Research statement

Steven Charlton

1. Introduction

The dilogarithm function—a generalisation of the logarithm—is one of the simplest special functions, and is defined by the power series

$$\mathrm{Li}_2(z) \coloneqq \sum_{k=1}^{\infty} \frac{z^k}{k^2} \,.$$

Having been first mentioned in correspondence between Bernoulli and Leibniz, this function has been actively studied by many prominent mathematicians since. Spence, Abel and Kummer discovered and rediscovered its remarkable five-term functional equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2\Big(\frac{x}{1-y}\Big) - \mathrm{Li}_2\Big(\frac{y}{1-x}\Big) + \mathrm{Li}_2\Big(\frac{xy}{(1-x)(1-y)}\Big) = -\log(1-x)\log(1-y) \,.$$

Lobachevsky showed that volumes of hyperbolic orthoschemes can be computed by the dilogarithm. Remarkably the dilogarithm also found important applications in the computation of scattering amplitudes.

My interest in these functions stems from a conjecture due to Zagier, the rough form of which is as follows.

Conjecture 1 (Zagier, [44])**.** *The value of the Dedekind zeta function $\zeta_F(n)$ of number field F, at an integer $n \geq 2$, can be expressed via (a single-valued version of) the polylogarithm Li_n.*

Beĭlinson and Deligne [1] have put this conjecture into the context of the theory of mixed Tate motives. Zagier's conjecture has since spawned a vast programme by Goncharov (see the recent Bourbaki talk [25]; therein some of my work is discussed in the context of this programme), towards understanding the structure of mixed Tate motives over a field F, in particular through understanding the properties of the *multiple polylogarithm* functions

$$\mathrm{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d) \coloneqq \sum_{1 \leq k_1 < k_2 < \cdots < k_d} \frac{x_1^{k_1} \cdots x_d^{k_d}}{k_1^{n_1} \cdots k_d^{n_d}} \,, \quad |x_i| < 1 \,.$$

Here d is called the depth, and $w = n_1 + \cdots + n_d$ the weight; these quantities give a useful benchmark for how complicated the function is. I am therefore mainly interested in the properties of these functions from the viewpoint *mixed Tate motives* over F and *Goncharov's programme* (broadly *number theory*, with connections to *algebraic* and *arithmetic geometry*). I am also interested in mixed Tate motives over rings such as $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{2}]$, whose periods essentially give the multiple zeta values (multiple polylogarithms at $x_i = 1$), or respectively the alternating multiple zeta values (multiple polylogarithms at $x_i = \pm 1$, also related to multiple t values).

As I explain more fully in §2, computational experiments have been the major instrument of all of my work. It is well-known that the (multiple) polylogarithms satisfy many increasingly-complicated functional equations, of which the five-term relation is one of the simplest; in order to conceptually understand these identities and properties, it becomes important to explicitly find them first, which requires (because of the vast scope) computer experimentation. I have developed many refined routines for computing with multiple polylogarithms and multiple zeta values, which have been useful in many projects, such as a recent project of a very computational nature [15] wherein we (myself, with Gangl, Lai, Xu, and Zhao) proved some conjectural identities of Z.-W. Sun, that were equivalent to certain explicit identities amongst special values of polylogarithms.

Although computer experiments have been the major instrument in all of my work, in what follows however I will concentrate on the conceptual meaning of my results. I will also sketch some further research project based on them, that I intend to carry out. In §2, I discuss my work ([11], and with various of Gangl, Radchenko [18] and Rudenko [19]) around *Goncharov's depth conjecture*, which predicts the existence of certain depth-reduction identities on multiple polylogarithms. In §3 I explain my work (with Gangl and Radchenko [16]) on explicit formulas for the *Grassmannian and Aomoto polylogarithms* (which play a role in the story around Zagier's conjecture) in terms of multiple polylogarithms. In §4 I explain some of my earlier work [11, 14, 10] on *multiple zeta values*, i.e. the special values $\zeta(n_1,\ldots,n_d) \coloneqq \mathrm{Li}_{n_1,\ldots,n_d}(1,\ldots,1)$ of multiple polylogarithms. In particular, I discuss the *block decomposition* which I used to unify some previously unconnected conjectures; I also indicate where and how the block decomposition has inspired work of other mathematicians. This lead to a recent project (with Keilthy [21]), where we were able to connect the *period polynomial* relations of double zeta values, to the block degree 2 part of the *motivic Lie algebra.* Finally in §5 I explain my *motivic linear independence* results [12] on *multiple t values* (an 'odd' variant of multiple zeta values), and its use in some joint work with Hoffman [20] on a *symmetry result* for multiple t values.

