Twisty puzzles and group theory

Steven Charlton

15 December 2015

Outline

1 Introduction

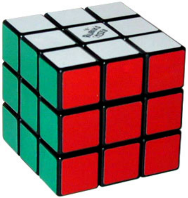
2 Rubik's Cube

3 Commutators and algorithms

4 Impossible positions

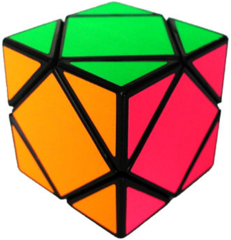
5 Extra stuff

Classification - Turning



Face turning
(Rubik's cube)

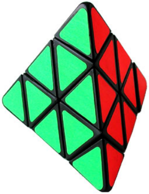
Edge turning
(Helicopter cube)



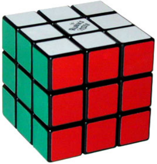
Vertex turning
(Skewb)

Pictures are from `twistypuzzles.com` museum
Each picture links to the puzzle's entry in the museum

Classification - Shape



Tetrahedron (Pyraminx)



(Rubik's) Cube

(Face Turning) Octahedron

Doecahedron (Megaminx)

Icosahedron (Icosaix)

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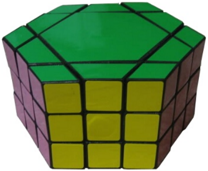
Classification - Shape



(Hexagonal) Prism

30 Sided (Dayan Gem VI)

Dipyramid (Jumblix)

Huh? (Septic Twist)

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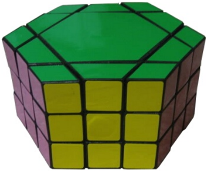
Classification - Shape



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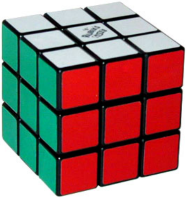
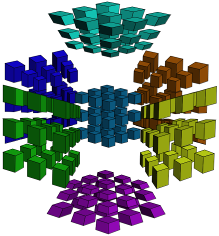
Classification - Dimension

2D
(Geranium)



3D
(Rubik's cube)



4D
(Rubik's Tesseract?)

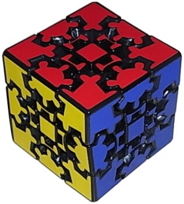
'2D' and '3D' pictures from `twistypuzzles.com` museum, pictures link to the puzzle's entry in the museum
'4D' picture based on screenshot from Magic Cube 4D program available at `http://superliminal.com/cube/cube.htm`

Classification - Bandaging

Bandaged (Fuse Cube)

Constrained Cube

Latch Cube



Gear Cube

Circles
(Crazy Tetrahedron)

Jumbling
(Curvy Copter)

'Jumbling' picture is my own
Other pictures from `twistypuzzles.com` museum
Each picture links to the puzzle's entry in the museum

Jumbling

Jumbles
(Curvy Copter)

Jumbled

Unbandage once
(Curvy Copter Plus)

- Infinitely many cuts to fully unbandage

'Jumbles' and 'Jumbled' pictures are my own
'Unbandaged once' picture from twistypuzzles.com museum
Each picture links to the puzzle's entry in the museum

Rubik's Cube - Structure

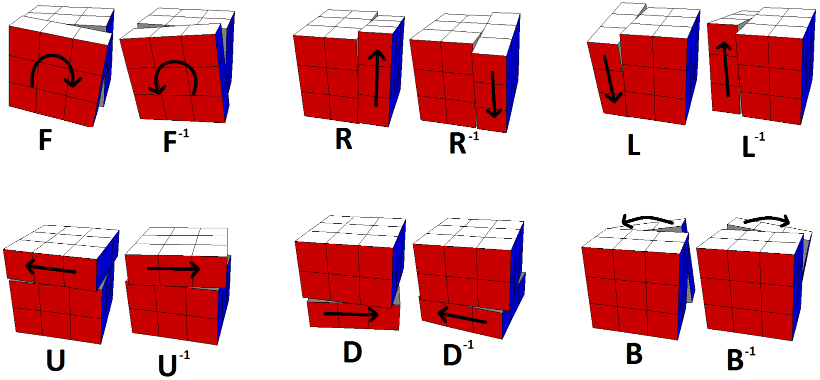
- 1 core
- 8 edge pieces
- 12 corner pieces
- 54 stickers

