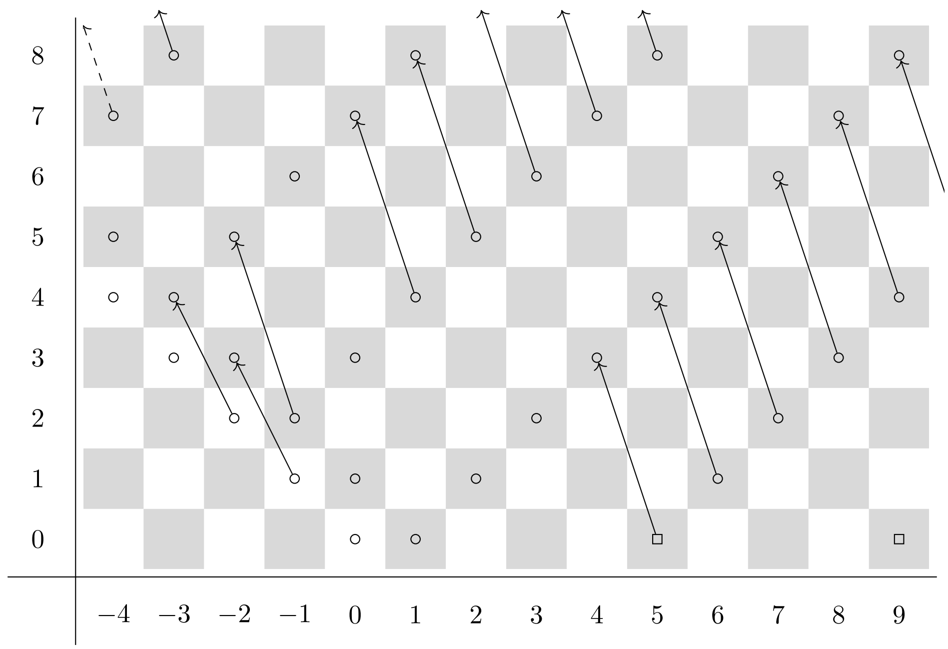


FIGURE 7. The E_3-page of the Picard spectral sequence for KO, as in [GL21, Figure 7.2].

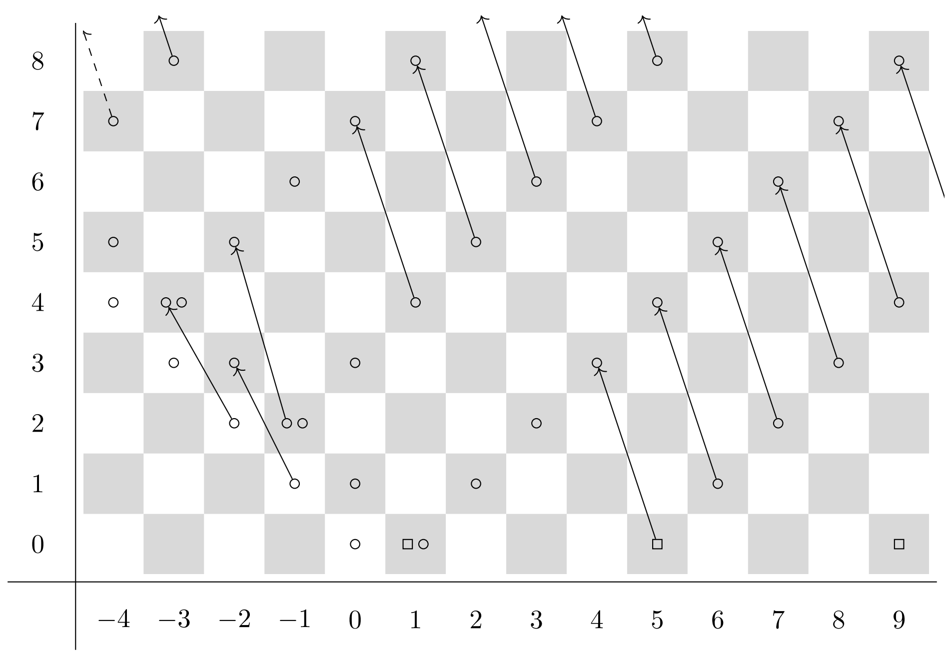


FIGURE 8. The Picard spectral sequence for KO_2.

APPENDIX A. DÉCALAGE RESULTS

We make the derivation of the descent spectral sequence a little more explicit, and relate it to the spectral sequence obtained from the covering $\mathbb{G} \twoheadrightarrow *$. For clarity most of this section is written in a general context, but the main result is Lemma A.3, which will be used to relate the descent spectral sequence to the K-local E-Adams spectral sequence through décalage.

Let $\mathcal{C}$ be a site, and write $\mathcal{A} \coloneqq \mathcal{S}h(\mathcal{C}, \mathcal{S}p)$. Given any object $\mathcal{F} \in \mathcal{A}$, there are two natural filtrations one can consider:

(1) The usual t-structure on $\mathcal{S}p$ induces one on $\mathcal{A}$, defined by the property that $\mathcal{F}' \in \mathcal{A}$ is t-truncated if and only if $\mathcal{F}'(X) \in \mathcal{S}p_{\leq t}$ for every $X \in \mathcal{C}$. One can therefore form the Postnikov tower

$$\mathcal{F} \simeq \lim(\cdots \to \tau_{\leq t}\mathcal{F} \to \cdots)$$

in $\mathcal{A}$, and obtain a tower of spectra on taking global sections.

(2) Suppose $U \twoheadrightarrow *$ is a covering of the terminal object. Since A is a sheaf, we can recover $\Gamma\mathcal{F}$ as the limit of its Čech complex for the covering, and this has an associated tower. Explicitly, write $\mathrm{Tot}_q = \lim_{\mathbf{\Delta}_{\leq q}}$ so that

$$\Gamma\mathcal{F} \simeq \mathrm{Tot}\,\mathcal{F}(U^\bullet) \simeq \lim(\cdots \to \mathrm{Tot}_0\,\mathcal{F}(U^\bullet)).$$

Any tower of spectra $X = \lim(\cdots \to X_t \to \cdots)$ gives rise to a (conditionally convergent) spectral sequence, as in Lemma 2.4. Respectively, in the cases above these read

$$E_2^{s,t} = \pi_{t-s}\Gamma\tau_t\mathcal{F} \implies \pi_{t-s}\Gamma\mathcal{F}, \tag{58}$$

and

$$\check{E}_2^{p,q} = \pi_{q-p} f_q \mathcal{F}(U^\bullet) \implies \pi_{q-p}\Gamma\mathcal{F}, \tag{59}$$

where f_q denotes the fibre of the natural transformation $\mathrm{Tot}_q \to \mathrm{Tot}_{q-1}$. In each case, the d_r differential has bidegree $(r+1, r)$ in the displayed grading.

For our purposes, the first spectral sequence, whose E_2-page and differentials are both defined at the level of truncations, is useful for interpreting the descent spectral sequence: for example, we use this in Theorem 4.4. On the other hand, the second spectral sequence is more easily evaluated once we know the value of a proétale sheaf on the free $\mathbb{G}$-sets. It will therefore be important to be able to compare the two, and this comparison is made using the *décalage* technique originally due to Deligne in [Del71]. The following material is well known (see for example [Lev15]), but we include an exposition for convenience and to fix indexing conventions. The décalage construction of [Hed21], which 'turns the page' of a spectral sequence by functorially associating to a filtered spectrum its *decalée* filtration, is closely related but not immediately equivalent.

