# Finiteness and Internal $\infty$-Categories

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I will start the talk with four sentences, the first three of which cannot be refuted, but the last one is up to debate.

1. Defining $\infty$-categories requires providing data that depends on the natural numbers.
2. Our definition of natural numbers depends on the mathematical foundation that we have chosen.
3. This creates a dependency of our notion of $\infty$-categories on our chosen mathematical foundation.
4. This dependency matters in very concrete ways!

The goal of the talk is to see how an internal notion of finiteness influences our definition of internal $\infty$-category in a way that makes an actual difference. The work in this direction is very fluent and changing and any discussion or feedback is much appreciated.

## 1 From Categories to Internal $\infty$-Categories

I will take for granted that you know what categories are and that they are used extensively. However, I do want to mention that categories can be internalized to internal categories.

**Definition 1.1.** Let $\mathcal{C}$ be a category with finite limits. An internal category is a diagram of the following form.

$$\mathcal{O} \underset{t}{\overset{s}{\leftleftarrows}} \mathcal{M} \xleftarrow{c} \mathcal{M} \times_{\mathcal{O}} \mathcal{M}$$

along with the evident equalities

- $sc = s\pi_1$, $tc = t\pi_2$.
- $c(c \times id) = c(id \times c) : \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \to \mathcal{M}$.
- $id = c(u \times id) = id = c(id \times u) : \mathcal{M} \to \mathcal{M}$.

Internal categories are relevant in many settings.

**Example 1.2.** Let $\mathcal{C}$ be a category along with a fully faithful inclusion $\mathcal{D}\text{isc} : \mathcal{S}\text{et} \to \mathcal{C}$ and a good interaction between pullbacks and coproducts. Then a category enriched over $(\mathcal{C}, \times, 1)$ is given by an internal category with discrete set of objects.

**Example 1.3.** If $\mathcal{C} = \mathcal{S}\text{et}$ we get small categories.

**Example 1.4.** If $\mathcal{C} = \mathcal{C}\text{at}$ we get double categories.

**Example 1.5.** If $\mathcal{C} = \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S}\mathrm{et})$, then a category object is a presheaf valued in categories. The same applies to a category of sheaves.

Further examples includes various existing groupoids in the literature, such as Lie groupoids, or Hopf algebroids.

On the other side we can generalize from categories to $\infty$-categories. $\infty$-categories are generalizations of categories which include homotopical data, meaning associativity and identity holds up to chosen homotopies. There are a variety of models to make such a notion precise, the most prominent ones being quasi-categories and complete Segal spaces. For now let's just take for granted that every $\infty$-category has a notion of objects, mapping spaces (or Kan complexes, or anima, ... ) and all other standard categorical notions.

**Definition 1.6.** An $\infty$-category is a category where the relevant categorical data (associativity, unitality) is given via coherent structures rather than properties.

In particular, we can define Grothendieck $\infty$-topoi, as a generalization of Grothendieck topoi, which (more or less) corresponds to a higher categorical category of sheaves. Beyond that higher category theory has proven useful all over mathematics, such as algebraic topology, derived geometry, representation theory, ... .

What we would like is to combine these two, internal categories and $\infty$-categories into one. Higher category is nice, but what we would like is again an internal version analogous to internal categories and use that in a variety of settings. However, this requires more effort, meaning we need to have a precise way to characterize the data of an $\infty$-category and then internalize that. Here we will use the notion of complete Segal spaces.

**Definition 1.7.** A complete Segal space is a simplicial space $W : \mathbb{\Delta}^{op} \to \mathcal{S}$ that satisfies two collections of finite limit conditions.

- **Segal:** For all $n \geq 2$, $W_n \to W_1 \times_{W_0} ... \times_{W_0} W_1$ is an equivalence.
- **Completeness:** The morphims $W_0 \to W_3 \times_{W_1 \times_{W_0} W_1 \times_{W_0} W_1} W_1$ is an equivalence.

Complete Segal spaces are a model of $\infty$-categories, introduced by Rezk [Rez01] and studied by many [JT07, Toë05]. While there are many other models of $\infty$-categories, the benefit of this particular model is that it can be internalized fairly easily.

**Definition 1.8.** Let $\mathcal{C}$ be an $\infty$-category with finite limits. A complete Segal object is a simplicial object $W : \mathbb{\Delta}^{op} \to \mathcal{C}$ that satisfies two collections of finite limit conditions.

