Complete Segal Objects & Univalent Maps

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1) Complete Segal Spaces (CSS)

Higher Category:

- Category with "higher" morphism
- Different models:
  - I) Simplicial Categories
  - II) Quasi-Categories
  - III) Complete Segal Spaces (CSS)

Def/ A simplicial space is a map

$X_\bullet : \Delta^{op} \to \mathcal{S}$ (spaces) which we denote by $s\mathcal{S}$.

We want it to look like category:

F(1) "walking arrow" Categorical Direction $\longrightarrow$

$\Delta[1]$ "walking isomorphism" homotopical direction $\downarrow$

$$\begin{array}{ccccccc}
X_{00} & \leftrightarrows & X_{10} & \leftrightarrows & X_{20} & \cdots \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & \\
X_{01} & \leftrightarrows & X_{11} & \leftrightarrows & X_{21} & \cdots \\
\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow & \\
X_{02} & \leftrightarrows & X_{12} & \leftrightarrows & X_{22} & \cdots \\
\vdots & & \vdots & & \vdots &
\end{array}$$

Objects Morphisms Compositions

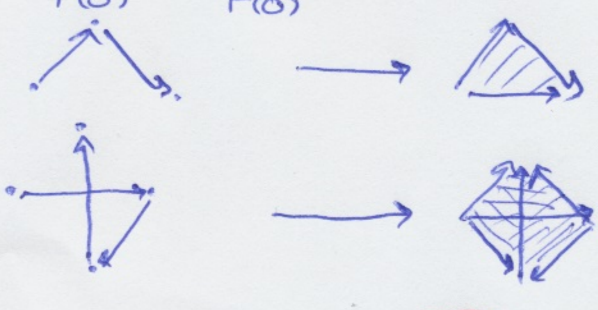
We need to add conditions to make it look the right way.

Def/ (Segal Condition) A simplicial space $X_\bullet$ satisfies the <u>Segal condition</u> if:

$$F(1) \coprod_{F(0)} \cdots \coprod_{F(0)} F(1) \longrightarrow F(n) \qquad n \geq 2$$

$n=2$

$n=3$



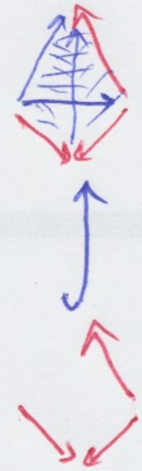
"spine of a simplex"

$$X_n \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

the simplicial map induced by the spine is an equivalence.

Def/ (Completeness Condition) A simplicial space $X_\bullet$ satisfies <u>completeness</u> <u>condition</u> if:

$$\begin{tikzcd}
X_0 \arrow[dr, "{(s_0,s_0)}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_{noequiv} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_3 \arrow[d] \\
& X_1 \arrow[r] & X_1 \overset{s\ s}{\times}_{X_0} X_1 \overset{t\ t}{\times}_{X_0} X_1
\end{tikzcd}$$



$$f \longmapsto (id_x, f, id_y)$$

the map (s_0, s_0) is an equivalence.

Def/ A simplicial space $W_\bullet$ is a <u>CSS</u> if

I) It is <u>Reedy</u> <u>Fibrant</u> (technical condition)

II) It satisfies <u>Segal</u> <u>condition</u>

III) It satisfies <u>completeness</u> <u>condition</u>

CSS behaves like a category.

It has objects, morphisms, compositions ...

Obj/ $\mathrm{Obj}\,W = \mathrm{Hom}(*, W)$

Mor/ $x, y \in \mathrm{Obj}\,W$

"map (x,y) is the following pullback"

$$\begin{tikzcd}
\mathrm{map}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_1 \arrow[d, "{(s,t)}"] \\
* \arrow[r, "{x,y}"'] & W_0 \times W_0
\end{tikzcd}$$

Comp/ $x, y, z \in \mathrm{Obj}\,W$

"μ is the composition map"

$$\begin{tikzcd}
\mathrm{map}(x,z) & \mathrm{map}(x,y,z) \arrow[l, "\mu"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right, "\cong"'] & X_2 \arrow[d] \arrow[dd, bend right, "\cong"'] \\
& * \arrow[r, "{(x,y,z)}"'] & X_0 \times X_0 \times X_0 \\
& \mathrm{map}(x,y) \times \mathrm{map}(y,z) \arrow[r] & X_1 \underset{X_0}{\times} X_1
\end{tikzcd}$$

Mantra/ None of the definitions we made relied on the fact that we are working with spaces except for the existence of <u>limits</u>.