2. Depth reductions, and Goncharov's motivic Lie algebra

Goncharov [31, §7.3] has described a beautiful conjectural picture for understanding the structure of higher depth multiple polylogarithms. In Goncharov's picture, the multiple polylogarithms are lifted to motivic versions using their iterated integral representation

$$\begin{aligned}\mathrm{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d) &= (-1)^d I_{n_1,\ldots,n_d}\Big(\frac{1}{x_1\cdots x_d}, \frac{1}{x_2\cdots x_d}, \ldots, \frac{1}{x_d}\Big) \\ &\coloneqq (-1)^d \int_0^1 \frac{\mathrm{d}t}{t-(x_1\cdots x_d)^{-1}} \circ \underbrace{\frac{\mathrm{d}t}{t}\circ\cdots\circ\frac{\mathrm{d}t}{t}}_{n_1-1} \circ\cdots\circ \frac{\mathrm{d}t}{t-(x_d)^{-1}} \circ \underbrace{\frac{\mathrm{d}t}{t}\circ\cdots\circ\frac{\mathrm{d}t}{t}}_{n_d-1} \,.\end{aligned}$$

The motivic multiple polylogarithms then form a Hopf algebra graded by weight (recall: the weight of $\mathrm{Li}_{n_1,\ldots,n_d}$ is $n_1+\cdots+n_d$), with a coproduct Δ given by an explicit combinatorial formula (often depicted as a semi-circular polygon). For simplicity, we consider this modulo products and can obtain a Lie coalgebra $\mathcal{L}_\bullet(F)$ of multiple polylogarithms with values in some field F, generated by (formal) symbols $\mathrm{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d)$, $x_i \in F$ subject to the (implicit) functional equations of multiple polylogarithms, with a coproduct (cobracket) $\Delta\colon \mathcal{L}_\bullet(F) \to \bigwedge^2 \mathcal{L}_\bullet(F)$. The depth d of $\mathrm{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d)$ defines a filtration on $\mathcal{L}_\bullet(F)$, with associated graded $\mathrm{gr}_\bullet \mathcal{L}_\bullet(F)$; we have the following.

Conjecture 2 (Goncharov, [31], §7)**.** *A linear combination of multiple polylogarithms has depth $\leq d$ if and only if the d-th iterated truncated coproduct $\overline{\Delta}^{[d]}$ vanishes. More, $\mathrm{gr}_\bullet \mathcal{L}_\bullet(F)$ is cofree, cogenerated by depth 1 polylogarithms.*

The weight 2 and 3 part of this conjecture has been long known: it explains the surprising fact that the *a priori* more complicated 2-variable function $\mathrm{Li}_{1,1}(x,y)$ can be expressed via Li_2 alone

$$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\Big(\frac{y(x-1)}{1-y}\Big) - \mathrm{Li}_2\Big(\frac{y}{y-1}\Big) - \mathrm{Li}_2(xy)\,, \quad |xy|<1, |y|<1\,,$$

similar results are known for $\mathrm{Li}_{1,1,1}, \mathrm{Li}_{2,1}$ and $\mathrm{Li}_{1,2}$ in terms of Li_3.

In general Conjecture 2 is very difficult to investigate, because it makes no claims on the form or complexity of the depth $\leq d$ terms. It might very well happen that Conjecture 2 has a non-explicit proof, but even if that is so, finding the explicit reduction formulae would be highly desirable for applications. One has no real option then except—for a fixed combination—to experimentally test many candidate depth $\leq d$ terms, generated by a combination of intuition, insight, brute force and hope for a reduction. Importantly, one needs to match the singularities of the original combination, which forces certain factorisation properties upon $z_i - z_j$ in any potential terms $\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_j)$, however since one can (or even must) introduce spurious singularities which eventually cancel, generating such candidates is extremely open ended.

The weight 4 case was proven by Gangl [27], after many years of such computer experimentation, wherein he found a 122-term expression for $I_{3,1}(—, z)$ applied to the five-term relation for Li_2 in terms of Li_4 only. Since $\overline{\Delta} I_{3,1}(x,y) = \mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$, Gangl's result can be used to rewrite any combination with vanishing coproduct via Li_4. It was only many years later that Goncharov and Rudenko [30] were able to understand and conceptually (re)drive Gangl's result, after introducing a fundamental new type of functional equation for weight 4 multiple polylogarithms.

A surprising consequence of Conjecture 2 is that every depth d multiple polylogarithm in weight n should be expressible via a single function $\mathrm{Li}_{n-d+1,1,\ldots,1}$. In my thesis [11], I had already found indication of this in weight 5, by trying to match the coproduct of $I_{3,2}(x,y)$ directly with $I_{4,1}$ terms. As the depth 2 terms were designed to be very simple, all of the complexity of the reduction was forced into the Li_5 terms. With some routines made available by Radchenko for generating candidate Li_5 terms, and my own improvements to various 'symbol' ($\otimes$-invariant) routines, allowing much faster calculations with greater scope, I was able to show the following.

Theorem 3 (Appendix B, [11])**.** *The function $I_{3,2}(x,y)$ can be expressed via six $I_{4,1}$ terms and 141 Li_5 terms. (A typical Li_5 argument in this expression is of the form $-\frac{(1-x)(x-y)^3}{x(1-y)^2y}$.)*

I had also found, later, some expressions for $I_{4,2}$ in terms of $I_{5,1}$, but surprisingly with the arguments of $I_{4,2}$ being (x^2, y^2), and other terms involving x or y in an impossible to eliminate way. Originally only a curiosity, with Gangl, Radchenko and Rudenko [19], we were able to understand the underlying structure and generalise it. In particular, we proved the $d=2$ case of the previously mentioned surprising consequence of Conjecture 2.