- **Segal:** For all $n \geq 2$, $W_n \to W_1 \times_{W_0} ... \times_{W_0} W_1$ is an equivalence.
- **Completeness:** The morphims $W_0 \to W_3 \times_{W_1 \times_{W_0} W_1 \times_{W_0} W_1} W_1$ is an equivalence.

Complete Segal objects are also called Rezk objects [RV17] and can also be called internal $\infty$-categories, to be more "model-independent". Let's look at some examples.

**Example 1.9.** If $\mathcal{C} = \mathcal{S}$, then we get complete Segal spaces i.e. $\infty$-categories.

**Example 1.10.** If $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$ itself, then we get a (bad) notion of double $\infty$-categories, which however is a good way towards 2-fold complete Segal spaces, a prominent model of $(\infty, 2)$-categories due to Barwick [Bar05].

There is one interesting example that plays an important role in this talk. Let us make the following cool observation.

**Example 1.11.** Let $\mathcal{S}$ be the $\infty$-category of spaces and consider it as a complete Segal space. This means we have a functor $\mathcal{S} : \mathbb{\Delta}^{op} \to \hat{\mathcal{S}}$ i.e. $\mathcal{S}$ is itself a simplicial object in $\hat{\mathcal{S}}$ (up to size issues). The value of $\mathcal{S}$ is given by $\mathcal{S}_n = (\mathcal{S}^{[n]})^{\simeq}$. Moreover, the simplicial object has the cool property that for a given object $X$, we have an equivalence $\mathrm{Map}(X, \mathcal{S}) \simeq (\hat{\mathcal{S}}_{/X})^{sm}$. In other words, there is a simplicial object in $\mathcal{S}$, which classifies $\mathcal{S}$ itself!

**Definition 1.12.** Let $\mathcal{C}$ be a finitely complete $\infty$-category. Then $\mathcal{C}$ admits an *internalization* if there exists an internal $\infty$-category $\underline{\mathcal{C}}$ and an equivalence $(\mathcal{C}_{/-})^{sm} \simeq \mathrm{Map}(-, \underline{\mathcal{C}})$.

The previous example states that $\mathcal{S}$ permits an internalization. This observation generalizes to a wide range of settings and is in fact the key internal $\infty$-category we are interested in.

**Example 1.13.** Let $\mathcal{G}$ be a Grothendieck $\infty$-topos. This means $\mathcal{G}$ is a category of sheaves on a Grothendieck site (up to hypercompleteness issues) $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{C}, J)$. Then an internal $\infty$-category is a sheaf of $\infty$-categories, $\mathcal{S}\mathrm{hv}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}, J)$. We can now generalize the previous argument to get a similar internal sheaf $\underline{\mathcal{G}}$ with the universal property $\mathrm{Map}(X, \underline{\mathcal{G}}) \simeq \mathcal{G}_{/X}$. Concretely, it is given by $\underline{\mathcal{G}}(c) = \mathcal{G}_{/Y_c}$.

The way I like to think about this is that in certain $\infty$-categories (and concretely $\infty$-topoi) we can internalize an $\infty$-category in itself. Those internalizations are magic!

Internal $\infty$-categories in a category of sheaves has a well-defined category theory i.e. limits, colimits, adjunctions, left Kan extensions, ... due to Martini and Wolf [Mar21, MW21]. For example adjunctions in $\mathcal{G}$ are characterized 2-categorically given by the data $(\mathcal{C}, \mathcal{D}, F, G, c, u)$, where $\mathcal{C}, \mathcal{D}$ are internal $\infty$-categories, $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ are internal functors and $c, u$ are internal natural transformations satisfying the triangle identities. By definition $\underline{\mathcal{G}}$ has all colimits if the evident inclusion functor $\underline{\mathcal{G}} \to \underline{\mathcal{G}}^I$ has a left adjoint, where $I$ is another internal $\infty$-category. Now, we have the following result about this internal $\infty$-category.

**Theorem 1.14** (Martini, Wolf)**.** *Let $\mathcal{G}$ be a Grothendieck $\infty$-topos, then its internalization $\infty$-category $\underline{\mathcal{G}}$ is internally cocomplete and in fact the free cocompletion of the terminal internal $\infty$-category.*

What happens when we move beyond $\infty$-categories of sheaves to places where the internal logic changes in unexpected manners?

