Formalizing Double-Categories

Nima Rasekh

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This talk is about what looking at categorical structures in formal (and particularly univalent) mathematics can teach us about categories. The new parts are joint work with Benedikt Ahrens, Page North and Niels van der Weide [WRAN23].

1 What is formalization of mathematics?

Formalization of mathematics puts mathematical definitions, lemmas, theorems, ... in a language a computer can understand and then check. This roughly proceeds along the following lines.

[Mathematical statements] $\rightsquigarrow$ translated to $\rightsquigarrow$ [Type theory (syntax) & Category theory (semantics)] $\rightsquigarrow$ entered into computers $\rightsquigarrow$ [Programming code e.g. Lean, Coq, ...] $\rightsquigarrow$ via proof assistants $\rightsquigarrow$ [Formally verified mathematics]

The key facts about type theory is that we have types X, terms $x : X$ and a variety of type constructors following some rules. Instead of going through all the rules explicitly let's do an example.

Words	Math	Type Theory
An associative magma is a set with an associative multiplication.	$(X, m : X \times X \to X, \forall x, y, z \in X,\ m(m(x, y), z) = m(x, m(y, z)))$	$\text{AssocMag} := \sum(X : U)(m : X \times X \to X), \prod(xyz : X), m(mxy)z = mx(myz)$
An associative magma has an identity.	$(M \in AssocMag, \exists e \in M, \forall x \in M(m(e, x) = x) \wedge (m(x, e) = x))$	$AssocMagId := \sum(M : AssocMag)(e : M), \prod(x : M), (mex = x) \times (mxe = x)$
Identities in associative magmas are unique.	$\forall M \in Assocmag, \forall e, e' \in AssocMagId\ M(e = e')$	$\prod(M : AssocMag)\ (e\ e' : AssocMagId\ M), (e = e')$

Now, as we mentioned type theories have various constructor rules, which tell us how to construct "terms" and give us a "proof", which can be verified by a computer. So, formalization can help check advanced mathematical proofs. Important examples includes the Liquid Tensor Experiment in the proof assistant Lean [BCc21] or the computation of homotopy groups of spheres in cubical Agda [LM23].

2 Identity Types

The goal of this talk is to focus on how pursuing type theory and formalization can actually help our mathematical understanding. The key fact here are identity types. Commonly type theories fall into two directions:

Identity Types

⇙ ⇘

Extensional: Identities are unique (Lean, ...)

Intensional: Identities are structure (Coq UniMath, ...)

In an extensional setting, the identity types in a type like AssocMag do not carry additional information. But in the intensional setting, things can get very interesting. In particular if we have some control over the identities of the universe.

Let's look at examples that contrasts these two perspectives.

Example 2.1. The category of sets is a model for extensional types, whereas groupoids are intensional.

Type Theory	Set Theory	Groupoid Theory
Types	Sets	Groupoids
Terms	Elements	Objects
$x = y$	$\{* \mid x = y\}$	Hom(x,y)
U	Set of all sets	Groupoid of all groupoids

Remark 2.2. There is a subtlety here. We really want to use the 2-groupoid of groupoids here instead of the 1-groupoid, as we want that the identities of groupoids correspond to the groupoid of equivalences. This property is known as *univalence*, which we need to get the desired results.

Here is a question: How does working in intensional (and even univalent) type theories effect the structure of a category?

3 Categories in Type Theories

A category is defined as usual in a type theoretical setting.

$$
\begin{aligned}
\mathcal{C}\mathrm{at} := \sum\ & (O : U) \\
& (M : O \times O \to U) \\
& (m : \prod(x\ y\ z : O), (M\ x\ y \to M\ y\ z \to M\ x\ z)) \\
& (e : \prod(x : O), (M\ x\ x)), \\
& \quad (rid : \prod(x\ y : O)(f : M\ x\ y), m\ f\ (e\ x) = f) \\
& \times (lid : \prod(x\ y : O)(g : M\ x\ y), m\ (e\ y)\ g = g) \\
& \times (assoc : \prod(x\ y\ z\ w : O)(f : M\ x\ y)(g : M\ y\ z)(h : M\ z\ w), (m\ (m\ f\ g)h = m\ f\ (m\ g\ h))) \\
& \times (homsets : \prod(x\ y : O), (isaset\ M\ x\ y)).
\end{aligned}
$$

Here I used implicit notation for m.