Theorem 4 (Theorem 2, [19])**.** *Every depth 2 multiple polylogarithm $\mathrm{Li}_{a,b}$ can be expressed via $\mathrm{Li}_{a+b-1,1}$ with arguments being Laurent monomials in $\sqrt[N]{x}$, $\sqrt[N]{y}$, for some N.*

(This leads to a much shorter reduction for $I_{3,2}$ to $I_{4,1}$, but involves 12-th roots of unity. In weight 5, other expressions with no roots of unity follow from the geometric identities [17] we found with Gangl and Radchenko, the more general quadrangular identities [38] found independently by Matveiakin and Rudenko.)

In the same work as our depth 2 reduction, we showed the surjectivity part of Conjecture 2. Write $\mathcal{B}_n(F)$ for the subspace of $\mathcal{L}_\bullet(F)$ spanned by depth 1 polylogarithms $\mathrm{Li}_n(x)$, then we have the following.

Theorem 5 (Theorem 5, [19])**.** *If F is quadratically closed, then the following map is surjective,*

$$\overline{\Delta}^{[d-1]} \colon \operatorname{gr}_d \mathcal{L}_\bullet(F) \to \operatorname{coLie}\Big(\bigoplus_{n\geq 2} \mathcal{B}_n(F)\Big) .$$

Strikingly (as pointed out to us by Goncharov), this result is enough to iteratively reduce Conjecture 2 to the case $d = 1$, as we showed in Corollary 6 [19].

Project. We want to generalise Theorem 4 to higher depth. We still expect some expression involving only Laurent monomials; as a starting point we have some depth 3 reductions to $\mathrm{Li}_{a,b,1}$. These reductions in general seem to be related to the geometry of modular varieties [32].

In another direction, Gangl, Radchenko and I investigated the implications of Conjecture 2 for so-called Nielsen polylogarithms $S_{n,p}(x) \coloneqq \mathrm{Li}_{\overbrace{1,\ldots,1}^{p-1},n+1}(1,\ldots,1,x)$. These functions represent somehow the simplest higher depth polylogarithms (appearing non-trivially first in weight 5), and provide a good starting point for investigating Goncharov's depth conjecture.

We showed the Nielsen polylogarithm $S_{3,2}(x)$, which has coproduct $\mathrm{Li}_2(x) \wedge \zeta(3)$, satisfies the five-term relation modulo explicit Li_5 terms, giving a one-variable weight 5 analogue of Gangl's $I_{3,1}$ reduction. The following is an equivalent, symmetrised form.

Theorem 6 (Theorem 16, [18])**.** *The Nielsen polylogarithm $S_{3,2}$ satisfies the following identity*

$$\begin{aligned} &\mathrm{Alt}_5\, 11 S_{3,2}(\mathrm{cr}(x_1,x_2,x_3,x_4)) = \\ &\quad \mathrm{Alt}_5 \Big(15\, \mathrm{Li}_5(\mathrm{r}_1(x_1,\ldots,x_5)) - 9\, \mathrm{Li}_5(\mathrm{r}_2(x_1,\ldots,x_5)) + \mathrm{Li}_5(\mathrm{r}_3(x_1,\ldots,x_5))\Big) \quad (\text{mod products}) , \end{aligned}$$

where $\mathrm{cr}(a,b,c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$ is the classical cross-ratio, and $\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3$ are certain explicit 'higher-ratios', the most complicated of which is $\mathrm{r}_3(x_1,\ldots,x_5) \coloneqq -\frac{(x_1-x_2)^3(x_1-x_5)(x_3-x_4)^2(x_3-x_5)}{(x_1-x_3)^3(x_1-x_4)(x_2-x_4)(x_2-x_5)^2}$.

We also investigated weight 6, where the function $S_{4,2}(x)$, whose coproduct is $\mathrm{Li}_3(x) \wedge \zeta(3)$, should satisfy trilogarithm identities modulo explicit Li_6 terms. So far, anything beyond the three-term relation and a certain 1-variable infinite algebraic family (already given by me in [11, Propositions 7.6.10, and 7.6.12]) seems to be out of reach. Nevertheless, we could directly show some non-trivial consequences of "$S_{4,2}(x)$ satisfies trilogarithm identities" do indeed hold. We gave a evaluations in terms of Li_6 for $S_{4,2}(-1)$ and $S_{4,2}(\phi^{-2})$, $\phi = \frac{1+\sqrt{5}}{2}$, which would otherwise follow from the three-term reduction and a hypothetical (so-called) duplication reduction.

Theorem 7 (§7.3, [18])**.** *The special value $S_{4,2}(-1) = \mathrm{Li}_{1,5}(1,-1)$ has the following expression,*

$$S_{4,2}(-1) = \tfrac{1}{13}\big(\tfrac{1}{3}\,\mathrm{Li}_6(-\tfrac{1}{8}) - 162\,\mathrm{Li}_6(-\tfrac{1}{2}) - 126\,\mathrm{Li}_6(\tfrac{1}{2})\big) \quad (\text{mod products}) ,$$

where the products are of the form $\zeta(n)^i \log(2)^j$.

Whilst we could understand and derive *most* of the structure of this evaluation via Brown's extension [6, 7, 8] of the motivic framework, to fix the coefficient of $\zeta(6)$ we needed a numerical identity. This entailed finding a complicated expression for the related $\mathrm{Li}_{5,1}(-x,-1)$ via 9 Nielsen and 117 Li_6 terms, modulo products, lifting this to an analytic identity verifiable by differentiation, and analytically continuing to the region $x = 1$.