Recall that any tower dualises to a filtration (this will be recounted below); the proof is cleanest in the slightly more general context of *bifiltered* spectra, and so we will make the connection between (58) and (59) explicit after proving a slightly more general result.

We suppose therefore that X is a spectrum equipped with a (complete and decreasing) bifiltration. That is, we have a diagram of spectra $X^{t,q} : \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}} \to \mathcal{S}p$ such that $X = \mathrm{colim}_{p,q} X^{t,q}$. We will write $X^{-\infty,q} := \mathrm{colim}_t X^{t,q}$ for any fixed q, and likewise $X^{t,-\infty} := \mathrm{colim}_q X^{t,q}$ for fixed t. Finally, write $X^{t/t',q/q'} := \mathrm{cofib}(X^{t',q'} \to X^{t,q})$.

Proposition A.1. *Let X be a bifiltered spectrum, and suppose that for all t and q we have $\pi_s X^{t/t+1,q/q+1} = 0$ unless $s = t - q$. Then there is an isomorphism*

$$^1E_2^{s,t} \simeq {}^2E_3^{2s-t,s}, \tag{60}$$

where the left-hand side is the spectral sequence for the filtration $X = \mathrm{colim}_t X^{t,-\infty}$ and the right-hand for $X = \mathrm{colim}_q X^{-\infty,q}$.

This isomorphism is compatible with differentials, and so extends to isomorphisms

$$^1E_r^{s,t} \simeq {}^2E_{r+1}^{2s-t,s}$$

for $1 \leq r \leq \infty$.

Proof. We begin with the isomorphism (60): it is obtained by considering the following trigraded spectral sequences, which converge to the respective E_2-pages.

$$E_2^{s,t,q} = \pi_{q-s} X^{t/t+1,q+1/q} \implies \pi_{q-s} X^{t/t+1,-\infty} = {}^1E_2^{s+t-q,t}, \tag{61}$$

$$E_2^{s,t,q} = \pi_{t-s} X^{t+1/t,q+1/q} \implies \pi_{t-s} X^{-\infty,q+1/q} = {}^2E_2^{s+q-t,q}. \tag{62}$$

The d_r differentials have (s,t,q)-tridegrees $(r+1, 0, r)$ and $(r+1, r, 0)$ respectively. Both spectral sequences therefore take a very simple form, because we have assumed each object $X^{t+1/t,q+1/q}$ is Eilenberg-Mac Lane. They are displayed in Fig. 9.

(A) Spectral sequence (61) at fixed t. The y-intercept is $s = t$.

(B) Spectral sequence (62) at fixed q. The y-intercept is $s = q$.

FIGURE 9. The trigraded spectral sequences computing ${}^1E_2^{*,t}$ and ${}^2E_2^{*,q}$ respectively.

In particular, the first collapses after the E_2-page, so that ${}^1E_2^{*,t}$ is the cohomology of the complex

$$\cdots \to \pi_{t-q}X^{t/t+1,q/q+1} \to \pi_{t-q-1}X^{t/t+1,q+1/q+2} \to \cdots \tag{63}$$

whose differentials are induced by the composites

$$X^{t+1/t,q/q+1} \to \Sigma X^{t+1/t,q+1} \to \Sigma X^{t+1/t,q+1/q+2}.$$

More precisely, ${}^1E_2^{s,t} = \pi_{t-s}X^{t/t+1,-\infty} \cong H^s\left(\pi_{t-*}X^{t/t+1,*/*+1}\right)$.

The second spectral sequence is even simpler, collapsing immediately to give

$${}^2E_2^{2s-t,s} = \pi_{t-s}X^{-\infty,s/s+1} \cong \pi_{t-s}X^{t/t+1,s/s+1}.$$

We can further identify the first differential on ${}^2E_2^{2s-t,s}$: it is induced by the map $X^{-\infty,2s-t/2s-t+1} \to \Sigma X^{-\infty,2s-t+1/2s-t+2}$, and so the identification ${}^1E_2^{s,t} \cong {}^2E_3^{2s-t,s}$ follows from the commutative diagram below, in which the top row is part of the complex (63).

$$\begin{tikzcd}
\pi_{t-s}X^{t/t+1,s/s+1} \arrow[r] \arrow[d, "\sim"'] & \pi_{t-s}\Sigma X^{t/t+1,s+1} \arrow[r] \arrow[d] & \pi_{t-s}\Sigma X^{t/t+1,s+1/s+2} \arrow[d, "\sim"] \\
\pi_{t-s}X^{-\infty,s/s+1} \arrow[r] & \pi_{t-s}\Sigma X^{-\infty,s+1} \arrow[r] & \pi_{t-s}\Sigma X^{-\infty,s+1/s+2}
\end{tikzcd}$$

We next argue that this extends to an isomorphism of spectral sequences ${}^1E_r^{s,t} \cong {}^2E_{r+1}^{2s-t,s}$. To do so we will give a map of exact couples as below:

$$\left(\begin{tikzcd}
{}^1D_2^{s,t} \arrow[rr] & & {}^1D_2^{s,t} \arrow[dl] \\
& {}^1E_2^{s,t} \arrow[ul] &
\end{tikzcd}\right) \to \left(\begin{tikzcd}
{}^2D_3^{2s-t,s} \arrow[rr] & & {}^2D_3^{2s-t,s} \arrow[dl] \\
& {}^2E_3^{2s-t,s} \arrow[ul] &
\end{tikzcd}\right)$$

By definition of the derived couple on the right-hand side, this amounts to giving maps

$$\pi_{t-s}X^{t,-\infty} \to \mathrm{im}(\pi_{t-s}X^{-\infty,s} \to \pi_{t-s}X^{-\infty,s-1})$$

for all s and t, subject to appropriate naturality conditions. To this end we claim that the first map in each of the spans below induces an isomorphism on π_{t-s}:

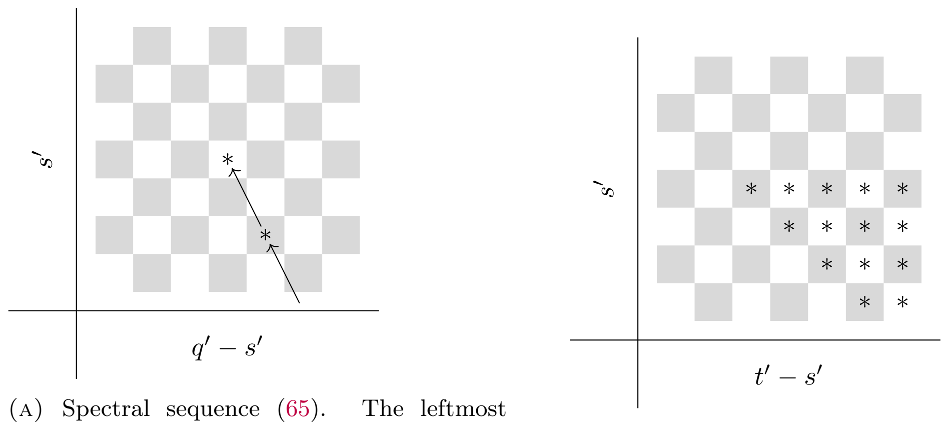
$$X^{t,-\infty} \leftarrow X^{t,s-1} \to X^{-\infty,s-1}.$$

Indeed, writing $C := \mathrm{cofib}(X^{t,s-1} \to X^{t,-\infty})$, we have a filtration $C = \operatorname{colim} C^{t'}$, where $t' \geq t$ and $C^{t'} = \mathrm{cofib}(X^{t',q-1} \to X^{t',-\infty})$. The resulting spectral sequence reads

$$\pi_{t'-s'}C^{t'/t'+1} \implies \pi_{t'-s'}C \qquad (t' \geq t), \tag{64}$$

and its E_2-page is in turn computed by a trigraded spectral sequence

$$\pi_{q'-s'}X^{t'/t'+1,q'/q'+1}X \implies \pi_{q'-s'}C^{t'/t'+1} \qquad (q' \leq s-2). \tag{65}$$



(A) Spectral sequence (65). The leftmost nonzero entry is $(q' - s', s') = (t' - s + 2, 2s - t' - 4)$.