Let $\mathcal{C}, \mathcal{D} : \mathcal{C}\mathrm{at}$. What can we say about the type $\mathcal{C} = \mathcal{D}$? First of all unwinding it from the definition of identity types, the data of such a term $i : \mathcal{C} = \mathcal{D}$ is given by

- $i_O : O_\mathcal{C} = O_\mathcal{D}$
- $i_M\ x\ y : M\ x\ y = M\ (e_O x)\ (e_O y)$
- with compatibilities which unwind to functoriality.

So, the key is to check what it means for objects and for morphisms to be identified. Let us look how this manifests in groupoids to get a better intuition.

Example 3.1. Let $\mathcal{C}, \mathcal{D}$ be a category in a classical sense. Then we can construct a category in groupoids, where all groupoids are discrete denoted $N\mathcal{C}, N\mathcal{D}$. Then what is $N\mathcal{C} = N\mathcal{D}$? It unwinds precisely to an isomorphism of categories, as our identities of sets are precisely bijections.

Example 3.2. Let $\mathcal{C}, \mathcal{D}$ again be categories in a classical sense. Then let $N^{Rezk}\mathcal{C}$ be the category in groupoids given by:

- Objects are given by the underlying groupoid $\mathcal{C}^{\simeq}$.
- Morphisms are given by the underlying groupoid of the category of arrows $(Arr\mathcal{C})^{\simeq}$.
- The remaining structure follows from the categorical structure in $\mathcal{C}$.

Then what would it mean to have $N^{Rezk}\mathcal{C} = N^{Rezk}\mathcal{D}$? Following the argument above, it is saying:

- The underlying groupoids are equivalent.
- The underlying groupoids of arrow categories are equivalent.

This information precisely corresponds to being an equivalence of categories. Notice there is some mathematics here that was kind of known, but first explicated by Rezk [Rez01].

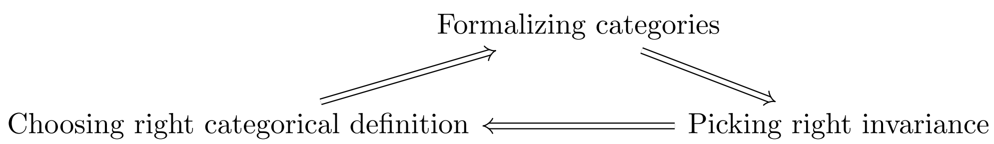
So, we can get two completely diverging mathematical behaviors from categories. What should we do? The solution is to distill the properties that resulted in this diverging behavior and add it to the definition. This was first recognized by Rezk [Rez01] under the title "completeness" and then adopted to the type theoretical setting under the name "univalence" by [AKS15].

Proposition 3.3. *Let* $\mathcal{C}at_{\mathcal{S}et} = \sum(\mathcal{C} : \mathcal{C}at), (isaset\ O_{\mathcal{C}})$. *Then identities in* $\mathcal{C}at_{\mathcal{S}et}$ *correspond to isomorphisms.*

Proposition 3.4 ([AKS15])**.** *Let* $\mathcal{C}at_{\mathcal{U}niv} = \sum(\mathcal{C} : \mathcal{C}at), (isuniv\ \mathcal{C})$. *Then identities in* $\mathcal{C}at_{\mathcal{U}niv}$ *correspond to equivalences.*

Here are some upshots:

1. Using intensional type theories creates an opening for incorporating much more diverse notions of equality. We can in one foundation both talk about isomorphisms of categories and equivalences of categories.
2. More precisely, we can choose our "universe of categories" based on what we want the category to be invariant under. Hence, if we want prove a sentence of the form "For every category ..." we choose the universe based on the invariance of the mathematical statement.
3. Achieving this relies on what I call the category theory/univalent foundations feedback loop:



4 Digression: 2-Categories

There is a similar story about 2-categories. A 2-category is a 1-category along with a set of 2-morphisms that compose associatively and unitaly.