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2) Complete Segal Objects (CSO)

$$\begin{tikzcd}
\text{Category} \arrow[r, hook] \arrow[d] & \text{Higher Category (like CSS)} \arrow[d] \\
\text{Category Objects} \arrow[r, hook] & \text{Higher Category Objects (CSO)}
\end{tikzcd}$$

Idea: Define a higher categorical object internal to a given higher category.

For the rest of the talk let W be a CSS with finite limits.

Def/ A <u>complete</u> <u>Segal</u> <u>object</u> (CSO) in W is a simplicial object $\Omega_\bullet : \Delta^{op} \to W$ such that it satisfies:

I) Segal Condition: $\Omega_n \xrightarrow{\simeq} \Omega_1 \times_{\Omega_0} \cdots \times_{\Omega_0} \Omega_1$

II) Completeness Condition: $\Omega_0 \xrightarrow{\simeq} \Omega_1 \underset{\Omega_1 \overset{s\,s}{\times}_{\Omega_0} \Omega_1 \overset{t\,t}{\times}_{\Omega_0} \Omega_1}{\times} \Omega_3$

3) Examples of CSO

Ex/ $W = \mathcal{S}$

Complete Segal "space" = "complete Segal space"

Ex/ $W = Set$

Complete Segal set = Category without non-trivial automorphism

Similar thing happens in HoTT

Ex/ $\mathcal{W} = CSS$

Complete Segal CSS = higher double category

Ex/ If $\Omega_\bullet$ is a CSO such that $s_0 : \Omega_0 \xrightarrow{\simeq} \Omega_1$, then we get a ∞-groupoid object.

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4) Representable Maps

Def/ For every object $x \in \mathcal{W}$ we have a map

$$\begin{aligned} \rho_x : \mathcal{W}^{op} &\longrightarrow \mathcal{S} \\ y &\longmapsto map(y, x) \end{aligned}$$

and a map $F : \mathcal{W}^{op} \to \mathcal{S}$ is <u>represented</u> by x if $F \simeq \rho_x$.

(This suggests following definition)

Def/ For every CSO $\Omega_\bullet : \Delta^{op} \to \mathcal{W}$ there is a map

$$\begin{aligned} \rho_{\Omega_\bullet} : \mathcal{W}^{op} &\longrightarrow CSS \\ y &\longmapsto map(y, \Omega_\bullet) \end{aligned}$$

and a map $F : \mathcal{W}^{op} \to CSS$ is <u>represented</u> by $\Omega_\bullet$ if $F \simeq \rho_{\Omega_\bullet}$

Special Case: We are interested in the case where the map

$$\begin{array}{l} W_{/-} : W^{op} \longrightarrow CSS \\ \quad x \longmapsto W_{/x} \end{array} \iff \begin{array}{c} W^{F(1)} \\ \downarrow t \\ W \end{array} \text{ Cartesian Fibration}$$

is represented by an $\underline{\text{CSO}}$ $\Omega_\bullet$.

Remark/ This statement has set-theoretical nuances which we ignore for the purpose of this talk.

Ex/ This special case happens in many situations: Let $\mathcal{X}$ be an $\underline{\infty\text{-topos}}$ (presentable ∞-category with descent). By Lurie we know there is an object Ω_0 such that for every $x \in \mathcal{X}$

$$(\mathcal{X}_{/x})^{eq} \simeq map(x, \Omega_0)$$

where left side is the maximal sub ∞-groupoid. This argument generalizes to arbitrary n:

$$(\mathcal{X}^{\Delta[n]}_{/x})^{eq} \simeq map(x, \Omega_n)$$

which gives a CSO, $\Omega_\bullet$, representing $W_{/-}$.