(B) The E_2-page of (64). The leftmost nonzero entry is $(t' - s', s') = (t - s + 2, s - 2)$.

FIGURE 10. Truncated spectral sequences computing $\pi_* C$.

Spectral sequence (65) can be thought of as a truncation of Figure 9a; it is in turn displayed in Figure 10a. the form of its E_2-page implies that $C^{t'/t'+1}$ is $(t' - s + 2)$-connected, so that (64) takes the form displayed in Figure 10b.

In particular, C is $t - s + 1$-connected and so the map $X^{t,s-1} \to X^{t,-\infty}$ is $(t - s + 1)$-connected. Applying an identical analysis to $X^{t,s} \to X^{t,-\infty}$ shows that this map has $(t - s)$-connected cofibre, and so induces a surjection on π_{t-s}; the diagram below therefore produces the requisite map ${}^1D_2^{s,t} \to {}^2D_3^{2s-t,s}$.

$$\begin{tikzcd}
\pi_{t-s} X^{t,s} \arrow[r, two heads] \arrow[d] & \pi_{t-s} X^{t,s-1} \arrow[r, "\sim"] \arrow[d] & \pi_{t-s} X^{t,-\infty} \arrow[dl, "\exists"] \\
\pi_{t-s} X^{-\infty,s} \arrow[r] & \pi_{t-s} X^{-\infty,s-1} &
\end{tikzcd}$$

We will not show that this indeed defines a map of exact couples, since this result is well-known: for example, see [Lev15, §6]. □

We now relate this to the context of Postnikov towers and cobar complexes. If $X \simeq \lim_t(\cdots \to X_t \to \cdots)$ is a convergent tower of spectra with $\operatorname{colim} X_t = 0$, we can form a *dual* filtered spectrum

$$X^t \coloneqq \operatorname{fib}(X \to X_{t-1}).$$

Then $\operatorname{colim} X^t \xrightarrow{\sim} X$, and we get another spectral sequence

$$\pi_{t-s} f^t X \implies \pi_{t-s} X,$$

where once again $X^{t/t+1} \coloneqq \operatorname{cofib}(X^{t+1} \to X^t)$. Observe that $X^{t/t+1} \simeq X_{t/t-1} \coloneqq \operatorname{fib}(X_t \to X_{t-1})$, by the octahedral axiom:

$$\begin{tikzcd}
X^{t+1} \arrow[r] \arrow[dr] & X^t \arrow[d] \arrow[r] & X_{t/t-1} \arrow[dl] \\
 & X \arrow[r] \arrow[dl] & X_t \arrow[dll] \\
X_{t-1} & &
\end{tikzcd}$$

Lemma A.2. *The spectral sequences for (X_t) and (X^t) agree.*

Proof. The observation above is that the E_2-pages agree. To show that the entire spectral sequences match it is enough to show that the differentials d_2 do, in other words that the outer diagram below commutes.

$$\begin{tikzcd}
X_{t/t-1} = X^{t/t+1} \arrow[r] \arrow[d] & \Sigma X^{t+1} \arrow[d] \\
X_t \arrow[r] \arrow[ur, dashed] & \Sigma X_{t+1/t} = \Sigma X^{t+1/t+2}
\end{tikzcd}$$

The dashed arrow is given by applying the 3×3-lemma [Nee00] to obtain the diagrams below; note that each triangle above only anticommutes, and so the outer square is commutative.

$$\begin{tikzcd}
\Sigma^{-1}X^{t+1} \arrow[r] \arrow[d] & \Sigma^{-1}X^{t} \arrow[r] \arrow[d] & \Sigma^{-1}X^{t/t+1} \arrow[r] \arrow[d] & X^{t+1} \arrow[d] \\
\Sigma^{-1}X \arrow[r, equal] \arrow[d] & \Sigma^{-1}X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[d] \\
\Sigma^{-1}X_t \arrow[r] \arrow[d] & \Sigma^{-1}X_{t-1} \arrow[r] \arrow[d] & X_{t/t-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "(-1)"] & X_t \arrow[d, dotted] \\
X^{t+1} \arrow[r] & X^{t} \arrow[r] & X^{t/t+1} \arrow[r] & \Sigma X^{t+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{t-1} \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "(-1)"] & \Sigma X_{t/t-1} \arrow[r] \arrow[d] & \Sigma X_t \arrow[r] \arrow[d] & \Sigma X_{t-1} \arrow[d] \\
\Sigma X^{t} \arrow[r] \arrow[d] & \Sigma X^{t/t+1} \arrow[r] \arrow[d] & \Sigma^2 X^{t+1} \arrow[r] \arrow[d] & \Sigma^2 X^{t} \arrow[d] \\
\Sigma X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \Sigma^2 X \arrow[r, equal] \arrow[d] & \Sigma^2 X \arrow[d] \\
\Sigma X_{t-1} \arrow[r] & \Sigma^2 X_{t/t-1} \arrow[r] & \Sigma^2 X_t \arrow[r] & \Sigma^2 X_{t-1}
\end{tikzcd}$$

□

By dualising the Postnikov and Tot-towers, it will therefore suffice to verify that the induced bifiltration satisfies the assumptions of Proposition A.1.

Lemma A.3. *Let $\mathcal{F}$ be a sheaf of spectra on a site $\mathcal{C}$, and let $X \twoheadrightarrow *$ be a covering of the terminal object. Suppose that for every t and every $q > 0$ we have $\Gamma(X^q, \tau_t\mathcal{F}) = \tau_t\Gamma(X^q, \mathcal{F})$. Then there is an isomorphism between the descent and Bousfield-Kan spectral sequences, up to reindexing: for all r,*

$$E_r^{s,t} \cong \check{E}_{r+1}^{2s-t,s}.$$

Proof. We form the bifiltrations $(\Gamma\mathcal{F})_{t,q} = \operatorname{Tot}_q \Gamma(U^\bullet, \tau_{\leq t}\mathcal{F})$. Then

$$(\Gamma\mathcal{F})_{t,-\infty} = \lim_q \operatorname{Tot}_q \Gamma(U^\bullet, \tau_{\leq t}\mathcal{F}) = \operatorname{Tot}\Gamma(U^\bullet, \tau_{\leq t}\mathcal{F}) = \Gamma\tau_{\leq t}\mathcal{F},$$

while

$$(\Gamma\mathcal{F})_{-\infty,q} = \lim_t \operatorname{Tot}_q \Gamma(U^\bullet, \tau_{\leq t}\mathcal{F}) = \operatorname{Tot}_q \Gamma(U^\bullet, \lim_t \tau_{\leq t}\mathcal{F}) = \operatorname{Tot}_q \Gamma(U^\bullet, \mathcal{F}).$$