Together these results are some of the best evidence so far for Goncharov's depth conjecture, but also indicate how complicated the generic reductions are likely to be.

Project. Extend the identity in Theorem 6 to the 2-variable function $I_{4,1}(x,y) + I_{4,1}(x,y^{-1})$, whose coproduct is $\mathrm{Li}_2(x) \wedge \mathrm{Li}_3(y)$. This would be the key part of the depth conjecture in weight 5. We already have some progress, via simpler combinations (two-term symmetries I previously found [11, §7.4]), certain interesting degenerations, and syzygies between combinations we expect to reduce individually.

Project. Further investigations into weight 6 reductions. As a starting point, $S_{4,2}(x)$ under trilogarithm identities should reduce to depth 1. Likewise $I_{5,1}(x,y) \pm I_{5,1}(x,y^{-1})$ should reduce to depth 1 under Li_4 identities (with $+$), or Li_3 identities (with $-$). Already Matveiakin and Rudenko [38] have reduced $I_{4,1,1}(x,y,z)$ under dilogarithm identities to depth 2, modulo some still missing symmetry result which we will investigate.

3. Explicit formulas for Aomoto and Grassmannian polylogarithms

Grassmannian polylogarithms—analytic functions on configurations of $2k$ points in k space—have been studied since [28] as a type of geometric generalisation of the dilogarithm. Another geometric generalisation, expressing the dilogarithm via Chen's iterated integral construction [22] occurs in the work of Aomoto. Here a polylogarithm function $\mathcal{A}_{n-1}(L;M)$ on pairs of n-simplices L, M is defined, by integrating a differential form ω_M with log-singularities on M over the simplex Δ_L with boundary L.

Goncharov [29] has defined a Grassmannian m-logarithm $\mathrm{Gr}_m(v_1,\dots,v_{2m})$ as a the skew-symmetrisation under permutations of $v_1,\dots,v_{2m}$ of a primitive of the 1-form

$$\Omega(v_1,\dots,v_{2m}) = \mathrm{Alt}_{2m}\,\mathcal{A}_{m-1}(v_1,\dots,v_m;v_{n+1},\dots,v_{2m})\,\mathrm{d}\log\Delta(v_{m+1},\dots,v_{2m})\,.$$

In [29, Theorem 1.1], Goncharov shows that $\Omega(v_1,\dots,v_{2m})$ is closed, hence $\mathrm{Gr}_m(v_1,\dots,v_{2m})$ is well-defined. He [29, §4, Theorem 4.2] then computes the symbol ($\otimes^m$-invariant) of the Grassmannian polylogarithm to be

$$2(-1)^m(m!)^2\,\mathrm{Alt}_{2m}\,\Delta(v_1,\dots,v_m)\otimes\Delta(v_2,\dots,v_{m+1})\otimes\cdots\otimes\Delta(v_m,\dots,v_{2m-1})\,,$$

and shows [29, Theorem 4.3] that this symbol is integrable (i.e. that it should be expressible via a Chen iterated integral).

On a rational variety one can—in principle—integrate any integrable symbol to multiple polylogarithm function, however an algorithmic approach is so far only known when the symbol alphabet satisfies certain conditions (c.f. 'linear reducibility' in [9, §3],[24, §8.5]). Even if an algorithmic approach is possible, one would have little control over the structure of the final result; symmetries present in the original symbol might be neglected, or unnecessarily complicated combinations of high depth functions with many product terms are easily generated. The question of integrating the symbol of $\mathrm{Gr}_m(v_1,\dots,v_{2m})$, and expressing the Grassmannian polylogarithm via classical iterated integrals $I(a;x_1,\dots,x_n;b)$ is decidedly non-trivial. The (implicit) expression of $\mathrm{Gr}_4(v_1,\dots,v_8)$ via weight 4 iterated integrals (i.e. ultimately $I_{3,1}$ and I_4) was an important step in the recent proof [30] of Zagier's conjecture on $\zeta_F(4)$.

With Gangl and Radchenko [16], we were able to find a surprisingly simple expression for both the Grassmannian polylogarithm, and the Aomoto polylogarithm, of generic configurations. Each of them is expressed as a single-term under their natural symmetry. First, it is convenient to introduce some notation for certain ratios of Plücker coordinates (determinants of collections of m points), namely define

$$\rho_i \coloneqq \frac{\Delta(v_i,v_{i+1},\dots,v_{i+m-2},v_1)}{\Delta(v_i,v_{i+1},\dots,v_{i+m-2},v_{2m})}$$

Then we showed the following.