## 2 Internal $\infty$-Categories and Natural Numbers

Up to here we have exclusively focused on higher categories. How does finiteness enter the picture? Let's think somewhat more philosophically. In a given framework $\mathcal{C}$, an $\infty$-category in $\mathcal{C}$ is given by an assignment:

- For all natural numbers $n$, an object of $n$-fold compositions $X_n$.
- For all natural numbers $n$, coherent composition maps $X_n \to X_1$.

This stands in stark contrast with the definition of a 1-category, where we only need the object of 2-fold compositions and 3-fold compositions, $X_2$ and $X_3$, but no further. The definition of an internal $\infty$-category we built in a notion of "all finite natural numbers". So, what does it mean so say for all finite natural numbers? What is a natural number in the first place?

- What is a natural number? A natural number is an element in the natural numbers!
- What are the natural numbers? The answer might surprise you.

In a category, there are now a variety of ways to define natural numbers internally in a way that are equivalent to each other, known as Lawvere, Freyd and Peano natural numbers, but that all characterizes natural numbers universally (if it exists) [Joh02].

**Example 2.1.** Let $\mathcal{S}$ be the $\infty$-category of spaces. Then $\mathbb{N}$ are the natural numbers.

**Example 2.2.** More generally, let $\mathcal{C}$ be an $\infty$-category with terminal object and infinite coproducts, then $\coprod_{\mathbb{N}} 1$ is the object of natural numbers.

I want to make the case that there are unexpected examples. Before I start, notice this is one instance of a "filter quotient construction".

**Example 2.3.** For this recall the following. There exists an $\infty$-category $\prod_{\mathcal{F}} \mathcal{S}$ described as follows

- Objects are sequences of spaces $(X_n)_{n\in\mathbb{N}}$
- Morphisms are equivalence classes of the Kan complex $\coprod_{n\in\mathbb{N}} \prod_{n>N} \mathrm{Map}_{\mathcal{S}}(X_n, Y_n)$ where the equivalence relation is given by eventual equality.

This category is an $\infty$-categorical lift of an analogous 1-categorical construction $\prod_{\mathcal{F}} \mathcal{S}\mathrm{et}$, which is a well known way to get interesting topoi which play a role in non-standard analysis [AJ82, Pal97].

Now, let's come back to the original questions. What is the object of natural numbers? It is the constant diagram $(\mathbb{N})_{\mathbb{N}}$. What is a natural number? It is a morphism $(1)_{\mathbb{N}} \to (\mathbb{N})_{\mathbb{N}}$. Now, unwinding the definitions, a natural number is a sequence of natural numbers up to eventual equality.

This means there are constant sequences, which correspond to the ones we would expect, but there are also non-constant sequences which sit disjoint from the constant ones, such as the diagonal. Indeed, the following pullback is empty

$$\begin{array}{ccc} \emptyset & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ 1 & \xrightarrow{n} & \mathbb{N} \end{array}$$

So, this means our natural numbers look like

$$1, 2, 3, ..., d, ...$$

Given that, let's say I want to internalize $\prod_{\mathcal{F}} \mathcal{S}$ in itself, I now face a dilemma. When I say "for all $n$ there should be an object of $n$-fold compositions $X_n$", where is the "for all $n$" indexed over, because there are two possible reasonable choices which do not coincide. How do we proceed? If we took the classical external notion of $\mathbb{N}$, then that would simplify things and we could recycle existing results about internal $\infty$-categories and presumably use work of Martini and Wolf.

## 3 A Case for the Internal Indexing

I want to make an argument for the other side! Namely, that we should in fact take an internal view towards finiteness and hence adjust our notion of internal $\infty$-categories. Let's start the debate with a conjecture.

**Conjecture 3.1.** *Let $\mathcal{E}$ be locally Cartesian closed finitely cocomplete $\infty$-category with a "universe", meaning an object $\mathcal{U}$ in $\mathcal{E}$ with the universal property* $\mathrm{Map}(-, \mathcal{U}) \simeq (\mathcal{E}_{/-})^{\simeq}$*. Then*

1. *there is an internalization* $\underline{\mathcal{E}} : \Delta^{op} \to \mathcal{E}$ *with* $\underline{\mathcal{E}}_0 \simeq \mathcal{U}$.