Badly made/drawn cubes

- Many more at `http://badlydrawnrubikscubes.tumblr.com`

Pictures link to posts at `http://badlydrawnrubikscubes.tumblr.com`

Moves



- Turn clockwise when looking at it

Pictures based on screenshots from pCubes program available at
`http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054`

As a group

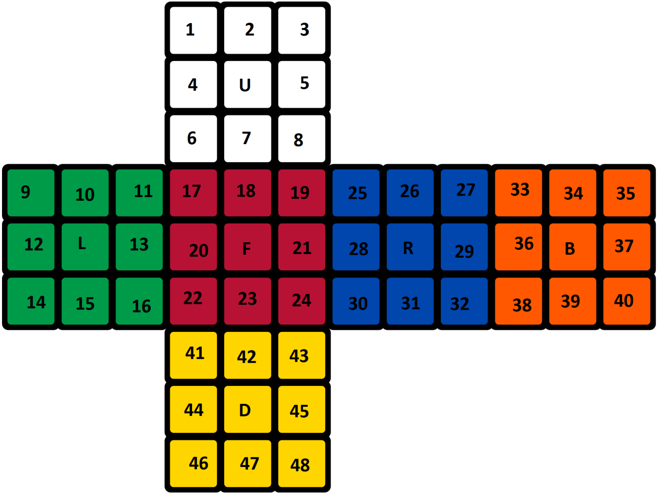
Operation: compose sequences of moves

- Identity - do nothing move
- Inverses - undo a move
- Associativity - do each sequence in turn

$$(A_1 \cdots A_k \circ B_1 \cdots B_\ell) \circ C_1 \cdots C_m = \\ A_1 \cdots A_k B_1 \cdots B_\ell C_1 \cdots C_m = \\ A_1 \cdots A_k \circ (B_1 \cdots B_\ell \circ C_1 \cdots C_m)$$

As a group (again)

- Each move *permutes* the stickers



Picture based on `https://commons.wikimedia.org/wiki/File:Rubik%E2%80%99s_cube_colors.svg` by NI74

Rubik's cube - Permutations

$$U = (1\ 3\ 8\ 6)(2\ 5\ 7\ 4)(9\ 33\ 25\ 17)(10\ 34\ 26\ 18)(11\ 35\ 27\ 19)$$

$$L = (9\ 11\ 16\ 14)(10\ 13\ 15\ 12)(1\ 17\ 41\ 40)(4\ 20\ 44\ 37)(6\ 22\ 46\ 35)$$

$$\vdots$$

- Rubik's cube is a subgroup of S_{48}
- Generated by U, L, F, R, B, D

Commutators and Algorithms

Theorem

Let $\sigma, \tau \in S_X$. Let $\mathrm{supp}(\sigma)$ be those elements permuted by σ. If

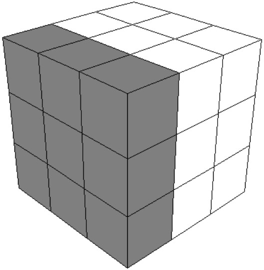
$$\#(\mathrm{supp}(\sigma) \cap \mathrm{supp}(\tau)) = 1\,,$$

then $\sigma\tau\sigma^{-1}\tau^{-1} = [\sigma, \tau]$ is a 3-cycle.

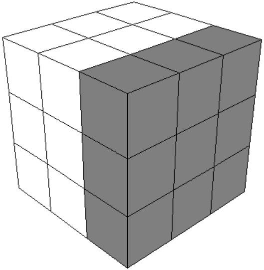
- Can view X as pieces (rather than stickers)
- Can look at edges, corners, . . . , separately

Cycle edges

- Look at $\sigma = F$ and $\tau = R^{-1}$



F



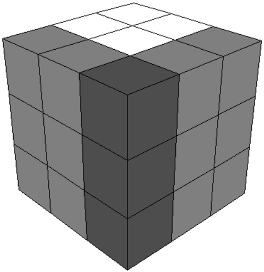
R

- Expect $[F, R^{-1}] = FR^{-1}F^{-1}R$ is a 3-cycle of edges (ignore corners)