Applying Proposition A.1 to the dual filtration, we need to verify that

$$\operatorname{Tot}^{q/q+1} \Gamma(X^\bullet, \tau_t\mathcal{F})$$

is Eilenberg-Mac Lane. But recall that for any cosimplicial spectrum $B^\bullet$ we have

$$\operatorname{fib}(\operatorname{Tot}_q B^\bullet \to \operatorname{Tot}_{q-1} B^\bullet) \simeq \Omega^q N^q B^\bullet,$$

where N^q denotes the fibre of the map from X^q to the q-th matching spectrum; $N^q B^\bullet$ is a pointed space with

$$\pi_j N^q B^\bullet = \pi_j B^q \cap \ker s^0 \cap \ker s^{q-1}.$$

In the case of pointed spaces, this fact is [BK87, Prop. X.6.3]; the proof, which appears also as [GJ09, Lemma VIII.1.8], works equally well for a cosimplicial spectrum[5]. By abuse of notation, we also denote this

[5] Note that the inductive argument there applies in the 'cosimplicial' direction, i.e., in the notation of *loc. cit.* one shows for any fixed t that $N^{n,k}\pi_t X = \ker(\pi_t X \to M^{n,k}\pi_t X)$ for $k, n \leq s$.

group by $N^q\pi_j B^\bullet$. Thus

$$\begin{aligned}\pi_j\,\mathrm{Tot}^{q/q+1}\,\Gamma(X^\bullet,\tau_t\mathcal{F}) &\simeq \pi_j\Omega^q N^q\Gamma(X^\bullet,\tau_t\mathcal{F})\\ &\simeq N^q\pi_{j+q}\Gamma(X^\bullet,\tau_t\mathcal{F})\\ &\simeq N^q\pi_{j+q}\tau_t\Gamma(X^\bullet,\mathcal{F})\\ &\subset \pi_{j+q}\tau_t\Gamma(X^\bullet,\mathcal{F}) = \begin{cases}\pi_t\Gamma(X^\bullet,\mathcal{F}) & j=t-q\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$ □

Remark A.4. On the starting pages, one has

$$E_2^{s,t} = \pi_{t-s}\Gamma\tau_t\mathcal{F} \simeq H^s(\mathcal{C},\pi_t\mathcal{F})$$

and

$$\check{E}_3^{2s-t,s} = H^s(\pi_{t-*}\Omega^* N^*\Gamma(X^\bullet,\mathcal{F})) = H^s(N^*\pi_t\Gamma(X^\bullet,\mathcal{F})) \simeq \check{H}^s(X \twoheadrightarrow *,\pi_t\mathcal{F}).$$

In particular, note that our assumption implies that the Čech-to-derived functor spectral sequence collapses.

Appendix B. The Adams spectral sequence at height one

In this appendix, we compute the $K(1)$-local E_1-Adams spectral sequences at all primes. The results are again well-known, and implicit in the computations of [Rav84, MRW77]. At the prime two we give a different perspective to [BGH22] on the computation, which illustrates how one can make use of the Postnikov tower of a sheaf of spectra on $B\mathbb{G}_{\text{proét}}$: this approach may be useful in higher height examples that are more computationally challenging. In particular, we use the finite resolution of the $K(1)$-local sphere, and as such our only input is knowledge of the HFPSS for the conjugation action on KU, displayed in Fig. 12.

B.1. Odd primes.

At odd primes, the multiplicative lift gives a splitting

$$\mathbb{Z}_p^\times \simeq (1+p\mathbb{Z}_p)\times\mu_{p-1} \simeq \mathbb{Z}_p\times\mu_{p-1},$$

with the second isomorphism given by the p-adic logarithm. A pair $(a,b)\in\mathbb{Z}_p\times\mu_{p-1}$ therefore acts on $\pi_t\mathrm{E}$ as $(b\exp_p(a))^t$.

Lemma B.1. *The starting page of the* K*-local* E*-Adams spectral sequence is*

$$E_2^{s,t} = H^s(\mathbb{Z}_p^\times,\pi_t\mathrm{E}) = \begin{cases}\mathbb{Z}_p & t=0 \text{ and } s=0,1\\ \mathbb{Z}/p^{\nu_p(t')+1} & t=2(p-1)t'\neq 0 \text{ and } s=1\end{cases} \tag{66}$$

and zero otherwise. In particular, it collapses immediately to E_∞.

Proof. We will use the Lyndon-Hochschild-Serre spectral sequence [SW00, Theorem 4.2.6] for the inclusion $\mathbb{Z}_p\simeq 1+p\mathbb{Z}_p\hookrightarrow\mathbb{Z}_p^\times$, which reads

$$H^i(\mu_{p-1},H^j(\mathbb{Z}_p,\pi_t KU_p)) \implies H^{i+j}(\mathbb{Z}_p^\times,\pi_t KU_p).$$

Since everything is (p)-local, taking μ_{p-1} fixed-points is exact. The spectral sequence therefore collapses, and what remains is to compute $\mathbb{Z}_p$-cohomology.

By [SW00, §3.2], the trivial pro-p module $\mathbb{Z}_p$ admits a projective resolution

$$0\to\mathbb{Z}_p[[\mathbb{Z}_p]]\xrightarrow{\zeta-1}\mathbb{Z}_p[[\mathbb{Z}_p]]\to\mathbb{Z}_p\to 0$$

in the (abelian) category $\mathfrak{C}_p(\mathbb{Z}_p)$ of pro-p continuous $\mathbb{Z}_p^\times$-modules; here we write ζ for a topological generator, and $\mathbb{Z}_p[[G]] \coloneqq \lim_{U<_oG}\mathbb{Z}_p[G/U]$ for the *completed* group algebra of a profinite group G. In particular,

$$H^j(\mathbb{Z}_p,\pi_{2t}KU_p) = H^j\left(\mathbb{Z}_p\xrightarrow{\exp_p(\zeta)^t-1}\mathbb{Z}_p\right) \simeq \begin{cases}\mathbb{Z}_p & t=0 \text{ and } j=0,1\\ \mathbb{Z}_p/p^{\nu_p(t)+1} & t\neq 0 \text{ and } j=1\end{cases}$$

where for the final isomorphism we have used the isomorphism $\exp_p : p^{j-1}\mathbb{Z}_p/p^j\mathbb{Z}_p \simeq 1+p^j\mathbb{Z}_p^\times/1+p^{j+1}\mathbb{Z}_p^\times$ to obtain $\nu_p(\exp_p(\zeta)^t-1) = \nu_p(t\zeta)+1 = \nu_p(t)+1$. The μ_{p-1}-action has fixed points precisely when it is trivial, i.e. when $p-1\mid t$, which gives the stated form. □

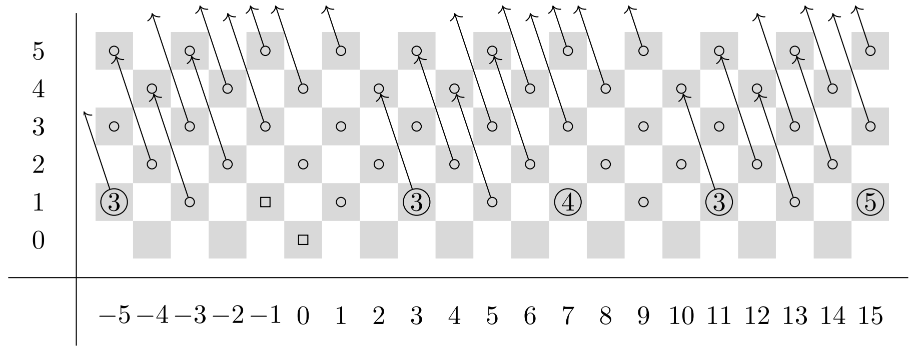


FIGURE 11. The E_3-page of the descent spectral sequence for E at $p = 2$.