2. $\underline{\mathcal{E}}$ *is internally cocomplete and in fact the free cocompletion of the terminal internal* $\infty$*-category.*

Can we prove it? The first part is possible and has in fact already been done in [Ras21] generalizing work in [GK17]. But, everything beyond the first part runs into problems, and I want to suggest that one needs to rethink the definition to fix things. That is the goal of the remainder of the talk.

For the rest of the talk I want to explicitly focus on the our main example, namely $\prod_{\mathcal{F}} \hat{\mathcal{S}}$. Before we start analyzing things, let's review some key properties:

1. It is a finitely (co)complete locally Cartesian closed $\infty$-category.

2. The $\infty$-category is given as the underlying $\infty$-category of a simplicial model structure (in the sense of Quillen).

3. The projection functor $\prod_{\mathbb{N}} \hat{\mathcal{S}} \to \prod_{\mathcal{F}} \hat{\mathcal{S}}$ preserves and reflects everything finite (limits, colimits, Cartesian closure), but rarely anything infinite.

4. Restricting to constant diagrams we get a functor $P_{\mathcal{F}} : \hat{\mathcal{S}} \to \prod_{\mathcal{F}} \hat{\mathcal{S}}$, which also preserves finite limits and colimits as well as various universal objects, such as the natural number object, which is given by $P_{\mathcal{F}}(\mathbb{N})$ and universe by $P_{\mathcal{F}}(\mathcal{S}^{\simeq})$.

I now claim that the following statements all hold simultaneously:

1. There exist a *naive internalization* $\underline{\mathcal{S}} : \Delta^{op} \to \prod_{\mathcal{F}} \mathcal{S}$, along with the universal property $(\prod_{\mathcal{F}} \mathcal{S})_{/-} \simeq \mathrm{Map}(-, \underline{\mathcal{S}})$.

2. The naive internalization $\underline{\mathcal{S}}$ is not internally cocomplete.

3. There is a more internal notion of an $\infty$-category based on the internal natural numbers and we can in particular construct a *genuine internalization* $\underline{\underline{\mathcal{S}}}$.

4. The genuine internalization $\underline{\underline{\mathcal{S}}}$ is in fact "more internally cocomplete" and the "more internal cocompletion of the point".

Let me go through the claims one by one.

**1.** The first one is easy: We take the internalization $\underline{\mathcal{S}}$ in $\hat{\mathcal{S}}$ and project it via $P_{\mathcal{F}}$ i.e. post-compose $\underline{\mathcal{S}} : \Delta^{op} \to \hat{\mathcal{S}} \to \prod_{\mathcal{F}} \hat{\mathcal{S}}$ i.e. it looks like the simplicial object $(\underline{\mathcal{S}}, \underline{\mathcal{S}}, \ldots)$. Done!

**2.** Moving on to the second claim. Here we make the following observation, If $\underline{\mathcal{S}}$ has all internal colimits, then $\prod_{\mathcal{F}} \hat{\mathcal{S}}$ has at least all geometric realizations. Why is that? Because we in particular need an "internal groupoidification morphism" $\underline{s\mathcal{S}} \to \underline{\mathcal{S}}$ in $(\prod_{\mathcal{F}} \mathcal{S})^{\Delta^{op}}$. Indeed, the colimit is described as the universal morphism filling the following diagram

$$
\begin{array}{ccccc}
\mathcal{LF}\mathrm{ib}_{/I} & \xleftarrow[\simeq]{} & [I, \underline{\mathcal{S}}] & \xrightarrow{\mathrm{colim}_I} & \underline{\mathcal{S}} \\
\big\downarrow & & & \nearrow_{(-)^{grpd}} & \\
\underline{s\mathcal{S}}_{/I} & \xrightarrow{pr} & \underline{s\mathcal{S}} & &
\end{array}
$$

Here

1. $\underline{s\mathcal{S}}$ by definition classifies simplicial objects, meaning we have an equivalence $((\prod_{\mathcal{F}} \hat{\mathcal{S}})_{/X})^{\Delta^{op}} \simeq \mathrm{Map}(X, \underline{s\mathcal{S}})$.