Pictures based on screenshots from pCubes program available at
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054

Cycle edges

- Look at $\sigma = F$ and $\tau = R^{-1}$



Overlap of F and R

- Expect $[F, R^{-1}] = FR^{-1}F^{-1}R$ is a 3-cycle of edges (ignore corners)

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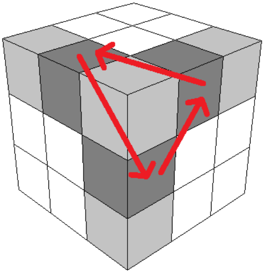
Cycle edges

- Look at $\sigma = F$ and $\tau = R^{-1}$



3 cycle of edges

- Expect $[F, R^{-1}] = FR^{-1}F^{-1}R$ is a 3-cycle of edges (ignore corners)

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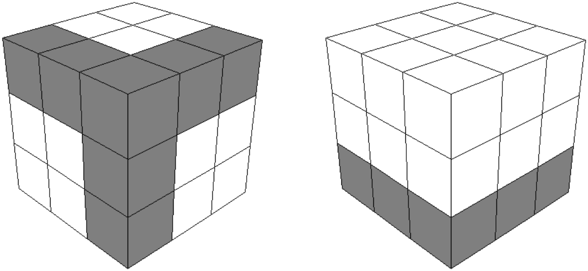
Cycle corners

- Look at $\sigma = [F, R^{-1}]$, $\tau = D$



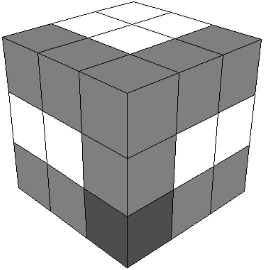
$[F, R^{-1}]$ D

- Expect $[[F, R^{-1}], D] = (FR^{-1}F^{-1}R)D$ $(R^{-1}FRF^{-1})D^{-1}$ is a 3-cycle of corners

Pictures based on screenshots from pCubes program available at
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054

Cycle corners

- Look at $\sigma = [F, R^{-1}]$, $\tau = D$



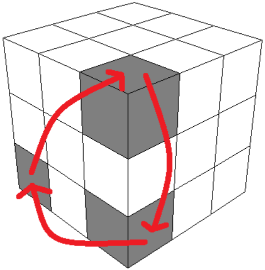
Overlap of $[F, R^{-1}]$ and D

- Expect $[[F, R^{-1}], D] = (FR^{-1}F^{-1}R)D(R^{-1}FRF^{-1})D^{-1}$ is a 3-cycle of corners

Pictures based on screenshots from pCubes program available at
`http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054`

Cycle corners

- Look at $\sigma = [F, R^{-1}]$, $\tau = D$



3 cycle of corners

- Expect $[[F, R^{-1}], D] = (FR^{-1}F^{-1}R)D(R^{-1}FRF^{-1})D^{-1}$ is a 3-cycle of corners

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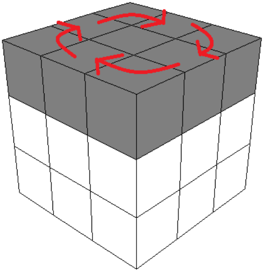
Edge/corner parity

When solving edges first

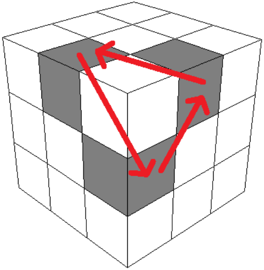
- May need to swap two edges (odd perm)
- Can't do this with 3-cycles (always even)



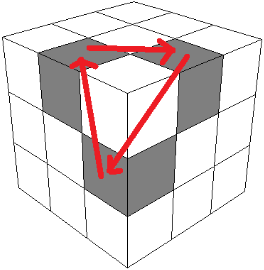
- U cycles 4 edges (and 4 corners)
- Changes parity of edges