2-categorical equivalences	corresponding 2-category
isomorphism	strict 2-category
essentially surjective local isomorphism	2-category with underlying univalent 1-category
equivalence	univalent bicategory[AFMvdW19]

Remark 4.1. Notice in the last item to get a proper notion it was not enough to make a univalence assumption on a strict 2-category. In fact, we needed to relax the 2-category structure to a bicategory, which relaxes unitality and associativity.

This is an excellent example of the above mentioned feedback loop. However, note here we could benefit from the fact that weak 2-category theory aka bicategory theory, has been developed quite extensively, going back decades (for example [Bén67]). As we shall see in double category theory the situation is far more tricky.

5 Why Double Categories?

We now want to use this insight for the study of double categories. Let us do a quick stop and recall why we care about double categories in the first place.

- Have been used to define limits of 2-categories [Gra20, cM22].
- Relevant in applied category theory (Lenses, ...) [Cla23].
- Various applications in computer science [BCV22, DM13].

So, what we want is a formalization of double categories with the following aims.

1. Formalizing result involving double categories.
2. Using the category theory/univalent foundations feedback loop to develop appropriate double categorical structures for a given double categorical equality.

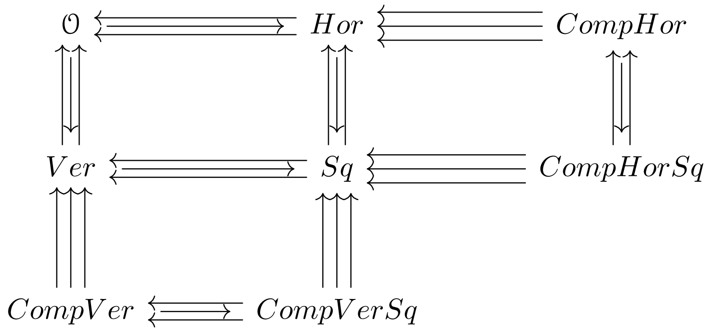
6 Defining Double Categories

Definition 6.1. A double category is a category object in categories.

More explicitly we have

- Objects
- Horizontal morphisms depending on objects
- Vertical morphisms depending on objects
- Squares depending on horizontal/vertical morphisms
- Identities for objects, horizontal and vertical morphisms
- Compositions of horizontal, vertical morphisms, squares

Pictorially we can depict this information as follows:



where the compositions are subject to appropriate notions of associativity and unitality.

For categories we witnessed two relevant notions of equality (isomorphism and equivalence). We hence need to determine relevant notions of equivalences for double categories.

1. Isomorphism: Bijection on objects, hor/ver morphisms, squares.

2. Horizontal Equivalence: Equivalence of objects & horizontal morphisms, vertical morphisms & squares.

3. Vertical Equivalence: Flip of the previous one.

4. Gregarious Equivalence: A double functor that is surjective on objects (up to a technical notion of equivalence), surjective on morphisms (up to isomorphism), fully faithful on squares.

The case for isomorphisms is clear. We similarly can assume all levels are sets to get the desired behavior. Let us focus on the case of horizontal equivalences. Here we use the previous section to deduce that the categories induced by $(\mathcal{O}, Hor)$ and (Ver, Sq) must be univalent. This implies that the category $(\mathcal{O}, Ver)$ must be defined in a weaker manner, meaning associativity and unitality must be defined weakly. This results in an asymmetrical notion of double category where the horizontal direction is strict, but the vertical direction is weak. This notion has already been studied in the literature and is known as a pseudo double category [Gra20, JY21].