2. It comes with a full subcategory of "internal left fibrations".

3. There is a "Grothendieck" or "unstraightening" construction, establishing an equivalence $\mathcal{LF}\mathrm{ib}_{/I} \simeq [I, \underline{\mathcal{S}}]$.

In other words, the colimit is described as the groupoidification of the total space of the corresponding left fibration. The existence of such a functor $(-)^{grpd} : \underline{s\mathcal{S}} \to \underline{\mathcal{S}}$, representably implies the existence of the colimit $\mathrm{colim} : ((\prod_{\mathcal{F}} \hat{\mathcal{S}})_{/X})^{\mathbb{\Delta}^{op}} \to (\prod_{\mathcal{F}} \hat{\mathcal{S}})_{/X}$. Now, of course the claim is that $\prod_{\mathcal{F}} \hat{\mathcal{S}}$ does not have geometric realization!

**Example 3.2.** $\prod_{\mathcal{F}} \hat{\mathcal{S}}$ does not have a geometric realization, which means $\underline{\mathcal{S}}$ cannot have all internal colimits. Indeed, let $(S^0)^{\bullet} : \mathbb{\Delta}^{op} \to \prod_{\mathcal{F}} \hat{\mathcal{S}}$ be the simplicial object given by $((S^0)^{\bullet})_n = (S^0)^n$. Then the colimit of the simplicial diagram is given by the sequential colimit

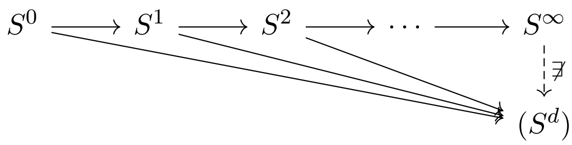
$$\mathrm{colim}_{\mathbb{\Delta}^{op}_{\leq 0}} (S^0)^{\bullet} \to \mathrm{colim}_{\mathbb{\Delta}^{op}_{\leq 1}} (S^0)^{\bullet} \to \mathrm{colim}_{\mathbb{\Delta}^{op}_{\leq 2}} (S^0)^{\bullet} \to ...$$

which we can evaluate to the sequential colimit

$$S^0 \to S^1 \to S^2 \to ....$$

Notice, this is a projectively cofibrant diagram and so with some adjustment the existence of the $\infty$-categorical colimit can be reduced to the non-existence of the strict colimit, which easily does not exist. If $C$ is a colimit, then $C_k$ is a colimit for $S^0_k \to S^1_k \to S^2_k \to ...$, which stabilizes after $k$ steps, meaning $C_k \cong (S^\infty)_k$ i.e. $C = S^\infty$.

Now, the following cocone does not factor




Now that claim two is established, we are motivated to pursue claim three.

**3.** Let's move on to claim three. Before we can proceed, we have to first say what it means to be "more internal". I will give some very general ideas and gloss over details.

**Definition 3.3.** Let $\mathcal{C}$ be a finitely complete $\infty$-category with natural number object $\mathbb{N}$. An internal category of simplices is a category object $\mathbb{\Delta}_{\mathcal{C}} : \mathbb{\Delta}^{op} \to \mathcal{C}$ with objects $\mathbb{N}$ and morphisms order preserving morphisms.

**Definition 3.4.** Let $\mathcal{C}$ be as before along with naive internalization $\underline{\mathcal{C}} : \mathbb{\Delta}^{op} \to \mathcal{C}$. A more internal simplicial object in $\underline{\mathcal{C}}$ is a functor of internal $\infty$-categories $W : \mathbb{\Delta}^{op}_{\mathcal{C}} \to \underline{\mathcal{C}}$, meaning a natural transformation of simplicial objects.

Recall that $\mathrm{Map}(X, \underline{\mathcal{C}}) \simeq \mathcal{C}_{/X}$ and all $\mathcal{C}_{/X}$ are finitely complete. So, I can representably define finite limits in $\underline{\mathcal{U}}$.

**Definition 3.5.** A more internal simplicial object $W$ is a more internal $\infty$-category if it satisfies the Segal and completeness condition.

- Segal: For all $n : \mathbb{N}$, $W_{n+1} \to W_n \times_{W_0} W_1$.
- Completeness: As before.

