GROUP THEORY AND TWISTY PUZZLES

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1. Introduction

Motivation: Pretty much everyone has heard of the Rubik's cube. Most people have owned/played with one, and probably have gotten frustrated at being unable to solve the thing. The main goal of this talk is to see some of the mathematics behind this puzzle, and to convince you that you can all figure out your own solution with some knowledge of group theory.

Types of puzzles: A rough classification that twisty puzzlers have developed breaks down puzzles into a few classes

- Doctrine puzzles: puzzles like the Rubik's cube where every possible move is always available, and after each move the puzzle has the same shape
- Bandaged puzzles: puzzles which have some constraints on the allowed moves at various stages. For example by gluing pieces together like in the Siamese cube, or by adding constraining centres like in the 90° cube, or gears like in the gear cube.
- Jumbling puzzles: a subset of bandaged puzzles, these are puzzles which can't be unbandaged into (shapemods) of doctrine puzzles without creating infinitely many pieces. For example the helicopter cube, and Bermuda cubes jumble.

For this talk, I'll focus mainly on doctrine puzzle, and more specifically the Rubik's cube.

2. Rubik's cube as a group

View the moves U, D, L, R, F, B as permutations of the 54 stickers. This shows the Rubik's cube is a subgroup of S_{54}. Indeed S_{48} by ignoring the centre stickers.

3. Commutators and building algorithms

The main tool for this is the following theorem. The idea is that if two moves overlap by a small amount, the commutator only has a small result.

Theorem 1. *Let $\sigma, \tau \in \operatorname{Sym}(X)$. Let $\operatorname{supp}(\sigma)$ be those elements permuted by σ. If*

$$\#(\operatorname{supp}(\sigma) \cap \operatorname{supp}(\tau)) = 1 \,,$$

then $\sigma\tau\sigma^{-1}\tau^{-1}$ is a 3-cycle.

Proof. Write τ in disjoint cycle notation. If $\operatorname{supp}(\sigma) \cap \operatorname{supp}(\tau)$ has size 1, then only one of the cycles in τ interacts with σ. The others are disjoint. Since disjoint cycles commute, we can assume $\tau = (\tau_1 \cdots \tau_n)$ is an n-cycle.

We know that $\sigma\tau\sigma^{-1}$ conjugates τ by σ in $\operatorname{Sym}(X)$, and gives the following

$$\sigma\tau\sigma^{-1} = (\sigma(\tau_1) \cdots \sigma(\tau_n)) \,.$$

Then suppose $\tau_1 \in \operatorname{supp}(\sigma) \cap \operatorname{supp}(\tau)$. (Can shift around τ cycle until this holds.) This results in

$$\sigma\tau\sigma^{-1} = (\sigma(\tau_1)\tau_2 \cdots \tau_n) \,.$$

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Finally we compute

$$\sigma\tau\sigma^{-1}\tau^{-1} = (\sigma(\tau_1)\tau_2\cdots\tau_n)(\tau_n\tau_{n-1}\cdots\tau_1) = (\tau_1\sigma(\tau_1)\tau_2)\,,$$

since $\sigma(\tau_1) \notin \mathrm{supp}(\tau)$ (otherwise $\sigma(\tau) \in \mathrm{supp}(\sigma) \cap \mathrm{supp}(\tau)$, which we assumed contains only one element). □

So if we can find sequences of moves which intersect in only one element, commutating gives us 3-cycles. Here are some examples.

4. Impossible states

We know how to make small changes to the cube, but are these moves enough? Will we ever need to swap *two* edges, or rotate *one* corner?

Suppose someone mischievous (like my office mate…) comes along, see my cube and decides to scramble it and play tricks on me. While scrambling it, he pops out one of the edges, and puts it back in the wrong way. When I come along and try to solve it, I can get as far as this but can't seem to finish solving it. What do I do…

Well… at this point I realise what he's done, and just pop the edge out and replace it correctly. I know this is an impossible state to reach. But how?

Proposition 2. *The following state is impossible on a Rubik's cube. (One flipped edge.)*

Proof. As a permutation of the *edge* stickers, every generating permutation is even, but this permutation is odd. Therefore it can't be reached. □

Proposition 3. *The following state is impossible on a Rubik's cube. (Two swapped edges.)*

Proof. Every generating move can be viewed as a permutation of the corners pieces, and a permutation of the edges pieces. The parity of corners and edges is always equal. If the corners are in the solved state, then the edges must be in an even permutation. But two swapped edges is odd. Therefore this can't be reached. □

Proposition 4. *The following state is impossible on a Rubik's cube. (One rotated corner.)*

Proof. This is more delicate. A parity argument won't work because there are three possible states (0, 1/3, 2/3 of a turn) rather than just two.

Add the following reference markings to the cube. On the white (U) face and the yellow (D) face. Now compute the following total

$$\text{total twist} = \sum \text{number of steps around a white or yellow sticker is} \pmod 3$$

Then we check that total twist is invariant. Under D or U rotations this is clear, the references markings don't change. For F, B, R, L (And wlog F) the reference markings are $-1, 1, -1, 1$ out of place. So the total changes by 0.

Since total twist is 0 for the solved cube, and ± 1 for the cube with a rotated corner, we see those states are impossible to reach. □

Proposition 5. *The following state is impossible on a super-Rubik's cube. (One centre rotated by* 90°*.)*

Proof. Add four spots to each centre (at the edges, say). Then each generating rotation is a 4-cycle of these and a 4-cycle of the corners (or edges). If the corners are in the solved state they have even parity. So the centres must have even parity. But a 90° rotated centre is an odd permutation. So it can't be obtained. □

5. M12

We compute $f_1 = (2,3,4,5,6)$ and $f_2 = (1,6,7,8,3)$. So $f_2^{-1} = (1,3,8,7,6)$, and conclude

$$f_1 f_2^{-1} = (1,4,5,6)(2,3,8,7) \in M_{12}$$

References

[Joy08] David Joyner. *Adventures in group theory: Rubik's Cube, Merlin's machine, and other mathematical toys*. JHU Press, 2008.

[Joy96] David Joyner. "Mathematics of the Rubiks cube". Spring semester 1996–7. 1996. URL: `http://www.permutationpuzzles.org/rubik/webnotes/rubik.pdf`.