Theorem 8 (Theorems 4, and 5, [16])**.** *i) The symbol of the Grassmannian polylogarithm* $\mathrm{Gr}(v_1,\dots,v_{2m})$ *is equal to the symbol of*

$$(-1)^m\frac{m!(m-1)!}{2m-1}\,\mathrm{Alt}_{2m}\,I(0;0,\rho_{m+1},\rho_m,\dots,\rho_3;\rho_2)\,.$$

ii) The symbol of the Aomoto polylogarithm $\mathcal{A}_{m-1}(v_1,\dots,v_m;v_{m+1},\dots,v_{2m})$ *is equal to the symbol of*

$$\frac{(-1)^{m-1}}{m^2}\,\mathrm{Alt}_{m,m}\,I(0;\rho_{m+1},\rho_m,\dots,\rho_3,\rho_2)\,.$$

These results, particularly our formula for the Grassmannian polylogarithm, directly inspired Matveiakin and Rudenko [38] to define the notion of a cluster Grassmannian polylogarithm, and investigate its properties. In particular, as more conceptual construction that the quadrangular polylogarithms introduced earlier [40]. Already they have used these functions to construct a part of a bi-Grassmannian n-logarithm cocycle, whose existence was conjectured by Goncharov [29, Conjecture 4.4]. They have also made some significant progress towards the depth 3 part of Goncharov's depth conjecture in weight 6 [38, Theorem 1.4].

For the Aomoto polylogarithm, we even obtain a refined expression, by needing to symmetrise over only $\mathrm{Alt}_{m-1,m-1}$, i.e. permuting $v_2,\dots,v_m$ and permuting $v_{m+1},\dots,v_{2m-1}$. This has led us to introduce the following weight n function defined on configurations of $2m$ vectors in $\mathbb{C}^m$,

$$\mathcal{A}_{n,m}(v_1,\dots,v_m;v_{m+1},\dots,v_{2m}) \coloneqq \mathrm{Alt}_{m-1,m-1}\,I(0;\underbrace{0,\dots,0}_{n-m+1},\rho_{m+1},\dots,\rho_3;\rho_2)$$

which simultaneously generalises the Grassmannian and the Aomoto polylogarithm. More precisely, we have Gr_m is a rational multiple of $\mathrm{Alt}_{2m}\,\mathcal{A}_{m,m}$ and $\mathcal{A}_{m-1}$ is $(-1)^{m-1}$ times $\mathcal{A}_{m-1,m}$. We calculated the symbol of this function, and showed it is of a particularly simple type, consisting only of Plücker coordinates.

Theorem 9 (Remark 8, [16])**.** *The symbol of* $\mathcal{A}_{n,m}(v_1,\dots,v_m;v_{m+1},\dots,v_{2m})$ *is Plücker.*

4. The block decomposition of multiple zeta values

Multiple zeta values (MZV's) are the special values of multiple polylogarithms at $x_i = 1$, namely

$$\zeta(n_1,\dots,n_d) \coloneqq \mathrm{Li}_{n_1,\dots,n_d}(1,\dots,1) = \sum_{1\le k_1<k_2<\cdots<k_d}\frac{1}{k_1^{n_1}\cdots k_d^{n_d}}\,.$$

They arise as periods of mixed Tate motives over $\mathbb{Z}$, and have a surprisingly intricate structure all of their own, with many disparate identities (proven or conjectural), and surprising connections to other fields.

I introduced [11, 14] the block decomposition $\mathrm{bl}(k_1, \ldots, k_d)$ of a multiple zeta value $\zeta(k_1, \ldots, k_d)$, in order to understand, unify and generalise a number of disparate conjectural MZV identities. Write $w = yx^{k_1-1} \cdots yx^{k_d-1}$ as the word describing the Hopf algebra [35] or iterated integral representation [31] of the MZV (with $x \leftrightarrow 0, y \leftrightarrow 1$). Then factor xwy (where x and y correspond to the bounds of the integral) into alternating words of maximal length, by deconcatenating at a repeated letter xx or yy, and record the lengths of these words,

$$\begin{aligned}(2,3,4,5) \rightsquigarrow xwy &= \overbrace{xyxyx}^{5} \mid xyx \mid x \mid xyx \mid x \mid x \mid xy\,, \text{ so} \\ \mathrm{bl}(2,3,4,5) &= (5,3,1,3,1,1,2)\end{aligned}$$

From the block decomposition, one can recover the original MZV, so this map is invertible (with some parity condition on the domain). Based on numerical and symbolic experiments, I posed the following conjecture, which generalises and unifies the Cyclic Insertion Conjecture in [3] with a conjecture by Hoffman [2, Eqns. 9.1–9.5]. These *a priori* unrelated conjectures are both shadows of the same deeper structure.

Conjecture 10 (Conjecture 6.3 [14]; Hirose and Sato Theorem 11 [33] if $b_i \geq 2$)**.** *For any block decomposition* $(b_1, \ldots, b_r)$ *with* $\mathrm{bl}^{-1}(b_1, \ldots, b_r) = (k_1, \ldots, k_d)$*, define* $\zeta_{\mathrm{bl}}(b_1, \ldots, b_r) = (-1)^d \zeta(k_1, \ldots, k_d)$*. Then*

$$\sum_{i=1}^{r} \zeta_{\mathrm{bl}}(b_i, \ldots, b_r, b_1, \ldots, b_{i-1}) = \zeta_{\mathrm{bl}}(b_1 + \cdots + b_r + 2) + \textit{explicit products}\,.$$

Moreover, if $(b_i, b_{i+1}) \neq (1,1)$*, then no products occur.*

With Brown's motivic framework [6], I was able to show a symmetrisation holds

Theorem 11 (Corollary 5.13 [14])**.** *Summing over all permutations gives*

$$\sum_{\sigma \in S_r} \zeta_{\mathrm{bl}}(b_{\sigma(1)}, \ldots, b_{\sigma(i)}) \in \zeta_{\mathrm{bl}}(b_1 + \cdots + b_r + 2)\mathbb{Q}\,.$$

This theorem is enough to show Hoffman's identity up to a rational, and improve the results of [3, 4] towards the Cyclic Insertion Conjecture, as well as to generate a number of surprisingly structured new MZV identities.