Orient edges

- Use two different 3-cycles



$FR^{-1}FR$



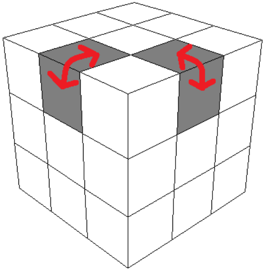
$U^{-1}RUR^{-1}$

- $(FR^{-1}FR)(U^{-1}RUR^{-1})$ flips 2 edges

Pictures based on screenshots from pCubes program available at
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054

Orient edges

- Use two different 3-cycles



Flipping 2 edges

- $(FR^{-1}FR)(U^{-1}RUR^{-1})$ flips 2 edges

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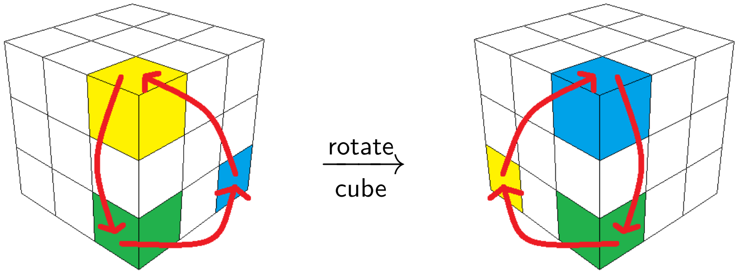
Orient corners

- Use two different 3-cycles



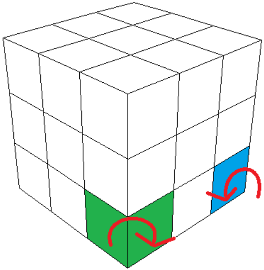
$[[F, R^{-1}], D^{-1}]$ $[[F, R^{-1}], D]$

- $[[F, R^{-1}], D^{-1}][[R, D^{-1}], F]$ twists 2 corners

Pictures based on screenshots from pCubes program available at
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054

Orient corners

- Use two different 3-cycles



Twisting corners in opposite directions

- $[[F, R^{-1}], D^{-1}][[R, D^{-1}], F]$ twists 2 corners

Pictures based on screenshots from pCubes program available at
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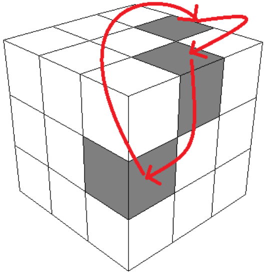
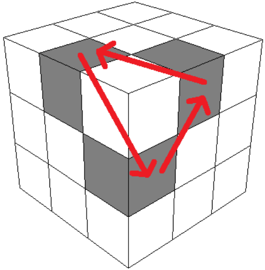
Conjugation

- How to solve far apart pieces?



$\rightarrow$



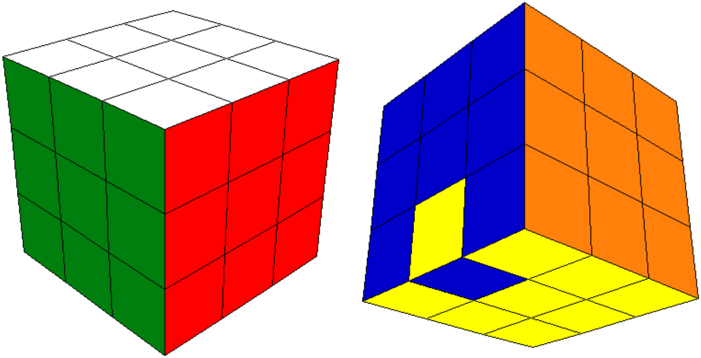
Conjugate by U

- Use a set-up move to conjugate

Pictures based on screenshots from pCubes program available at
`http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054`

Flipping one edge

- Flipping one edge is impossible



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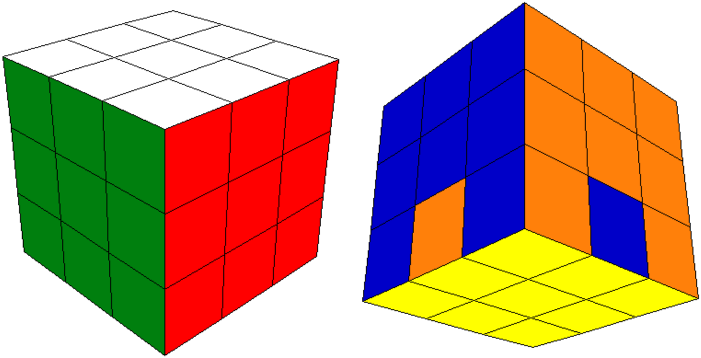
Swapping two edges