Motivating Questions:

This motivates the following questions:

I) Is it a locally Cartesian closed ∞-category?

II) Does it have finite colimits?

III) Does it have descent?

IV) Does representability and presentability give us an ∞-topos?

5) Univalence

Precise definition of univalence quite abstract so we give the following:

Def/ Let $\mathcal{C}$ be a presentable locally Cartesian closed ∞-category. A map $p: X \to S$ is univalent:

- If $f \simeq g \longrightarrow f^*X \simeq g^*X$ (always true)
- If $f^*X \simeq g^*X \longrightarrow f \simeq g$ (true if p univalent)

$$\begin{tikzcd} f^*X \simeq g^*X \arrow[rr] \arrow[rr, bend right] \arrow[d] & & X \arrow[d, "p"] \\ T \arrow[rr, shift left] \arrow[rr, shift right] \arrow[rr, phantom, "\simeq"] & & S \end{tikzcd}$$

Corollary/ $p: X \to S$ is univalent iff it is the pullback of a universal map along a monomorphism.

This is all based on work by Gepner-Kock.

This allows us to generalize univalent maps:

Def/ Let W be a CSS such that W_{1-} is represented by Ω_0. This gives us a universal map $p: \Sigma_0 \to \Omega_0$ classifying objects. Now a map is <u>univalent</u> if it is the monomorphic pullback of the universal map.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_0 \arrow[d, "p"] \\ S \arrow[r, tail] & \Omega_0 \end{tikzcd}$$

This definition suggest "minimal conditions" which a category needs to satisfy in order to define univalent maps, namely the existence of an <u>object classifier</u>.

This suggests an interesting generalization: <u>n-univalent maps</u>.

The intuitive definition is the following:

Def/ A chain of maps $X_n \to X_{n-1} \to \cdots \to S$ is <u>n-univalent</u>:

- $f \simeq g \to$ equivalence of chains (always true)
- equivalence of chains $\to f \simeq g$ (<u>n-univalence</u>)

$$\begin{tikzcd} f^*X_n \arrow[r, phantom, "\simeq"] \arrow[d] & g^*X_n \arrow[d] & X_n \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ f^*X_1 \arrow[r, phantom, "\simeq"] \arrow[d] & g^*X_1 \arrow[d] & X_1 \arrow[d] \\ f^*X_0 \arrow[r, phantom, "\simeq"] \arrow[dr] & g^*X_0 \arrow[d] & X_0 \arrow[d] \\ & T \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "\simeq"] & S \end{tikzcd}$$

The precise definition should be about pullbacks of n-universal map in cases where W_{1-} is classified.

Remark/ n-univalent maps are vastly more complicated.

For example:

In the category of sets there are four univalent maps:

I) $\begin{tikzcd} \varnothing \arrow[d] \\ \varnothing \end{tikzcd}$ II) $\begin{tikzcd} \varnothing \arrow[d] \\ * \end{tikzcd}$ III) $\begin{tikzcd} * \arrow[d] \\ * \end{tikzcd}$ IV) $\begin{tikzcd} * \arrow[d] \\ \{*,*\} \end{tikzcd}$

But even $\underline{2}$-$\underline{\text{univalent}}$ maps are more diverse. Among them are the following:

I) $\begin{tikzcd} * \arrow[d] \\ \{*,*\} \arrow[d] \\ * \end{tikzcd}$ II) $\begin{tikzcd} & & * \arrow[dl] \\ \{* \arrow[d] & * \arrow[dl] & *\} \arrow[d] \\ \{* & & *\} \end{tikzcd}$ III) $\begin{tikzcd} & * \arrow[d] & \\ & \{* \arrow[dl] & *\} \arrow[dl] \\ \{* & * & *\} \end{tikzcd}$

Motivating Questions:

I) How can we classify $\underline{n}$-$\underline{\text{univalent}}$ maps?

II) What does an n-univalent map tell me about the underlying category?