I [10] have also used the block decomposition structure to reinterpret a result of Zhao [45], by giving a direct construction of his iteratively argument string which relates multiple zeta star values (with $\leq$ instead of $<$), and so-called alternating multiple zeta-half values (with characters $(-1)^{n_i}$ weighting with $\frac{1}{2}$ when $n_i = n_{i+1}$). For example $\zeta^\star(2,3,4,5) = -2^7 \zeta^{1/2}(3,3,1,3,1,1,\overline{2})$, based on $\mathrm{bl}(2,3,4,5)$ above, where the first argument of the block decomposition must always be decreased by 2.

Already the idea of the block decomposition has spawned further work by other mathematicians in many directions. In one direction: these conjectures and this new structure inspired work by Hirose and Sato [33]. They defined a new *block shuffle* product structure on MZV's, $\sqcup\!\sqcup_{\mathrm{bl}}$ between block decompositions, and showed the beautifully simple identity $\zeta_{\mathrm{bl}}(b_1 \sqcup\!\sqcup_{\mathrm{bl}} b_2) = 0$. Moreover, they were able to show that for $b_i \geq 2$ my conjecture (hence Hoffman's conjecture, and the Cyclic Insertion Conjecture) follows from this identity.

In another direction: Brown [5] observed that the block decomposition actually gives rise to a so-called *block filtration*, which is motivic, and which coincides with the well-known coradical filtration on motivic MZV's. Therefore, the block decomposition gives a simple to calculate combinatorial description of the coradical degree. Moreover, this also gives the level filtration (i.e. number of 3's) used in Brown's proof [6] that the Hoffman elements $\zeta(k_1, \ldots, k_d)$, $k_i \in \{2,3\}$ are a motivic basis.

Finally: Keilthy [37, 36] further developed this viewpoint, by lifting Brown's filtration to $\mathbb{Q}\langle e_0, e_1\rangle$ in such a way as to retain a non-trivial associated graded. Keilthy constructed a block-graded Lie algebra $\mathfrak{bg}$ encoding relations between block-graded MZV's modulo products, and showed $\mathfrak{bg} \cong \mathrm{Lie}[\sigma_3, \sigma_5, \ldots]$ is free, and therefore (non-canonically) isomorphic to the motivic Lie algebra $\mathfrak{g}^{\mathfrak{m}}$. This is in stark contrast to the depth-graded Lie algebra $\mathfrak{dg}$, which is not free, and has quadratic relations and extra generators in depth 4, encoding the period polynomial relations of cusp forms. Keilthy also represented this Lie algebra (hence relations among block-graded MZV's) through a subspace of commutative polynomials, satisfying certain *block relations* coming from the shuffle regularisation, a dihedral symmetry, and a differential equation. Keilthy also proved a version of the above conjecture in the block-graded, i.e. modulo terms of lower block degree.

At this point, I re-enter the picture. We want to understand the motivic Lie algebra $\mathfrak{g}^{\mathfrak{m}}$, and by Keilthy's work we can do this by understanding the block graded Lie algebra $\mathfrak{bg}$, with degree 2 being the next non-trivial case. The depth filtration is a subfiltration of the block filtration, so one expects the double zeta values and their famous period polynomial relations [26] to manifest somehow in the world of block degree 2 MZV's. Keilthy and I gave an evaluation for $\zeta(\{2\}^a, 4, \{2\}^b)$ (with block decomposition $(2a+3, 1, 2b+2)$ of degree 2, and $\{2\}^a$ denoting a repetitions of $2, \ldots, 2$) via double zeta values. At the outset, it is not at all obvious such a result exists, and on the way we had to deal with a number of non-trivial steps, including the discovery of an explicit Galois descent for alternating MZV's of the form $\zeta(\overline{\mathrm{ev}}, \overline{\mathrm{ev}})$ from the plethora of MZV relations. Nevertheless, we obtained the full version the following explicit identity by analytic means, and then gave a direct motivic proof.

Theorem 12 (Lemma 4.1, and Theorem A.7, [21])**.** *The following evaluation holds*

$$\zeta(\{2\}^a, 4, \{2\}^b) \;=\; 4(-1)^n \Bigg[-\zeta(2a+2, 2b+2) - \zeta(2a+3, 2b+1) \\ + \sum_{j=1}^{2n+3} 2^{j-4-2n} \left(\binom{2n+3-j}{2b+1} - \binom{2n+3-j}{2a+1} \right) \zeta(j, 2n+4-j) \Bigg] \quad (\text{mod explicit products})\,.$$

From this we obtained [21, Corollary 1.3] a surprisingly simple expression for the double zeta values of weight $2n+4$ in terms of block degree 2

$$\zeta(2a+1, 2n-2a+3) = \frac{(-1)^n}{4} \sum_{i=a}^{n-a} \zeta(\{2\}^i, 4, \{2\}^{n-i}) \quad (\text{mod products})\,,$$

and could finally connect the period polynomial relations to the block relations. More generally we obtained the following.