- Swapping two edges is impossible



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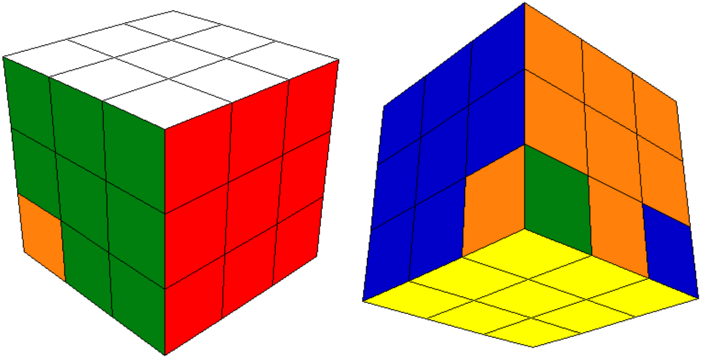
Swapping two edges

- Swapping two corners is impossible



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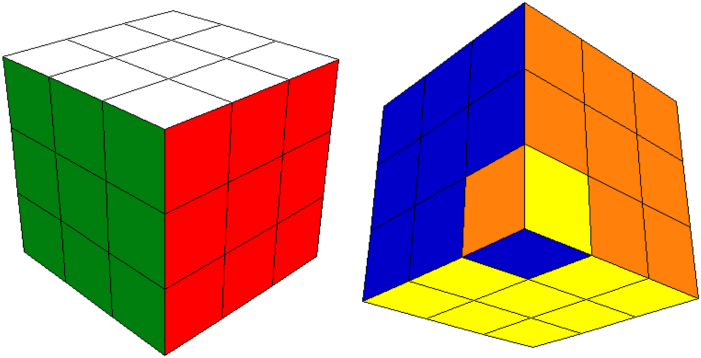
Rotating one corner

- Rotating one corner is impossible



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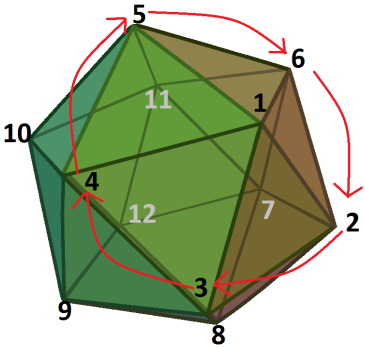
Number of positions

- Place corners in $8!$ ways
- Place edges in $12!/2$ ways
- Orient edges in 2^{11} ways
- Orient corners in 3^7 ways

Total number of positions

$$8! \times 12!/2 \times 2^{11} \times 3^7$$
$$= 43\,252\,003\,274\,489\,856\,000$$

M_{12} from a puzzle

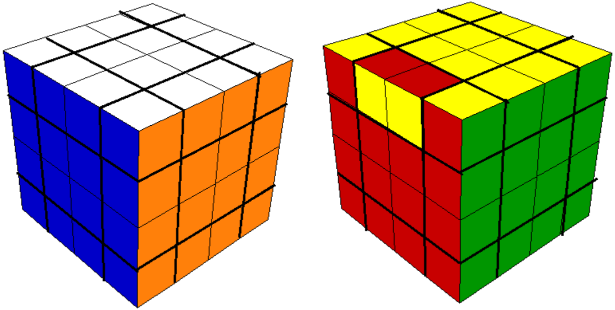


f_1

- $f_1 = (2\ 3\ 4\ 5\ 6) \in S_{12}$
- $M_{12} = \{\, f_i f_j^{-1} \mid 1 \leq i, j \leq 12 \,\}$

$4 \times 4 \times 4$ parity

- Solve by reducing to a $3 \times 3 \times 3$
- Can get one flipped edge!



- Solution: rotating middle layer is an *odd* permutation

Pictures based on screenshots from pCubes program available at
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=27054

Conclusions

- Twisty puzzles involve maths
- Twisty puzzles are fun
- Group theory is actually useful
- You can figure out how to solve a Rubik's cube