So, how can we get such an object in our particular example of $\prod_{\mathcal{F}} \hat{\mathcal{S}}$? This one is also reasonably straightforward, but confusing. Here we need the useful functor $(P_{\mathcal{F}})_* : \hat{\mathcal{S}}^{\mathbb{\Delta}^{op}} \to (\prod_{\mathcal{F}} \hat{\mathcal{S}})^{\mathbb{\Delta}^{op}}$.

- Thinking of $\mathbb{\Delta}$ as a complete Segal space, we can define $\underline{\mathbb{\Delta}} : \mathbb{\Delta}^{op} \to \prod_{\mathcal{F}} \hat{\mathcal{S}}$ as $(P_{\mathcal{F}})_*(\mathbb{\Delta})$.
- Recall that we have $\underline{\mathcal{S}} : \mathbb{\Delta}^{op} \to \hat{\mathcal{S}}$ and based on this we defined $\underline{\underline{\mathcal{S}}}$ as $(P_{\mathcal{F}})_*(\underline{\mathcal{S}}) : \mathbb{\Delta}^{op} \to \prod_{\mathcal{F}} \hat{\mathcal{S}}$.
- What we want is a map $\underline{\underline{\mathcal{S}}} : (P_{\mathcal{F}})_*(\mathbb{\Delta}^{op}) \to (P_{\mathcal{F}})_*(\underline{\mathcal{S}})$, which we can obtain by thinking of $\underline{\mathcal{S}}$ as a map of complete Segal spaces (i.e. natural transformation of simplicial spaces) and then applying $(P_{\mathcal{F}})_*$.

Essentially what we are doing is the following. We take the complete Segal space, thinking of it as a morphism in $\hat{\mathcal{S}}^{\mathbb{\Delta}^{op}}$

$$\underline{\mathcal{S}} : \mathbb{\Delta}^{op} \to \hat{\mathcal{S}}$$

Then we parametrize it, thinking of it as a morphism in $(\prod_{\mathbb{N}} \hat{\mathcal{S}})^{\mathbb{\Delta}^{op}}$

$$\underline{\underline{\mathcal{S}}} = (\underline{\mathcal{S}}, \underline{\mathcal{S}}, \underline{\mathcal{S}}, ...) : (\mathbb{\Delta}^{op}, \mathbb{\Delta}^{op}, \mathbb{\Delta}^{op}, ...) \to (\hat{\mathcal{S}}, \hat{\mathcal{S}}, \hat{\mathcal{S}}, ...)$$

and finally we just take its class in the filter quotient $(\prod_{\mathbb{N}} \hat{\mathcal{S}})^{\mathbb{\Delta}^{op}}$.

Notice this object looks as one would expect:

- We have $(\mathbb{\Delta}_{\mathcal{C}})_0 = \mathbb{N} = (\mathbb{N}, \mathbb{N}, \mathbb{N}, ...)$.
- We have $(\underline{\mathcal{U}})_0 = (\hat{\mathcal{S}}^{\simeq}, \hat{\mathcal{S}}^{\simeq}, \hat{\mathcal{S}}^{\simeq}, ...)$ is the "object classifier".
- The map
$$(\mathbb{N}, \mathbb{N}, \mathbb{N}, ...) \to (\hat{\mathcal{S}}^{\simeq}, \hat{\mathcal{S}}^{\simeq}, \hat{\mathcal{S}}^{\simeq}, ...)$$
takes an elements $(a_1, a_2, ...)$ to $(\mathcal{S}_{a_1}, \mathcal{S}_{a_2}, ...)$ and we have composition maps
$$(\mathcal{S}_{a_1}, \mathcal{S}_{a_2}, ...) \to (\mathcal{S}_1, \mathcal{S}_1, ...)$$
Now, the fact that we have "additional natural numbers" means we have "additional compositions".
- Notice we in fact have
$$(\mathcal{S}_{a_1+1}, \mathcal{S}_{a_2+1}, ...) \simeq (\mathcal{S}_{a_1}, \mathcal{S}_{a_2}, ...) \times_{(\mathcal{S}_0, \mathcal{S}_0, ...)} (\mathcal{S}_1, \mathcal{S}_1, ...)$$
meaning the Segal condition holds and gives us new composition maps.

**4.** Finally, we now confirm the claim that $\underline{\underline{\mathcal{S}}}$ is "more internally cocomplete". One can make a general argument, but here I want to focus on the one case we cared about.