B.2. $p = 2$. What changes at even primes? In this case the multiplicative lift is instead defined on $(\mathbb{Z}/4)^\times$, and provides a splitting

$$\mathbb{Z}_2^\times \cong (1 + 4\mathbb{Z}_2) \times C_2 \cong \mathbb{Z}_2 \times C_2.$$

This implies the following more complicated form for the starting page, since $\mathrm{cd}_2(C_2) = \infty$.

Lemma B.2. *The starting page of the descent spectral sequence for the action of $\mathbb{G}$ on E is given by*

$$E_2^{s,t} = H^s(\mathbb{Z}_2^\times, \pi_t \mathrm{E}) = \begin{cases} \mathbb{Z}_2 & t = 0 \text{ and } s = 0, 1 \\ \mathbb{Z}/2 & t \equiv_4 2 \text{ and } s \text{ odd} \\ \mathbb{Z}/2 & t \equiv_4 0 \text{ and } s > 1 \text{ odd} \\ \mathbb{Z}/2^{\nu_2(t)+2} & 0 \neq t \equiv_4 0 \text{ and } s = 1 \end{cases} \tag{67}$$

and zero otherwise. The result is displayed in Fig. 11, which is reproduced for convenience of the reader.

Proof. We will again use the Lyndon-Hochschild-Serre spectral sequence for the inclusion $\mathbb{Z}_2 \simeq 1+4\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2^\times$, which reads

$$H^i(C_2, H^j(\mathbb{Z}_2, \pi_t KU_2)) \implies H^{i+j}(\mathbb{Z}_2^\times, \pi_t KU_2).$$

Since C_2 is 2-torsion, we will have higher C_2-cohomology contributions. Nevertheless, the computation of $\mathbb{Z}_2$-cohomology is identical to the odd-prime case, except that now one has $\nu_2(\exp_2(\zeta)^t - 1) = \nu_2(t) + 2$. Thus

$$H^j(\mathbb{Z}_2, \pi_t KU_2) = H^j\left(\mathbb{Z}_2 \xrightarrow{\exp_p(\zeta)^t - 1} \mathbb{Z}_2\right) \simeq \begin{cases} \mathbb{Z}_2 & t = 0 \text{ and } j = 0, 1 \\ \mathbb{Z}_2/2^{\nu_2(t)+2} & t \neq 0 \text{ and } j = 1 \end{cases}$$

For $t \neq 0$, the E_2-page of the Lyndon-Hochschild-Serre is therefore concentrated in degrees $j = 1$, and so collapses. This yields

$$H^s(\mathbb{Z}_2^\times, \pi_t KU_2) \simeq H^{s-1}(C_2, \mathbb{Z}/2^{\nu_2(t)+2}),$$

which accounts for most of the groups in (67). For $t = 0$, it instead takes the form

$$E_2^{*,*} = \mathbb{Z}_2[x^{(0,1)}, y^{(2,0)}]/(2x, 2y, x^2) \implies H^{i+j}(\mathbb{Z}_2^\times, \pi_0 KU_2).$$

In particular, the entire spectral sequence is determined by the differential $d_2(x) = 0$, which implies that all other differentials vanish by multiplicativity. To deduce this differential, note that the edge map $H^1(\mathbb{Z}_2^\times, \pi_0 KU_2) \to H^0(C_2, H^1(\mathbb{Z}_2, \pi_0 KU_2))$ can be interpreted as the restriction map

$$\mathrm{Hom}(\mathbb{Z}_2^\times, \mathbb{Z}_2) \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2),$$

along $\exp_2 : \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2^\times$. This is an isomorphism since $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}_2) = 0$, so d_2 must act trivially on bidegree $(0, 1)$. $\square$

Our next task is to compute the differentials. In the rest of the appendix we will prove the following:

Proposition B.3. *The differentials on the third page are as displayed in Fig. 11. The spectral sequence collapses at E_4 with a horizontal vanishing line.*

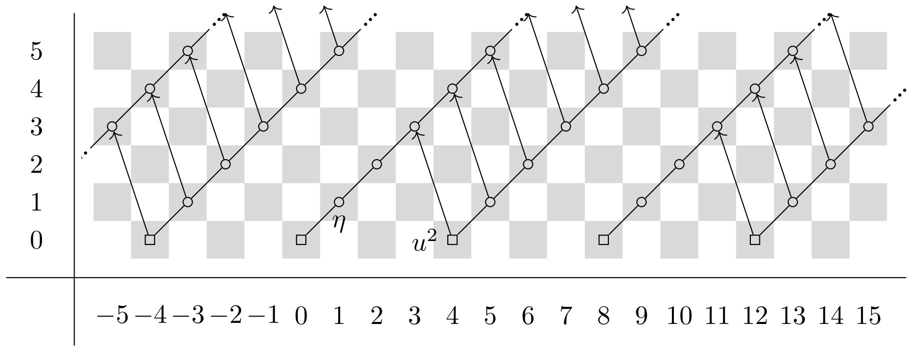


FIGURE 12. The HFPSS for the C_2-Galois extension $KO \to KU$. The class η represents η in the Hurewicz image in $\pi_* KO$, and towers of slope one are related by η-multiplications. In particular, η^4 cannot survive to E_∞ since $\eta^4 = 0 \in \pi_*\mathbb{S}$. The only option is $d_3(u^2\eta) = \eta^4$, which implies the rest by multiplicativity.

We will compute these differentials by comparing to the HFPSS for the conjugation action on KU_2 (Fig. 12), which reads

$$E_2^{s,t} = H^s(C_2, \pi_* KU_2) = \mathbb{Z}_2[\eta, u^{\pm 2}]/2\eta \implies \pi_{t-s} KO_2.$$

The following result folklore:

Lemma B.4. *The* K*-local sphere fits in a fibre sequence*

$$\mathbf{1}_{\mathrm{K}} \to KO_2 \xrightarrow{\psi^5 - 1} KO_2.$$

Proof. We first consider the map $\psi^5 - 1 : KU_2 \to KU_2$. Certainly ψ^5 acts trivially on $KU_2^{h(1+4\mathbb{Z}_2)} := \Gamma(\mathbb{Z}_2^\times/1 + 4\mathbb{Z}_2, \underline{\mathrm{E}})$, so there is a map

$$KU_2^{h(1+4\mathbb{Z}_2)} \to \mathrm{fib}(KU_2 \xrightarrow{\psi^5 - 1} KU_2) \tag{68}$$

induced by the inclusion of fixed points $KU_2^{h(1+4\mathbb{Z}_2)} \to KU_2$. As observed in Lemma B.2, the HFPSS computing $\pi_* KU_2^{h(1+4\mathbb{Z}_2)}$ collapses at E_2 with horizontal vanishing above filtration one, and one therefore sees that (68) must be an equivalence by computing homotopy groups of $\mathrm{fib}(\psi^5 - 1)$ using the exact sequence.