This suggests a connection between identities of pseudo double categories and horizontal equivalences that we confirm in our work [WRAN23].

Theorem 6.2 (van der Weide–R.–Ahrens–North)**.** *Identities in the universe of univalent horizontal (vertical) pseudo double categories corresponds to horizontal (vertical) equivalences.*

This leaves us with the last case, which is in fact not settled and is work in progress. Here is a reasonable conjecture, building on work of Verity, who defines a very general notion of double categories, called double bicategories [Ver11], and ideas of Campbell [Cam20].

Conjecture 6.3. *For a version of Verity double bicategories, identities coincide with gregarious equivalences.*

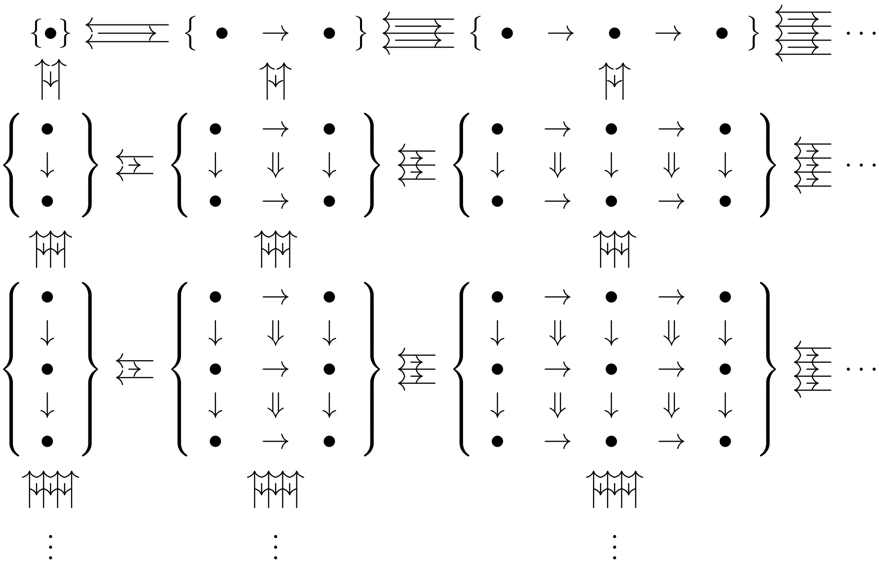
One challenge that impedes any progress is the lack of existing results, meaning we need to develop a lot of just pure category theory to be able to get to the desired result.

7 Implications for double ∞-category theory

Let us end this talk with analysis of double ∞-category theory. Let $\mathbb{\Delta}$ be the simplex category (the category of non-empty finite linear orders). A (non-complete) double ∞-category can be defined as a functor

$$X : \mathbb{\Delta}^{op} \times \mathbb{\Delta}^{op} \to s\mathcal{S}\mathrm{et}$$

that satisfies some technical conditions (namely Reedy fibrancy and the Segal condition), that make it look as follows.



In the homotopical literature, one relevant question is what a good notion of "complete double ∞-category" is (i.e. univalent double ∞-category). Various notions have been proposed, but in particular a symmetric notion of univalence has been missing from the literature, as can be seen from the following sample of results:

1. Bidirectional completeness [JFS17].
2. One directional completeness [Hau16].
3. One directional completeness + local completeness.
4. Completeness based on gregarious equivalences [Cam20].

Our work would not just provide interesting results regarding double categories. Rather it could also be directly applied to double ∞-category theory, by appropriately adopting the conditions.

8 Summary

1. Formalization translates mathematical statements into ones that can be checked by a computer via proof assistants.
2. Formalization (in particular in the univalent setting) can actually contribute to concrete theoretical developments.
3. An relevant example are double categories. Equivalences of double categories and their corresponding appropriately chosen weak notions are not studied extensively, and a univalent perspective can be a powerful guide towards appropriate definitions.
4. Beyond applications to double category theory, the results also have valuable implications for double ∞-category theory.

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