ERRATUM TO:
THE LOW-DIMENSIONAL HOMOLOGY OF FINITE-RANK COXETER GROUPS

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Abstract. We resolve three computational errors in Appendix A of the named paper, and furthermore we include some clarifications to Definition 1.3 and Theorem B. We are grateful to Giles Gardam and Jean-Pierre Serre for pointing the errors out, and for helpful conversations.

Appendix A

In [Boy20, Appendix A] we use Theorems A and B of the paper to calculate the second and third integral homology of the finite irreducible Coxeter groups, as classified by Coxeter [Cox33]. There were three errors in the calculations, all of which were computational i.e. errors in my calculations as opposed to errors in the theorems.

We give the full correct results in the table below. The three errors occurred in the groups $H_2(D_4;\mathbb{Z})$, $H_3(D_6;\mathbb{Z})$, and $H_3(E_7;\mathbb{Z})$. All of these groups were missing a factor of $\mathbb{Z}_2$ in the original paper. Gardam has verified these computations with the GAP package HAP ([GAP21],[Ell21]), and the output and code can be found on the zenodo repository [Gar21].

W	$H_1(W;\mathbb{Z})$	$H_2(W;\mathbb{Z})$	$H_3(W,\mathbb{Z})$
$\mathbf{A}_n$ $n \geq 1$	$\mathbb{Z}_2$	$0 \quad n \leq 2$ $\mathbb{Z}_2 \quad n \geq 3$	$\mathbb{Z}_2 \quad n=1$ $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \quad n=2$ $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \quad n=3,4$ $\mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \quad n \geq 5$
$\mathbf{B}_n$ $n \geq 2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \quad n=2$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad n=3$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad n \geq 4$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4 \quad n=2$ $\mathbb{Z}_2^4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \quad n=3$ $\mathbb{Z}_2^5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^2 \quad n=4$ $\mathbb{Z}_2^6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^2 \quad n=5$ $\mathbb{Z}_2^7 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^2 \quad n \geq 6$
$\mathbf{D}_n$ $n \geq 4$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad n=4$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad n \geq 5$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^3 \quad n=4$ $\mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^2 \quad n=5$ $\mathbb{Z}_2^4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^2 \quad n=6$ $\mathbb{Z}_2^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^2 \quad n \geq 7$
$\mathbf{I}_2(p)$ $p \geq 5$	$\mathbb{Z}_2 \quad p$ odd $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad p$ even	$0 \quad p$ odd $\mathbb{Z}_2 \quad p$ even	$\mathbb{Z}_2 \oplus \mathbb{Z}_p \quad p$ odd $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p \quad p$ even
$\mathbf{F}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_4$
$\mathbf{H}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
$\mathbf{H}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$
$\mathbf{E}_6$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$
$\mathbf{E}_7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$
$\mathbf{E}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$

Clarifications for Theorem B

The following definition is an alternative to [Boy20, Definition 1.3]. It is more precise than the definition in the paper, where some of the diagram definitions are not coherent to those unfamiliar with the proof.

Definition. Let $\mathcal{D}_W$ be a Coxeter diagram corresponding to the Coxeter system (W, S). Write $m(A)$ for $m(s,t)$ when $A = \{s,t\}$ is a two element subset of S.

- Let $\mathcal{D}_{A_2}$ be the diagram with
  vertices: the subsets A of S with $|A| = 2$ and $m(A) = 3$
  edges: the pairs (A, A') such that $|A \cap A'| = 1$ and $m(A \cup A' - A' \cap A) = 2$.
- Let $\mathcal{D}_{\bullet\ \bullet\overset{2r}{-}\bullet}$ be the diagram with
  vertices: the subsets B of S with $|B| = 3$, such that there exists an $s \in B$ with i) $m(s,b) = 2$ for all $b \in B - \{s\}$, and ii) $m(B - \{s\})$ is even.
  edges: the pairs (B, B') such that there exist $s \in B$ and $s' \in B'$ satisfying i) and ii) with $B - \{s\} = B' - \{s'\}$ and $m(s,s')$ odd.
- Let $\mathcal{D}_{A_3}$ be the diagram with
  vertices: pairs (t, A) with $t \in S$ and $A \subseteq S - \{t\}$ with $|A| = 2$, such that $m(A) = 2$ and $m(t,a) = 3$ for all $a \in A$
  edges: pairs $((t,A),(t',A'))$ such that $t' \in A$, $t \in A'$, $|A \cup A' - \{t,t'\}| = 2$, and $m(A \cup A' - \{t,t'\}) = 2$.
- Let $\mathcal{D}^{\square}_{\bullet\bullet}$ be the CW-complex formed from the diagram $\mathcal{D}_{\bullet\bullet}$ by attaching a 2-cell to every square.

A couple of points of further clarification for those wishing to compute groups using Theorem B: in Theorem B the sum over $W(H_3) \subset W$, means the sum over the subsets of S of type H_3, as opposed to subgroups of W isomorphic to $W(H_3)$. The same holds for the sum over $W(B_3) \subset W$. And finally my convention is for ∞ to be neither even or odd.

References

[Boy20] Rachael Boyd. The low-dimensional homology of finite-rank coxeter groups. *Algebraic Geometric Topology*, 20(5):2609–2655, Nov 2020.

[Cox33] H. S. M. Coxeter. The complete enumeration of finite groups of the form $R_i^2 = (R_iR_j)^{k_{ij}} = 1$. *Journal of the London Mathematical Society*, s1-10(1):21–25, 1933.

[Ell21] G. Ellis. HAP, homological algebra programming, Version 1.30. https://gap-packages.github.io/hap, Apr 2021. Refereed GAP package.

[GAP21] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021.

[Gar21] Giles Gardam. Low-dimensional homology of finite Coxeter groups. https://doi.org/10.5281/zenodo.4758008, May 2021.