Taking fixed points for the C_2 action now yields the result:

$$\begin{aligned} \mathbf{1}_{\mathrm{K}} &\simeq (KU_2^{h(1+4\mathbb{Z}_2)})^{hC_2} \\ &\simeq \mathrm{fib}(KU_2 \xrightarrow{\psi^5 - 1} KU_2)^{hC_2} \\ &\simeq \mathrm{fib}(KU_2^{hC_2} \xrightarrow{\psi^5 - 1} KU_2^{hC_2}) \\ &\simeq \mathrm{fib}(KO_2 \xrightarrow{\psi^5 - 1} KO_2). \end{aligned}$$

□

Proof (Proposition B.3). The previous lemma gives the diagram

$$\begin{tikzcd} \Sigma^{-1} KO_2 \arrow[r] \arrow[d] & \mathbf{1}_{\mathrm{K}} \arrow[r] \arrow[d] & KO_2 \arrow[d] \\ \Sigma^{-1} KU_2 \arrow[r] & KU_2^{h(1+4\mathbb{Z}_2)} \arrow[r] & KU_2 \end{tikzcd} \tag{69}$$

in which the top row is obtained as C_2-fixed points of the bottom. The HFPSS for the middle map,

$$E_2^{s,t} = H^s(C_2, \pi_t KU_2^{h(1+4\mathbb{Z}_2)}) \implies \pi_{t-s} \mathbf{1}_{\mathrm{K}}, \tag{70}$$

is very closely to the descent spectral sequence; it is displayed in Fig. 13. In fact, in Lemma B.5 we will show that the two spectral sequences are isomorphic (including differentials), up to a certain filtration shift. To infer the differentials in Fig. 11, it is therefore enough to compute the differentials in Fig. 13.

Next observe that Lemma B.4 implies existence of exact sequences

$$0 \to \pi_{2t} KU_2^{h(1+4\mathbb{Z}_2)} \to \mathbb{Z}_2 \xrightarrow{5^t-1} \mathbb{Z}_2 \to \pi_{2t-1} KU_2^{h(1+4\mathbb{Z}_2)} \to 0,$$

in which the leftmost term vanishes for $t \neq 0$ and the middle map is null for $t = 0$. Taking C_2-cohomology yields isomorphisms

$$H^*(C_2, \pi_0 KU_2^{h(1+4\mathbb{Z}_2)}) \xrightarrow{\sim} H^*(C_2, \pi_0 KU_2) \tag{71}$$

$$H^*(C_2, \pi_0 KU_2) \xrightarrow{\sim} H^*(C_2, \pi_{-1} KU_2^{h(1+4\mathbb{Z}_2)}), \tag{72}$$

and an exact sequence

$$\begin{aligned} 0 \to H^{s-1}(C_2, &\pi_{2t-1} KU^{h(1+4\mathbb{Z}_2)}) \to H^s(C_2, \pi_{2t} KU_2) \\ &\to H^s(C_2, \pi_{2t} KU_2) \to H^s(C_2, \pi_{2t-1} KU^{h(1+4\mathbb{Z}_2)}) \to 0 \end{aligned}$$

for $s \geq 1$ (and $t \neq 0$). The middle terms are either both zero or both $\mathbb{Z}/2$, and in the latter case we obtain the following further isomorphisms:

$$H^{s-1}(C_2, \pi_{2t-1} KU_2^{h(1+4\mathbb{Z}_2)}) \xrightarrow{\sim} H^s(C_2, \pi_{2t} KU_2) \simeq \mathbb{Z}/2, \tag{73}$$

$$\mathbb{Z}/2 \simeq H^s(C_2, \pi_{2t} KU_2) \xrightarrow{\sim} H^s(C_2, \pi_{2t-1} KU_2^{h(1+4\mathbb{Z}_2)}). \tag{74}$$

For $s = 0$ and $t \neq 0$ even, we instead have an exact sequence

$$0 \to \mathbb{Z}_2 \xrightarrow{5^t-1} \mathbb{Z}_2 \to H^0(C_2, \pi_{2t-1} KU_2^{h(1+4\mathbb{Z}_2)}) \to 0. \tag{75}$$

Equations (71) to (75) compute the effect of the maps in (69) on spectral sequences. Namely:

(1) The map $\Sigma^{-1} KU_2 \to KU_2^{h(1+4\mathbb{Z}_2)}$ induces a (filtration preserving) surjection from $E_2(KU)$ onto the unfilled classes in Fig. 13.
(2) The map $KU_2^{h(1+4\mathbb{Z}_2)} \to KU_2$ is injective on the solid classes in Fig. 13. It induces a filtration-preserving isomorphism on the subalgebras in internal degree $t = 0$, and away from this increases filtration by one.

The differentials in Fig. 13 follow almost immediately: on unfilled classes, they are images of differentials in the HFPSS for KU_2, and on most solid classes they are detected by differentials in the HFPSS for KU_2. We are left to determine a small number of differentials on classes with internal degree t close to zero.

(1) The exact sequence induced by Lemma B.4 shows that the map $\mathbf{1}_{\mathrm{K}} \to KO_2$ induces an isomorphism on π_0. As a result, the unit in the HFPSS for $KU^{h(1+4\mathbb{Z}_2)}$ is a permanent cycle.
(2) Write $u^{-2}\eta^2$ for the generator in bidegree $(s,t) = (2,0)$, which maps to a class of the same name in the HFPSS for KU_2. This cannot survive in the HFPSS for $KU_2^{h(1+4\mathbb{Z}_2)}$: one sees that $\pi_{-2}\mathbf{1}_{\mathrm{K}} = 0$ using the fibre sequence of Lemma B.4 and the fact that $\pi_{-2} KO_2 = \pi_{-1} KO_2 = 0$. The only possibility is a nonzero d_2 since all other possible differentials occur at E_4 or later, when there are no possible targets left (by virtue of the known d_3-differentials). Likewise, this implies

$$d_2((u^{-2}\eta^2)^{2j+1}) \equiv_2 (u^{-2}\eta^2)^{2j} d_2(u^{-2}\eta^2) \neq 0$$

by the Leibniz rule.
(3) Write z for the generator in bidegree $(s, t-s) = (0, -3)$, which is detected in filtration one by $u^{-2}\eta$. As above one computes that $\pi_{-3}\mathbf{1}_{\mathrm{K}} = 0$, and so this class must die; the only option is a nonzero d_4 on z. In fact, when j is even the exact sequence

$$\pi_{4j-2} KO_2 \to \pi_{4j-3}\mathbf{1}_{\mathrm{K}} \to \pi_{4j-3} KO_2$$

implies that $\pi_{4j-3}\mathbf{1}_{\mathrm{K}} = 0$, so that $(u^{-2}\eta^2)^{2j} z$ supports a d_4 by the same argument. When j is odd the sequence only gives a bound $|\pi_{4j-3}\mathbf{1}_{\mathrm{K}}| \leq 4$, but this is already populated by elements in filtration $s \leq 2$ (which survive by comparison to the HFPSS for KU_2). Thus $(u^{-2}\eta^2)^{2j} z$ supports a d_4 in these cases too.

After this, the spectral sequence collapses by sparsity. □

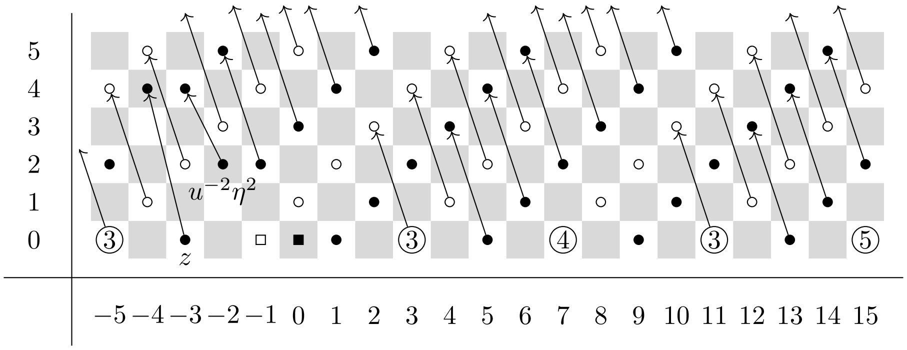


FIGURE 13. The HFPSS for C_2 acting on $KU^{h(1+4\mathbb{Z}_2)}$. Solid classes are detected in the HFPSS for KU_2 by the map $KU_2^{h(1+4\mathbb{Z}_2)} \to KU_2$, with a filtration shift of one in internal degrees $t \neq 0$. Unfilled classes are in the image of the HFPSS for KU_2 under $\Sigma^{-1}KU_2 \to KU_2^{h(1+4\mathbb{Z}_2)}$. The only differentials not immediately determined by this are the d_2 and d_4 differentials on classes in internal degree $t = 0$ and -3 respectively, which are treated at the end of Proposition B.3.