Theorem 13 (Proposition 3.5, [21])**.** *All relations among even weight double zeta values modulo products arise from the block relations.*

In particular, the period polynomial relations on double zeta values are a consequence of the block relations; moreover we showed how arise in very explicit way [21, Proposition 3.6].

Project. We expect a similar story to play out in higher block degree, once we understand how to express values $\zeta(2, \ldots, 2, 2k, 2, \ldots, 2)$ via depth $2k-2$ MZV's. The explicit identity for $\zeta(2, \ldots, 2, 4, 2, \ldots, 2)$ in Theorem 12 already gives us a much better idea about the possible structure of such evaluations; we aim to a direct motivic proof of the key identities for $\zeta(2, \ldots, 2, 2k, 2, \ldots, 2)$ as it seems much less feasible to give an analytic identity in the higher cases.

5. The structure of multiple t values

Multiple t values (MtV's) are a recently (re)-introduced 'odd variant' [34] of multiple zeta values, given by summing over only odd denominators

$$t(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{(2n_1-1)^{k_1} \cdots (2n_d-1)^{k_d}}\,.$$

They are both very similar, and markedly different to MZV's. They can be expressed via alternating multiple zeta values (i.e. multiple polylogarithms at $x_i = \pm 1$), so arise as periods of mixed Tate motives over $\mathbb{Z}[\frac{1}{2}]$. The function version of these objects, so-called *alternating polylogarithms* [40, §6.5] already play an important role in the computation of volumes of orthoschemes [40, Theorem 6.22].

5.1. Motivic MtV's

Murakami [39] calculated the motivic derivations D_{2r+1} on motivic versions $t^{\mathfrak{m}}(k_1, \ldots, k_d)$ in the case when all $k_i > 1$, in order to prove the surprising result that $t^{\mathfrak{m}}(k_1, \ldots, k_d)$, $k_i \in \{2, 3\}$ form a basis for motivic multiple zeta values. I extended this computation to all motivic multiple t values; a very pleasing result of this is that D_1 acts by deconcatenation, and hence conceptually explains Hoffman's empirically observed [34, Conjecture 2.1] 'derivation with respect to log(2)'

Proposition 14 (Proposition 5.8, [12])**.** *D_1 acts by deconcatenation of leading/trailing 1's*

$$D_1 t^{\mathfrak{m}}(k_1, \ldots, k_d) = \delta_{k_1=1} 2 \log^{\mathfrak{l}}(2) \otimes t^{\mathfrak{m}}(k_2, \ldots, k_d) - \delta_{k_d=1} \log^{\mathfrak{l}}(2) \otimes t^{\mathfrak{m}}(k_1, \ldots, k_{d-1})\,.$$

I established an evaluation for the stuffle-regularised $t(2, \ldots, 2, 1, 2, \ldots, 2)$ in terms of Riemann zeta values and log(2), following ideas from Zagier [43] and Murakami [39]. In contrast with these previous results, I had to overcome some difficulties caused by *regularisation*, i.e. since $t(2, \ldots, 2, 1)$ is divergent the natural generating series $\sum_{a,b=0}^{\infty} t(\{2\}^a, 1, \{2\}^b) x^{2a} y^{2b}$ does not initially make sense, but this can be handled using the Evans-Stanton/Ramanujan asymptotic of hypergeometric series.

With the motivic derivations, I could lift this analytic evaluation to a motivic evaluation, holding for notions of shuffle regularised and stuffle regularised motivic MtV's. By understanding the arithmetic of the coefficients in my evaluation of $t^{\mathfrak{m}}(2, \ldots, 2, 1, 2, \ldots, 2)$, together with Murakami's evaluation of $t^{\mathfrak{m}}(2, \ldots, 2, 3, 2, \ldots, 2)$, I established some motivic linear independence results on motivic MtV's.

Theorem 15 (Corollaries 7.15, 8.19, and 8.26, [12])**.** *The following hold*

i) The so-called Saha elements $\{t^{\mathfrak{m}}(k_1, \ldots, k_{d-1}, k_d + 1) \mid k_i \in \{1, 2\}\}$ *are linearly independent.*

ii) The shuffle regularised Hoffman-type elements $\{t^{\mathfrak{m}}(k_1, \ldots, k_d) \mid k_i \in \{1, 2\}\}$ *are linearly independent.*

iii) The stuffle regularised Hoffman-type elements $\{t^{*,\mathfrak{m}}(k_1,\ldots,k_d) \mid k_i \in \{1,2\}\}$, *with* $t^{*,\mathfrak{m}}(1) = \frac{1}{2}\log^{\mathfrak{m}}(2)$ *or with* $t^{*,\mathfrak{m}}(1) = \log^{\mathfrak{m}}(2)$, *are linearly independent.*

Some comments on this are in order. Firstly, motivic multiple t values are contained in motivic alternating multiple zeta values, and it is know [23] that alternating motivic MZV's have dimension F_{N+1} in weight N, where $F_k = F_{k-1} + F_{k-2}$, $F_0 = F_1 = 1$ are the Fibonacci numbers. Hence for dimensional reasons the results in ii) and iii) show that the shuffle/stuffle regularised Hoffman elements $t^{(*),\mathfrak{m}}(k_1,\ldots,k_d)$, $k_i \in \{1,2\}^\times$ are a basis for both regularised MtV's and alternating MZV's, and hence regularised MtV's have dimension F_{N+1}.