**Example 3.6.** $\underline{\mathcal{S}}$ is cocomplete, meaning we have a left adjoint colim : $\underline{s\mathcal{S}} \to \underline{\mathcal{S}}$. This induces a level-wise left adjunction

$$(\text{colim}, \text{colim}, ...) : (\underline{s\mathcal{S}}, \underline{s\mathcal{S}}, ...) \to (\underline{\mathcal{S}}, \underline{\mathcal{S}}, ...)$$

in $\prod_{\mathbb{N}} \hat{\mathcal{S}}$, which upon taking classes gives us the desired left adjoint

$$\text{colim} : \underline{\underline{s\mathcal{S}}} \to \underline{\underline{\mathcal{S}}}$$

of genuine internalizations in $\prod_{\mathcal{F}} \hat{\mathcal{S}}$.

How does that interact with our original counter-example? In the new case we get an "internal sequential colimit"

$$S^0 \to S^1 \to S^2 \to ...,$$

which is now indexed by all sphere $S^n$ and not just some of them.

The more general case goes as follows: Fix an $\infty$-category $\mathcal{C}$ with appropriate conditions (finite limits and colimits, locally Cartesian closure) and with given internal simplices $\mathbb{\Delta}_{\mathcal{C}}$ and internalization $\underline{\mathcal{C}}$.

**Definition 3.7.** An adjunction of more internal $\infty$-categories is given as a tuple $(W, V, F, G, u, c)$, where

- $W, V : \mathbb{\Delta}^{op}_{\mathcal{C}} \to \underline{\underline{\mathcal{U}}}$ are more internal $\infty$-categories.
- Internal functors $F : W \to V$, $G : V \to W$.
- Internal natural transformations $u : W \times \Delta^1 \to W$, $c : V \times \Delta^1 \to V$.
- The triangle identities hold.

that satisfies the triangle identity.

For a more internal $\infty$-category $I : \mathbb{\Delta}^{op}_{\mathcal{C}} \to \underline{\underline{\mathcal{C}}}$, and the given more internal universe $\underline{\underline{\mathcal{C}}}$ we representably define $(\underline{\underline{\mathcal{C}}})^I$ via the Cartesian closure in $\mathcal{C}$.

**Definition 3.8.** For a given diagram $I \to \mathcal{C}$, colimits are defined as left adjoints to $\underline{\underline{\mathcal{U}}} \to \underline{\underline{\mathcal{U}}}^I$.

**Theorem 3.9.** *The more internal $\infty$-category $\underline{\underline{\mathcal{S}}}$ is internally cocomplete.*

*Proof.* Fix a more internal diagram

$$(I_1, I_2, I_3, ...) : (\mathbb{\Delta}^{op}, \mathbb{\Delta}^{op}, \mathbb{\Delta}^{op}, ...) \to (\hat{\mathcal{S}}, \hat{\mathcal{S}}, \hat{\mathcal{S}}, ...).$$

Then using the fact that $\underline{\underline{\mathcal{S}}}$ is cocomplete in the $\infty$-topos $\prod_{\mathbb{N}} \mathcal{S}$ by work of Martini and Wolf, it follows that the adjunction data exists and we can descend it down to $\prod_{\mathcal{F}} \mathcal{S}$ to get the desired result. □

## 4 Where does that leave us? More Internal $\infty$-Categories?

The implication is that unlike category theory, $\infty$-category theory is fundamentally tied to the underlying logic we are working with and in particular what it means to be finite, which is a fact that has been largely ignored up until now. This concern is not just theoretical. Indeed, what I would like is to do algebraic topology in settings other than spaces. We have already seen significant development towards algebraic topology in Grothendieck $\infty$-topoi, however, beyond that results are sparse, and what this talk demonstrates is that somewhat generalizing the definition of an $\infty$-category is necessary to make any progress.

This at least gives us a definition of an internal $\infty$-category. What is left is to further develop this definition:

1. We can define internal $\infty$-categories, but it has no worked out category theory. This part does not face any conceptual challenges, however this doesn't mean it's going to be easy.
2. We knew that we can lift a universe to an internal $\infty$-category, but this result isn't good enough anymore, so what we need is a completely new result about this and in fact it's not even clear if it holds at all.

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