To conclude, we must show that the two spectral sequences

$$H^*(\mathbb{G}, \pi_* KU_2) \implies \pi_* \mathbf{1}_{\mathrm{K}} \tag{76}$$

$$H^*(C_2, \pi_* KU_2^{h(1+4\mathbb{Z}_2)}) \implies \pi_* \mathbf{1}_{\mathrm{K}} \tag{77}$$

are isomorphic, up to a shift in filtration.

Lemma B.5. *For each $s \geq 0$ and t there are isomorphisms*

$$H^{s+1}(\mathbb{G}, \pi_t KU_2) \simeq H^s(C_2, \pi_{t-1} KU_2^{h(1+4\mathbb{Z}_2)}) \qquad (t \neq 1),$$
$$H^s(\mathbb{G}, \pi_0 KU_2) \simeq H^s(C_2, \pi_0 KU_2).$$

These are compatible with differentials, and together yield a (filtration-shifting) isomorphism of spectral sequences between (76) and (77).

Remark B.6. In other words, when passing from the HFPSS for $KU_2^{h(1+4\mathbb{Z}_2)}$ (77) to the descent spectral sequence (76), the filtration shift is precisely by one away from internal degree $t = 0$, and zero otherwise.

Proof. Note first that the E_1-pages are abstractly isomorphic:

$$H^{i+j}(\mathbb{G}, \pi_{2t} KU_2) \simeq H^i(C_2, H^j(H, \pi_{2t} KU_2)) \simeq H^i(C_2, \pi_{2t-j} KU_2^{h(1+4\mathbb{Z}_2)}), \tag{78}$$

where $j = 1$ unless $t = 0$, in which case we also have $j = 0$. The first isomorphism is given by the Lyndon-Hochschild-Serre spectral sequence, which collapses with each degree of the abutment concentrated in a single filtration; the second isomorphism comes from the same fact about the homotopy fixed point spectral sequence for the action of $1 + 4\mathbb{Z}_2$ on KU_2. Note that in both cases the abutment is sometimes in positive filtration, and this will account for the shift.

Next recall that the descent spectral sequence comes from global sections of the Postnikov tower for $\underline{\mathrm{E}} \in \widehat{\mathcal{S}h}(B\mathbb{G}_{\text{proét}}, \mathcal{S}p)$; on the other hand, the C_2-fixed points spectral sequence is induced by global sections of the sheaf $j_*\underline{\mathrm{E}} \in \widehat{\mathcal{S}h}(B(C_2)_{\text{proét}}, \mathcal{S}p)$, where $j : \mathbb{G} \to C_2$ is the quotient by the open subgroup $1 + 4\mathbb{Z}_2$. Since each $j_*\tau_{\leq t}\underline{\mathrm{E}}(T) = \tau_{\leq t}\underline{\mathrm{E}}(\mathrm{res}^{C_2}_{\mathbb{G}} T)$ is t-truncated, we obtain a map of towers in $\widehat{\mathcal{S}h}(B(C_2)_{\text{proét}}, \mathcal{S}p)$,

$$\begin{tikzcd}
j_*\underline{\mathrm{E}} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \tau_{\leq t} j_*\underline{\mathrm{E}} \arrow[r] \arrow[d] & \tau_{\leq t-1} j_*\underline{\mathrm{E}} \arrow[r] \arrow[d] & \cdots \\
j_*\underline{\mathrm{E}} \arrow[r] & \cdots \arrow[r] & j_*\tau_{\leq t}\underline{\mathrm{E}} \arrow[r] & j_*\tau_{\leq t-1}\underline{\mathrm{E}} \arrow[r] & \cdots
\end{tikzcd}$$

Taking global sections of the top row yields the HFPSS (77), while the bottom yields the descent spectral sequence (76) (bearing in mind that $\Gamma(C_2/C_2, j_*(-)) \simeq \Gamma(\mathbb{G}/\mathbb{G}, -)$).

To proceed, we compute the sections of these towers. Note that any cover of $C_2 \in B(C_2)_{\text{proét}}$ must split, and so sheafification on $B(C_2)_{\text{proét}}$ preserves any multiplicative presheaf when restricted to the generating sub-site Free_{C_2}, i.e. any $\mathcal{F}$ satisfying $\mathcal{F}(S \sqcup S') \simeq \mathcal{F}(S) \times \mathcal{F}(S')$. Thus

$$\Sigma^t \pi_t j_* \underline{\mathrm{E}} : C_2 \mapsto \Sigma^t \pi_t \underline{\mathrm{E}}(\mathbb{Z}_2^\times / 1 + 4\mathbb{Z}_2) = \Sigma^t \pi_t KU_2^{h(1+4\mathbb{Z}_2)}$$

This sheaf is therefore zero unless t is odd or zero. On the other hand,

$$\pi_s \Gamma j_* \Sigma^t \pi_t \underline{\mathrm{E}} : C_2 \mapsto \pi_s \Gamma(C_2, j_* \Sigma^t \pi_t \underline{\mathrm{E}}) = \pi_s \Gamma(\mathbb{Z}_2^\times / 1 + 4\mathbb{Z}_2, \Sigma^t \pi_t \underline{\mathrm{E}}) = H^{t-s}(1 + 4\mathbb{Z}_2, \pi_t KU_2).$$

This is zero unless t is even and $s = 1$, or $t = s = 0$. In particular, for $t \neq 0$ we have

$$\Sigma^{2t-1} \pi_{2t-1} j_* \underline{\mathrm{E}} \simeq \Sigma^{2t-1} \pi_{2t-1} j_* \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}} \simeq j_* \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}},$$

while $j_* \pi_0 \underline{\mathrm{E}}$ has homotopy concentrated in degrees $\{-1, 0\}$. Note that this also implies that $\Gamma j_* \tau_{\leq 2t} \underline{\mathrm{E}}$ is $(2t-1)$-truncated for $t \neq 0$, since $\pi_{2t} \Gamma j_* \tau_{\leq 2t} \underline{\mathrm{E}} = \pi_{2t} \Gamma j_* \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}} = 0$.

The isomorphisms (78) can be interpreted as arising from the two trigraded spectral sequences

$$\begin{aligned} H^i(C_2, H^j(1 + 4\mathbb{Z}_2, \pi_{2t} KU_2)) &\implies H^{i+j}(\mathbb{G}, \pi_{2t} KU_2), \\ H^i(C_2, H^j(1 + 4\mathbb{Z}_2, \pi_{2t} KU_2)) &\implies H^i(C_2, \pi_{2t-j} KU_2^{h(1+4\mathbb{Z}_2)}) \end{aligned} \tag{79}$$

coming from the bifiltration $\Gamma \underline{\mathrm{E}} = \Gamma \tau_{\leq j} j_* \tau_{\leq 2t} \underline{\mathrm{E}}$ (c.f. Proposition A.1). At a fixed t, the first is associated to the filtration $\Gamma \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}} = \Gamma j_* \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}} = \lim_j \Gamma \tau_{\leq j} j_* \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}}$, or equivalently to the resolution of $H^*(1 + 4\mathbb{Z}_2, \pi_{2t} KU_2)$ by acyclic C_2-modules, and is therefore the LHSSS. On the other hand, the second is C_2-cohomology applied pointwise to the HFPSS for the $1+4\mathbb{Z}_2$-action (at a fixed j); in particular, both collapse at E_1.