Second, Saha [41] conjectured that the elements $t(k_1,\ldots,k_{d-1},k_d+1)$, $k_i \in \{1,2\}$ are a basis for the convergent multiple t values, i.e. those with last argument > 1. The result in i) above gives the (motivic) linear independence part of this conjecture; at the moment it is not clear how to show Saha's elements actually span the space of convergent MtV's. We do not have a good enough upper bound yet (i.e. F_N) on the dimension of convergent MtV's; the upper bounds in the case of MZV's and alternating MZV's were obtained by some very deep arithmetic geometry [42], [23], that would be highly non-trivial to extend.

Project. It is an on-going idea with A. Keilthy to utilise the deconcatenation property of D_1 to try to characterise motivically the convergent MtV's, in order to obtain the necessary upper bound by more elementary means. By tensoring with $\mathbb{F}_2$, the kernel of $D_1 \otimes \mathbb{F}_2$ contains the convergent MtV's. We therefore expect that by understanding the relations "modulo 2", we can make progress towards the dimension of convergent MtV's.

Project. I have already given an evaluation [13] for the alternating MtV $t(\overline{1},\ldots,\overline{1},1,\overline{1},\ldots,\overline{1})$, which should be the key identity for showing a similar motivic linear independence of the alternating MtV's $t^{\mathfrak{m}}(k_1,\ldots,k_d)$, $k_i \in \{1,\overline{1}\}$. However, we need to regularise with $t(1) = \frac{1}{2}\log(2)$ to get a basis, as with $t(1) = \log(2)$ already $t(\overline{1},1) = t(1,\overline{1})$. Since alternating MtV's are $\mathbb{Q}(\mathrm{i})$-linear combinations of level $N = 4$ MZV's, much of the motivic framework must be reworked to allow $\mathbb{Q}(\mathrm{i})$-coefficients, before being able to tackle linear independence.

5.2. Symmetries of MtV's

Hoffman conjectured that MtV's satisfy a extra symmetry unlike MZV's, namely

$$t(k_1,\ldots,k_d) + (-1)^{k_1+\cdots+k_d} t(k_d,\ldots,k_1) \equiv 0 \quad (\text{mod products of } t\text{'s})\,.$$

A similar identity for MZV's typically involves many lower depth irreducible terms, which seemingly and mysterious are not present in the MtV identity.

With Hoffman, [20], we found a generalisation and proof of this. The first main ingredient is a(n underappreciated) proof Goncharov's gave of a multiple polylogarithm inversion theorem [31], which could be modified to multiple t values. However this identity only allowed us to give an explicit generating series identity expressing the alternating MtV's (k_i with or without bars, denoting an extra sign $(-1)^{n_i}$ in numerator of an MtV) via products of alternating t's and products of alternating ζ's:

$$t(\overset{(-)}{k_1},\ldots,\overset{(-)}{k_d}) + (-1)^{k_1+\cdots+k_d}(-1)^{\#\{\text{bars}\}} t(\overset{(-)}{k_d},\ldots,\overset{(-)}{k_1}) = \text{products of alternating } t\text{'s and } \zeta\text{'s.}$$

Unexpectedly, the motivic results from Murakami [39] and my motivic results in Theorem 15 above come to the rescue. We can always re-write alternating MZV's via multiple t values of the form $t(k_1,\ldots,k_d)$, $k_i \in \{1,2\}$, and so all products of MZV's are actually products of MtV's. Hence Hoffman's form of the symmetry conjecture holds and we have the following.

Theorem 16 (Theorem 2.21, Corollary 2.25, [20]). *The following symmetry result holds for alternating multiple t values, for any choice of barred entries*

$$t(\overset{(-)}{k_1},\ldots,\overset{(-)}{k_d}) + (-1)^{k_1+\cdots+k_d}(-1)^{\#\{bars\}} t(\overset{(-)}{k_d},\ldots,\overset{(-)}{k_1}) \equiv 0 \quad (\text{mod products of (alternating) } t\text{'s})$$

It is rather surprising and disconcerting that one needs to appeal to deep motivic basis results in order to convert MZV's to MtV's for this theorem. Our proof actually works with any root of unity (i.e. MtV's of level N), modulo products of MtV's level N and products of level N MZV's. Presumably the decomposability actually always holds modulo products of level N MtV's only, but how can one show this without appealing to a motivic basis result to rewrite level N MZV's via MtV's?

Project. It would be interesting to find a more direct proof of this symmetry result, without appealing to an abstract motivic result to rewrite zeta values via t values. This might be a reflection of some distinguished new structure in the Hopf algebra of multiple t values. In the opposite direction, at least, Hoffman and I are investigating a conjectural identity relating MtV's to certain interpolated zeta-half values; we have some partial results. If proven in general we would already obtain an explicit expression for MtV's in terms of MZV's.

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