For $t \neq 0$, the towers therefore look as in the following diagram, in which we have identified in both rows those consecutive layers with zero fibre, i.e. we run the tower 'at double speed'.

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_{\leq 2t-1} j_* \underline{\mathrm{E}} \arrow[rr] \arrow[dd] & & \tau_{\leq 2t-3} j_* \underline{\mathrm{E}} \arrow[r] \arrow[dl] \arrow[dd] & \cdots \\
& & \tau_{2t-1} j_* \underline{\mathrm{E}} \arrow[ul] \arrow[dd] & & \\
\cdots \arrow[r] & j_* \tau_{\leq 2t} \underline{\mathrm{E}} \arrow[rr] & & j_* \tau_{\leq 2t-2} \underline{\mathrm{E}} \arrow[r] \arrow[dl] & \cdots \\
& & j_* \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}} \arrow[ul] & &
\end{tikzcd}$$

A large but routine diagram verifies that the map $\Sigma^{2t-1} \pi_{2t-1} j_* \underline{\mathrm{E}} \to j_* \Sigma^{2t} \pi_{2t} \underline{\mathrm{E}}$ agrees on homotopy with (78).

Near zero, we have instead the diagram

$$\begin{tikzcd}
\tau_{\leq 1} j_* \underline{\mathrm{E}} \arrow[rr] \arrow[dd] & & \tau_{\leq 0} j_* \underline{\mathrm{E}} \arrow[rr] \arrow[dl] \arrow[dd] & & \tau_{\leq -1} j_* \underline{\mathrm{E}} \arrow[rr] \arrow[dl] \arrow[dd] & & \tau_{\leq -2} j_* \underline{\mathrm{E}} \arrow[dl] \arrow[dd] \\
& \Sigma \pi_1 j_* \underline{\mathrm{E}} \arrow[ul] \arrow[dd] & & \pi_0 j_* \underline{\mathrm{E}} \arrow[ul] \arrow[dd] & & \Sigma^{-1} \pi_{-1} j_* \underline{\mathrm{E}} \arrow[ul] \arrow[dd] \arrow[dll, dashed, bend left, "\beta"] & \\
j_* \tau_{\leq 1} \underline{\mathrm{E}} \arrow[rr, equal] & & j_* \tau_{\leq 0} \underline{\mathrm{E}} \arrow[rr] \arrow[dl] & & j_* \tau_{\leq -1} \underline{\mathrm{E}} \arrow[rr, equal] \arrow[dl] & & j_* \tau_{\leq -2} \underline{\mathrm{E}} \arrow[dl] \\
& 0 \arrow[ul] & & j_* \pi_0 \underline{\mathrm{E}} \arrow[ul] & & 0 \arrow[ul] &
\end{tikzcd}$$

To see that the dashed arrow exists, note that

$$d_1 : \Sigma^{-1} \pi_{-1} j_* \underline{\mathrm{E}} \to \Sigma \pi_0 j_* \underline{\mathrm{E}}$$

is zero on homotopy: in the proof Proposition B.3 we computed the only nontrivial d_2 differentials, which have source in internal degree $t = 0$. As a map between Eilenberg-Mac Lane objects, it is in fact null, so we can lift as below:

$$\begin{tikzcd}
\Sigma^{-1}\pi_{-1}j_*\underline{\mathrm{E}} \arrow[rr, "0", "d_1"'] \arrow[dr] \arrow[dd, dashed, "\exists\alpha"'] & & \Sigma\pi_0 j_*\underline{\mathrm{E}} \\
& \tau_{\leq -1} j_*\underline{\mathrm{E}} \arrow[ur] & \\
\tau_{\leq 0} j_*\underline{\mathrm{E}} \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^{-1}\pi_{-1}j_*\underline{\mathrm{E}} \arrow[rrr] \arrow[dr, "\alpha"] \arrow[dd, dashed, "\exists\beta"'] & & & 0 \arrow[dd] \\
& \tau_{\leq 0} j_*\underline{\mathrm{E}} \arrow[dr] & & \\
j_*\pi_0\underline{\mathrm{E}} \arrow[rr] & & j_*\tau_{\leq 0}\underline{\mathrm{E}} \arrow[r] & j_*\tau_{\leq -1}\underline{\mathrm{E}}
\end{tikzcd}$$

We must show that evident map $\pi_0 j_*\underline{\mathrm{E}} \to j_*\pi_0\underline{\mathrm{E}}$ and the map β agree with the respective compositions of edge maps. To deduce this, it is enough to contemplate the diagrams below, in which the dashed arrow are equivalences and the dotted arrows admit right inverses.

$$\begin{tikzcd}
\pi_0 j_*\underline{\mathrm{E}} \arrow[rrr] \arrow[dr] \arrow[dd, squiggly, bend right] & & & \tau_{\leq 0} j_*\underline{\mathrm{E}} \arrow[ddl] \arrow[dddd] \\
\pi_0 j_*\tau_{\geq 0}\underline{\mathrm{E}} \arrow[u, dashed] \arrow[d] & \pi_0 j_*\tau_{\leq 0}\underline{\mathrm{E}} \arrow[dr] & & \\
\pi_0 j_*\pi_0\underline{\mathrm{E}} \arrow[ur] \arrow[dr] \arrow[dd, squiggly] & & \tau_{\leq 0} j_*\tau_{\leq 0}\underline{\mathrm{E}} & \\
& \tau_{\leq 0} j_*\Sigma^p\pi_p\underline{\mathrm{E}} \arrow[ur] & & \\
j_*\pi_0\underline{\mathrm{E}} \arrow[rrr] \arrow[ur, dashed] & & & j_*\tau_{\leq 0}\underline{\mathrm{E}} \arrow[uul, dashed]
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^{-1}\pi_{-1} j_*\underline{\mathrm{E}} \arrow[rrr] \arrow[dr] \arrow[dd, squiggly, bend right] & & & \tau_{\leq 0} j_*\underline{\mathrm{E}} \arrow[ddl] \arrow[dddd] \\
\Sigma^{-1}\pi_{-1} j_*\tau_{\geq 0}\underline{\mathrm{E}} \arrow[u, dashed] \arrow[d] & \Sigma^{-1}\pi_{-1} j_*\tau_{\leq 0}\underline{\mathrm{E}} \arrow[dr] & & \\
\Sigma^{-1}\pi_{-1} j_*\pi_0\underline{\mathrm{E}} \arrow[ur] \arrow[dr] \arrow[dd, squiggly] & & \tau_{\leq -1} j_*\tau_{\leq 0}\underline{\mathrm{E}} & \\
& \tau_{\leq -1} j_*\pi_0\mathcal{E} \arrow[ur] & & \\
j_*\pi_0\underline{\mathrm{E}} \arrow[rrr] \arrow[ur, dotted] & & & j_*\tau_{\leq 0}\underline{\mathrm{E}} \arrow[uul, dotted]
\end{tikzcd}$$

The squiggly arrows are the edges maps from the trigraded spectral sequences, which are isomorphisms thanks to the collapse of the two trigraded spectral sequences. $\square$

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